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Multiple solutions for a singular fractional Kirchhoff problem with variable exponents

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Abstract: In this work, we studied the multiplicity of solutions for a Kirchhoff problem involving the $\kappa(\xi)$-fractional derivative and critical exponent. More precisely, we transformed the studied problem into an integral equation that lead to the study of the critical point for the energy functional; after that, we presented and proved some properties related to this functional and demonstrated that the energy functional satisfied the geometry of the mountain pass geometry. Finally, by applying the mountain pass theorem for the even functional, we proved that this functional admitted infinitely many critical points, which means that the studied problem has infinitely many solutions.

Keywords: ψ-Hilifer fractional derivative; mountain pass geometry; concentration-compactness principle; variational methods
Mathematics Subject Classification: 31B30, 35J35, 35J60

1. Introduction

In this paper, we are concerned with the existence of infinitely many solutions for the following fractional problem:

$$\begin{cases} -\mathcal{S}\left(\int_\Lambda \mathcal{L}^{\varpi,\varepsilon;\psi}(\xi,\varphi)\,d\xi\right)\ {}^{\mathrm{H}}\mathbb{D}_T^{\varpi,\varepsilon;\psi}\left(\mathbf{L}^{\varpi,\varepsilon;\psi}(\xi,\varphi)\right) = a(\xi)h(\varphi), \text{ in } \Lambda, \\ \varphi = 0 \quad \text{on } \partial\Lambda, \end{cases} \tag{1.1}$$

where $\Lambda = (0,T)\times(0,T)$,

$$\mathbf{L}^{\varpi,\varepsilon;\psi}(\xi,\varphi) = \left|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\varepsilon;\psi}\varphi\right|^{\kappa_1(\xi)-2}\ {}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\varepsilon;\psi}\varphi + \mu(\xi)\left|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\varepsilon;\psi}\varphi\right|^{\kappa_2(\xi)-2}\ {}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\varepsilon;\psi}\varphi, \tag{1.2}$$

$$\mathcal{L}^{\varpi,\varepsilon;\psi}(\xi,\varphi)=\frac{1}{\kappa_1(\xi)}\left|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\varepsilon;\psi}\varphi\right|^{\kappa_1(\xi)}+\frac{\mu(\xi)}{\kappa_2(\xi)}\left|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\varepsilon;\psi}\varphi\right|^{\kappa_2(\xi)}, \tag{1.3}$$

and ${}^{\mathrm{H}}\mathbb{D}_{T}^{\varpi,\varepsilon;\psi}(\cdot)$ and ${}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\varepsilon;\psi}(\cdot)$ are ψ-Hilfer fractional partial derivatives of order $\frac{1}{\kappa_1^+}<\varpi<1$ and type $0\leq\nu\leq 1$. $\kappa_1,\kappa_2\in C_+(\bar{\Lambda})$, $1<\kappa_1^-=\inf_{\Lambda}\kappa_1(\xi)\leq\kappa_2^+=\sup_{\Lambda}\kappa_2(\xi)<2$.

$\mathcal{S}$, h and a are functions that satisfy appropriate conditions which will be fixed later.

We note that fractional calculus can model several phenomena in sciences and engineering; one can see the works of [1] (application in mechanics), [2] (application in engineering), [3] (application in viscoelasticity), [4] (application in dynamical systems), and [5] (application in modeling blood alcohol concentration), and other applications can be found in [6–8]. Due to the importance of fractional calculus in several fields, many researchers have concentrated on the development of new fractional operators. Recently, there have many papers dealing with the ψ-Hilifer fractional derivative and p-Laplacian operator, we cite for instance the papers of Alsaedi and Ghanmi [9] (Variational method combined with different versions of the mountain pass theorem), Ezati and Nymouradi [10] (genus properties in critical point theory), Venterler et al. [12–14] (Nehari manifold in fractional Sobolev spaces), and the references [15–18] (existence and stability of solutions). The first important point in our study is that we consider a Kirchhoff problem with critical exponent. We also note that Kirchhoff's problem was introduced by Kirchhoff [19]; it is noted that Kirchhoff-type problems generally refer to mathematical models related to the analysis of electrical circuits using Kirchhoff's laws, which are, in fact, fundamental principles in circuit theory. The second important point in our study is that we consider a fractional Kirchhoff problem with a critical exponent. It is noted that to manipulate the critical growth, we use the concentration-compactness principle which is introduced by Lions [20]; this principle is particularly crucial when considering equations involving critical exponent. For a more comprehensive understanding of this topic, we suggest referring to the references of Azorero and Alonso [21] (multiplicity of solutions for an elliptic problem with a critical exponent or nonsymmetric term), Bahri and Lions [22] (compactness issues of the variational formulation for a nonlinear Maxwell-Dirac system), Ghanmi et al. [23] (combination of variational techniques with a truncation argument for a singular fractional Kirchhoff problem), and Rabinowitz [24] (Minimax methods in critical point theory).

The study of problems involving variable exponents has been extensively studied by several authors; we cite, for instance, the papers of Dai et al. [25, 26] (combination of a direct variational approach and the theory of the variable exponent Sobolev spaces), Ambrosio and Isernia [27] (combination of penalization techniques with Ljusternik-Schnirelmann theory), and Fiscella and Pucci [28] (combination of the variational method with the mountain pass theorem). The $p(\xi)$-Laplacian possesses more complex nonlinearity which raises some of the essential difficulties; for example, in [29], the authors consider the Kirchhoff's fractional $\kappa(\xi)$-Laplacian problem without critical exponent:

$$\begin{cases}\mathcal{S}\left(\int_{\Lambda}\frac{1}{\kappa(\xi)}\left|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\varepsilon;\psi}\varphi\right|^{\kappa(\xi)}d\xi\right)\mathbf{L}_{\kappa(\xi)}^{\varpi,\varepsilon;\psi}u &= \}(\xi,\varphi),\ \ in\ \Lambda=[0,T]\times[0,T],\\ u &= 0,\ \ on\ \partial\Lambda.\end{cases} \tag{P_2}$$

The authors employed variational methods, the mountain pass theorem, and the Fountain theorem to establish the existence and multiplicity of solutions for problem (P_2).

Motivated by the results presented in reference [29], our paper aims to contribute a study of a more general problem; precisely, we consider a double-phase Kirchhoff problem involving critical nonlinearity. To manipulate the critical exponent, we use a concentration-compactness principle. We note that the double-phase problem (1.1) is inspired by various models in mathematical physics. For example, consider the fourth-order relativistic operator

$$\varphi \longmapsto div\left(\frac{|\nabla\varphi|^2}{(1-|\nabla\varphi|^4)^{\frac{3}{4}}}\nabla\varphi\right),$$

which characterizes a wide range of phenomena in relativistic quantum mechanics. If we apply Taylor's formula to the function $y^2(1-y^4)^{-\frac{3}{4}}$, then we can approximate the last operator by the following double-phase operator

$$\varphi \longmapsto \Delta_4\varphi + \frac{3}{4}\Delta_8\varphi.$$

Our study provides a generalization and improvement of other aforementioned references in the literature. More precisely, in the case when $\mu \equiv 0$; we obtain the result of Sousa et al. [29]. Moreover, if $\kappa_1 = \kappa_2 = p$ constant, then we obtain the result of Nouf et al. [11].

To prove the existing result of this work, we introduce in the next section (Section 2) some results on the functional space and we present the main tool used in the proof (symmetric version of the mountain pass theorem); in Section 3, we present and prove the main result of this work.

2. Preliminaries

In this section, we begin by presenting a functional space and some of its properties. After that, we recall some results about the modular of this space, and we finish this part by recalling the main tool for proving the main result of this work. Since we study a double-phase problem, we introduced the function $\mathcal{M}$, defined on $\overline{\Lambda}\times[0,\infty[$, by:

$$\mathcal{M}(\xi,t) = t^{\kappa_1(\xi)} + \mu(\xi)t^{\kappa_2(\xi)}.$$

Associated to the function $\mathcal{M}$, we define the following modular:

$$\rho_{\mathcal{M}}(\varphi) = \int_\Lambda \mathcal{M}(\xi,|\varphi|)\,d\xi = \int_\Lambda |\varphi|^{\kappa_1(\xi)} + \mu(\xi)|\varphi|^{\kappa_2(\xi)}\,d\xi.$$

Next, we denote by $B(\Lambda,\mathbb{R})$ the set of all Borel measurable functions, and we define the following functional space:

$$\mathcal{L}^{\mathcal{M}}(\Lambda) = \{\varphi \in B(\Lambda,\mathbb{R}) : \rho_{\mathcal{M}}(\varphi) < \infty\}.$$

We endow the last space with the following norm:

$$|\varphi|_{\mathcal{L}^{\mathcal{M}}(\Lambda)} = \inf\left\{\varpi > 0 : \rho_{\mathcal{M}}(\frac{\varphi}{\varpi}) \leq 1\right\}.$$

We recall from [13] that $\left(\mathcal{L}^{\mathcal{M}}(\Lambda), |\cdot|_{\mathcal{L}^{\mathcal{M}}(\Lambda)}\right)$ is a Banach space, which generalizes the Lebegue space with variable exponent; precisely, if $\mu(.) \equiv 0$, then the space $\mathcal{L}^{\mathcal{M}}(\Lambda)$ is reduced to the space $\mathcal{L}^{\kappa_1(.)}(\Lambda)$. Also, we recall from [13] that the fractional Sobolev space is defined as follows:

$$\mathcal{H}^{\varpi,\varepsilon;\mu}_{\mathcal{M}}(\Lambda) = \left\{\varphi \in L^{\mathcal{M}}(\Lambda) : \left|\mathbb{D}^{\varpi,\varepsilon;\mu}_{0+}\varphi\right| \in L^{\mathcal{M}}(\Lambda)\right\}$$

with the norm

$$\|\varphi\| = \|\varphi\|_{L^{\mathcal{M}}(\Lambda)} + \left\|\mathbb{D}_{0+}^{\varpi,\varepsilon;\mu}\varphi\right\|_{L^{\mathcal{M}}(\Lambda)}.$$

Denote by $\mathcal{H}_{\mathcal{M},0}^{\varpi,\varepsilon;\mu}(\Lambda)$ the subspace of $\mathcal{H}_{\mathcal{M}}^{\varpi,\varepsilon;\mu}(\Lambda)$ defined as the closure of $C_0^\infty(\Lambda)$ with respect to the norm $\|.\|$.

In the rest of this paper, we adopt the following notations: the letter j will denote one of the integers 1 or 2, θ_j and T_j are nonnegative constants such that $\theta_j < T_j$, $\varpi_j \in (0,1)$, $I_j = [\theta_j, T_j]$. Next, we put $\Lambda = I_1 \times I_2$, in the case when $\theta_1 = \theta_2 = 0$ and $T_1 = T_2 = T$, and we obtain the set Λ introduced in our problem. Let μ be a positive function such that $\mu'(\xi_j) > 0$ for all $\xi_j \in [\theta_j, T_j]$, and let $\varpi = (\varpi_1, \varpi_2)$ with $0 < \varpi_j \le 1$), and $\varepsilon = (\varepsilon_1, \varepsilon_2)$ with $0 \le \nu_j \le 1$. We recall from [12] the following definition of the μ-Hilfer fractional derivatives.

$$\mathbb{D}_{\theta}^{\varpi,\varepsilon;\mu}\varphi(\xi_1,\xi_2) = \mathbf{I}_{\theta}^{\varepsilon(1-\varpi),\mu}\left(\frac{1}{\mu'(\xi_1)\mu'(\xi_2)}\left(\frac{\partial^2}{\partial\xi_1\partial\xi_2}\right)\right)\mathbf{I}_{\theta}^{(1-\varepsilon)(1-\varpi),\mu}\varphi(\xi_1,\xi_2), \tag{2.1}$$

and

$$\mathbb{D}_{T}^{\varpi,\varepsilon;\mu}\varphi(\xi_1,\xi_2) = \mathbf{I}_{T}^{\varepsilon(1-\varpi),\mu}\left(-\frac{1}{\mu'(\xi_1)\mu'(\xi_2)}\left(\frac{\partial^2}{\partial\xi_1\partial\xi_2}\right)\right)\mathbf{I}_{T}^{(1-\varepsilon)(1-\varpi),\mu}\varphi(\xi_1,\xi_2), \tag{2.2}$$

where $\mathbf{I}_{\theta}^{\alpha;\mu}$ and $\mathbf{I}_{T}^{\alpha;\mu}$ are the μ-Riemann-Liouville fractional partial integrals (see also [12]). For simplicity, in the rest of this work, $\mathbb{D}_{\theta}^{\varpi,\varepsilon;\mu}\varphi(\xi_1,\xi_2)$, $\mathbb{D}_{T}^{\varpi,\varepsilon;\mu}\varphi(\xi_1,\xi_2)$, and $\mathbf{I}_{\theta}^{\varpi;\mu}\varphi(\xi_1,\xi_2)$ will be denoted, respectively, by $\mathbb{D}_{\theta}^{\varpi,\varepsilon;\mu}\varphi$, $\mathbb{D}_{T}^{\varpi,\varepsilon;\mu}\varphi$, and $\mathbf{I}_{\theta}^{\varpi;\mu}\varphi$.

To manipulate any minimizing sequence for the energy functional, we need some properties of the functional space; precisely, we have the following result.

Proposition 2.1. *[29] $\mathcal{L}^{\mathcal{M}}(\Lambda)$ and $\mathcal{H}_{\mathcal{M},0}^{\varpi,\varepsilon;\mu}(\Lambda)$ are reflexive Banach spaces which are, in addition, separable.*

To transform the main equation in the integral equation defined by the functional energy, we need to use the Hölder inequality which holds in our functional space. Precisely, we have the following result.

Proposition 2.2. *[29–31]*

(1) For each $\varphi_1 \in \mathcal{L}^{\kappa(\xi)}(\Lambda)$ and $\varphi_2 \in \mathcal{L}^{\kappa'(\xi)}(\Lambda)$ (with $\frac{1}{\kappa(\xi)} + \frac{1}{\kappa'(\xi)} = 1$), we have

$$\left|\int_{\Lambda}\varphi_1\varphi_2\,d\xi\right| \le \left(\frac{1}{\kappa^-} + \frac{1}{(\kappa')^-}\right)|\varphi_1|_{\kappa(\xi)}|\varphi_2|_{\kappa'(\xi)}.$$

(2) For each p_1 and p_2 in $C_+(\overline{\Lambda})$ with $p_1 \le p_2$ in $\overline{\Lambda}$, we have a continuous embedding

$$\mathcal{L}^{p_2(\xi)}(\Lambda) \hookrightarrow \mathcal{L}^{p_1(\xi)}(\Lambda),$$

where $C_+(\overline{\Lambda}) = \left\{r \in C(\overline{\Lambda}) : 1 < r^- \le r^+ < \infty\right\}$.

Since we shall work in the space $\mathcal{H}_{\mathcal{M},0}^{\varpi,\varepsilon;\mu}(\Lambda)$, we use an equivalent norm, so we need to present the following important inequality.

Proposition 2.3. *[29, 32] For any $\varphi \in \mathcal{H}_{\kappa(\xi)}^{\varpi,\varepsilon;\mu}(\Lambda)$, $\exists\, c > 0$ such that*

$$\|\varphi\|_{\mathcal{L}^{\kappa(\xi)}(\Lambda)} \le c\left\|\mathbb{D}_{0+}^{\varpi,\varepsilon;\mu}\varphi\right\|_{\mathcal{L}^{\kappa(\xi)}(\Lambda)}.$$

From the last inequality, we can consider the space $\mathcal{H}^{\varpi,\varepsilon;\mu}_{\mathcal{M},0}(\Lambda)$ with the following norm:

$$\|\varphi\| = \left\|\mathbb{D}^{\varpi,\varepsilon;\mu}_{0+}\varphi\right\|_{\mathcal{L}^{\mathcal{M}}(\Lambda)}.$$

The following proposition highlights further properties related to the functional space.

Proposition 2.4. *[29] If $q \in C_+(\overline{\Lambda})$ with $q(\xi) < \kappa^*(\xi)$ for all $\xi \in \overline{\Lambda}$, then the embedding from $\mathcal{H}^{\varpi,\varepsilon;\mu}_{\mathcal{M}}(\Lambda)$ into $\mathcal{L}^{q(\xi)}(\Lambda)$ is continuous; moreover, this embedding is compact. Here, $\kappa^*(\xi)$ is defined as follows:*

$$\kappa^*(\xi) = \begin{cases} \frac{2\kappa(\xi)}{2-\varpi\kappa(\xi)}, & if\ \ \kappa(\xi) < 2, \\ \\ \infty, & if\ \ \kappa(\xi) \geq 2. \end{cases}$$

For simplicity, let us denote

$$\Gamma^{\varpi,\varepsilon}(\varphi) = \int_\Lambda \left|{}^{\mathrm{H}}\mathbb{D}^{\varpi,\varepsilon;\psi}_{0+}\varphi\right|^{\kappa_1(\xi)} + \mu(\xi)\left|{}^{\mathrm{H}}\mathbb{D}^{\varpi,\varepsilon;\psi}_{0+}\varphi\right|^{\kappa_2(\xi)} d\xi.$$

The following proposition provides important properties of the functional $\Gamma^{\varpi,\varepsilon}$:

Proposition 2.5. *[29] For each $\varphi \in \mathcal{H}^{\varpi,\varepsilon;\mu}_{\mathcal{M},0}(\Lambda)$, the following statements hold:*

(1) $\Gamma^{\varpi,\varepsilon}(\varphi) < 1\ (= 1, > 1) \iff \|\varphi\| < 1\ (= 1, > 1)$.
(2) $\min\left(\|\varphi\|^{\kappa_1^-}, \|\varphi\|^{\kappa_1^+}\right) \leq \Gamma^{\varpi,\varepsilon}(\varphi) \leq \max\left(\|\varphi\|^{\kappa_1^-}, \|\varphi\|^{\kappa_1^+}\right)$.

The next proposition provides properties of the functional $\rho_{\mathcal{M}}$:

Proposition 2.6. *[30, 33] For all $\varphi \in \mathcal{L}^{\mathcal{M}}(\Lambda)$, we have*

(1) $|\varphi|_{\mathcal{L}^{\mathcal{M}}(\Lambda)} < 1\ (= 1, > 1) \iff \rho_{\mathcal{M}}(\varphi) < 1\ (= 1, > 1)$.
(2) $\min\left(|\varphi|^{\kappa_2^-}_{\mathcal{L}^{\mathcal{M}}(\Lambda)}, |\varphi|^{\kappa_2^+}_{\mathcal{L}^{\mathcal{M}}(\Lambda)}\right) \leq \rho_{\mathcal{M}}(\varphi) \leq \max\left(|\varphi|^{\kappa_2^-}_{\mathcal{L}^{\mathcal{M}}(\Lambda)}, |\varphi|^{\kappa_2^+}_{\mathcal{L}^{\mathcal{M}}(\Lambda)}\right)$.

The next proposition relates the norms of a function in variable exponent Lebesgue spaces with its pointwise behavior.

Proposition 2.7. *[30, 33] Let p be a measurable function in $\mathcal{L}^\infty(\mathbb{R}^N)$, and let q be a measurable function such that for each $\xi \in \mathbb{R}^N$, we have $\kappa(\xi)q(\xi) \geq 1$, then for each nontrivial function φ in $\mathcal{L}^{q(\xi)}(\mathbb{R}^N)$, we have*

$$\min\left(|\varphi|^{q^-}_{\kappa(\xi)q(\xi)}, |\varphi|^{q^+}_{\kappa(\xi)q(\xi)}\right) \leq \||\varphi|^{\kappa(\xi)}|_{q(\xi)} \leq \max\left(|\varphi|^{q^-}_{\kappa(\xi)q(\xi)}, |\varphi|^{q^+}_{\kappa(\xi)q(\xi)}\right).$$

Theorem 2.1. *(Symmetric mountain pass theorem [34]) Let X be an infinite dimensional real Banach space. Let $\mathcal{J} \in C^1(X, \mathbb{R})$, satisfying the following conditions:*

(1) $\mathcal{J}$ is an even functional such that $\mathcal{J}(0) = 0$.
(2) $\mathcal{J}$ satisfies the Palais-Smale condition.
(3) There exist positive constants η and ρ, such that if $\|u\| = \eta$, then, $\mathcal{J}(u) \geq \rho$.
(4) For each finite dimensional subspace $X_1 \subset X$, the set $\{u \in X_1, \mathcal{J}(u) \geq 0\}$ is bounded in X. Then, $\mathcal{J}$ has an unbounded sequence of critical values.

3. Main results

In this section, we present and prove the main result of this work. For this, we assume the following hypotheses:

($\mathbf{C_1}$) We assume that the functions a and h are measurable for which there is a positive constant c_1, and two functions P, q in $C_+(\overline{\Lambda})$ such that for each $(\xi, \varphi) \in \Lambda \times \mathbb{R}$, we have

$$a(\xi) \in \mathcal{L}^{\frac{P(\xi)}{P(\xi)-q(\xi)}}(\Lambda), \quad h(\varphi) \leq c_1 \, |\varphi|^{q(\xi)-1} \, ,$$

and

$$\kappa_2^+ < q(\xi) < P(\xi) < \kappa_2^*(\xi). \tag{3.1}$$

($\mathbf{C_2}$) We assume that for some positive constants m_0, M_0, we have

$$m_0 \leq \mathcal{S}(t) \leq M_0, \;\; \forall \; t \geq 0.$$

($\mathbf{C_3}$) We assume that for any $t \geq 0$, we have

$$\widehat{\mathcal{S}}(t) := \int_0^t \mathcal{S}(s) \, ds \geq (1 - \omega)\mathcal{S}(t)t,$$

for some satisfying $0 < \omega \leq 1 - \frac{1}{\kappa_1^+}$.

($\mathbf{C_4}$) There exist $M_1 > 0$ and $\theta > \frac{\kappa_2^+}{1-\omega}$, such that for any $\xi \in \Lambda$, and any $|\varphi| \geq M_1$, we have

$$0 < \theta a(\xi)H(\varphi) := \theta a(\xi) \int_0^\varphi h(s) \, ds \leq a(\xi)h(\varphi)\varphi.$$

($\mathbf{C_5}$) We have

$$h(-\varphi) = -h(\varphi), \;\; \forall \, \varphi \in [0, T].$$

We note that, if we define a function h by:

$$h(\varphi) = |\varphi|^{q(\xi)-2}\varphi,$$

with $q^- > \frac{\kappa_2^+}{1-\omega}$, and a is a positive function in $\mathcal{L}^{\frac{P(\xi)}{P(\xi)-q(\xi)}}(\Lambda)$, then assumptions ($\mathbf{C_1}$), ($\mathbf{C_4}$), and ($\mathbf{C_5}$) hold. Next, we define a weak solution related to problem (1.1).

Definition 3.1. *A function $\varphi \in \mathcal{H}_{\mathcal{M},0}^{\varpi,\varepsilon;\psi}(\Lambda)$ is a weak solution for the Eq* (1.1) *if, for any $v \in X$, we have*

$$\mathcal{S}\left(\int_\Lambda \mathcal{L}^{\varpi,\varepsilon;\psi}(\xi,\varphi)d\xi\right) \int_\Lambda \mathbf{L}^{\varpi,\varepsilon;\psi}(\xi,\varphi) \;\; {}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\varepsilon;\psi} v(\xi) \, d\xi = \int_\Lambda a(\xi)h(\varphi)v(\xi) \, d\xi.$$

The main result of this work is the following theorem.

Theorem 3.1. *Under the hypotheses* ($\mathbf{C_1}$)–($\mathbf{C_5}$)*, the problem* (1.1) *admits infinitely many weak solutions.*

To prove the last theorem, we begin by defining on $\mathcal{H}^{\varpi,\varepsilon;\psi}_{\mathcal{M},0}(\Lambda)$ the functional $\mathcal{J}$ by:

$$\mathcal{I}(\varphi) \;=\; L(\varphi) - J(\varphi),$$

where L and J are defined on $\mathcal{H}^{\varpi,\varepsilon;\psi}_{\mathcal{M},0}(\Lambda)$ by

$$L(\varphi) = \widehat{\mathcal{S}}\left(\int_{\Lambda} \mathcal{L}^{\varpi,\varepsilon;\psi}(\xi,\varphi(\xi))d\xi\right),$$

and

$$J(\varphi) = \int_{\Lambda} a(\xi)H(\varphi(\xi))\,d\xi.$$

It is proved in [33] that the functional L is of class C^1, and for each $(\varphi, v) \in \left(\mathcal{H}^{\varpi,\varepsilon;\psi}_{\mathcal{M},0}(\Lambda)\right)^2$, we have

$$\begin{aligned} < L^{'}(\varphi), v > \;&=\; \mathcal{S}\left(\int_{\Lambda} \mathcal{L}^{\varpi,\varepsilon;\psi}(\xi,\varphi)d\xi\right)\int_{\Lambda} \mathbf{L}^{\varpi,\varepsilon;\psi}(\xi,\varphi)\ \ {}^{\mathrm{H}}\mathbb{D}^{\varpi,\varepsilon;\psi}_{0+} v\,d\xi \\ :=&\; \mathcal{S}\left(\int_{\Lambda} \mathcal{L}^{\varpi,\varepsilon;\psi}(\xi,\varphi)d\xi\right) < \mathcal{L}^{'}(\varphi), v > . \end{aligned}$$

The functional $\mathcal{L}^{'}$ satisfies the following properties.

Proposition 3.1. *[33] Then, the following statements hold:*

(1) The operator $\mathcal{L}^{'} : \mathcal{H}^{\varpi,\varepsilon;\psi}_{\mathcal{M},0}(\Lambda) \to \mathcal{H}^{\varpi,\varepsilon;\psi}_{\mathcal{M},0}(\Lambda)^*$ is continuous, strictly monotone, and bounded.
(2) The mapping $\mathcal{L}^{'}$ is of type (S_+), which means that any sequence φ_n that converges weakly to φ in $\mathcal{L}^{'} : \mathcal{H}^{\varpi,\varepsilon;\psi}_{\mathcal{M},0}(\Lambda)$ and satisfies in addition,

$$\limsup_{n\to\infty} < \mathcal{L}^{'}(\varphi_n) - \mathcal{L}^{'}(\varphi), \varphi_n - \varphi > \leq 0,$$

converges strongly to φ in $\mathcal{H}^{\varpi,\varepsilon;\psi}_{\mathcal{M},0}(\Lambda)$.

Remark 3.1. *It can be shown using* $(\mathbf{C_1})$*, Propositions 2.5 and 2.7, and the Hölder inequality that $J \in C^1(\mathcal{L}^{'} : \mathcal{H}^{\varpi,\varepsilon;\psi}_{\mathcal{M},0}(\Lambda), \mathbb{R})$. Furthermore, for any $(\varphi, v) \in \left(\mathcal{H}^{\varpi,\varepsilon;\psi}_{\mathcal{M},0}(\Lambda)\right)^2$, we have*

$$< J^{'}(\varphi), v > = \int_{\Lambda} a(\xi)h(\varphi(\xi))v(\xi)\,d\xi.$$

So, Proposition 3.1 and Remark 3.1 imply that $\mathcal{J} \in C^1(\mathcal{H}^{\varpi,\varepsilon;\psi}_{\mathcal{M},0}(\Lambda), \mathbb{R})$. Moreover, for each $(\varphi, v) \in \left(\mathcal{H}^{\varpi,\varepsilon;\psi}_{\mathcal{M},0}(\Lambda)\right)^2$, we get

$$< \mathcal{J}'(\varphi), v > \;=\; \mathcal{S}\left(\int_{\Lambda} \mathcal{L}^{\varpi,\varepsilon;\psi}(\xi,\varphi)d\xi\right) < \mathcal{L}^{'}(\varphi), v > - \int_{\Lambda} a(\xi)h(\varpi(\xi))v(\xi)\,d\xi.$$

By the last equation, we deduce that weak solutions of system (1.1) are critical points of $\mathcal{J}$.
Next, we present an important property of the function in the variational method.

Lemma 3.1. *Assume that* $(\mathbf{C_1})$–$(\mathbf{C_3})$ *are satisfied. Then, there exist positive constants* η *and* ρ*, such that if* $\|\varphi\| = \eta$*, then,* $\mathcal{J}(\varphi) \geq \rho$*,*

Proof. Let $\varphi \in X$, with $\|\varphi\| < 1$. Under the hypothesis $(\mathbf{C_1})$, we have for all $\xi \in \Omega$,

$$H(\varphi) \leq \frac{c_1}{q(\xi)}|\varphi|^{q(\xi)}. \tag{3.2}$$

Since $1 < p(\xi) < \kappa_2^*(\xi)$, and according to Proposition 2.4, we obtain the existence of $c_3 > 0$, such that

$$|\varphi|_{\mathcal{L}^{p(\xi)}(\Omega)} \leq c_3 \|\varphi\|. \tag{3.3}$$

On the other hand, under hypotheses $(\mathbf{C_2})$ and $(\mathbf{C_3})$ and by Proposition 2.5, we get,

$$\begin{aligned} L(\varphi) &= \widehat{\mathcal{S}}\left(\int_\Omega \mathcal{L}(\xi,\varphi)d\xi\right) \\ &\geq (1-\omega)\mathcal{S}\left(\int_\Omega \mathcal{L}(\xi,\varphi)d\xi\right)\int_\Omega \mathcal{L}(\xi,\varphi)d\xi \\ &\geq \frac{(1-\omega)m_0}{\kappa_2^+}\Gamma(\varphi) \geq \frac{(1-\omega)m_0}{\kappa_2^+}\|\varphi\|^{\kappa_2^+}. \end{aligned} \tag{3.4}$$

Now, by (3.2)–(3.4) and using Propositions 2.2, 2.5, and 2.7, we obtain,

$$\begin{aligned} \mathcal{J}(\varphi) &= \widehat{\mathcal{S}}\left(\int_\Omega \mathcal{L}(\xi,\varphi)d\xi\right) - \int_\Omega a(\xi)H(\varphi)d\xi, \\ &\geq \frac{(1-\omega)m_0}{\kappa_2^+}\|\varphi\|^{\kappa_2^+} - \frac{c_3}{q^-}|a|_{\mathcal{L}^{\frac{P(\xi)}{P(\xi)-q(\xi)}}(\Omega)}\|\varphi\|^{q^-} \\ &\geq \|\varphi\|^{\kappa_2^+}\left(\frac{(1-\omega)m_0}{\kappa_2^+} - \frac{c_3}{q^-}|a|_{\mathcal{L}^{\frac{p(\xi)}{p(\xi)-q(\xi)}}(\Omega)}\|\varphi\|^{q^-}\right) \\ &\geq \|\varphi\|^{\kappa_2^+}\left(\frac{(1-\omega)m_0}{\kappa_2^+} - t\|\varphi\|^{q^- - \kappa_2^+}\right), \end{aligned}$$

where

$$t = \frac{c_3}{q^-}|a|_{\mathcal{L}^{\frac{p(\xi)}{p(\xi)-q(\xi)}}(\Omega)}.$$

Since q^- is greater than κ^+, we can choose $\|\varphi\| = \eta$ to be sufficiently small such that

$$\frac{(1-\omega)m_0}{\kappa_2^+} - t\eta^{q^- - \kappa_2^+} > 0.$$

Finally, we conclude that

$$\mathcal{J}(\varphi) \geq \eta^{\kappa_2^+}\Big(\frac{(1-\omega)m_0}{\kappa_2^+} - t\,\eta^{q^- - \kappa_2^+}\Big) := \rho > 0.$$

□

Lemma 3.2. *Assume that* $(\mathbf{C_1})$–$(\mathbf{C_3})$ *are satisfied. Then,* $\mathcal{J}$ *is coercive on* $\mathcal{H}^{\varpi,\varepsilon;\psi}_{\mathcal{M},0}(\Lambda)$.

Proof. Let $\varphi \in \xi$, with $\|\varphi\| > 1$. By (3.2)–(3.4) and using Propositions 2.2, 2.5, and 2.7, we obtain,

$$\begin{aligned}\mathcal{J}(\varphi) &= \widehat{\mathcal{S}}\left(\int_\Lambda \mathcal{L}^{\varpi,\varepsilon;\psi}(\xi,\varphi)d\xi\right) - \int_\Lambda a(\xi)H(\varphi)\,d\xi \\ &\geq \frac{(1-\omega)m_0}{\kappa_2^+}\|\varphi\|^{\kappa_1^+} - \frac{c_3}{q^-}|a|_{\mathcal{L}^{\frac{P(\xi)}{P(\xi)-q(\xi)}}(\Lambda)}\|\varphi\|^{q^-}.\end{aligned}$$

Since $q^- < \kappa_1^+$, then $\mathcal{J}$ is coercive and bounded from below on $\mathcal{H}^{\varpi,\varepsilon;\psi}_{\mathcal{M},0}(\Lambda)$. □

Next, we present the following lemma which establishes an important convergence result.

Lemma 3.3. *Assume that conditions* $(\mathbf{C_1})$–$(\mathbf{C_4})$ *are satisfied and let* $\{\varphi_n\}$ *be a Palais-Smale sequence in* $\mathcal{H}^{\varpi,\varepsilon;\psi}_{\mathcal{M},0}(\Lambda)$*. Then,* $\{\varphi_n\}$ *admits a subsequence that converges strongly in* $\mathcal{H}^{\varpi,\varepsilon;\psi}_{\mathcal{M},0}(\Lambda)$*.*

Proof. We begin by fixing a Palais-Smale sequence $\{\varphi_n\}$ in $\mathcal{H}^{\varpi,\varepsilon;\psi}_{\mathcal{M},0}(\Lambda)$. This means that, for some real number c, we have

$$\mathcal{J}(\varphi_n) \to c, \text{ and } \mathcal{J}'(\varphi_n) \to 0.$$

From Lemma 3.2, we can prove that $\{\varphi_n\}$ is bounded in the reflexive space $\mathcal{H}^{\varpi,\varepsilon;\psi}_{\mathcal{M},0}(\Lambda)$. So, up to a subsequence, we have $\varphi_n \rightharpoonup \varphi$ weakly in $\mathcal{H}^{\varpi,\varepsilon;\psi}_{\mathcal{M},0}(\Lambda)$. Since $P(\xi) < \kappa^*_\varpi(\xi)$, then by Proposition 2.4, we get the strongly convergence of φ_n to φ in $\mathcal{L}^{P(\xi)}(\Lambda)$.

Next, we prove the strong convergence in $\mathcal{H}^{\varpi,\varepsilon;\psi}_{\mathcal{M},0}(\Lambda)$. For this, we begin by remarking that from hypothesis $(\mathbf{C_1})$, Propositions 2.4 and 2.7, and the Hölder inequality, we have

$$\begin{aligned}\int_\Lambda a(\xi)h(\varphi_n)(\varphi_n-\varphi)\,d\xi &\leq \int_\Lambda c_1|a(\xi)||\varphi_n|^{q(\xi)-1}|\varphi_n-\varphi|\,d\xi \\ &\leq c_1|\varphi_n-\varphi|_{\mathcal{L}^{P(\xi)}}|a(\xi)|_{\mathcal{L}^{\frac{P(\xi)}{P(\xi)-q(\xi)}}}\,||\varphi_n|^{q(\xi)-1}|_{\mathcal{L}^{\frac{P(\xi)}{q(\xi)-1}}} \\ &\leq c_1|\varphi_n-\varphi|_{\mathcal{L}^{P(\xi)}}|a(\xi)|_{\mathcal{L}^{\frac{P(\xi)}{P(\xi)-q(\xi)}}}\max\left(||\varphi_n|^{q^+-1}|_{\mathcal{L}^{P(\xi)}}, ||\varphi_n|^{q^--1}|_{\mathcal{L}^{P(\xi)}}\right) \\ &\leq c_1|\varphi_n-\varphi|_{\mathcal{L}^{P(\xi)}}|a(\xi)|_{\mathcal{L}^{\frac{P(\xi)}{P(\xi)-q(\xi)}}}\max\left(\|\varphi_n\|^{q^+-1}|, \|\varphi_n\|^{q^--1}|\right).\end{aligned}$$

Hence, we have

$$\lim_{n\to\infty}\int_\Lambda a(\xi)h(\varphi_n)(\varphi_n-\varphi)\,d\xi = 0. \tag{3.5}$$

Now, using the fact that $< \mathcal{J}'(\varphi_n), \varphi_n-\varphi >\to 0$, we obtain that

$$< L'(\varphi_n), \varphi_n - u > \;=\; \mathcal{S}\left(\int_\Lambda \mathcal{L}^{\varpi,\varepsilon;\psi}(\xi,\varphi_n)d\xi\right) < \mathcal{L}^{'}(\varphi_n), \varphi_n - u >\to 0.$$

On the other hand, from hypothesis $(\mathbf{C_2})$, we know $\mathcal{S}\left(\int_\Lambda \mathcal{L}^{\varpi,\varepsilon;\psi}(\xi,\varphi_n)d\xi\right) \neq 0$, so we get

$$< \mathcal{L}^{'}(\varphi_n), \varphi_n-\varphi >=\to 0.$$

Hence, we deduce that

$$\lim_{n\to\infty} < \mathcal{L}^{'}(\varphi_n) - \mathcal{L}^{'}(\varphi), \varphi_n-\varphi >= 0.$$

Finally, Proposition 3.1, implies that $\varphi_n \to \varphi$ strongly in $\mathcal{H}^{\varpi,\varepsilon;\psi}_{\mathcal{M},0}(\Lambda)$. □

Now, we establish the following lemma that provides a key result regarding the boundedness of a set under certain hypotheses.

Lemma 3.4. *Under hypotheses* $(\mathbf{C_1})$–$(\mathbf{C_4})$*, if F is a finite dimensional subspace of X, then the set*

$$T = \{\varphi \in F, \ \ such \ \ that \ \ \mathcal{J}(\varphi) \geq 0\},$$

is bounded in X.

Proof. Let $\varphi \in T$. By $(\mathbf{C_4})$, there exists a constant $A > 0$ such that

$$a(\xi)H(\varphi) \geq A \mid \varphi \mid^{\theta} . \tag{3.6}$$

Then, By $(\mathbf{C_2})$, (3.6), and Proposition 2.5, we have:

$$\begin{aligned} \mathcal{J}(\varphi) &\leq \widehat{S}\Big(\int_{\Lambda} \mathcal{L}^{\varpi,\nu;\psi}(\xi,\varphi)d\xi\Big) - \int_{\Lambda} a(\xi)H(\varphi)d\xi \\ &\leq M_0\Big(\int_{\Lambda} \mathcal{L}^{\varpi,\nu;\psi}(\xi,\varphi)d\xi\Big) - A\int_{\Lambda} |\varphi|^{\theta}d\xi \\ &\leq C(\|\varphi\|^{\kappa_2^+} + \|\varphi\|^{\kappa_1^-}) - A|\varphi|^{\theta}_{\mathcal{L}^{\theta}}. \end{aligned}$$

Since F is a finite subspace, we know that the norms $|.|_{\mathcal{L}^{\theta}}$ and $\|.\|$ are equivalent. So, we get the existence of $k > 0$ that satisfies:

$$\|\varphi\|^{\theta} \leq k|\varphi|^{\theta}_{\mathcal{L}^{\theta}}.$$

Therefore, we have

$$\mathcal{J}(\varphi) \leq C(\|\varphi\|^{\kappa_2^+} + \|\varphi\|^{\kappa_1^-}) - \frac{A}{k}\|\varphi\|^{\theta}.$$

Hence, since $\kappa_1^- < \kappa_2^+ < \theta$, we deduce that T is bounded in X. □

Proof of Theorem 3.1. We observe that $\mathcal{J}(0) = 0$, and due to $(\mathbf{C_5})$, the functional $\mathcal{J}$ is even. Furthermore, Lemmas 3.1, 3.3, and 3.4 establish the fulfillment of all conditions stated in Theorem 2.1. Consequently, we can conclude that the conclusion of Theorem 2.1 holds, which means that the problem (1.1) possesses an unbounded sequence of nontrivial solutions. □

4. Conclusions

In this paper, we proved the multiplicity of solutions for a double-phase Kirchhoff-type problem involving ψ-Hilfer derivatives and variable exponents. More precisely, we transformed the question of the existence of solutions for such a problem to the existence of critical points for the associated functional energy. After that, we proved that the functional energy satisfies all conditions of the symmetric mountain pass theorem, so the conclusion of this theorem leads to the existence of infinitely many nontrivial solutions. We note that in the case $\mu \equiv 0$, we obtain the result of Sousa et al. [29].
We will generalize this study by studying the following cases:
(1) Perturbed this equation by a singular nonlinearity.
(2) Consider the same problem in Orlicz spaces.

Author contributions

Najla Alghamdi: Conceptualization, writing-review & editing; Abdeljabbar Ghanmi: Conceptualization, resources. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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