AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(1): 839–857.
DOI: 10.3934/math.2025040
Received: 24 October 2024
Revised: 25 December 2024
Accepted: 30 December 2024
Published: 15 January 2025

Research article

Asymptotic behavior of non-autonomous stochastic Boussinesq lattice system

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Abstract: In this paper, we investigate the existence of a random uniform exponential attractor for the non-autonomous stochastic Boussinesq lattice equation with multiplicative white noise and quasi-periodic forces. We first show the existence and uniqueness of the solution of the considered Boussinesq system. Then, we consider the existence of a uniform absorbing random set for a jointly continuous non-autonomous random dynamical system (NRDS) generated by the system, and make an estimate on the tail of solutions. Third, we verify the Lipschitz continuity of the skew-product cocycle defined on the phase space and the symbol space. Finally, we prove the boundedness of the expectation of some random variables and obtain the existence of a random uniform exponential attractor for the considered system.

Keywords: random uniform exponential attractor; Boussinesq lattice equations; multiplicative white noise; quasi-periodic forces
Mathematics Subject Classification: 37L55, 35B41, 35B40

1. Introduction

It is well known that attractors are an important part of describing the long-time asymptotic behavior of infinite-dimensional dynamical systems, such as global attractors, pullback or uniform attractors for deterministic autonomous and non-autonomous dynamical systems, see [1–6]. Random attractor, random pullback or uniform attractors for autonomous and non-autonomous random dynamical systems; see [7–19] and the references therein. However, the dimension of the attractor may be infinite. This means that the asymptotic behavior of the dynamical systems may not be described with finite independent parameters. Moreover, the rate at which the attractor attracts trajectories may be very slow, so that the attractor may be unstable under some small perturbations, this brings some difficulties to practical application and numerical simulations. For these reasons, Eden et al. in [20] introduced the concept of the exponential attractor, which is a compact and positive invariant set with a finite fractal dimension attracts any trajectories exponentially for deterministic autonomous dynamical systems. Since then, the concept of an exponential attractor extended to deterministic non-autonomous dynamical systems, autonomous and non-autonomous random dynamical systems (NRDS), such as exponential attractors, pullback or uniform exponential attractors (see [21–27]), random exponential attractors and random pullback exponential attractors (see [28–32]).

Recently, Han and Zhou in [33] defined the random uniform exponential attractor for NRDS and established the existence criterion of the random uniform exponential attractor for a joint continuous NRDS by introducing a skew-product cocycle on the extended space and applied it to the non-autonomous stochastic first-order lattice system and FitzHugh-Nagumo lattice system with quasi-periodic forces and multiplicative noise. Using this criterion, the random uniform exponential attractor for the non-autonomous stochastic Schrödinger lattice system and discrete long wave-short wave resonance system in [34, 35] is obtained, respectively.

In this paper, we consider the existence of a random uniform exponential attractor for the following non-autonomous stochastic Boussinesq lattice system with quasi-periodic forces and multiplicative white noise:

$$\begin{cases} \ddot{u}_j + \delta\dot{u}_j + \alpha(Au)_j + \beta(Bu)_j + \lambda u_j - \frac{k}{3}(D(D^*u)^3)_j = f_j(\widetilde{\sigma}(t)) + au_j \circ \dot{W}, \\ u_j(0) = u_{j,0}, \dot{u}_j(0) = u_{1j,0}, \end{cases} \quad t > 0, \tag{1.1}$$

where $u_j = u_j(t) \in \mathbb{R}, j = (j_1, j_2, \cdots, j_N) \in \mathbb{Z}^N$, ($\mathbb{R}$ and $\mathbb{Z}$ are the sets of real and integer numbers); $\alpha, \delta, \lambda, k, a$ are positive constants, $\beta \in \mathbb{R}$; $\mathbb{T}^m$ is the m- dimensional torus, $\widetilde{\sigma}(t) = (\mathbf{x}t + \sigma)mod(\mathbb{T}^m), \sigma \in \mathbb{T}^m, t \in \mathbb{R}, \mathbf{x} = (x_1, \cdots, x_m) \in \mathbb{R}^m$ is a fixed vector, and $x_1, \cdots, x_m$ are rationally independent numbers; $f(\widetilde{\sigma}) = (f_j(\widetilde{\sigma}))_{j\in\mathbb{Z}^N} \in C(\mathbb{T}^m, l^2)$; $W(t)$ is a two-sided real-valued Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}, \mathcal{F}$ is the Borel σ-algebra on Ω generated by the compact open topology, and $\mathbb{P}$ is the Wiener measure on $(\Omega, \mathcal{F})$. A, B, D, and D^* are all linear coupled operators; "$\circ$" means the sense of Stratonovich in the stochastic term.

The Boussinesq equation is one of the mathematical models describing wave propagation, which is widely used in the fields of ocean engineering, coastal protection, and marine resources development, such as wave motion, swell, and tide, and is also used in physics and mechanics, such as nonlinear elastic beam systems, thermomechanical phase transitions, and some Hamiltonian mechanics.

For the autonomous lattice dynamical system (1.1) without multiplicative white noise and quasi-periodic forces ($j \in \mathbb{Z}, f_j(\widetilde{\sigma}(t)) = f_j \in \mathbb{R}, a = 0$), Abdallah in [1] obtained the existence and upper semi-continuity of the global attractor, Zhao and Zhou in [25] proved the existence of the exponential attractor, then, they in [26] obtained the existence of the pullback and uniform exponential attractor for non-autonomous Boussinesq lattice system (1.1)

$$j \in \mathbb{Z}, \alpha = \alpha_j(t), \beta = \beta_j(t), f_j(\widetilde{\sigma}(t)) = f_j \in \mathbb{R}, a = 0$$

and further proved the existence of the random attractor for the non-autonomous stochastic Boussinesq lattice system in [17]. As we are aware, there are no results on the random uniform exponential attractor for the non-autonomous stochastic Boussinesq lattice system with quasi-periodic forces and multiplicative white noise. Motivated by [33–35], we will consider the existence of a random uniform exponential attractor for the system (1.1). The time-dependent external force term of the system (1.1) is $f_j(\widetilde{\sigma})$, where the time symbol $\widetilde{\sigma}(t) = (\mathbf{x}t + \sigma)mod(\mathbb{T}^m) \in \mathbb{T}^m$ on the finite-dimensional torus $\mathbb{T}^m$ is taken as the parameter. Thus, the solutions of this system (1.1) generates a NRDS, denoted $\{\Phi(t,\omega,\sigma)\}_{t\geq 0,\omega\in\Omega,\sigma\in\mathbb{T}^m}$, which can be regarded as a family of autonomous random dynamical system with the parameter σ. We investigate that the random uniform exponential attractor is a family of single-parameter random sets, which involve three properties: random compactness, finite fractal dimensionality, and uniform exponential attraction.

This paper is organized as follows: In Section 2, we introduce some basic concepts and assumptions of the coefficients and external force term of the system (1.1). In Section 3, we apply the existence criterion of a random uniform exponential attractor in [33] to the considered system (1.1). In Section 4, we make conclusions and discussion.

2. Preliminaries

In this section, we recall some concepts that can be obtained directly from [8, 12, 33–35], and make some assumptions about a and $f_j(\widetilde{\sigma}(t))$.

Let $l^2 = \{u = (u_j)_{j\in\mathbb{Z}^N} : j = (j_1, j_2, \cdots, j_N) \in \mathbb{Z}^N,\ u_j \in \mathbb{R},\ \sum_{j\in\mathbb{Z}^N} u_j^2 < \infty\}$ be a Hilbert space with the inner product and norm are defined as:

$$(u,v) = \sum_{j\in\mathbb{Z}^N} u_j v_j, \quad \|u\|^2 = (u,u) = \sum_{j\in\mathbb{Z}^N} |u_j|^2, \ \ \forall u,v \in l^2.$$

Define the linear operators $A, B, D, D^* : l^2 \to l^2$ as follows:

$$A = A_1 + A_2 + \cdots + A_N, \quad B = B_1 + B_2 + \cdots + B_N,$$

for all $u = (u_j)_{j\in\mathbb{Z}^N} \in l^2,\ j = (j_1, j_2, \cdots, j_N) \in \mathbb{Z}^N,\ i = 1, 2, \cdots, N$,

$$\begin{aligned}
(A_i u)_j &= u_{(j_1,j_2,\cdots,j_{i-1},j_i+2,j_{i+1},\cdots,j_N)} - 4u_{(j_1,j_2,\cdots,j_{i-1},j_i+1,j_{i+1},\cdots,j_N)} \\
&\quad + 6u_{(j_1,j_2,\cdots,j_{i-1},j_i,j_{i+1},\cdots,j_N)} - 4u_{(j_1,j_2,\cdots,j_{i-1},j_i-1,j_{i+1},\cdots,j_N)} + u_{(j_1,j_2,\cdots,j_{i-1},j_i-2,j_{i+1},\cdots,j_N)}, \\
(B_i u)_j &= u_{(j_1,j_2,\cdots,j_{i-1},j_i+1,j_{i+1},\cdots,j_N)} - 2u_{(j_1,j_2,\cdots,j_{i-1},j_i,j_{i+1},\cdots,j_N)} + u_{(j_1,j_2,\cdots,j_{i-1},j_i-1,j_{i+1},\cdots,j_N)}, \\
(D_i u)_j &= u_{(j_1,j_2,\cdots,j_{i-1},j_i+1,j_{i+1},\cdots,j_N)} - u_{(j_1,j_2,\cdots,j_{i-1},j_i,j_{i+1},\cdots,j_N)}, \\
(D_i^* u)_j &= u_{(j_1,j_2,\cdots,j_{i-1},j_i,j_{i+1},\cdots,j_N)} - u_{(j_1,j_2,\cdots,j_{i-1},j_i-1,j_{i+1},\cdots,j_N)}.
\end{aligned}$$

Then $A_i = B_i^2,\ B_i = D_i D_i^* = D_i^* D_i$, where D_i^* is the conjugate operator of D_i, this is, for all $u = (u_j)_{j\in\mathbb{Z}^N}, v = (v_j)_{j\in\mathbb{Z}^N} \in l^2,\ i = 1, 2, \cdots, N$, then

$$(D_i u, v) = -(u, D_i^* v), \ \ (B_i u, v) = -(D_i u, D_i v), \ \ (A_i u, v) = (B_i u, B_i v).$$

Endowed with the inner products and norms on l^2 as: for any $u = (u_j)_{j\in\mathbb{Z}^N}, v = (v_j)_{j\in\mathbb{Z}^N} \in l^2$,

$$(u,v)_\lambda = \lambda \sum_{j\in\mathbb{Z}^N} u_j v_j, \ \ \|u\|_\lambda = (u,u)_\lambda = (\lambda \sum_{j\in\mathbb{Z}^N} u_j^2)^{\frac{1}{2}}.$$

Let $l_\lambda^2 = (l^2, (\cdot,\cdot)_\lambda, \|\cdot\|_\lambda)$ and $E = l_\lambda^2 \times l^2$, it is obvious that the norm $\|\cdot\|_\lambda$ in l_λ^2 and the usual norm $\|\cdot\|$ in l^2 are equivalent, and E is a Hilbert space with the inner product $(\cdot,\cdot)_E$ and the norm $\|\cdot\|_E$: for $\varphi^{(i)} = (u^{(i)}, v^{(i)}) = (u_j^{(i)}, v_j^{(i)})_{j\in\mathbb{Z}^N} \in E,\ i = 1, 2,$

$$(\varphi^{(1)}, \varphi^{(2)})_E = (u^{(1)}, u^{(2)})_\lambda + (v^{(1)}, v^{(2)}), \ \ \|\varphi\|_E^2 = (\varphi,\varphi)_E = \|u\|_\lambda^2 + \|v\|^2,$$

and $\mathcal{B}(E)$ is the Borel $\sigma-$algebra of E.

Let $\mathbb{T}^m$ be the m–dimensional torus

$$\mathbb{T}^m = \{\sigma = (\sigma_1, \cdots, \sigma_m) : \sigma_l \in [-\pi, \pi], \forall l = 1, \cdots, m\},$$

where $(\sigma_1, \cdots, \sigma_{l-1}, -\pi, \sigma_{l+1}, \cdots, \sigma_m) \sim (\sigma_1, \cdots, \sigma_{l-1}, \pi, \sigma_{l+1}, \cdots, \sigma_m), \forall l = 1, \cdots, m$, and the norm in $\mathbb{T}^m$ is given by

$$\|\sigma\|_{\mathbb{T}^m} = (\sum_{l=1}^{m} \sigma_l^2)^{\frac{1}{2}}, \ \ \forall \sigma = (\sigma_1, \cdots, \sigma_m) \in \mathbb{T}^m.$$

Let $\mathbf{x} = (x_1, \cdots, x_m) \in \mathbb{R}^m$ be a fixed vector such that $x_1, \cdots, x_m$ are rationally independent. For $t \in \mathbb{R}$, define

$$\vartheta_t \sigma = (\mathbf{x}t + \sigma) mod(\mathbb{T}^m), \ \sigma \in \mathbb{T}^m,$$

then $\{\vartheta_t\}_{t\in\mathbb{R}}$ is a translation group on $\mathbb{T}^m$ with $\vartheta_t \mathbb{T}^m = \mathbb{T}^m$ and $(t, \sigma) \to \vartheta_t \sigma$ is continuous for $\forall t \in \mathbb{R}$, $\mathcal{B}(\mathbb{T}^m)$ denotes the Borel $\sigma-$algebra of $\mathbb{T}^m$.

Define the extended space $\mathbb{E} = \mathbb{T}^m \times E$ with norm

$$\|\Upsilon\|_{\mathbb{E}} = (\|\sigma\|_{\mathbb{T}^m}^2 + \|\varphi\|_E^2)^{\frac{1}{2}}, \ \ \forall \Upsilon = \{\sigma\} \times \{\varphi\} \in \mathbb{E}, \ \ \varphi = (u, v) \in E,$$

and the Borel $\sigma-$algebra $\mathcal{B}(\mathbb{E})$. Norm $\|\cdot\|_{\mathbb{E}}$ induces a metric.

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t \omega\}_{t\in\mathbb{R}})$ be an ergodic metric dynamical system [8]. The two groups $\{\theta_t\}_{t\in\mathbb{R}}$ and $\{\vartheta_t\}_{t\in\mathbb{R}}$ are said to be base flows [12]. Hereafter, for simplicity, we identify "a.e. $\omega \in \Omega$" as "$\omega \in \Omega$ ".

Definition 2.1. *[33] A continuous NRDS on E with base flows $\{\theta_t\}_{t\in\mathbb{R}}$ on Ω and $\{\vartheta_t\}_{t\in\mathbb{R}}$ on $\mathbb{T}^m$ is defined as a mapping $\varphi(t, \omega, \sigma, u) : \mathbb{R}^+ \times \Omega \times \mathbb{T}^m \times E \to E$ satisfying*
(i) φ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(\mathbb{T}^m) \times \mathcal{B}(E), \mathcal{B}(E))$ measurable;
(ii) $\varphi(0, \omega, \sigma, \cdot)$ is the identity on E for each $\sigma \in \mathbb{T}^m$ and $\omega \in \Omega$;
(iii) $\forall t, s \geq 0, \omega \in \Omega, \sigma \in \mathbb{T}^m$, $\varphi(t + s, \omega, \sigma, \cdot) = \Phi(t, \theta_s \omega, \vartheta_s \sigma, \cdot) \circ \varphi(s, \omega, \sigma, \cdot)$;
(iv) $\forall t \in \mathbb{R}^+, \omega \in \Omega, \sigma \in \mathbb{T}^m$, $\varphi(t, \omega, \sigma, \cdot)$ is continuous.

A NRDS is said to be jointly continuous in $\mathbb{T}^m$ and E if the mapping $\varphi(t, \omega, \cdot, \cdot)$ is continuous for each $t \in \mathbb{R}^+$ and $\omega \in \Omega$.

Definition 2.2. *[8] A (autonomous) random set $D = D(\cdot)$ in E is a multi-valued map $D : \Omega \to 2^E \backslash \emptyset$ such that for each $u \in E$, the map $\omega \to dist_E(u, D(\omega))$ (distance in E between u and $D(\omega)$) is measurable. It is said that the (autonomous) random set is bounded (respectively, closed or compact) if $D(\omega)$ is bounded (respectively, closed or compact) for $\omega \in \Omega$.*

Given two random sets D_1, D_2, we write $D_1 \subseteq D_2$ if $D_1(\omega) \subseteq D_2(\omega)$ for all $\omega \in \Omega$.

Definition 2.3. *[8] A random set $D(\cdot)$ in E is called tempered with respect to $\{\theta_t\}_{t\in\mathbb{R}}$, if for $\omega \in \Omega, \gamma > 0$, $\lim_{t\to+\infty} e^{-\gamma t}\|D(\theta_{-t}\omega)\|_E = 0$, where $\|D(\omega)\|_E = \sup_{x\in D(\omega)} \|x\|_E$.*

Let $\mathcal{D} = \mathcal{D}(E)$ be the collection of all tempered bounded random sets of E.

Definition 2.4. *[33] A random set $\{\mathcal{M}(\omega)\}_{\omega\in\Omega}$ in E is called a $\mathcal{D}(E)-$random uniform exponential attractor for the continuous NRDS $\{\varphi(t,\omega,\sigma)\}_{t\geq 0,\omega\in\Omega,\sigma\in\mathbb{T}^m}$ on E with base flows $\{\theta_t\}_{t\in\mathbb{R}}$ and $\{\vartheta_t\}_{t\in\mathbb{R}}$ if there is a set of full measure $\widetilde{\Omega}\in\mathcal{F}$ such that for every $\omega\in\widetilde{\Omega}$, it holds that*
(i) $\mathcal{M}(\omega)$ is a compact set;
(ii) There exists a random variable $\xi_\omega<\infty$ such that $dim_f\mathcal{M}(\omega)\leq\xi_\omega$, where $dim_f\mathcal{M}(\omega)$ is the fractal dimension of $\mathcal{M}(\omega)$;
(iii) There exists a constant $b>0$ such that for any $B\in\mathcal{D}(E)$, there exist random variables $\bar{t}_B(\omega)\geq 0$, $\overline{Q}(\omega,\|B\|_E)>0$ satisfying $\sup_{\sigma\in\mathbb{T}^m} dist_E(\varphi(t,\theta_{-t}\omega,\vartheta_{-t}\sigma)B(\theta_{-t}\omega),\mathcal{M}(\omega))\leq\overline{Q}(\omega,\|B\|_E)e^{-bt},\ t\geq\bar{t}_B(\omega)$.

For the given jointly continuous NRDS φ, introduce a mapping $\pi:\mathbb{R}^+\times\Omega\times\mathbb{E}\to\mathbb{E}$ by

$$\pi(t,\omega,\{\sigma\}\times\{\mathbb{E}\})=\{\vartheta_t\sigma\}\times\{\varphi(t,\omega,\sigma,x)\}.$$

Then the mapping π satisfying: (i) π is $(\mathcal{B}(\mathbb{R}^+)\times\mathcal{F}\times\mathcal{B}(\mathbb{E}),\mathcal{B}(\mathbb{E}))-$measurable; (ii) $\pi(0,\omega,\Upsilon)=\Upsilon,\forall\omega\in\Omega,\Upsilon\in\mathbb{E}$; (iii) the cocycle property $\pi(t+s,\omega,\Upsilon)=\pi(t,\theta_s\omega,\pi(s,\omega,\Upsilon)),\forall t,s\geq 0,\omega\in\Omega,\Upsilon\in\mathbb{E}$.

The π is called the skew-product cocycle on the extended space $\mathbb{E}$. Note that π is continuous; that is, the mapping $\Upsilon\to\pi(\cdot,\cdot,\Upsilon)$ is continuous in $\mathbb{E}$ if and only if φ is jointly continuous in $\mathbb{T}^m$ and E. Let $\mathcal{D}_{\mathbb{E}}=\{\mathbb{B}:\mathbb{B}=\mathbb{T}^m\times B=\{\mathbb{T}^m\times B(\omega)\}_{\omega\in\Omega}\ and\ B\in\mathcal{D}(E)\}$ be some class of random sets in $\mathbb{E}$.

In order to study the existence of a uniform exponential attractor, we need to make the following assumptions:

(H1) $f(\sigma)=(f_j(\sigma))_{j\in\mathbb{Z}^N}\in C(\mathbb{T}^m,l^2)$, that is, for any $f(\sigma)\in C(\mathbb{T}^m,l^2)$, $\|f(\sigma)\|_C^2=\max_{\sigma\in\mathbb{T}^m}\|f(\sigma)\|^2<\infty$, and for any $\epsilon>0$, there exists $I(\epsilon)\in\mathbb{N}$ such that $\max_{\sigma\in\mathbb{T}^m}\sum_{\|j\|>I(\epsilon)}f_j^2(\sigma)<\epsilon$, where $\|j\|=\max\{|j_i|,i=1,\cdots,N\}$;

(H2) There exists $d=(d_j)_{j\in\mathbb{Z}^N}\in l^2$ and $d_j>0$ such that

$$|f_j(\sigma_1)-f_j(\sigma_2)|\leq d_j\|\sigma_1-\sigma_2\|_{\mathbb{T}^m};$$

(H3) $\frac{\varepsilon}{2}-(\frac{2a}{\sqrt{\pi\delta}}+\frac{12\sqrt{2}\alpha+4\beta}{\sqrt{\lambda}}+\frac{2a\varepsilon}{\sqrt{\lambda\pi\delta}}+\frac{a^2}{2\delta\sqrt{\lambda}})>0$, where $\varepsilon=\frac{\lambda\delta}{2\lambda+\delta^2}>\frac{48\sqrt{2}\alpha+16|\beta|}{\sqrt{\lambda}}$.

3. Random uniform exponential attractor for non-autonomous stochastic Boussinesq lattice system

The system (1.1) can be rewritten in the following vector form:

$$\begin{cases} \ddot{u}+\delta\dot{u}+\alpha Au+\beta Bu+\lambda u-\frac{k}{3}D(D^*u)^3=f(\widetilde{\sigma}(t))+au\circ\dot{W}(t), & t>0,\\ u(0)=u_0,\ \dot{u}(0)=u_{1,0}, \end{cases}\tag{3.1}$$

where

$$u=(u_j(t))_{j\in\mathbb{Z}^N},Au=((Au(t))_j)_{j\in\mathbb{Z}^N},Bu=((Bu(t))_j)_{j\in\mathbb{Z}^N},$$

$$D^*u=((D^*u(t))_j)_{j\in\mathbb{Z}^N},f(\widetilde{\sigma}(t))=(f_j(\widetilde{\sigma}(t)))_{j\in\mathbb{Z}^N}.$$

Let $z(\theta_t\omega)=-\delta\int_{-\infty}^0 e^{\delta s}\theta_t\omega(s)ds,\ t\in\mathbb{R},\omega\in\Omega$ be the Ornstein–Uhlenbeck stationary processes, and a stationary solution of Itô equation $dz(\theta_t\omega)+z(\theta_t\omega)dt=dW(t)$, where $W(t)=w(t)$. It follows from [8, 12] that $z(\theta_t\omega)$ is continuous in t and has the following properties:

$$\begin{cases} \lim\limits_{t\to\pm\infty} \frac{|z(\theta_t\omega)|}{t} = \lim\limits_{t\to\pm\infty} \frac{\int_0^t z(\theta_s\omega)ds}{t} = \lim\limits_{t\to\pm\infty} e^{-\epsilon|t|}|z(\theta_t\omega)| = 0, \ \forall\epsilon > 0, \\ \lim\limits_{t\to\pm\infty} \frac{\int_0^t |z(\theta_s\omega)|^r ds}{t} = \mathbf{E}[|z(\omega)|^r] = \frac{\Gamma(\frac{1+r}{2})}{\sqrt{\pi\delta^r}}, \ \forall r > 0, \\ \mathbf{E}[e^{\epsilon\int_\tau^{\tau+t}|z(\theta_s\omega)|ds}] \le e^{\frac{\epsilon}{\sqrt{\delta}}t}, \ 0 < \epsilon^2 \le \delta^3, t \ge 0, \\ \mathbf{E}[e^{\epsilon\int_\tau^{\tau+t}|z(\theta_s\omega)|^2 ds}] \le e^{\frac{\epsilon}{\delta}t}, \ 0 < 2\epsilon \le 1, t \ge 0, \end{cases} \tag{3.2}$$

where $\Gamma(\cdot)$ is the Gamma function, "$\mathbf{E}$" denotes the expection.

Let $v = \dot{u} + \varepsilon u - auz(\theta_t\omega)$ and $\varphi = (u, v)^T$, where u is the solution of the system (3.1). Then the system (3.1) is equivalent to the following random system without the white noise term:

$$\begin{cases} \dot{\varphi} + \Lambda\varphi = F(\varphi, \theta_t\omega), \\ \varphi_0 = (u_0, v_0)^T = (u_0, u_{1,0} + \varepsilon u_0 - au_0 z(\omega))^T, \end{cases} \quad t > 0 \tag{3.3}$$

where

$$\Lambda\varphi = \begin{pmatrix} \varepsilon u - v \\ \lambda u - \varepsilon(\delta - \varepsilon)u + (\delta - \varepsilon)v \end{pmatrix}, \tag{3.4}$$

$$F(\varphi, \theta_t\omega) = \begin{pmatrix} az(\theta_t\omega)u \\ -\alpha(Au) - \beta(Bu) + \frac{1}{3}k(D(D^*u)^3) + (2a\varepsilon z(\theta_t\omega) - a^2z^2(\theta_t\omega))u - az(\theta_t\omega)v + f(\widetilde{\sigma}(t)) \end{pmatrix}. \tag{3.5}$$

3.1. A jointly continuous NRDs generated by system (3.3)

Lemma 3.1. *Let (H1)–(H3) hold, then*
(i) For all $\omega \in \Omega, t \in [0, T], T > 0, \varphi_0(\omega) \in E$, the system (3.3) has a unique solution $\varphi(\cdot, \omega, \sigma, \varphi_0(\omega)) \in C([0, +\infty), E)$ and the solution φ is measurable in ω;
(ii) Let $\varphi^{(i)}(\cdot, \omega, \sigma, \varphi_0^{(i)}(\omega))$ be the solution of system (3.3) with $\sigma_i \in \mathbb{T}^m$ and $\varphi_0^{(i)}(\omega) \in E, i = 1, 2,$ $T > 0$ is fixed, then there exists a constant $s(T, \omega) > 0$ such that for all $t \in [0, T]$,

$$\|\varphi^{(1)}(t, \omega, \sigma_1, \varphi_0^{(1)}(\omega)) - \varphi^{(2)}(t, \omega, \sigma_1, \varphi_0^{(2)}(\omega))\|_E^2 \le e^{s(T,\omega)t}(\|\varphi_0^{(1)} - \varphi_0^{(2)}\|_E^2 + \|\sigma_1 - \sigma_2\|_{\mathbb{T}^m}^2).$$

Proof. It is easy to verify that for all $\omega \in \Omega$, $F(\varphi, \theta_t\omega)$ is continuous in t and φ. Let Q be a bounded set in E; then there exists a positive constant $L(Q)$ depending on Q, such that for all $\varphi^{(i)} = (u^{(i)}, v^{(i)})^T \in Q$, $i = 1, 2$, $\|\varphi^{(i)}\|_E \le L(Q)$. For all $\omega \in \Omega, t \in [0, T]$, it follows that

$$\begin{aligned} &\|F(\varphi^{(1)}, \theta_t\omega) - F(\varphi^{(2)}, \theta_t\omega)\|_E \\ &\le a|z(\theta_t\omega)|\|u^{(1)} - u^{(2)}\|_\lambda + \alpha\|A(u^{(1)} - u^{(2)})\| + |\beta|\|B(u^{(1)} - u^{(2)})\| + \frac{k}{3}\|D(D^*u^{(1)})^3 - D(D^*u^{(2)})^3\| \\ &+ (2\varepsilon a|z(\theta_t\omega)|\|u^{(1)} - u^{(2)}\| + a^2z^2(\theta_t\omega)\|u^{(1)} - u^{(2)}\| + a|z(\theta_t\omega)|\|v^{(1)} - v^{(2)}\| \\ &\le \left[a\max_{t\in[0,T]}|z(\theta_t\omega)| + \frac{12\sqrt{2}\alpha + 4|\beta| + 2\varepsilon a\max\limits_{t\in[0,T]}|z(\theta_t\omega)| + a^2\max\limits_{t\in[0,T]}z^2(\theta_t\omega)}{\sqrt{\lambda}} + \frac{16k}{\sqrt{\lambda^3}}L^2(Q)\right]\|\varphi^{(1)} - \varphi^{(2)}\|_E. \end{aligned}$$

Thus, F satisfies the local Lipschiz condition in φ; by the standard theory of ordinary differential equations, we obtain that there exists a $T_{max} \le +\infty$ such that the system (3.3) has a unique solution

$\varphi(t) \in C([0, T_{max}), E)$ satisfying that $\limsup_{t\to T_{max}} \|\varphi\|_E = +\infty$ if $T_{max} \leq +\infty$. Next, we prove that this local solution is a global one.

Let $T \in [0, T_{max})$, taking the inner product of (3.3) with φ in E, we have

$$\frac{1}{2}\frac{d}{dt}\|\varphi\|_E^2 + (\Lambda\varphi, \varphi)_E = (F(\varphi, \theta_t\omega), \varphi)_E, \tag{3.6}$$

where

$$(\Lambda\varphi, \varphi)_E \geq \frac{\varepsilon}{2}\|\varphi\|_E^2 + \frac{\delta}{2}\|v\|^2. \tag{3.7}$$

$$\begin{aligned}(F(\varphi, \theta_t\omega), \varphi)_E = {} & (az(\theta_t\omega)u, u)_\lambda - \alpha(Au, v) - \beta(Bu, v) + \frac{1}{3}k(D(D^*u)^3, v) \\ & - az(\theta_t\omega)(v - 2\varepsilon u + az(\theta_t\omega)u, v) + (f(\widetilde{\sigma}(t), v),\end{aligned} \tag{3.8}$$

where

$$\begin{cases} (az(\theta_t\omega)u, u)_\lambda \leq a|z(\theta_t\omega)|\|u\|_\lambda^2, \\ -\alpha(Au, v) - \beta(Bu, v) \leq \frac{6\sqrt{2}\alpha+2|\beta|}{\sqrt{\lambda}}\|\varphi\|_E^2, \\ \frac{1}{3}k(D(D^*u)^3, v) = -\frac{k}{12}\frac{d}{dt}\|(D^*u)^2\|^2 - \frac{k(\varepsilon-az(\theta_t\omega))}{3}\|(D^*u)^2\|^2, \\ -az(\theta_t\omega)(v - 2\varepsilon u + az(\theta_t\omega)u, v) \leq a|z(\theta_t\omega)|\|v\|^2 + \frac{2a\varepsilon|z(\theta_t\omega)|+a^2z^2(\theta_t\omega)}{2\sqrt{\lambda}}\|\varphi\|_E^2, \\ (f(\widetilde{\sigma}(t), v) \leq \frac{1}{2\delta}\|f\|_C^2 + \frac{\delta}{2}\|v\|^2. \end{cases} \tag{3.9}$$

Combining (3.6)–(3.9), we obtain that

$$\frac{d}{dt}(\|\varphi\|_E^2 + \frac{k}{6}\|(D^*u)^2\|^2) \leq (-\varepsilon + \rho(\theta_t\omega))(\|\varphi\|_E^2 + \frac{k}{6}\|(D^*u)^2\|^2) + \frac{1}{\delta}\|f\|_C^2, \tag{3.10}$$

where

$$\rho(\theta_t\omega) = 2a|z(\theta_t\omega)| + \frac{12\sqrt{2}\alpha + 4|\beta|}{\sqrt{\lambda}} + \frac{2a\varepsilon|z(\theta_t\omega)| + a^2z^2(\theta_t\omega)}{\sqrt{\lambda}}. \tag{3.11}$$

Applying Gronwall's inequality in (3.10) over $[0, t](0 \leq t < T_{max})$, we obtain

$$\|\varphi\|_E^2 + \frac{k}{6}\|(D^*u)^2\|^2 \leq e^{\int_0^t(-\varepsilon+\rho(\theta_s\omega))ds}(\|\varphi_0\|_E^2 + \frac{k}{6}\|(D^*u_0)^2\|^2) + \int_0^t e^{\int_l^t(-\varepsilon+\rho(\theta_s\omega))ds}\frac{1}{\delta}\|f\|_C^2 dl. \tag{3.12}$$

We further obtain that

$$\|\varphi\|_E^2 \leq e^{\int_0^T(-\varepsilon+\rho(\theta_s\omega))ds}(\|\varphi_0\|_E^2 + \frac{k}{6}\|(D^*u_0)^2\|^2) + \frac{1}{\delta}\|f\|_C^2 \int_0^T e^{\int_l^T(-\varepsilon+\rho(\theta_s\omega))ds} dl < +\infty. \tag{3.13}$$

Thus, the statement (i) holds.

(ii) Let $\varphi^{(i)}(t, \omega) = (u^{(i)}, v^{(i)})^T = \varphi(t, \omega, \sigma_i, \varphi_0^{(i)}(\omega)),\ i = 1, 2,\ \widetilde{\varphi} = (\widetilde{u}, \widetilde{v})^T = \varphi^{(1)}(t, \omega) - \varphi^{(2)}(t, \omega)$, then

$$\begin{cases} \dot{\widetilde{\varphi}} + \Lambda\widetilde{\varphi} = F(\varphi^{(1)}, \theta_t\omega) - F(\varphi^{(2)}, \theta_t\omega), \\ \widetilde{\varphi}_0 = \varphi_0^{(1)} - \varphi_0^{(2)}, \end{cases} \quad t > 0. \tag{3.14}$$

Taking the inner product of (3.14) with $\widetilde{\varphi}$ in E, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\widetilde{\varphi}\|_E^2 + (\Lambda\widetilde{\varphi}, \widetilde{\varphi})_E = (F(\varphi^{(1)}, \theta_t\omega) - F(\varphi^{(2)}, \theta_t\omega), \widetilde{\varphi})_E, \tag{3.15}$$

where

$$(\Lambda\widetilde{\varphi},\widetilde{\varphi})_E \geq \frac{\varepsilon}{2}\|\widetilde{\varphi}\|_E^2 + \frac{\delta}{2}\|\widetilde{v}\|^2. \tag{3.16}$$

and

$$\begin{aligned}(F(\varphi^{(1)},\theta_t\omega) - F(\varphi^{(2)},\theta_t\omega),\widetilde{\varphi})_E &= (az(\theta_t\omega)\widetilde{u},\widetilde{u})_\lambda + \frac{1}{3}k(D((D^*u^{(1)})^3 - (D^*u^{(2)})^3),\widetilde{v})\\ &- \alpha(A\widetilde{u},\widetilde{v}) - \beta(B\widetilde{u},\widetilde{v}) - az(\theta_t\omega)(\widetilde{v} - 2\varepsilon\widetilde{u} + az(\theta_t\omega)\widetilde{u},\widetilde{v}) + (f(\widetilde{\sigma_1}(t)) - f(\widetilde{\sigma_2}(t)),\widetilde{v}).\end{aligned} \tag{3.17}$$

By (H2), we derive each term on the right-hand side of (3.17),

$$(az(\theta_t\omega)\widetilde{u},\widetilde{u})_\lambda \leq a|z(\theta_t\omega)|\|\widetilde{u}\|_\lambda^2, \tag{3.18}$$

$$\begin{aligned}\frac{1}{3}k(D((D^*u^{(1)})^3 - (D^*u^{(2)})^3),\widetilde{v}) &\leq \frac{4Nk^2}{9\delta}\|(D^*u^{(1)})^3 - (D^*u^{(2)})^3)\|^2 + \frac{\delta}{4}\|\widetilde{v}\|^2\\ &\leq \frac{64N^2k^2}{\delta}(\|u^{(1)}\|^2 + \|u^{(2)}\|^2)^2\|u^{(1)} - u^{(2)}\|^2 + \frac{\delta}{4}\|\widetilde{v}\|^2\\ &\leq \frac{256N^2k^2}{\lambda^3\delta}L^4(Q)\|\widetilde{\varphi}\|_E^2 + \frac{\delta}{4}\|\widetilde{v}\|^2,\end{aligned} \tag{3.19}$$

$$-\alpha(A\widetilde{u},\widetilde{v}) - \beta(B\widetilde{u},\widetilde{v}) \leq \frac{6\sqrt{2}\alpha + 2|\beta|}{\sqrt{\lambda}}\|\widetilde{\varphi}\|_E^2, \tag{3.20}$$

$$-az(\theta_t\omega)(\widetilde{v} - 2\varepsilon\widetilde{u} + az(\theta_t\omega)\widetilde{u},\widetilde{v}) \leq a|z(\theta_t\omega)|\|\widetilde{v}\|^2 + \frac{2a\varepsilon|z(\theta_t\omega)| + a^2z^2(\theta_t\omega)}{2\sqrt{\lambda}}\|\widetilde{\varphi}\|_E^2, \tag{3.21}$$

$$(f(\widetilde{\sigma_1}(t)) - f(\widetilde{\sigma_2}(t)),\widetilde{v}) \leq \frac{\|d\|^2}{\delta}\|\sigma_1 - \sigma_2\|_{\mathbb{T}^m}^2 + \frac{\delta}{4}\|\widetilde{v}\|^2. \tag{3.22}$$

Summing up (3.18)–(3.22) and combining (3.15)–(3.17), we obtain

$$\frac{d}{dt}\|\widetilde{\varphi}\|_E^2 \leq (-\varepsilon + \rho(\theta_t\omega) + \frac{512N^2k^2}{\lambda^3\delta}L^4(Q))\|\widetilde{\varphi}\|_E^2 + \frac{2\|d\|^2}{\delta}\|\sigma_1 - \sigma_2\|_{\mathbb{T}^m}^2,$$

which implies

$$\frac{d}{dt}(\|\widetilde{\varphi}\|_E^2 + \|\sigma_1 - \sigma_2\|_{\mathbb{T}^m}^2) \leq (\varepsilon + \rho(\theta_t\omega) + \frac{512N^2k^2}{\lambda^3\delta}L^4(Q) + \frac{2\|d\|^2}{\delta})(\|\widetilde{\varphi}\|_E^2 + \|\sigma_1 - \sigma_2\|_{\mathbb{T}^m}^2). \tag{3.23}$$

Applying Gronwall's inequality in (3.22) over $[0,t](0 \leq t < T)$, then

$$\|\widetilde{\varphi}\|_E^2 \leq e^{\int_0^t(\varepsilon+\rho(\theta_s\omega)+\frac{512N^2k^2}{\lambda^3\delta}L^4(Q)+\frac{2\|d\|^2}{\delta})ds}(\|\widetilde{\varphi_0}\|_E^2 + \|\sigma_1 - \sigma_2\|_{\mathbb{T}^m}^2).$$

Let $s(T,\omega,Q) = \varepsilon + \max\limits_{0\leq s\leq T}\rho(\theta_s\omega) + \frac{512N^2k^2}{\lambda^3\delta}L^4(Q) + \frac{2\|d\|^2}{\delta}$.

Thus, the statement (ii) holds. The proof is completed. □

From Lemma 3.1, we know that the solution $\varphi(t,\omega,\sigma,\varphi_0(\omega))$ of the system (3.3) satisfies cocycle definition. Thus, we can define a jointly continuous NRDS $\Phi : \mathbb{R}^+ \times \Omega \times \mathbb{T}^m \times E \to E$,

$$\Phi(t,\omega,\sigma,\varphi_0) = \Phi(t,\omega,\sigma)\varphi_0 = \varphi(t,\omega,\sigma,\varphi_0(\omega)), \quad t > 0.$$

3.2. Uniformly absorbing set

Lemma 3.2. *Assume that (H1)–(H3) hold. Then for all $\omega \in \Omega$ and $D \in \mathcal{D}(E)$, there exist a $T_D(\omega) \geq 0$ and a tempered random variable $R_0(\omega)$, such that $\forall t \geq T_D(\omega)$,*

$$\|\varphi(t, \theta_{-t}\omega, \sigma, \varphi_0(\theta_{-t}\omega))\|_E^2 + \frac{\varepsilon}{2}\int_0^t e^{\int_l^t(-\frac{\varepsilon}{2}+\rho(\theta_s\omega))ds}(\|\varphi(l, \theta_{-t}\omega, \sigma, \varphi_0(\theta_{-t}\omega))\|_E^2 + \frac{k}{6}\|(D^*u)^2\|^2)dl \leq R_0^2(\omega)$$

holds uniformly for $\sigma \in \mathbb{T}^m$.

Proof. Similar to the derivation of (3.10), then

$$\frac{d}{dt}(\|\varphi\|_E^2 + \frac{k}{6}\|(D^*u)^2\|^2) + \frac{\varepsilon}{2}\|\varphi\|_E^2 \leq (-\frac{\varepsilon}{2} + \rho(\theta_t\omega))(\|\varphi\|_E^2 + \frac{k}{6}\|(D^*u)^2\|^2) + \frac{1}{\delta}\|f\|_C^2. \tag{3.24}$$

Using Gronwall's inequality in (3.24) over $[0, t]$, we obtain

$$\begin{aligned}
&\|\varphi\|_E^2 + \frac{k}{6}\|(D^*u)^2\|^2 + \frac{\varepsilon}{2}\int_0^t e^{-\int_l^t(\frac{\varepsilon}{2}-\rho(\theta_s\omega))ds}\|\varphi(l, \omega, \sigma, \varphi_0(\omega))\|_E^2 dl \\
&\leq e^{-\int_0^t(\frac{\varepsilon}{2}-\rho(\theta_s\omega))ds}(\|\varphi_0\|_E^2 + \frac{k}{6}\|(D^*u_0)^2\|^2) + \frac{1}{\delta}\|f\|_C^2 \int_0^t e^{-\int_l^t(\frac{\varepsilon}{2}-\rho(\theta_s\omega))ds}dl.
\end{aligned}$$

For $t \geq 0$, replacing ω by $\theta_{-t}\omega$, we have

$$\begin{aligned}
&\|\varphi(t, \theta_{-t}\omega, \sigma, \varphi_0(\theta_{-t}\omega))\|_E^2 + \frac{\varepsilon}{2}\int_0^t e^{-\int_l^t(\frac{\varepsilon}{2}-\rho(\theta_{s-t}\omega))ds}\|\varphi(l, \theta_{-t}\omega, \sigma, \varphi_0(\theta_{-t}\omega))\|_E^2 dl \\
&\leq e^{-\int_{-t}^0(\frac{\varepsilon}{2}-\rho(\theta_s\omega))ds}(\|\varphi_0\|_E^2 + \frac{k}{6}\|(D^*u_0)^2\|^2) + \frac{1}{2}R_0^2(\omega),
\end{aligned} \tag{3.25}$$

where

$$R_0^2(\omega) = \frac{2}{\delta}\|f\|_C^2 \int_{-\infty}^0 e^{-\int_l^0(\frac{\varepsilon}{2}-\rho(\theta_s\omega))ds}dl. \tag{3.26}$$

By (H3), we have

$$\lim_{t\to+\infty} e^{-\int_{-t}^0(\frac{\varepsilon}{2}-\rho(\theta_s\omega))ds}(\|\varphi_0\|_E^2 + \frac{k}{6}\|(D^*u_0)^2\|^2) = 0.$$

Since $\rho(\omega)$ is tempered with respect to $\{\theta_t\}_{t\in\mathbb{R}}$, we know that $R_0^2(\omega)(< \infty)$ is also tempered. The proof is completed. □

According to Lemma 3.2, we can obtain that the random set

$$B_0 = \{B_0(\omega) = \{\varphi \in E : \|\varphi\| \leq R_0(\omega)\}, \omega \in \Omega\} \in \mathcal{D}$$

is a uniformly (with respect to $\sigma \in \mathbb{T}^m$) bounded closed absorbing set for Φ, then there exists a $T_{B_0(\omega)} \geq 0$ such that $\Phi(t, \theta_{-t}\omega, \sigma, B_0(\theta_{-t}\omega)) \subseteq B_0(\omega)$ for any $t \geq T_{B_0(\omega)}$.

3.3. Tail estimation of solutions

Choosing an increasing smooth function $\mu \in \mathbf{C}'(\mathbb{R}_+, [0,1])$, such that

$$\begin{cases} \mu(s) = 0, & 0 \le s \le 1, \\ 0 \le \mu(s) \le 1, & 1 \le s \le 2, \\ \mu(s) = 1, & 2 \le s < +\infty, \end{cases} \quad |\mu'(s)| \le \mu_0, \ \forall s \in \mathbb{R}^+, \ \mu_0 > 0.$$

Lemma 3.3. *Assume that (H1) and (H3) hold, and let $\varphi(t,\omega,\sigma,\varphi_0(\omega))$ be the solution of (3.3) with $(\sigma,\varphi_0(\omega)) \in \mathbb{T}^m \times B_0(\omega)$. Then for every $\omega \in \Omega$, $J \in \mathbb{N}$, and for any $\nu > 0$, there exists $T_\nu(\omega) > 0$ such that*

$$\sum_{j\in\mathbb{Z}^N} \mu(\frac{\|j\|}{J}) \|\varphi_j(t,\theta_{-t}\omega,\sigma,\varphi_0(\theta_{-t}\omega))\|_E^2 \le \nu + c_2(\frac{1}{J} + \gamma_{1,J}) R_0^2(\omega), \ t > T_\nu(\omega), \tag{3.27}$$

where c_2 and $\gamma_{1,J}$ are given in the proof below.

Proof. Taking the inner product of (3.3) with $\phi(t) = (x,y)^T = (x_j,y_j)^T_{j\in\mathbb{Z}^N} = (\mu(\frac{\|j\|}{J})\varphi_j(t))_{j\in\mathbb{Z}^N}$ in E, where $x_j = \mu(\frac{\|j\|}{J})u_j, \ y_j = \mu(\frac{\|j\|}{J})v_j$, we have

$$\frac{1}{2}\frac{d}{dt}\sum_{j\in\mathbb{Z}^N} \mu(\frac{\|j\|}{J})\|\varphi_j\|_E^2 + \sum_{j\in\mathbb{Z}^N}\mu(\frac{\|j\|}{J})(\frac{\varepsilon}{2}\|\varphi_j\|_E^2 + \frac{\delta}{2}|v_j|^2) \le (F(\varphi,\theta_t\omega),\phi)_E, \tag{3.28}$$

where

$$\begin{aligned} (F(\varphi,\theta_t\omega),\phi)_E &= (az(\theta_t\omega)u,x)_\lambda - \alpha(Au,y) - \beta(Bu,y) + \frac{1}{3}k(D(D^*u)^3,y) \\ &- (az(\theta_t\omega)v,y) + ((2a\varepsilon z(\theta_t\omega) - a^2z^2(\theta_t\omega))u,y) + (f(\widetilde{\sigma}(t)),y). \end{aligned} \tag{3.29}$$

By calculation, we obtain the following estimates:

$$(az(\theta_t\omega)u,x)_\lambda - (az(\theta_t\omega)v,y) \le a|z(\theta_t\omega)| \sum_{j\in\mathbb{Z}^N} \mu(\frac{\|j\|}{J})\|\varphi_j\|_E^2, \tag{3.30}$$

$$-\alpha(Au,y) - \beta(Bu,y) \le \frac{6\sqrt{2}\alpha + 2|\beta|}{\sqrt{\lambda}} \sum_{j\in\mathbb{Z}^N}\mu(\frac{\|j\|}{J})\|\varphi_j\|_E^2 + \frac{8\mu_0 NK}{J\sqrt{\lambda}}\|\varphi\|_E^2, \tag{3.31}$$

where $K = \max\{\alpha, |\beta|\}$,

$$\begin{aligned} \frac{1}{3}k(D(D^*u)^3,y) &= -\frac{k}{3}\sum_{j\in\mathbb{Z}^N}\mu(\frac{\|j\|}{J})(\frac{1}{4}\frac{d}{dt}(D^*u)_j^4 + (\varepsilon - az(\theta_t\omega))(D^*u)_j^4) \\ &- \frac{k}{3}\sum_{j\in\mathbb{Z}^N}(D^*u)_j^3(\mu(\frac{\|j\|}{J}) - \mu(\frac{\|j-1\|}{J}))v_{j-1} \\ &\le -\frac{k}{3}\sum_{j\in\mathbb{Z}^N}\mu(\frac{\|j\|}{J})\left(\frac{1}{4}\frac{d}{dt}(D^*u)_j^4 + (\varepsilon - az(\theta_t\omega))(D^*u)_j^4\right) + \frac{k\mu_0 N}{3J\sqrt{\lambda^3}}\|\varphi\|_E^4, \end{aligned} \tag{3.32}$$

$$((2a\varepsilon z(\theta_t\omega) - a^2z^2(\theta_t\omega))u,y) \le \frac{2a\varepsilon|z(\theta_t\omega)| + a^2z^2(\theta_t\omega)}{2\sqrt{\lambda}}\sum_{j\in\mathbb{Z}^N}\mu(\frac{\|j\|}{J})\|\varphi_j\|_E^2, \tag{3.33}$$

$$(f(\widetilde{\sigma}(t)),y)\leq\frac{1}{2\delta}\sum_{j\in\mathbb{Z}^N}\mu(\frac{\|j\|}{J})f_j^2(\widetilde{\sigma}(t))+\frac{\delta}{2}\sum_{j\in\mathbb{Z}^N}\mu(\frac{\|j\|}{J})|v_j|^2. \tag{3.34}$$

From (3.28)–(3.34), we obtain

$$\begin{aligned}\frac{d}{dt}\sum_{j\in\mathbb{Z}^N}\mu(\frac{\|j\|}{J})(\|\varphi_j\|_E^2+\frac{k}{6}(D^*u)_j^4)\leq(-\varepsilon+\rho(\theta_t\omega))\sum_{j\in\mathbb{Z}^N}\mu(\frac{\|j\|}{J})(\|\varphi_j\|_E^2+\frac{k}{6}(D^*u)_j^4)\\+\frac{1}{\delta}\sum_{j\in\mathbb{Z}^N}\mu(\frac{\|j\|}{J})f_j^2(\widetilde{\sigma}(t))+\frac{c_1}{J}\|\varphi\|_E^2,\end{aligned} \tag{3.35}$$

where

$$c_1=\frac{16\mu_0NK}{\sqrt{\lambda}}+\frac{2k\mu_0N}{3\sqrt{\lambda^3}}R_0^2(\omega).$$

By (3.25), we have

$$\begin{aligned}&\int_0^t e^{-\int_l^t(\frac{\varepsilon}{2}-\rho(\theta_{s-t}\omega))ds}\|\varphi(l,\theta_{-t}\omega,\sigma,\varphi_0(\theta_{-t}\omega)\|_E^2dl\\&\leq\frac{2}{\varepsilon}e^{-\int_{-t}^0(\frac{\varepsilon}{2}-\rho(\theta_s\omega))ds}(R_0^2(\theta_{-t}\omega)+\frac{8k}{3}\|u_0\|^4)+\frac{1}{\varepsilon}R_0^2(\omega).\end{aligned} \tag{3.36}$$

By applying Gronwall's inequality in (3.35) over $[0,t](t\geq 0)$ and replacing ω by $\theta_{-t}\omega$, we obtain that for any $J\in\mathbb{N}$,

$$\begin{aligned}&\sum_{j\in\mathbb{Z}^N}\mu(\frac{\|j\|}{J})\|\varphi_j(t,\theta_{-t}\omega,\sigma,\varphi_0(\theta_{-t}\omega))\|_E^2\\&\leq e^{-\int_{-t}^0(\frac{\varepsilon}{2}-\rho(\theta_s\omega))ds}(\|\varphi_0(\theta_{-t}\omega)\|_E^2+\frac{k}{6}\|D^*u_0\|^4)+\frac{1}{\delta}\sup_{\sigma\in\mathbb{T}^m}\sum_{\|j\|\geq J}f_j^2(\sigma)\\&+\frac{c_1}{J}\int_0^t e^{-\int_l^t(\frac{\varepsilon}{2}-\rho(\theta_{s-t}\omega))ds}\|\varphi(l,\theta_{-t}\omega,\sigma,\varphi_0(\theta_{-t}\omega))\|_E^2dl\int_0^t e^{-\int_l^t(\frac{\varepsilon}{2}-\rho(\theta_{s-t}\omega))ds}dl\\&\leq(1+\frac{2c_1}{J\varepsilon})e^{-\int_{-t}^0(\frac{\varepsilon}{2}-\rho(\theta_s\omega))ds}(R_0^2(\theta_{-t}\omega)+\frac{8k}{3}\|u_0\|^4)+\frac{c_1}{J\varepsilon}R_0^2(\omega)\\&+\frac{1}{2\|f\|_C^2}\sup_{\sigma\in\mathbb{T}^m}\sum_{\|j\|\geq J}f_j^2(\sigma)R_0^2(\omega)\\&\leq(1+\frac{2c_1}{J\varepsilon})e^{-\int_{-t}^0(\frac{\varepsilon}{2}-\rho(\theta_s\omega))ds}(R_0^2(\theta_{-t}\omega)+\frac{8k}{3}\|u_0\|^4)+c_2(\frac{1}{J}+\gamma_{1,J})R_0^2(\omega),\end{aligned} \tag{3.37}$$

where $c_2=\max\{\frac{c_1}{\varepsilon},\frac{1}{2\|f\|_C^2}\}$, $\gamma_{1,J}=\sup\limits_{\sigma\in\mathbb{T}^m}\sum\limits_{\|j\|\geq J}f_j^2(\sigma)$.

By (H3), $\lim\limits_{t\to+\infty}(1+\frac{2c_1}{J\varepsilon})e^{-\int_{-t}^0(\frac{\varepsilon}{2}-\rho(\theta_s\omega))ds}(R_0^2(\theta_{-t}\omega)+\frac{8k}{3}\|u_0\|^4)=0$. Thus, for any $\nu>0,\omega\in\Omega$, there exists $T_\nu(\omega)>0$ such that (3.27) holds. The proof is completed. □

3.4. Existence of a random uniform exponential Attractor

For every $\omega\in\Omega, s\geq 0,\nu>0$, set $T_0(\omega)=T(\omega,B_0)$ and

$$\mathbb{B}(\theta_{-s}\omega)=\overline{\cup_{t\geq\max\{T_0(\theta_{-s}\omega),T_0(\omega),T_0(\theta_{-T_\nu(\omega)}\omega)\}+T_\nu(\omega)}\pi(t,\theta_{-t-s}\omega)\mathbb{T}^m\times B_0(\theta_{-t-s}\omega)}, \tag{3.38}$$

where π is the skew-product cocycle generated by Φ and ϑ:

$$\pi(t,\theta_{-t-s}\omega)\mathbb{T}^m \times B_0(\theta_{-t-s}\omega) = \cup_{\sigma\in\mathbb{T}^m}(\vartheta_t\sigma)\times\varphi(t,\theta_{-t-s}\omega,\sigma)B_0(\theta_{-t-s}\omega).$$

It is easy to check from Lemma 3.2 and Lemma 3.3 that $\mathbb{B}$ has the following properties:

(A1) for every $\omega\in\Omega, \mathbb{B}(\omega)\subseteq\mathbb{T}^m\times B_0(\omega)$, the diameter of $\mathbb{B}(\omega)$ in $\mathbb{T}^m\times E$ is bounded by $(m(2\pi)^2 + 4R_0^2(\omega))^{\frac{1}{2}}$, where $R_0^2(\theta_t\omega)$ is continuous in $t\in\mathbb{R}$;

(A2) $\mathbb{B}(\omega)$ is positive invariant, i.e, for every $\omega\in\Omega, t\geq 0, \pi(t,\theta_{-t}\omega)\mathbb{B}(\theta_{-t}\omega)\subseteq\mathbb{B}(\omega)$ and by $P_E\mathbb{B}\subseteq B_0$, where P_E denotes the projection from $\mathbb{T}^m\times E$ to E;

(A3) $\mathbb{B}$ is pullback absorbing in $\mathcal{D}_{\mathbb{B}}$. Really, for all $\mathbb{D}\in\mathcal{D}_{\mathbb{B}}$, there exist $\widetilde{t}(\mathbb{D},\omega)>0$ such that $\pi(t,\theta_{-t}\omega)\mathbb{D}(\theta_{-t}\omega)\subseteq\mathbb{B}(\omega),\ t\geq\widetilde{t}(\mathbb{D},\omega)$;

(A4) for all $\{\sigma\}\times\{\varphi\}\in\mathbb{B}(\omega)$, the following is true.

$$\sum_{j\in\mathbb{Z}^N}\mu(\frac{\|j\|}{J})\|\varphi_j\|_E^2 \leq \nu + c_2(\frac{1}{J}+\gamma_{1,J})R_0^2(\omega). \tag{3.39}$$

For any $r\geq 0, t\geq 0, \omega\in\Omega, \{\sigma_i\}\times\{\varphi_0^{(i)}(\omega)\}\in\mathbb{B}(\omega), i=1,2$, let $\varphi^{(i)}(r)=\varphi^{(i)}(r,\theta_{-t}\omega,\sigma_i,\varphi_0^{(i)}(\theta_{-t}\omega)) = (u^{(i)},v^{(i)})^T$ and $\widetilde{\varphi}(r)=\varphi^{(1)}(r)-\varphi^{(2)}(r)=(\widetilde{u},\widetilde{v})^T$, then

$$\begin{cases} \frac{d\widetilde{\varphi}}{dr} + \Lambda\widetilde{\varphi} = F(\varphi^{(1)},\theta_{r-t}\omega) - F(\varphi^{(2)},\theta_{r-t}\omega), \\ \widetilde{\varphi}_0(\theta_{-t}\omega) = \varphi_0^{(1)}(\theta_{-t}\omega) - \varphi_0^{(2)}(\theta_{-t}\omega). \end{cases} \tag{3.40}$$

By (A2), we have

$$\varphi^{(i)}(r)\in B_0(\theta_{r-t}\omega), \quad \|\varphi^{(i)}(r)\|_E \leq R_0(\theta_{r-t}\omega), \quad i=1,2. \tag{3.41}$$

Lemma 3.4. *Assume that (H1)–(H3) hold. Then for all $r\geq 0, t\geq 0, \omega\in\Omega, J(\geq 1)\in\mathbb{N}, \{\sigma_i\}\times\{\varphi_0^{(i)}(\theta_{-t}\omega)\}\in\mathbb{B}(\theta_{-t}\omega), i=1,2$, there exist random variables $C_1(\omega), C_2(\omega), C_3(\omega)\geq 0$, such that*

$$\begin{aligned} &\|\pi(t,\theta_{-t}\omega)\{\sigma_1\}\times\{\varphi_0^{(1)}(\theta_{-t}\omega)\} - \pi(t,\theta_{-t}\omega)\{\sigma_2\}\times\{\varphi_0^{(2)}(\theta_{-t}\omega)\}\|_E^2 \\ &\leq e^{2\int_{-t}^0 C_1(\theta_s\omega)ds}(\|\sigma_1-\sigma_2\|_{\mathbb{T}^m}^2 + \|\varphi_0^{(1)}(\theta_{-t}\omega) - \varphi_0^{(2)}(\theta_{-t}\omega)\|_E^2), \end{aligned} \tag{3.42}$$

and

$$\sum_{\|j\|\geq 4J+1}\|\widetilde{\varphi}_j(t)\|_E^2 \leq (e^{\int_{-t}^0(-\frac{\varepsilon}{2}+C_2(\theta_s\omega)ds} + \frac{\delta_J}{2}e^{\int_{-t}^0 C_3(\theta_s\omega)ds})^2(\|\sigma_1-\sigma_2\|_{\mathbb{T}^m}^2 + \|\varphi_0^{(1)}(\theta_{-t}\omega) - \varphi_0^{(2)}(\theta_{-t}\omega)\|_E^2), \tag{3.43}$$

where δ_J is given in the proof below.

Proof. (i) Taking the inner product of (3.40) with $\widetilde{\varphi}(r)$ in E, we have

$$\frac{1}{2}\frac{d}{dr}\|\widetilde{\varphi}(r)\|_E^2 + (\Lambda\widetilde{\varphi}(r),\widetilde{\varphi}(r))_E = \Big(F(\varphi^{(1)}(r),\theta_t\omega) - F(\varphi^{(2)}(r),\theta_t\omega),\widetilde{\varphi}(r)\Big)_E. \tag{3.44}$$

Similar to (3.15)–(3.22) in Lemma 3.1, we obtain

$$\frac{d}{dt}(\|\widetilde{\varphi}(r)\|_E^2 + \|\sigma_1-\sigma_2\|_{\mathbb{T}^m}^2) \leq 2C_1(\theta_{r-t}\omega)(\|\widetilde{\varphi}(r)\|_E^2 + \|\sigma_1-\sigma_2\|_{\mathbb{T}^m}), \tag{3.45}$$

where

$$C_1(\theta_{r-t}\omega) = \frac{\varepsilon}{2} + \frac{\rho(\theta_t\omega)}{2} + \frac{256N^2k^2}{\lambda^3\delta}R_0^4(\theta_{r-t}\omega) + \frac{\|d\|^2}{\delta}. \tag{3.46}$$

Using Gronwall's inequality in (3.45) over $[0,t](t \geq 0)$ and replacing ω by $\theta_{-t}\omega$, we obtain

$$\|\varphi^{(1)}(t) - \varphi^{(2)}(t)\|_E^2 + \|\sigma_1 - \sigma_2\|_{\mathbb{T}^m}^2 \leq e^{\int_{-t}^0 2C_1(\theta_s\omega)ds}(\|\varphi_0^{(1)} - \varphi_0^{(2)}\|_E^2 + \|\sigma_1 - \sigma_2\|_{\mathbb{T}^m}^2). \tag{3.47}$$

Thus, (3.42) holds.
(ii) Let $I \in \mathbb{N}$, $\widetilde{\phi_j} = (\widetilde{x_j}, \widetilde{y_j})^T = \mu(\frac{\|j\|}{I})\widetilde{\varphi_j} = (\mu(\frac{\|j\|}{I})\widetilde{u_j}, \mu(\frac{\|j\|}{I})\widetilde{v_j})^T$, $\widetilde{\phi} = (\widetilde{x}, \widetilde{y})^T = (\widetilde{\phi_j})_{j\in\mathbb{Z}^N}$. Taking the inner product of (3.40) with $\widetilde{\phi}$ in E, we have

$$\frac{1}{2}\frac{d}{dt}\sum_{j\in\mathbb{Z}^N}\mu(\frac{\|j\|}{I})\|\widetilde{\varphi_j}\|_E^2 + \sum_{j\in\mathbb{Z}^N}\mu(\frac{\|j\|}{I})(\frac{\varepsilon}{2}\|\widetilde{\varphi_j}\|_E^2 + \frac{\delta}{2}|\widetilde{v_j}|^2) \leq (F(\varphi^{(1)}, \theta_{r-t}\omega) - F(\varphi^{(2)}, \theta_{r-t}\omega), \widetilde{\phi})_E, \tag{3.48}$$

where

$$\begin{aligned}(F(\varphi^{(1)}, \theta_{r-t}\omega) - F(\varphi^{(2)}, \theta_{r-t}\omega), \widetilde{\phi})_E &= (az(\theta_{r-t}\omega)\widetilde{u}, \widetilde{x})_\lambda - \alpha(A\widetilde{u}, \widetilde{y}) - \beta(B\widetilde{u}, \widetilde{y}) \\ &+ \frac{1}{3}k\left(D((D^*u^{(1)})^3 - (D^*u^{(2)})^3), \widetilde{y}\right) + \left((2a\varepsilon z(\theta_{r-t}\omega) - a^2z^2(\theta_{r-t}\omega))\widetilde{u}, \widetilde{y}\right) \\ &- (az(\theta_{r-t}\omega)\widetilde{v}, \widetilde{y}) + (f(\widetilde{\sigma_1}(r)) - f(\widetilde{\sigma_2}(r)), \widetilde{y}).\end{aligned} \tag{3.49}$$

By (H2) and (3.39), we have that for $\|j\| \geq 2J,\ J \in \mathbb{N}$,

$$\begin{cases} (az(\theta_{r-t}\omega)\widetilde{u}, \widetilde{x})_\lambda - (az(\theta_{r-t}\omega)\widetilde{v}, \widetilde{y}) \leq a|z(\theta_{r-t}\omega)|\sum\limits_{j\in\mathbb{Z}^N}\mu(\frac{\|j\|}{I})\|\widetilde{\varphi_j}\|_E^2, \\ -\alpha(A\widetilde{u}, \widetilde{y}) - \beta(B\widetilde{u}, \widetilde{y}) \leq \frac{6\sqrt{2}\alpha + 2|\beta|}{\sqrt{\lambda}}\sum\limits_{j\in\mathbb{Z}^N}\mu(\frac{\|j\|}{I})\|\widetilde{\varphi_j}\|_E^2 + \frac{8\mu_0NK}{I\sqrt{\lambda}}\|\widetilde{\varphi}\|_E^2, \\ \frac{1}{3}k(D((D^*u^{(1)})^3 - (D^*u^{(2)})^3), \widetilde{y}) = -\frac{1}{3}k((D^*u^{(1)})^3 - (D^*u^{(2)})^3, D^*\widetilde{y}) \\ \leq \frac{2k}{\sqrt{\lambda}}\sum\limits_{j\in\mathbb{Z}^N}\mu(\frac{\|j\|}{I})(\|\widetilde{\varphi_j}\|_E^2 + \|\widetilde{\varphi}_{j-1}\|_E^2)(|u_j^{(1)}|^2 + |u_j^{(2)}|^2) \\ \leq \frac{4k\nu}{\sqrt{\lambda^3}}\sum\limits_{j\in\mathbb{Z}^N}\mu(\frac{\|j\|}{I})\|\widetilde{\varphi_j}\|_E^2 + \left[\frac{2k\mu_0}{I\sqrt{\lambda^3}}(\nu + c_2(\frac{1}{J} + \gamma_{1,J})R_0^2(\theta_{r-t}\omega)) + \frac{4kc_2}{\sqrt{\lambda^3}}(\frac{1}{J} + \gamma_{1,J})R_0^2(\theta_{r-t}\omega)\right]\|\widetilde{\varphi}\|_E^2, \\ \left((2a\varepsilon z(\theta_{r-t}\omega) - a^2z^2(\theta_{r-t}\omega))\widetilde{u}, \widetilde{y}\right) \leq \frac{2a\varepsilon|z(\theta_{r-t}\omega)| + a^2z^2(\theta_{r-t}\omega)}{2\sqrt{\lambda}}\sum\limits_{j\in\mathbb{Z}^N}\mu(\frac{\|j\|}{I})\|\widetilde{\varphi_j}\|_E^2, \\ (f(\widetilde{\sigma_1}(r)) - f(\widetilde{\sigma_2}(r)), \widetilde{y}) \leq \frac{1}{2\delta}\sum\limits_{j\in\mathbb{Z}^N}\mu(\frac{\|j\|}{I})d_j^2\|\sigma_1 - \sigma_2\|_{\mathbb{T}^m}^2 + \frac{\delta}{2}\sum\limits_{j\in\mathbb{Z}^N}\mu(\frac{\|j\|}{I})|\widetilde{v_j}|^2\|\widetilde{\varphi}\|_E^2. \end{cases} \tag{3.50}$$

By (3.48)–(3.50), we obtain that for $I \geq 2J$,

$$\begin{aligned}\frac{d}{dt}\sum_{j\in\mathbb{Z}^N}\mu(\frac{\|j\|}{I})\|\widetilde{\varphi_j}\|_E^2 &\leq (-\varepsilon + \rho(\theta_{r-t}\omega) + \frac{8k\nu}{\sqrt{\lambda^3}})\sum_{j\in\mathbb{Z}^N}\mu(\frac{\|j\|}{I})\|\widetilde{\varphi_j}\|_E^2 + \frac{1}{\delta}\sum_{\|j\|\geq I}d_j^2\|\sigma_1 - \sigma_2\|_{\mathbb{T}^m}^2 \\ &+ [\frac{16\mu_0NK}{J\sqrt{\lambda}} + \frac{4k\mu_0}{J\sqrt{\lambda^3}}(\nu + c_2(\frac{1}{J} + \gamma_{1,J})R_0^2(\theta_{r-t}\omega)) + \frac{8kc_2}{\sqrt{\lambda^3}}(\frac{1}{J} + \gamma_{1,J})R_0^2(\theta_{r-t}\omega)]\|\widetilde{\varphi}\|_E^2 \\ &\leq (-\varepsilon + 2C_2(\theta_{r-t}))\sum_{j\in\mathbb{Z}^N}\mu(\frac{\|j\|}{I})\|\widetilde{\varphi_j}\|_E^2 + c_3\widetilde{\delta}_J(1 + R_0^2(\theta_{r-t}\omega))\|\widetilde{\varphi}\|_E^2,\end{aligned} \tag{3.51}$$

where

$$\widetilde{\delta_J} = \gamma_{2,J} + (\frac{1}{J}+1)(\frac{1}{J}+\gamma_{1,J}),\ \gamma_{2,J} = \sum_{\|j\|\geq I} d_j^2,\ c_3 = \frac{1}{\delta} + \frac{16\mu_0 NK}{J\sqrt{\lambda}} + \frac{4k\mu_0}{\sqrt{\lambda^3}}(\nu + c_2) + \frac{8kc_2}{\sqrt{\lambda^3}},$$

$$C_2(\omega) = \frac{\rho(\omega)}{2} + \frac{4k\nu}{\sqrt{\lambda^3}}. \tag{3.52}$$

By (3.47) and applying Gronwall's inequality in (3.51) over $[0,t]$, we have that for $I \geq 2J$,

$$\begin{aligned}\sum_{j\in\mathbb{Z}^N}\mu(\frac{\|j\|}{I})\|\widetilde{\varphi}_j\|_E^2 &\leq e^{\int_{-t}^0(-\varepsilon+2C_2(\theta_s\omega))ds}(\|\varphi_0^{(1)}-\varphi_0^{(2)}\|^2 + \|\sigma_1-\sigma_2\|_{\mathbb{T}^m}^2)\\ &+ \widetilde{\delta_J} e^{\int_{-t}^0(2C_1(\theta_s\omega)+2C_2(\theta_s\omega))ds}(\|\varphi_0^{(1)}-\varphi_0^{(2)}\|^2 + \|\sigma_1-\sigma_2\|_{\mathbb{T}^m}^2)\times\int_{-t}^0 c_3 e^{\varepsilon l}(1+R_0^2(\theta_l\omega))dl.\end{aligned} \tag{3.53}$$

Since for all $p \geq 0$, $\sqrt{p} \leq e^p$, it follows that

$$\begin{aligned}\int_{-t}^0 c_3 e^{\varepsilon l}(1+R_0^2(\theta_l\omega))dl &\leq (\int_{-t}^0 e^{2\varepsilon l}dl)^{\frac{1}{2}}(\int_{-t}^0 c_3^2(1+R_0^2(\theta_l\omega))dl)^{\frac{1}{2}}\\ &\leq \frac{1}{\sqrt{2\varepsilon}} e^{\int_{-t}^0 2c_3^2(1+R_0^4(\theta_l\omega))^2 dl}.\end{aligned}$$

By (3.53), it follows that for $I \geq 2J$,

$$\begin{aligned}\sum_{\|j\|\geq 4J}\|\widetilde{\varphi}_j\|_E^2 &\leq \sum_{j\in\mathbb{Z}^N}\mu(\frac{\|j\|}{I})\|\widetilde{\varphi}_j\|_E^2\\ &\leq \left(e^{\int_{-t}^0(-\varepsilon+2C_2(\theta_s\omega))ds} + \frac{\delta_J^2}{4}e^{\int_{-t}^0 2C_3(\theta_s\omega)ds}\right)(\|\varphi_0^{(1)}-\varphi_0^{(2)}\|^2 + \|\sigma_1-\sigma_2\|_{\mathbb{T}^m}^2),\end{aligned} \tag{3.54}$$

where $\delta_J^2 = \frac{4\widetilde{\delta_J}}{\sqrt{2\varepsilon}}$ and

$$C_3(\omega) = C_1(\omega) + C_2(\omega) + c_3^2(1+R_0^4(\omega)). \tag{3.55}$$

Thus, (3.43) holds. The proof is completed. □

Lemma 3.5. *Assume that the coefficient a and $\nu = \nu_0 > 0$ satisfy*

$$a < \min\left\{\frac{\varepsilon\sqrt{\delta}}{8}, \frac{\sqrt{\lambda\delta}}{8}, \frac{\sqrt{\delta\varepsilon}\lambda^{\frac{1}{4}}}{2}\right\}, \tag{3.56}$$

$$\frac{a}{\sqrt{\pi\delta}} + \frac{6\sqrt{2}\alpha + 2|\beta|}{\sqrt{\lambda}} + \frac{a\varepsilon}{\sqrt{\pi\lambda\delta}} + \frac{a^2}{4\delta\sqrt{\lambda}} + \frac{4k\nu_0}{\sqrt{\lambda^3}} < \frac{\varepsilon}{32}. \tag{3.57}$$

Then

$$0 \leq \boldsymbol{E}(C_2(\omega)) \leq \frac{\varepsilon}{32}, \quad 0 \leq \boldsymbol{E}(C_3^2(\omega)) < +\infty.$$

Proof. By (3.2), (3.11), (3.52), and (3.57), it is easy to have the following

$$\mathbf{E}(C_2(\omega)) = \frac{a}{\sqrt{\pi\delta}} + \frac{6\sqrt{2}\alpha + 2|\beta|}{\sqrt{\lambda}} + \frac{a\varepsilon}{\sqrt{\pi\lambda\delta}} + \frac{a^2}{4\delta\sqrt{\lambda}} + \frac{4k\nu_0}{\sqrt{\lambda^3}} < \frac{\varepsilon}{32}.$$

By (3.55), we have

$$\mathbf{E}(C_3^2(\omega)) \leq 4\left(\mathbf{E}(C_1^2(\omega)) + \mathbf{E}(C_2^2(\omega)) + c_3^4 + c_3^4\mathbf{E}(R_0^8(\omega))\right). \tag{3.58}$$

By (3.46), we know that

$$C_1^2(\omega) \leq \varepsilon^2 + \rho^2(\omega) + \frac{512^2N^4k^4}{\lambda^6\delta^2}R_0^8(\omega) + \frac{4\|d\|^4}{\delta^2},$$

$$C_2^2(\omega) = \left(\frac{\rho(\omega)}{2} + \frac{4k\nu_0}{\sqrt{\lambda^3}}\right)^2 \leq \frac{\rho^2(\omega)}{2} + \frac{32k^2\nu_0^2}{\lambda^3}.$$

By (3.26), (3.56), and H*ö*lder's inequality, we have

$$\begin{aligned}\mathbf{E}(R_0^8(\omega)) &= \frac{2^8}{\delta^4}\|f\|_C^8\mathbf{E}\left(\int_{-\infty}^0 e^{\frac{\varepsilon}{2}l+\int_l^0\rho(\theta_s\omega)ds}dl\right)^4\\ &\leq \frac{2^8}{\delta^4}\|f\|_C^8(\int_{-\infty}^0 e^{\frac{\varepsilon}{3}l}dl)^3\mathbf{E}(\int_{-\infty}^0 e^{\varepsilon l+\int_l^0 4\rho(\theta_s\omega)ds}dl)\\ &\leq \frac{2^8\cdot 3^3}{\delta^4\varepsilon^3}\|f\|_C^8\left(\frac{1}{\varepsilon-\frac{8a}{\sqrt{\delta}}} + \frac{1}{\varepsilon-\frac{48\sqrt{2}\alpha+16|\beta|}{\sqrt{\lambda}}} + \frac{1}{\varepsilon-\frac{8a\varepsilon}{\sqrt{\delta\lambda}}} + \frac{1}{\varepsilon-\frac{4a^2}{\sqrt{\lambda\delta}}}\right)\\ &< \infty.\end{aligned} \tag{3.59}$$

Thus,

$$0 \leq \mathbf{E}(R_0^4(\omega)) \leq \frac{1}{2}(1 + \mathbf{E}[R_0^8(\omega)]) < \infty. \tag{3.60}$$

$$\begin{aligned}\mathbf{E}[\rho^2(\omega)] &\leq 4\left(4a^2\mathbf{E}[|z(\omega)|^2] + \frac{(12\sqrt{2}\alpha + 4|\beta|)^2}{\lambda} + \frac{4a^2\varepsilon^2}{\lambda}\mathbf{E}[|z(\omega)|^2] + \frac{a^4}{\lambda}\mathbf{E}[|z(\omega)|^4]\right)\\ &= \frac{8a^2}{\delta} + 4\frac{(12\sqrt{2}\alpha + 4|\beta|)^2}{\lambda} + \frac{8a^2\varepsilon^2}{\lambda\delta} + \frac{3a^4}{\lambda\delta^2} < \infty.\end{aligned} \tag{3.61}$$

By (3.58)–(3.61), we have $\mathbf{E}[C_3^2(\omega)] < \infty$. The proof is completed. □

Theorem 3.1. *Assume that (H1)–(H3), (3.56), and (3.57) hold. Then* $\{\Phi(t,\omega,\sigma)\}_{t\geq 0,\omega\in\Omega,\sigma\in\mathbb{T}^m}$ *has a* $\mathcal{D}-$*random uniform exponential attractor* $\mathcal{A} = \{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ *with the following properties:*
(i) $\mathcal{A}$ *is a compact set of E and measurable in* ω*;*
(ii) There exists $J_0 \in \mathbb{N}$ *such that* $dim_f\mathcal{A}(\omega) \leq \frac{2[m+2(8J_0+1)]\ln(\frac{2\sqrt{m+2(8J_0+1)}}{\delta_{J_0}}+1)}{\ln\frac{4}{3}} < \infty, \forall\omega\in\Omega$*;*
(iii) For every $\omega\in\Omega, D\in\mathcal{D}$*, there exist* $\widetilde{T}(\omega,\mathbb{D}) \geq 0$ *and a tempered random variable* $\widetilde{h}(\omega) > 0$*, such that for any* $t \geq \widetilde{T}(\omega,\mathbb{D})$*,*

$$\sup_{\sigma\in\mathbb{T}^m} dist_E(\Phi(t,\theta_{-t}\omega,\vartheta_{-t}\sigma)D(\theta_{-t}\omega),\mathcal{A}(\omega)) \leq \widetilde{h}(\omega)e^{-\frac{\varepsilon\ln\frac{4}{3}}{64\ln 2}t},$$

where $\mathbb{D} = D\times\mathbb{T}^m$*.*

Proof. From Lemma 3.5, taking $t = t_0 = \frac{16ln2}{\varepsilon}$ in (3.42) and (3.43), it follows that

$$0 < t_0^2\Big(2\mathbf{E}[C_3^2(\omega)] + \frac{\varepsilon}{2}\mathbf{E}[C_3(\omega)]\Big) < +\infty.$$

Let

$$\kappa = \min\left\{\frac{1}{16},\ e^{-\frac{2}{\ln\frac{3}{2}}t_0^2\left(2\mathbf{E}[C_3^2(\omega)]+\frac{\varepsilon}{2}\mathbf{E}[C_3(\omega)]\right)}\right\}$$

be a finite positive constant. By (H1), when $J \to +\infty$, $\delta_J \to 0$, thus, we choose a large enough positive integer $J = J_0$ such that $\delta_J \leq \kappa$. Based on Theorem 2.1 in [33] and Theorem 2.6 in [35], it follows from Lemmas 3.1–3.5 that the proof of Theorem 3.1 is completed. □

4. Conclusions and discussion

In this paper, based on the existence criterion of a random uniform exponential attractor for non-autonomous random dynamical systems from Theorem 2.1 in [33] and Theorem 2.6 in [35], we proved the existence of a random exponential attractor for the non-autonomous stochastic Boussinesq lattice system with quasi-periodic forces and multiplicative white noise. The random uniform exponential attractor with finite fractal dimension is more stable than the random attractor. Therefore, the asymptotic behavior of the solution of the system (1.1) can be described by finite independent parameters. Applying the same idea, we can also consider the existence of a random exponential attractor for the non-autonomous stochastic Boussinesq lattice system with additive white noise. However, we do not need to restrict the coefficient of the random term to small enough, because the additive noise term is independent of the state variable. Inspired by [36, 37] and the references therein, we will consider the long-time asymptotic behavior of the non-autonomous stochastic Boussinesq lattice equation with nonlinear colored noise in future works.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The author would like to express their sincere thanks to the referee for his/her helpful comments and suggestions which greatly improved the presentation of this paper. The author also would like to thank the editors for their kind help. This work was supported by the Natural Science Research Key Project of the Education Department of Anhui Province (No.2024AH051359).

Conflict of interest

The author declares no conflict of interest regarding the publication of this paper.

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