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Research article

Continuous functions on primal topological spaces induced by group actions

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Abstract: If G is a group acting on a set X, then for any $a \in G$, the restriction $\phi_a : X \to X$ of the action to a induces a topology τ_a for X, called the primal topology induced by ϕ_a. First, we obtain a characterization of the normal subgroups in terms of the primal topologies. Later, we prove that some commutative relations among elements on the group G determine the continuity of maps among different primal spaces (X, τ_{ϕ_x}). In particular, we prove the continuity of some maps when $a, b, q \in G$ satisfy a quantum type relation, $ba = qab$, as is in the quaternion and Heisenberg groups.

Keywords: group action; primal topology; continuous function
Mathematics Subject Classification: Primary 54A10, 54C05; Secondary 54D05, 54D30

1. Introduction

Primal topological spaces were first introduced by Shirazi and Golestani [12] with the name functional Alexandroff spaces, and subsequently by Echi [2] in the following way: Given a set $X \neq \emptyset$ and a map $f : X \to X$, then the collection, $\tau_f = \{U \subset X : f^{-1}(U) \subset U\}$ is a topology for X, which is called the primal topology induced by f, thus producing the so-called primal space (X, τ_f). Many important issues of this space, especially compactness and connectedness, can be described in terms of the dynamics of its points with respect to the function f. Some further developments came later, especially with the work of Echi [2] and Echi and Turki [3]. The descriptions of the topological properties of primal spaces have been used recently in applications to problems in linear algebra and number theory, see Lazaar et al. [6], Lazaar and Sabri [7], Mejías et al. [8], Vielma and Guale [13], and Vielma et al. [14].

In addition, topologies induced by semigroup actions in a set, which was studied in [4], have been applied both in algebraic and topological contexts, as well as in some areas of computer science. The applications in algebra are established via Green's left quasi-order. In the field of topology, the main idea is to consider the relationship between Green's left quasi-order and principal topologies; see Richmond [9, 10]. Working similarly to those works, and to generalize, we consider the primary topologies induced by group actions and investigate the continuity of maps defined on such primal topological spaces, thereby obtaining some characterizations of homeomorphisms among spaces via properties of the acting groups.

Thus, we begin with an action Φ of a group G on a set X, that is, a map $\Phi : G \times X \to X$ which has some nice properties of compatibility with the group operation. Then, we consider the primal topology induced on X by map the $\phi_a : X \to X$ obtained when Φ is restricted to one specific $a \in G$; it turns out that the continuity of maps among different primal spaces are determined by relations among elements of the group. A very special situation is obtained when the group acts on itself with both a left translation and a conjugation. In that case, we obtain a characterization of normal subgroups in terms of the concepts from the topology.

Our presentation and contribution to the literature begins in Section 2, with an introduction to the basic facts about primal topological spaces and some results and conventions related to the notation. In Section 3, we set the context of the primal topologies induced by group actions with specific characterizations of normal subgroups in terms of the primal topologies. This part is followed by a quite long list of examples in Section 4, which illustrates a broad set of cases that are explained with the results presented in Section 5, where we explore the continuity of maps defined among this type of primal spaces and introduce some properties that are determined by commutation relations among elements of the groups involved. In particular, some conclusions about continuity of maps are derived when we sum the "quantum type" relation as $ba = qab$, which appears in interesting examples such as the quaternion group and the Heisenberg group.

Note that the problem of applying the techniques of this paper to topologies induced by actions of some richer algebraic structures as rings, modules, and algebras may be considered.

2. Primal topological spaces

In this section, we present the basic notions and standard notation related to primal topological spaces and their most important properties. In particular, we present the characterization of both minimal open and closed sets, as well as those of compact and connected sets.

The concept of a primal space was established by Shirazi and Golestani [12] in the following terms: If $X \neq \emptyset$ and $f : X \to X$ is a map, then the collection

$$\tau_f = \{U \subset X : f^{-1}(U) \subset U\}$$

is a topology for X, which is called the *primal topology on X induced by f*, and (X, τ_f) is said to be a *primal space*. In the contexts where no confusion arises, we simply denote (X, τ_f) by X_f. The following two facts are straightforward.

Remark 2.1. The map $f : X_f \to X_f$ is continuous.

Remark 2.2. If the function $f : X \to X$ is a bijective function, then X_f and $X_{f^{-1}}$ are homeomorphic.

Some general properties of primal topological spaces are determined by the dynamics of points concerning the function f. Let $\mathbb{N}$ denote the set of positive integers and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If $r \in \mathbb{N}_0$, then f^r is considered as the r-fold composite $f \circ \cdots \circ f$, where f^0 is the identity map, which is defined by $\Gamma_f(p)$ the *trajectory* of $p \in X$, and given by the following:

$$\Gamma_f(p) = \{y \in X : y = f^r(p) \text{ for } r \in \mathbb{N}_0\}.$$

Thus, it turns out that $\Gamma_f(p) = \overline{\{p\}}$ (the closure of $\{p\}$). If $f^r(p) = p$ for some $r > 0$, then we say that $\Gamma_f(p)$ is a *periodic trajectory* and p itself is referred to as a *periodic point*. If $\Gamma_f(p)$ contains just one element, then p is said to be a *fixed point*.

The basic open sets in X_f can also be described using trajectories. In fact, for any $p \in X$, it is known that the smallest open set containing p, which we denote as $\mathbb{P}_f(p)$, is given by the following:

$$\mathbb{P}_f(p) = \{y \in X : p = f^r(y) \text{ for } r \in \mathbb{N}_0\}.$$

Remark 2.3. In general, the primal space X_f is compact if and only if there exists a finite set $\{p_1, p_2, \ldots, p_n\} \subset X$ such that for all $x \in X$, it turns out that $x \in \mathbb{P}_f(p_i)$ and some $p_i \in \{p_1, p_2, \ldots, p_n\}$. On the other hand, the space X_f is connected if and only if for all $x \in X$, there exists a $p \in X$ such that $x \in \mathbb{P}_f(p) \cup \Gamma_f(p)$.

Next, we exhibit a few examples that illustrate the geometric aspects of primal spaces. The first two examples are well known, see Echi and Turki [3] and Shirazi and Golestani [12].

Example 2.4. If $X \neq \emptyset$ and $q \in X$, then consider the map $f : X \to X$, such that $f(x) = q$ for all $x \in X$ (constant). Then, for all $x \neq q$, we have that $\{x\} \in \tau_f$. Furthermore, the space (X, τ_f) satisfies the axiom T_0, but it is not a T_1 space.

Example 2.5. For any $X \neq \emptyset$, the primal topology induced by the identity map $\mathrm{id} : X \to X$, $\mathrm{id}(x) = x$, is the discrete topology. In this case, each $x \in X$ is a fixed point. This is the only case in which a primal space is T_1, because any set $\{x\}$ is closed if and only if $\Gamma_{\mathrm{id}}(x) = \{x\}$, which is true if and only if the map that induces the topology is the identity.

Example 2.6. Let $s : \mathbb{Z} \to \mathbb{Z}$ be the map defined by $s(x) = 1+x$. For all $x \in \mathbb{Z}$, we have $\mathbb{P}_s(x) \subset \mathbb{P}_s(x+1)$ and the space $(\mathbb{Z}, \tau_s)$ is connected, but it is not compact. This example is due to Dahane et al. [1].

Sometimes, in order to illustrate the properties of a topological primal space, we use a "diagram", where $a \to b$ means that b is the image of a. Thus, we have the following:

$$\cdots \to -2 \to -1 \to 0 \to 1 \to 2 \to \cdots . \qquad (\mathbb{Z}, \tau_s)$$

Example 2.7. Let $C : \mathbb{N} \to \mathbb{N}$ be the map defined by the following:

$$C(x) = \begin{cases} x/2, & \text{if } x \text{ is even,} \\ 3x + 1, & \text{if } x \text{ is odd.} \end{cases}$$

The primal space $(\mathbb{N}, \tau_C)$ has a unique periodic trajectory, namely $\Gamma_C(1) = \{1, 2, 3\}$. This space is closely related to the so-called "Collatz conjecture". In fact, deciding whether or not $(\mathbb{N}, \tau_C)$ is connected is equivalent to solving the famous Collatz problem, see Vielma and Guale [13] and Vielma et al. [14].

Example 2.8. If A is a square matrix of the order n considered as a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$, then A induces a primal topology τ_A on $\mathbb{R}^n$. Some properties of the space $(\mathbb{R}^n, \tau_A)$ are deduced from known facts about the matrix A. For instance, the space $(\mathbb{R}^n, \tau_A)$ is compact if and only if A is nilpotent. Another interesting result is that A and B are similar; then, the respective primal spaces $(\mathbb{R}^n, \tau_A)$ and $(\mathbb{R}^n, \tau_B)$ are homeomorphic, see Mejías et al. [8].

We conclude this section with three results about the relationship between the continuity of functions among primal topological spaces induced on the same set and the trajectories of some elements of its elements. One of the most important points about these results is that it motivates some ideas for more general cases once one considers the set of maps on a set as a semigroup.

Lemma 2.9. *Let X be a set and $\phi, \psi : X \to X$ be two maps. Let τ_ϕ and τ_ψ be the primal topologies induced by ϕ and ψ, respectively. If $\lambda : X_\phi \to X_\psi$ is a function such that $\lambda \circ \phi = \psi \circ \lambda$, then $\lambda(y) \in \Gamma_\psi(\lambda(x))$, for all $x, y \in X$ with $y \in \Gamma_\phi(x)$.*

Proof. Suppose that $x \in X$ and $y \in \Gamma_\phi(x)$. Then, there exists a $k \in \mathbb{N}_0$ such that $\phi^k(x) = y$. In this way, we have the following:

$$\lambda(y) = \lambda \circ \phi^k(x) = \psi^k \circ \lambda(x).$$

Therefore, $\lambda(y) \in \Gamma_\psi(\lambda(x))$. □

The map λ is known as a morphism of flows of X_ϕ to X_ψ. The next lemma is a particular case of result presented by Haouati and Lazaar [5].

Lemma 2.10. *Let X_ϕ, X_ψ be primal spaces and $\lambda : X_\phi \to X_\psi$ be a homeomorphism. Then, $\lambda(\Gamma_\phi(x)) = \Gamma_\psi(\lambda(x))$.*

Proof. Note that $\overline{\{x\}} = \Gamma_\phi(x)$ and $\overline{\{\lambda(x)\}} = \Gamma_\psi(\lambda(x))$. Then, since λ is a homeomorphism, the result is a direct consequence of the fact that for any set $A \subset X_\phi$, we have that $\lambda(\overline{A}) = \overline{\lambda(A)}$. □

Corollary 2.11. *Let X_ϕ, X_ψ be primal spaces and let $\lambda : X_\phi \to X_\psi$ be a homeomorphism. If $x, y \in X$, then $x \in \Gamma_\phi(y)$ is equivalent to $\lambda(x) \in \Gamma_\psi(\lambda(y))$.*

3. Primal topologies induced by group actions

The concept of a primal topology on a set X is based on a set-theoretical notion associated to a function $f : X \to X$, and the complexity of the topology depends on the properties of f.

Now, we turn our attention to the primal topologies induced by functions obtained by the action of groups as a generalization of the results of Mejías et al. [8] and, somehow motivated by the works of Richmond [9, 10], but working with groups rather than semigroups.

Thus, we consider the action of a group on a set and the primal topology induced by the function obtained when we consider the restriction of the action to a particular element of the group. A very special situation arises when the set is the group itself; in that case, we characterize the normal subgroups in terms of the topology.

Our research deals with the primal topologies induced on sets by a very specific types of maps: The actions of semigroups and, mainly, groups. In this section, we introduce some basic facts about those spaces. Recall that if X is a set and G is a semigroup, then an action of G on X is a map $\Phi : G \times X \to X$, such that for all $a, b \in G$ and for all $x \in X$, it turns out that

$$\Phi(a, \Phi(b, x)) = \Phi(ab, x).$$

Furthermore, if G is a monoid, then for an action of G on X, it is required that for all $x \in X$, it is verified that $\Phi(\epsilon, x) = x$. In that case, for each $a \in G$, the map $\phi_a : X \to X$, which is defined by $\phi_a(x) = \Phi(a, x)$, satisfies the following:

$$\phi_a(\phi_b(x)) = \phi_{ab}(x) \quad \text{and} \quad \phi_\epsilon(x) = x.$$

Thus, for each $a \in G$, we focus on the primal topology τ_{ϕ_a} induced by ϕ_a. As mentioned in Section 2, we use both (X, τ_{ϕ_a}) and X_{ϕ_a} to denote the corresponding primal space. Let us note that if G is a group and Φ is an action, then for all $a \in G$, ϕ_a is invertible and ${\phi_a}^{-1} = \phi_{a^{-1}}$.

It is well known that the arbitrary intersection of topologies is another topology. In the context of semigroup actions, such an intersection will be the trivial topology, if the action is transitive.

Theorem 3.1. *Let X be a nonempty set and $\mathcal{F}$ is the collection of all maps $\phi : X \to X$. If G is a semigroup and $\Phi : G \times X \to X$ is a transitive action, then*

$$\bigcap_{\phi \in \mathcal{F}} \tau_\phi = \{\emptyset, X\}.$$

Proof. Suppose that there exists a set $U \in \tau_\phi$ for all $\phi \in \mathcal{F}$, $U \neq \emptyset$, and $U \neq X$. Given $x \in U$, let us take $y \in X \setminus U$; then, there is $a \in G$ such that $\Phi(a, y) = x$, so $y \in \mathbb{P}_{\phi_a}(x)$. Since $\mathbb{P}_{\phi_a}(x)$ is the smallest open set containing x, then $y \in U$, which is a contradiction. □

The following examples show some spaces of particular interest. They play an important role when we consider some concrete cases, especially if G is a group, since we will use the notation given in them.

Example 3.2. If G is a semigroup and $a \in G$, then the left translation by a, which is denoted by $L_a : G \to G$, is given by the following:

$$L_a(x) = ax \quad \text{for all } x \in G,$$

which induces an action $\Phi : G \times G \to G$ defined by the following:

$$\Phi(a, x) = L_a(x).$$

When no confusion arises, we denote the topology by τ_a and the space G_{L_a} by G_a.

Example 3.3. Let G be a semigroup. For $a \in G$, a invertible, we define the conjugation $K_a : G \to G$ which is given for the following:

$$K_a(x) = axa^{-1}, x \in G.$$

In this special case, we denote the primal topology induced by K_a on G as κ_a, that is, $\kappa_a = \tau_{K_a}$. Note that $K_{a^{-1}} : G \to G$ is the inverse map of K_a.

With the conjugation K, it is possible to characterize the normal subgroups of a given group.

Theorem 3.4. *Let G be a group and H be a subgroup of G. Then, H is a normal subgroup if and only if H is a closed set in the space* (G,κ)*, where*

$$\kappa = \bigcap_{a\in G} \kappa_a.$$

Proof. By definition, $H \lhd G$ means that $K_a(H) \subset H$ for all $a \in G$. Hence, H is closed in the space (G,κ_a) for all $a \in G$. Therefore, H is closed in (G,κ).

Now, suppose that H is closed in the space (G,κ); then, H is closed in the space (G,κ_a) and $K_a(H) \subset H$ for all $a \in G$. Therefore, $H \lhd G$. □

In the following, we use the multiplicative notation for the semigroup operation and denote the unit by ϵ. One of the most basic problems in this context is to determine the topological properties of the trajectory of ϵ. We finish this section with three results about this issue.

Theorem 3.5. *If* $G = (G,\cdot,\epsilon)$ *is a group and* $\phi : G \to G$ *is a homomorphism, then* $\mathbb{P}_\phi(\epsilon)$ *is a subgroup of G, which is a closed set in the space* G_ϕ*.*

Proof. Clearly $\mathbb{P}_\phi(\epsilon) \neq \emptyset$, because $\phi(\epsilon) = \epsilon$. If $a \in \mathbb{P}_\phi(\epsilon)$, then there exists $r \in \mathbb{N}_0$ such that $\phi^r(a) = \epsilon$. Thus,

$$\phi^r(a^{-1}) = \epsilon \cdot \phi^r(a^{-1}) = \phi^r(a) \cdot \phi^r(a^{-1}) = \phi^r(aa^{-1}) = \epsilon.$$

Hence, $a^{-1} \in \mathbb{P}_\phi(\epsilon)$.

If $a, b \in \mathbb{P}_\phi(\epsilon)$, then there exist $r, s \in \mathbb{N}_0$ such that $\phi^r(a) = \phi^s(b) = \epsilon$. Then,

$$\phi^{r+s}(ab) = \phi^{r+s}(a) \cdot \phi^{r+s}(b) = \phi^s(\phi^r(a)) \cdot \phi^r(\phi^s(b)) = \phi^s(\epsilon) \cdot \phi^r(\epsilon) = \epsilon.$$

Thus, $ab \in \mathbb{P}_\phi(\epsilon)$. We conclude that $\mathbb{P}_\phi(\epsilon)$ is a nontrivial subgroup of G.

On the other hand, if $a \in \mathbb{P}_\phi(\epsilon)$ and $\phi^k(a) = \epsilon$ for a $k \in \mathbb{N}_0$, then $\phi^{k-1}(\phi(a)) = \epsilon$; thus $\phi(a) \in \mathbb{P}_\phi(\epsilon)$. Therefore, $\mathbb{P}_\phi(\epsilon) \subseteq \phi^{-1}(\mathbb{P}_\phi(\epsilon))$ and $\mathbb{P}_\phi(\epsilon)$ is a closed set. □

One may expect that whenever $\epsilon \in \mathbb{P}_{\phi_a}(b)$, we have that $b^{-1} \in \mathbb{P}_{\phi_a}(\epsilon)$, though such a claim has not been proven. However, the following lemma shows a positive result concerning this matter.

Lemma 3.6. *Let G be a group with* $a, b \in G$*, and* Φ *be an action of G on itself. If* ϕ_a *commutes with* L_b *and* $\phi_a(b) = \epsilon$*, then* $\phi_a(\epsilon) = b^{-1}$*.*

Proof. Since ϕ_a commutes with L_b, it also commutes with ${L_b}^{-1}$. Therefore,

$$\phi_a(\epsilon) = \phi_a(b^{-1}b) = \phi_a({L_b}^{-1}(b)) = {L_b}^{-1}(\phi_a(b)) = {L_b}^{-1}(\epsilon) = L_{b^{-1}}(\epsilon).$$

Thus, $\phi_a(\epsilon) = b^{-1}$. □

In the context of Lemma 3.6, note that we have $({\phi_a}^r(\epsilon))^{-1} = {\phi_{a^{-1}}}^r(\epsilon)$ for all $r \in \mathbb{N}_0$. In this case, the relationship between $\mathbb{P}_{\phi_a}(\epsilon)$ and $\Gamma_{\phi_a}(\epsilon)$ is even deeper, thus revealing its algebraic structure as the following proposition shows.

Lemma 3.7. *Let G be a group and* Φ *be an action of G into itself. Let* $a, b \in G$ *such that* ϕ_a *commutes with* L_b *and* $\phi_a(b) = \epsilon$*. If* $\mathbb{P}_{\phi_a}(\epsilon)$ *is a subgroup of G, then* $\Gamma_{\phi_a}(\epsilon)$ *is periodic.*

Proof. Clearly, $(\phi_a(\epsilon))^{-1} = {\phi_a}^{-1}(\epsilon)$. However, ${\phi_a}^{-1}(\epsilon) \in \mathbb{P}_{\phi_a}(\epsilon)$, which is a subgroup of G. Thus, $\phi_a(\epsilon) \in \mathbb{P}_{\phi_a}(\epsilon)$, that is, for some $r \in \mathbb{N}_0$, it turns out that $\phi_a^r(\epsilon) = \epsilon$. Therefore, $\Gamma_{\phi_a}(\epsilon)$ is periodic. □

Example 2.6 shows that the hypothesis $\mathbb{P}_{\phi_a}(\epsilon)$ is a subgroup in Lemma 3.7, which cannot be dropped; in that case, $a = 1$ and ϕ_a is the left translation.

4. A gallery of examples

Next, we introduce some concrete examples of primal topologies induced by the group actions. In each case, we describe some topological properties that can be proved directly; however, they can also be derived as consequences of some of the results presented in Section 5.

Example 4.1. For $n \in \mathbb{Z}$, let us consider the additive group $(\mathbb{Z}_n, +, 0)$ acting on itself by the left translation, that is, $L_a(x) = a + x$ for all $x \in \mathbb{Z}_n$. If n is a prime number, then the primal topological space $(\mathbb{Z}_n, \tau_{L_a})$ is connected, as illustrated in the following diagrams:

$$\begin{array}{ccccc} 0 & \rightarrow & 3 & \rightarrow & 1 \\ & \nwarrow & & & \downarrow \\ & & 2 & \leftarrow & 4 \end{array}, \qquad (\mathbb{Z}_5, \tau_{L_3})$$

$$\begin{array}{ccccc} 0 & \rightarrow & 4 & \rightarrow & 3 \\ & \nwarrow & & & \downarrow \\ & & 1 & \leftarrow & 2 \end{array}. \qquad (\mathbb{Z}_5, \tau_{L_4})$$

If n is not a prime number, then the space $(\mathbb{Z}_n, \tau_{L_a})$ may be not connected depending on whether or not a is prime with n and a does not divide n. For example, for $n = 6$ and $a = 4$, we have that 4 does not divide 6 but $(\mathbb{Z}_6, \tau_{L_4})$ is not connected. This situation is illustrated as follows:

$$\begin{array}{ccccccc} 0 & \rightarrow & 2 & & 1 & \rightarrow & 3 \\ & \nwarrow & \downarrow & & & \nwarrow & \downarrow \\ & & 4 & & & & 5 \end{array}, \qquad (\mathbb{Z}_6, \tau_{L_2})$$

$$0 \rightleftarrows 3 \quad 1 \rightleftarrows 4 \quad 2 \rightleftarrows 5, \qquad (\mathbb{Z}_6, \tau_{L_3})$$

$$\begin{array}{ccccc} 0 & \rightarrow & 5 & \rightarrow & 4 \\ \uparrow & & & & \downarrow \\ 1 & \leftarrow & 2 & \leftarrow & 3 \end{array}. \qquad (\mathbb{Z}_6, \tau_{L_5})$$

Example 4.2. We consider the general linear group $\mathrm{GL}_2(\mathbb{R})$ acting on $\mathbb{R}^2$ as indicated in Example 2.8, that is, $A \in \mathrm{GL}_2(\mathbb{R})$ is considered as a linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$. Then, A induces a primal topology τ_A on $\mathbb{R}^n$. For instance, let us note that if

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

then the subgroup $H = \langle A \rangle$ of $\mathrm{GL}_2(\mathbb{R})$ generated by A is not closed in the primal space $(\mathrm{GL}_2(\mathbb{R}), \kappa_B)$, because the trajectory of A is not contained in H. In fact,

$$K_B(A) = BAB^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \notin H.$$

We may also consider the primal topology induced by A as an element of the additive group of square matrices; however, that scenario is weaker than the other because of the commutativity of addition of the matrices.

Example 4.3. Let $\mathfrak{S}_3$ be the symmetric group of order 3, that is, the permutation group of the set $\{1, 2, 3\}$. Then,

$$\mathfrak{S}_3 = \{1, \sigma, \sigma^2, \rho, \sigma\rho, \sigma^2\rho\},$$

with the following relations:

$$\sigma^3 = 1, \quad \rho^2 = 1, \quad \rho\sigma = \sigma^2\rho,$$

namely permutations $\sigma = (1, 2, 3)$ and $\rho = (1, 2)$.

Next, we consider the different primal spaces induced on $\mathfrak{S}_3$ by the left translation. The following diagrams illustrate that primal spaces $(\mathfrak{S}_3, \tau_\sigma)$ and $(\mathfrak{S}_3, \tau_{\sigma^2})$ are homeomorphic, which is a consequence of the fact that $\sigma^{-1} = \sigma^2$:

$$\begin{array}{ccccccc} 1 & \to & \sigma & \quad & \rho & \to & \sigma\rho \\ & \nwarrow & \downarrow & & & \nwarrow & \downarrow \\ & & \sigma^2 & & & & \sigma^2\rho \end{array}, \qquad (\mathfrak{S}_3, \tau_\sigma)$$

$$\begin{array}{ccccccc} 1 & \to & \sigma^2 & \quad & \rho & \to & \sigma^2\rho \\ & \nwarrow & \downarrow & & & \nwarrow & \downarrow \\ & & \sigma & & & & \sigma\rho \end{array}. \qquad (\mathfrak{S}_3, \tau_{\sigma^2})$$

On the other hand, it is easy to prove that the spaces $(\mathfrak{S}_3, \tau_\sigma)$, $(\mathfrak{S}_3, \tau_{\sigma\rho})$, and $(\mathfrak{S}_3, \tau_{\sigma^2\rho})$ are homeomorphic:

$$1 \rightleftarrows \rho \qquad \sigma \rightleftarrows \sigma^2\rho \qquad \sigma^2 \rightleftarrows \sigma\rho, \qquad (\mathfrak{S}_3, \tau_\rho)$$

$$1 \rightleftarrows \sigma\rho \qquad \sigma \rightleftarrows \rho \qquad \sigma^2 \rightleftarrows \sigma^2\rho, \qquad (\mathfrak{S}_3, \tau_{\sigma\rho})$$

$$1 \rightleftarrows \sigma^2\rho \qquad \sigma \rightleftarrows \sigma^2\rho \qquad \sigma^2 \rightleftarrows \rho. \qquad (\mathfrak{S}_3, \tau_{\sigma^2\rho})$$

It is easy to verify that the left translation $L_\rho : (\mathfrak{S}_3, \tau_\sigma) \to (\mathfrak{S}_3, \tau_\sigma)$ is continuous, since $\tau_\sigma = \{\emptyset, \Gamma_\sigma(1), \Gamma_\sigma(\rho)\}, \mathfrak{S}_3\}$, $L_\rho(\Gamma_\sigma(1)) = \Gamma_\sigma(\rho)$, and $L_\rho(\Gamma_\sigma(\rho)) = \Gamma_\sigma(1)$.

We may also consider $\mathfrak{S}_3$ acting on itself by the conjugation K_g. Again, since $\sigma^{-1} = \sigma^2$, we have that the spaces $(\mathfrak{S}_3, \kappa_\sigma)$ and $(\mathfrak{S}_3, \kappa_{\sigma^2})$ are homeomorphic:

$$\begin{array}{cccccc} 1 & \sigma & \sigma^2 & \rho & \to & \sigma^2\rho \\ & & & & \nwarrow & \downarrow \\ & & & & & \sigma\rho \end{array}, \qquad (\mathfrak{S}_3, \kappa_\sigma)$$

$$\begin{array}{cccccc} 1 & \sigma & \sigma^2 & \rho & \to & \sigma\rho \\ & & & & \nwarrow & \downarrow \\ & & & & & \sigma^2\rho \end{array}. \qquad (\mathfrak{S}_3, \tau_{\sigma^2})$$

In this case, we have that the spaces $(\mathfrak{S}_3, \kappa_\sigma)$, $(\mathfrak{S}_3, \kappa_{\sigma\rho})$, and $(\mathfrak{S}_3, \kappa_{\sigma^2\rho})$ are homeomorphic, as illustrated in the following diagram:

$$1 \qquad \sigma \rightleftarrows \sigma^2 \qquad \rho \qquad \sigma\rho \rightleftarrows \sigma^2\rho, \qquad (\mathfrak{S}_3, \kappa_\rho)$$

$$1 \qquad \sigma \;\rightleftarrows\; \sigma^2 \qquad \sigma\rho \qquad \rho \;\rightleftarrows\; \sigma^2\rho, \tag{$\mathfrak{S}_3, \kappa_{\sigma\rho}$}$$

$$1 \qquad \sigma \;\rightleftarrows\; \sigma^2 \qquad \sigma^2\rho \qquad \rho \;\rightleftarrows\; \sigma\rho. \tag{$\mathfrak{S}_3, \kappa_{\sigma^2\rho}$}$$

Example 4.4. Let $\mathfrak{S}_3$ be the symmetric group of the set $\{1,2,3\}$, acting on $\mathbb{R}^3$, which is the same as taking the action on $\mathbb{R}^3$ by the subgroup $S_3(\mathbb{R})$ of the general linear group $\mathrm{GL}_3(\mathbb{R})$ generated by the following matrices:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let us note that for all $M \in S_3(\mathbb{R})$, the primal space $(\mathbb{R}^3, \tau_M)$ is not compact.

Example 4.5. Let us consider the following quaternion group:

$$\mathbb{H} = \{\pm\mathbf{1}, \pm\boldsymbol{i}, \pm\boldsymbol{k}, \pm\boldsymbol{j}\},$$

with the relations

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -\mathbf{1}.$$

We may easily prove, that for any $a \neq b$, $a, b \neq \pm\mathbf{1}$, the left translation $L_b : (\mathbb{H}, \tau_a) \to (\mathbb{H}, \tau_a)$ is continuous. For example, $\tau_{\boldsymbol{i}} = \{\emptyset, \Gamma_{\boldsymbol{i}}(\mathbf{1}), \Gamma_{\boldsymbol{i}}(\boldsymbol{j}), \mathbb{H}\}$, $L_{\boldsymbol{j}}(\Gamma_{\boldsymbol{i}}(\mathbf{1})) = \Gamma_{\boldsymbol{i}}(\boldsymbol{j})$, and $L_{\boldsymbol{j}}(\Gamma_{\boldsymbol{i}}(\boldsymbol{j})) = \Gamma_{\boldsymbol{i}}(\mathbf{1})$.

$$\begin{array}{ccccccc} \mathbf{1} & \to & \boldsymbol{i} & & \boldsymbol{j} & \to & \boldsymbol{k} \\ \uparrow & & \downarrow & & \uparrow & & \downarrow \\ -\boldsymbol{i} & \leftarrow & -\mathbf{1} & & -\boldsymbol{k} & \leftarrow & -\boldsymbol{j} \end{array}, \tag{$\mathbb{H}, \tau_{\boldsymbol{i}}$}$$

$$\begin{array}{ccccccc} \mathbf{1} & \to & \boldsymbol{j} & & \boldsymbol{k} & \to & \boldsymbol{i} \\ \uparrow & & \downarrow & & \uparrow & & \downarrow \\ -\boldsymbol{j} & \leftarrow & -\mathbf{1} & & -\boldsymbol{i} & \leftarrow & -\boldsymbol{k} \end{array}, \tag{$\mathbb{H}, \tau_{\boldsymbol{j}}$}$$

$$\begin{array}{ccccccc} \mathbf{1} & \to & \boldsymbol{k} & & \boldsymbol{i} & \to & \boldsymbol{j} \\ \uparrow & & \downarrow & & \uparrow & & \downarrow \\ -\boldsymbol{k} & \leftarrow & -\mathbf{1} & & -\boldsymbol{j} & \leftarrow & -\boldsymbol{i} \end{array}. \tag{$\mathbb{H}, \tau_{\boldsymbol{k}}$}$$

These diagrams suggest that all the primal spaces $(\mathbb{H}, \tau_g)$ are homeomorphic for $g \neq \pm\mathbf{1}$. In fact, the diagrams themselves indicate the respective homeomorphisms.

Again, let us consider the quaternion group $\mathbb{H}$ acting on itself, though now with a conjugation. Similar arguments can be introduced to find homeomorphisms among the different spaces $(\mathbb{H}, \kappa_g)$, for $g \neq \pm\mathbf{1}$. Let us note that $K_g(x) = x$, for $x = \pm\mathbf{1}, \pm\boldsymbol{g}$, Thus, in this case, the situation looks as follows:

$$\mathbf{1} \qquad -\mathbf{1} \qquad \boldsymbol{i} \qquad -\boldsymbol{i} \qquad \boldsymbol{j} \;\rightleftarrows\; -\boldsymbol{j} \qquad \boldsymbol{k} \;\rightleftarrows\; -\boldsymbol{k}, \tag{$\mathbb{H}, \kappa_{\boldsymbol{i}}$}$$

$$\mathbf{1} \qquad -\mathbf{1} \qquad \boldsymbol{j} \qquad -\boldsymbol{j} \qquad \boldsymbol{k} \;\rightleftarrows\; -\boldsymbol{k} \qquad \boldsymbol{i} \;\rightleftarrows\; -\boldsymbol{i}, \tag{$\mathbb{H}, \kappa_{\boldsymbol{j}}$}$$

$$\mathbf{1} \qquad -\mathbf{1} \qquad \boldsymbol{k} \qquad -\boldsymbol{k} \qquad \boldsymbol{i} \;\rightleftarrows\; -\boldsymbol{i} \qquad \boldsymbol{j} \;\rightleftarrows\; -\boldsymbol{j}. \tag{$\mathbb{H}, \kappa_{\boldsymbol{k}}$}$$

Let us note that $K_{\boldsymbol{k}} : (\mathbb{H}, \kappa_{\boldsymbol{i}}) \to (\mathbb{H}, \kappa_{\boldsymbol{j}})$ and $K_{\boldsymbol{k}} \circ K_{\boldsymbol{j}} : (\mathbb{H}, \kappa_{\boldsymbol{i}}) \to (\mathbb{H}, \kappa_{\boldsymbol{i}})$ are continuous functions and homeomorphisms. Additionally, let us note also that, in general, K_a does not commute with L_b, $a, b = \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, $a \neq b$. For example, $L_{\boldsymbol{i}}K_{\boldsymbol{j}}(\boldsymbol{i}) = \mathbf{1}$, though $K_{\boldsymbol{j}}L_{\boldsymbol{i}}(\boldsymbol{i}) = -\mathbf{1}$.

The next example brings one important mathematical object to the context of a primal topology. The n-th *Heisenberg group* $\mathfrak{H}_n(\mathbb{R})$ is considered as the set $\mathfrak{H}_n(\mathbb{R}) = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with the operation $*$ defined for all $a, b, a', b' \in \mathbb{R}^n$ and $c, c' \in \mathbb{R}$ as follows:

$$(a, b, c) * (a', b', c') = (a + a', b + b', c + c' + a' \cdot b),$$

where $\cdot$ on the right hand side of the equation represents the standard inner product on $\mathbb{R}^n$. A discrete version of the Heisenberg group may be obtained by considering $\mathbb{Z}$ instead of $\mathbb{R}$. Furthermore, in a similar fashion, we can take any commutative ring R instead of $\mathbb{R}$, see Semmes [11]. In particular, it makes sense to consider $\mathfrak{H}_n(\mathbb{Z}_p)$ for any $p \in \mathbb{N}$. In this paper, it is enough to consider $n = 1$ and $p = 2$ for the point that we want to illustrate.

Example 4.6. Let $\mathfrak{H} = \mathfrak{H}_1(\mathbb{Z}_2)$ be the group the Heisenberg associated to $\mathbb{Z}_2 = \{0, 1\}$, that is, the set of triplets (a, b, c) where $a, b, c \in \mathbb{Z}_2$, with the following operation:

$$(a, b, c) * (a', b', c') = (a + a', b + b', c + c' + a' \cdot b),$$

where $\cdot$ on the right hand side represents the usual product in $\mathbb{Z}_2$. It is easy to verify that $(\mathfrak{H}, *)$ is a noncommutative group with the identity $\epsilon = (0, 0, 0)$ and $(a, b, c)^{-1} = (-a, -b, a \cdot b - c)$.

Let us note that $\mathfrak{H}$ is generated by the elements $\alpha = (1, 0, 0), \beta = (0, 1, 0)$, and $\gamma = (0, 0, 1)$, with the following relations:

$$\alpha^2 = \beta^2 = \gamma^2 = \epsilon, \quad \alpha\gamma = \gamma\alpha, \quad \beta\gamma = \gamma\beta, \quad \beta\alpha = \gamma\alpha\beta.$$

Thus, $\mathfrak{H} = \{\epsilon, \alpha, \beta, \gamma, \alpha\beta, \beta\alpha, \alpha\gamma, \beta\gamma\}$. It is clear that $\mathfrak{H}$ is isomorphic to D_8, which is the dihedral group of order 8 (the group of symmetries of the square); however, in the context of this research, we prefer to stick to the the name "Heisenberg group" in order to consider the possibility of more general results.

The following diagrams illustrate the different primal spaces $\mathfrak{H}_a$, $a \neq \epsilon$, with $\mathfrak{H}$ acting on itself by the left translation $L_a(x) = ax$:

$$\epsilon \rightleftarrows \alpha \qquad \beta \rightleftarrows \alpha\beta \qquad \gamma \rightleftarrows \alpha\gamma \qquad \beta\alpha \rightleftarrows \beta\gamma, \tag{$\mathfrak{H}_\alpha$}$$

$$\epsilon \rightleftarrows \beta \qquad \alpha \rightleftarrows \beta\alpha \qquad \gamma \rightleftarrows \beta\gamma \qquad \alpha\beta \rightleftarrows \alpha\gamma, \tag{$\mathfrak{H}_\beta$}$$

$$\epsilon \rightleftarrows \gamma \qquad \alpha \rightleftarrows \alpha\gamma \qquad \beta \rightleftarrows \beta\gamma \qquad \alpha\beta \rightleftarrows \beta\alpha, \tag{$\mathfrak{H}_\gamma$}$$

$$\epsilon \rightleftarrows \alpha\gamma \qquad \alpha \rightleftarrows \gamma \qquad \beta \rightleftarrows \beta\gamma \qquad \alpha\beta \rightleftarrows \beta\gamma, \tag{$\mathfrak{H}_{\alpha\gamma}$}$$

$$\epsilon \rightleftarrows \beta\gamma \qquad \alpha \rightleftarrows \alpha\beta \qquad \beta \rightleftarrows \gamma \qquad \alpha\gamma \rightleftarrows \beta\alpha, \tag{$\mathfrak{H}_{\beta\gamma}$}$$

$$\begin{array}{ccccccc} \epsilon & \rightarrow & \alpha\beta & & \alpha & \rightarrow & \beta\gamma \\ \uparrow & & \downarrow & & \uparrow & & \downarrow \\ \beta\alpha & \leftarrow & \gamma & & \beta & \leftarrow & \alpha\gamma \end{array} . \tag{$\mathfrak{H}_{\alpha\beta}$}$$

It is easy to verify that the primal spaces $\mathfrak{H}_a$ with $a \notin \{\epsilon, \alpha\beta, \beta\alpha\}$ are homeomorphic to each other. On the other hand, $\mathfrak{H}_{\alpha\beta}$ and $\mathfrak{H}_{\beta\alpha}$ are homeomorphic, because $(\alpha\beta)^{-1} = \beta\alpha$. Additionally, let us note that $L_\beta : \mathfrak{H}_\alpha \to \mathfrak{H}_\alpha$ is not continuous, since $L_\beta(\mathbb{P}_\alpha(\epsilon)) = \{\beta, \beta\alpha\}$ is not connected.

Now, consider the different primal spaces generated by the conjugation on the Heisenberg group, $(\mathfrak{H}, \kappa_a)$, for $a = \alpha, \beta, \alpha\beta$. Note that $(\mathfrak{H}, \kappa_\gamma)$ is homeomorphic to $(\mathfrak{H}, \kappa_\epsilon)$ because γ commutes with all the elements of $\mathfrak{H}$, and $(\mathfrak{H}, \kappa_{\alpha\gamma})$ y $(\mathfrak{H}, \kappa_{\beta\gamma})$ are homeomorphic to $(\mathfrak{H}, \kappa_\alpha)$, because $K_{\alpha\gamma} = K_\alpha$.

$$\epsilon \quad \beta \quad \gamma \quad \beta\gamma \quad \alpha \rightleftarrows \alpha\gamma \quad \alpha\beta \rightleftarrows \beta\alpha, \qquad (\mathfrak{H}, \kappa_\beta)$$

$$\epsilon \quad \alpha \quad \gamma \quad \alpha\gamma \quad \beta \rightleftarrows \beta\gamma \quad \alpha\beta \rightleftarrows \beta\alpha, \qquad (\mathfrak{H}, \kappa_\alpha)$$

$$\epsilon \quad \gamma \quad \alpha\beta \quad \beta\alpha \quad \alpha \rightleftarrows \alpha\gamma \quad \beta \rightleftarrows \beta\gamma. \qquad (\mathfrak{H}, \kappa_{\alpha\beta})$$

Related to the remarks about Example 4.6, we consider the Heisenberg group associated to $\mathbb{Z}_3$. It is a non-abelian group of order 27 generated by three elements.

Example 4.7. The Heisenberg group $\mathfrak{H}_1(\mathbb{Z}_3)$ is generated by the elements $\alpha = (1, 0, 0)$, $\beta = (0, 1, 0)$, and $\gamma = (0, 0, 1)$ under the following relations:

$$\alpha^3 = \beta^3 = \gamma^3 = \epsilon, \quad \alpha\gamma = \gamma\alpha, \quad \beta\gamma = \gamma\beta, \quad \beta\alpha = \gamma\alpha\beta.$$

The following diagram describes the primal space $(\mathfrak{H}_1(\mathbb{Z}_3), \tau_\alpha)$:

$$\begin{array}{ccccccccc}
\epsilon & \rightarrow & \alpha & \beta & \rightarrow & \alpha\beta & \gamma & \rightarrow & \alpha\gamma \\
& \nwarrow & \downarrow & & \nwarrow & \downarrow & & \nwarrow & \downarrow \\
& & \alpha^2 & & & \alpha^2\beta & & & \alpha^2\gamma \\
\alpha\beta^2 & \rightarrow & \alpha^2\beta^2 & \alpha\beta^2\gamma & \rightarrow & \alpha^2\beta^2\gamma & \alpha\beta\gamma & \rightarrow & \alpha^2\beta\gamma \\
& \nwarrow & \downarrow & & \nwarrow & \downarrow & & \nwarrow & \downarrow \\
& & \beta^2 & & & \beta^2\gamma & & & \beta\gamma \\
\alpha\beta^2\gamma^2 & \rightarrow & \alpha^2\beta^2\gamma^2 & \alpha\gamma & \rightarrow & \alpha^2\gamma & \gamma^2 & \rightarrow & \alpha\gamma^2 \\
& \nwarrow & \downarrow & & \nwarrow & \downarrow & & \nwarrow & \downarrow \\
& & \beta^2\gamma^2 & & & \gamma & & & \alpha^2\gamma^2
\end{array}.$$

Similarly, we can verify that the primal space $(\mathfrak{H}_1(\mathbb{Z}_3), \tau_\beta)$ contains nine connected components and each of them is a cycle with three elements; therefore, the spaces are homeomorphic.

5. Commutative relations and continuity

Our main purposes in this work is to obtain links between properties of a group and the topological properties of the primal spaces that it induces by actions. Thus, it seems to be natural to look for characterizations of the primal spaces, which are determined by relations among some elements of the group. In that sense, we begin by showing that if a and b belong to a group G such that a is in the centralizer of b, then the spaces (X, τ_a) and (X, κ_a) are homeomorphic to (X, τ_b) and (X, κ_b), respectively. Before considering that situation, we will establish some general facts.

Example 5.1. Let us consider the following "rotation matrix":

$$A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

For $\theta = \pi/2$ and $\theta = \pi$, the corresponding matrices, which are considered as linear maps, induce the primal topologies $\tau_{A_{\pi/2}}$ and τ_{A_π} on $\mathbb{R}^2$, respectively, see Example 2.8. Note that any nonempty set in $\tau_{A_{\pi/2}}$ other than $\{(0,0)\}$ has at least four elements. Thus, the function $\iota : (\mathbb{R}^2, \tau_{A_{\pi/2}}) \to (\mathbb{R}^2, \tau_{A_\pi})$, which is defined by $\iota(x) = x$ for all x, is not continuous. In fact, if $\mathrm{e}_1 = (1,0)$, then $\mathrm{e}_1 = (0,1) \in \mathbb{R}^2$ and $\Gamma_{A_\pi}(\mathrm{e}_1) = \{\mathrm{e}_1, -\mathrm{e}_1\} \notin \tau_{A_{\pi/2}}$.

Lemma 5.2. *If G is a semigroup, then $a, b \in G$, and $R_b : G_a \to G_a$ is defined by the following:*

$$R_b(x) = xb.$$

Then, R_b is continuous.

Proof. We consider $V \in \tau_a$, that is, ${L_a}^{-1}(V) \subset V$. If $x \in {L_a}^{-1}({R_b}^{-1}(V))$, then $(ax)b = a(xb) \in V$. In other words, $xb \in {L_a}^{-1}(V) \subset V$. This implies that $x \in {R_b}^{-1}(V)$, so we conclude that ${L_a}^{-1}({R_b}^{-1}(V)) \subset {R_b}^{-1}(V)$. Therefore, ${R_b}^{-1}(V) \in \tau_a$. □

It is important to know what sort of relations among the elements of a group G acting on set X may allow us to decide whether or not two primal spaces are homeomorphic. In that order of ideas, Mejías et al. [8] proved that if A and B are two similar matrices of order n, that is, $A = PBP^{-1}$ for some P, then they induce homeomorphic topologies in $\mathbb{R}^n$. Besides this, as we saw in Example 4.3, the primal spaces $(\mathfrak{S}_3, \tau_\sigma)$, $(\mathfrak{S}_3, \tau_{\sigma\rho})$, and $(\mathfrak{S}_3, \tau_{\sigma^2\rho})$ are homeomorphic. This conclusion may be obtained from the following relations:

$$\sigma\rho = (\sigma\rho)\rho(\sigma\rho)^{-1}, \quad \text{and} \quad \sigma^2\rho = \sigma\rho\sigma^{-1},$$

where σ and ρ are the generators of the symmetric group $\mathfrak{S}_3$. Motivated by these examples, we have derived the following general result.

Theorem 5.3. *Let $\Phi : G \times X \to X$ be an action of a group G on a set X. Suppose that there exist $a, b, g \in G$ such that $a = gbg^{-1}$. Then, the primal spaces X_a and X_b are homeomorphic.*

Proof. We prove that the left translation $L_g : X_b \to X_a$ is a homeomorphism. If $V \in \tau_a$, then $a^{-1}(V) \subset V$ and

$$b^{-1}({L_g}^{-1}(V)) = b^{-1}(g^{-1}(V)) = (gb)^{-1}(V) = (ag)^{-1}(V) = g^{-1}(a^{-1}(V)).$$

Note that $g^{-1}(a^{-1}(V)) \subset g^{-1}(V) = {L_g}^{-1}(V)$. Thus, ${L_g}^{-1}(V) \in \tau_b$.

Now, let F be a closed set in X_a, that is, $a(F) \subset F$. Then, we have the following:

$$b(L_g(F)) = (bg)(F) = (ga)(F) = g(a(F)) \subset g(F) = L_g(F).$$

Then, L_g is a closed map, which means that ${L_g}^{-1} : X_a \to X_b$ is continuous; therefore, L_g is a homeomorphism. □

By applying the same hypothesis of Theorem 5.3 to the conjugation by g, K_g, we obtain that the topological primal spaces (X, κ_a) and (X, κ_b) are homeomorphic.

Theorem 5.4. *Let $\Phi : G \times X \to X$ be an action of a group G on a set X. Suppose that there exist $a, b, g \in G$ such that $a = gbg^{-1}$. Then, the conjugation by g, $K_g : (X, \kappa_b) \to (X, \kappa_a)$ is a homeomorphism.*

Proof. Let us prove that the conjugation by g, $K_g : (X, \kappa_b) \to (X, \kappa_a)$ is an open map. If $V \in \kappa_b$ and $x \in K_a{}^{-1}(K_g(V)) = K_{a^{-1}}(K_g(V))$, then $g^{-1}axa^{-1}g \in V$. However, $a = gbg^{-1}$, so $bg^{-1}xgb^{-1} \in V$, which means that $g^{-1}xg \in K_b^{-1}(V) \subset V$ because $V \in \kappa_b$. Hence, $x \in K_g(V)$ and $K_a{}^{-1}(K_g(V)) \subset K_g(V)$. In other words, $K_g(V) \in \kappa_a$. The same kind of argument shows that $K_g{}^{-1} : (X, \kappa_a) \to (X, \kappa_b)$ is open and, since K_g is a bijection, we conclude that K_g is a homeomorphism. □

With respect to Example 4.3, we can use Theorem 5.4 to prove that the spaces $(\mathfrak{S}_3, \kappa_\sigma)$, $(\mathfrak{S}_3, \kappa_{\sigma\rho})$, and $(\mathfrak{S}_3, \kappa_{\sigma^2\rho})$ are homeomorphic.

Let us note that an argument similar to that in the proofs of Theorems 5.3 and 5.4 gives us a result about the primal spaces X_{ϕ_a} and X_{ϕ_b} that involve an arbitrary action.

Theorem 5.5. *Let G be a semigroup and $\Phi : G \times X \to X$ be an action. If $a, b \in G$ with $ab = ba$, then the maps $\phi_a : X_{\phi_b} \to X_{\phi_b}$ and $\phi_b : X_{\phi_a} \to X_{\phi_a}$ are both continuous.*

Proof. Of course, it is enough to prove that $\phi_a : X_{\phi_b} \to X_{\phi_b}$ is continuous.

Let $V \in \tau_{\phi_b}$ and $x \in \phi_b{}^{-1}(\phi_a{}^{-1}(V))$; then, $\phi_a(\phi_b(x)) \in V$ and

$$\phi_b{}^{-1}(\phi_a(\phi_b(x))) \in \phi_b{}^{-1}(V).$$

However, a and b commute, thus $\phi_a\phi_b = \phi_b\phi_a$ and $\phi_a(x) \in \phi_b{}^{-1}(V) \subset V$. Therefore, $x \in \phi_a{}^{-1}(V)$, which means that $\phi_a{}^{-1}(V) \in \tau_{\phi_b}$. □

Corollary 5.6. *If G is a group and $a \in G$, then the left translation $L_a : G_a \to G_a$ is a homeomorphism.*

Proof. We know that L_a is continuous by Remark 2.1. Let us note that Theorem 5.5 implies that $L_{a^{-1}} = L_a{}^{-1} : G_a \to G_a$ is continuous. □

Corollary 5.7. *If G is a group and $a \in G$, then the conjugation $K_a : G_a \to G_a$ is continuous.*

Proof. We know that $K_a = L_a \circ R_{a^{-1}}$; therefore, we obtain the continuity of K_a by Theorem 5.5 and Lemma 5.2 applied to $b = a^{-1}$. □

It may seem that the hypothesis of the commutativity of a and b in Theorem 5.5 cannot be dropped. For instance, if we consider $A, B \in \mathrm{GL}_2(\mathbb{R})$, as in Example 2.8,

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

then $AB \neq BA$ and $\mathbb{P}_B(0, 1) = \{(0, 1)\} \in \tau_B$, but $A^{-1}(\mathbb{P}_B(0, 1)) = \{(-1, 1)\} \notin \tau_B$ because $(-1, 2) \in \mathbb{P}_B(-1, 1)$.

However, as the next theorem shows, the hypothesis of commutativity can be replaced by a sort of weaker condition.

Theorem 5.8. *Let $\Phi : G \times X \to X$ be an action of a semigroup on a set X. If $a, b, q \in G$ with $ba = qab$ and $\tau_a \subset \tau_q$, then $\phi_b : X_a \to X_a$ is continuous.*

Proof. Suppose that $V \in \tau_a = \tau_{\phi_a}$, which means that $\phi_a{}^{-1}(V) \subset V$. Then, if $x \in \phi_a{}^{-1}(\phi_b{}^{-1}(V))$, then we have $\phi_b(\phi_a(x)) = \phi_{ba}(x) = \phi_{qab}(x) \in V$. Thus, $\phi_{ab}(x) \in \phi_q{}^{-1}(V)$; from $\tau_a \subset \tau_q$, it turns out that $\phi_{ab}(x) \in V$. Then, $\phi_b(x) \in \phi_a{}^{-1}(V) \subset V$, hence $x \in \phi_b{}^{-1}(V)$. Thus, $\phi_a{}^{-1}(\phi_b{}^{-1}(V)) \subset \phi_b{}^{-1}(V)$, meaning that $\phi_b{}^{-1}(V) \in \tau_a$, so the map ϕ_b is continuous. □

Let us note that the hypothesis $\tau_a \subset \tau_q$ in Theorem 5.5 cannot be dropped. In fact, in the case of the Heisenberg group (Example 4.6), we have $\beta\alpha = \gamma\alpha\beta$, though $L_\beta : \mathfrak{H}_\alpha \to \mathfrak{H}_\alpha$ is not continuous.

Note that Theorem 5.5 is actually a particular case of Theorem 5.8 which takes $q = \epsilon$. Between these two results, we have the following corollary, which suits very nicely to the investigation of some specific examples.

Corollary 5.9. *Let $\Phi : G \times X \to X$ be an action of a semigroup on a set X. If $a, b \in G$ with $a^k b = ba$ for a number $k \in \mathbb{N}$, then $\phi_b : X_a \to X_a$ is continuous.*

As an application of Corollary 5.9, let us note that for the symmetric group $\mathfrak{S}_3$ (Example 4.3) from the relation $\rho\sigma = \sigma^2\rho$, we deduce that the left translation $L_\rho : (\mathfrak{S}_3, \tau_\sigma) \to (\mathfrak{S}_3, \tau_\sigma)$ is continuous.

Similarly, in Example 4.5, the equality $a^3 b = ba$ whenever $a \neq b$, $a, b \neq \pm\mathbf{1}$ and Corollary 5.9 imply that the left translation $L_b : (\mathbb{H}, \tau_a) \to (\mathbb{H}, \tau_a)$ is continuous.

Next, we consider other relevant consequences of Theorem 5.8.

Corollary 5.10. *Let $\Phi : G \times X \to X$ and $\Psi : G \times Y \to Y$ be actions of the semi-group G on some sets X and Y. Suppose that $a, b, q \in G$ with $ba = qab$ and $\tau_{\phi_a} \subset \tau_{\phi_q}$. If $f : X_{\phi_a} \to Y_{\psi_a}$ is a continuous function, then the composite $f \circ \phi_b : X_{\phi_a} \to Y_{\psi_a}$ is continuous.*

Corollary 5.11. *Let $\Phi : G \times X \to X$ and $\Psi : G \times Y \to Y$ be actions of the semi-group G on some sets X and Y. Suppose that $a, b, q \in G$ with $ba = qab$ and $\tau_{\phi_a} \subset \tau_{\phi_q}$. If $f : X_{\phi_a} \to Y_{\psi_b}$ is a continuous function, then the composite $f \circ \phi_b : X_{\phi_a} \to Y_{\psi_b}$ is continuous.*

Corollary 5.11 allows us to prove that the functions $K_{\boldsymbol{k}} : (\mathbb{H}, \kappa_{\boldsymbol{i}}) \to (\mathbb{H}, \kappa_{\boldsymbol{j}})$, $K_{\boldsymbol{j}} : (\mathbb{H}, \kappa_{\boldsymbol{k}}) \to (\mathbb{H}, \kappa_{\boldsymbol{i}})$, and $K_{\boldsymbol{k}} \circ K_{\boldsymbol{j}} : (\mathbb{H}, \kappa_{\boldsymbol{i}}) \to (\mathbb{H}, \kappa_{\boldsymbol{j}})$ in Example 4.5 are homeomorphisms.

Lemma 5.12. *Let $\Phi : G \times X \to X$ be an action of G on a set X. If $a, b \in G$ with $\tau_{\phi_b} \subset \tau_{\phi_{ab^{-1}}}$, then $\tau_{\phi_b} \subset \tau_{\phi_a}$.*

Proof. Let $V \in \tau_{\phi_b}$. Then, $V \in \tau_{\phi_{ab^{-1}}}$, which means that $(\phi_a {\phi_b}^{-1})^{-1}(V) \subset V$. Therefore,

$${\phi_a}^{-1}(V) \subset {\phi_b}^{-1}(V) \subset V,$$

because $V \in \tau_{\phi_b}$. In other words, $V \in \tau_{\phi_a}$. □

Corollary 5.13. *Let $\Phi : G \times X \to X$ be an action of G on a set X. If $a, b \in G$ with $\tau_{\phi_b} \subset \tau_{\phi_{ab^{-1}}}$, then the map $\iota : X_{\phi_a} \to X_{\phi_b}$ is continuous.*

Theorem 5.14. *Let $\Phi : G \times X \to X$ be an action of G on a set X. Let G be a group and $G \times X \to X$ be an action. If $a, b \in G$ commute and $\tau_{\phi_a}, \tau_{\phi_b} \subset \tau_{\phi_{ba^{-1}}}$, then X_{ϕ_a} and X_{ϕ_b} are homeomorphic.*

Proof. Let us consider the composite $\varphi = \phi_a \circ \iota : X_{\phi_a} \to X_{\phi_b}$:

$$X_{\phi_a} \xrightarrow{\iota} X_{\phi_b} \xrightarrow{\phi_a} X_{\phi_b}.$$

Obviously, φ is bijective. By Corollary 5.13, we have that both ι and ι^{-1} are continuous. On the other hand, by Theorem 5.5, we have that ϕ_a and ${\phi_a}^{-1}$ are continuous. Thus, both φ and φ^{-1} are continuous. □

6. Conclusions

A primal space (X, τ) is a topological structure constructed from a set-theoretical basis, where the topology τ is defined in terms of sets determined by a function $f : X \to X$, namely

$$\tau = \{U \subset X : f^{-1}(U) \subset U\}.$$

Therefore, it seems to be a natural option to investigate the properties of the topology τ when the function f has some associated structures other than the ones from the set theory. With that idea in mind, we considered functions that were induced by actions of a group G on a set X. It turns out that the algebraic structure has remarkable implications on the properties of τ and the other way around. In particular, we proved the following results:

1. If H is a subgroup of a group G, then H is normal if and only if H is a closed set in the space (G, κ), where
$$\kappa = \bigcap_{a \in G} \kappa_a.$$
2. If $\Phi : G \times X \to X$ is an action of a group G on a set X and if there are $a, b, g \in G$ such that $a = gbg^{-1}$, then the conjugation by g, $K_g : (X, \kappa_b) \to (X, \kappa_a)$ is a homeomorphism.
3. If $\Phi : G \times X \to X$ is an action of a group on a set X, $a, b, q \in G$ with $ba = qab$ and $\tau_a \subset \tau_q$, then $\phi_b : X_a \to X_a$ is continuous.

From these results, it makes sense to consider similar problems with some more complex algebraic structures such as rings, algebras, etc., and their eventual applications.

Author contributions

All authors contributed equally to the manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors have no conflict of interest to declare.

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