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Research article

Numerical solutions of multi-term fractional reaction-diffusion equations

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Abstract: In this paper, we have proposed a numerical approach based on generalized alternating numerical fluxes to solve the multi-term fractional reaction-diffusion equation. This type of equation frequently arises in the mathematical modeling of ultra-slow diffusion phenomena observed in various physical problems. These phenomena are characterized by solutions that exhibit logarithmic decay as time t approaches infinity. For spatial discretization, we employed the discontinuous Galerkin method with generalized alternating numerical fluxes. Temporal discretization was handled using the finite difference method. To ensure the robustness of the proposed scheme, we rigorously established its unconditional stability through mathematical induction. Finally, we conducted a series of comprehensive numerical experiments to validate the accuracy and efficiency of the scheme, demonstrating its potential for practical applications.

Keywords: fractional reaction-diffusion equation; time-fractional derivative; ultra-slow diffusion; stability
Mathematics Subject Classification: 65M12, 65M06, 35S10

1. Introduction

Fractional calculus, often regarded as a natural extension of classical calculus, has garnered significant attention over the past few decades. Its versatility has led to the widespread application of fractional-order partial differential equations (FPDEs) in addressing diverse scientific challenges across fields such as quantitative finance, engineering, biology, chemistry, and hydrology, among others [1, 2]. These equations offer powerful tools for modeling complex phenomena where classical models fall short. Fractional partial differential equations (FPDEs) have proven to be powerful tools for describing anomalous physical phenomena with greater accuracy than traditional integer-order equations. This unique capability has sparked significant interest in their investigation. Nevertheless, deriving analytical solutions for FPDEs poses considerable difficulties, particularly due to the complexity introduced by fractional derivatives. As a result, the focus has shifted toward the development and application of efficient numerical methods. These methods include finite volume (or element) approaches [3–6], finite difference schemes [7–15], meshless strategies [16], spectral techniques [17–20], discontinuous Galerkin methods [21–28], and collocation methods [29], among others.

The multi-term, time-fractional diffusion equation was introduced as an enhancement over the single-term model to improve the accuracy of describing anomalous diffusion processes. Its exceptional ability to capture anomalous diffusion in highly heterogeneous aquifers and intricate viscoelastic materials has made it a subject of extensive research from various perspectives [30]. This equation serves as a fractional extension of the classical diffusion equation: when $\alpha = 1$, it reduces to the classical model, inheriting some of its analytical characteristics. However, the inclusion of nonlocal fractional derivative terms introduces notable differences. Specifically, these terms result in limited spatial smoothing properties and slower asymptotic decay over time [31, 32], significantly influencing numerical analysis [33–35].

The discontinuous Galerkin (DG) method offers several advantages, particularly for reaction-diffusion equations. One of its key benefits is its flexibility in handling complex boundary conditions and non-smooth solutions, which are common in reaction-diffusion problems. The DG method also provides high accuracy even for problems with steep gradients or discontinuities, as it allows for discontinuities between elements while maintaining high-order accuracy within each element. Additionally, the method's inherent ability to handle local refinement and adaptivity makes it well-suited for capturing sharp interfaces or localized phenomena in reaction-diffusion models. The method also enjoys optimal stability properties, which are crucial for ensuring accurate solutions in stiff reaction-diffusion problems, particularly when dealing with highly varying reaction terms. In this paper, we investigate the numerical method based on generalized alternating numerical fluxes for the following multi-term, time-fractional reaction-diffusion equation:

$$\begin{aligned} &P_{\alpha,\alpha_1,\cdots,\alpha_l}(D_t)u(x,t) - \frac{\partial^2 u(x,t)}{\partial x^2} + \rho u(x,t) = f(x,t), \\ &\qquad x \in (a,b), t \in (0,T], \\ &\qquad u(x,0) = 0, \qquad x \in [a,b], \end{aligned} \tag{1.1}$$

where

$$P_{\alpha,\alpha_1,\cdots,\alpha_l}(D_t)u(x,t) = \left(D_t^{\alpha} + \sum_{i=1}^{l} d_i D_t^{\alpha_i} \right) u(x,t).$$

Here, $0 < \alpha_1 \leq \cdots \leq \alpha_l < \alpha < 1$ represents the fractional orders of the derivatives, and d_i ($i = 1, 2, \ldots, l$) are constant coefficients. $\rho > 0$ is the constant reaction rate. The Caputo fractional derivative D_t^{η} for $0 < \eta < 1$ is defined as in [36]:

$$D_t^{\eta} u(x,t) = \frac{1}{\Gamma(1-\eta)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^{\eta}}, \quad t > 0, \quad 0 < \eta < 1,$$

where $\Gamma(\cdot)$ denotes the Gamma function.

This work focuses exclusively on the mathematical formulation of the problem and does not impose specific boundary conditions. Consequently, the solution is assumed to be either periodic or compactly supported.

Model (1.1) was designed to refine the single-term model (1.3) for more accurately describing anomalous diffusion. For instance, in [37], a two-term fractional diffusion model was developed to capture solute transport dynamics by explicitly distinguishing between mobile and immobile solute states through fractional-order dynamics. Similarly, kinetic equations incorporating fractional derivatives of varying orders arise naturally when modeling subdiffusive motion in velocity fields. Discussions on their application to wave-type phenomena can also be found in [38]. Although several numerical schemes, such as finite difference methods [39–42], finite element methods [43–45], the matrix approach [46], and the discontinuous Galerkin method [47], have been proposed for problems involving multi-term fractional derivatives. The development of efficient and higher-order numerical methods remains a significant challenge in handling multi-term fractional derivative problems.

This paper is organized as follows: Section 2 introduces basic notations and theoretical preliminaries. In Section 3, we present a discontinuous Galerkin method based on generalized alternating numerical fluxes for the multi-term, time-fractional reaction-diffusion equation. Stability and convergence are rigorously proven using the mathematical induction approach in Section 4. Numerical experiments validating the proposed method are provided in Section 5, and the conclusions are given in Section 6.

2. Notations and auxiliary results

Consider a computational domain covered by the following mesh:

$$a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = b.$$

Each cell is represented as $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ for $j = 1, 2, \ldots, N$, with the corresponding cell lengths defined by $\Delta x_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$ for $1 \leq j \leq N$. We further define $h = \max_{1 \leq j \leq N} \Delta x_j$ as the largest cell length in the mesh.

At the cell interfaces, the values of u are denoted by $u^+_{j+\frac{1}{2}}$ and $u^-_{j+\frac{1}{2}}$, corresponding to the values from the right side of I_{j+1} and the left side of I_j, respectively.

To approximate solutions, we define the piecewise polynomial space V_h^k, consisting of polynomials of degree at most k within each cell I_j:

$$V_h^k = \{v : v \in P^k(I_j), x \in I_j, j = 1, 2, \ldots, N\}.$$

For any periodic function ω defined on $[a, b]$, the generalized Gauss-Radau projection [48, 49], denoted by $\mathcal{P}_\delta \omega$, is the unique element in V_h. Let $\omega^e = \mathcal{P}_\delta \omega - \omega$ represent the projection error. When $\delta \neq \frac{1}{2}$, it satisfies the following for $j = 1, 2, \ldots, N$:

$$\int_{I_j} \omega^e v \, dx = 0, \quad \forall v \in P^{k-1}(I_j), \quad \text{and} \quad (\omega^e)^{(\delta)}_{j+\frac{1}{2}} = 0. \tag{2.1}$$

We then have the subsequent result [49].

Lemma 2.1. *Let $\delta \neq \frac{1}{2}$. If $\omega \in H^{s+1}[a, b]$, the following inequality holds:*

$$\|\omega^e\| + h^{\frac{1}{2}} \|\omega^e\|_{L^2(\Gamma_h)} \leq C h^{\min(k+1, s+1)} \|\omega\|_{s+1}, \tag{2.2}$$

where the constant $C > 0$ is independent of h and ω. Here, Γ_h denotes the set of boundary points for all elements I_j, and

$$\|\omega^e\|_{L^2(\Gamma_h)} = \left(\frac{1}{2}\sum_{i=1}^{N}[((\omega^e)^+)^2_{i-\frac{1}{2}} + ((\omega^e)^-)^2_{i+\frac{1}{2}}]\right)^{\frac{1}{2}}.$$

In this paper, C denotes a generic positive constant whose value may vary in different contexts. The scalar inner product on $L^2(D)$ is denoted by $(\cdot,\cdot)_D$, with the associated norm represented as $\|\cdot\|_D$. When $D = \Omega$, the subscript D is omitted for simplicity.

3. The schemes

In this section, we present a detailed construction of the numerical scheme aimed at solving Eq (1.1). The process begins with the uniform division of the time interval $[0, T]$, where the time step size is denoted by $\Delta t = T/M$, with $M \in \mathbb{N}$. The mesh points are defined as $t_n = n\Delta t$ for $n = 0, 1, \ldots, M$.

$$\begin{aligned}
\frac{\partial^\eta v(x,t_n)}{\partial t^\eta} &= \frac{1}{\Gamma(1-\eta)}\int_0^{t_n}\frac{\partial v(x,s)}{\partial s}\frac{ds}{(t_n-s)^\eta} \\
&= \frac{1}{\Gamma(1-\eta)}\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\frac{\partial v(x,s)}{\partial s}\frac{ds}{(t_n-s)^\eta} \\
&= \frac{1}{\Gamma(1-\eta)}\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\frac{v(x,t_{i+1})-v(x,t_i)}{\Delta t}\frac{ds}{(t_n-s)^\eta} + R^n \\
&= \frac{(\Delta t)^{-\eta}}{\Gamma(2-\eta)}\sum_{i=1}^{n} b^\eta_{n-i}(v(x,t_i)-v(x,t_{i-1})) + R^n \\
&= \frac{(\Delta t)^{-\eta}}{\Gamma(2-\eta)}[v(x,t_n) + \sum_{i=1}^{n-1}(b^\eta_{n-i}-b^\eta_{n-i-1})v(x,t_i) - b_{n-1}v(x,t_0)] + R^n,
\end{aligned} \tag{3.1}$$

where $b^\eta_i = (i+1)^{1-\eta} - i^{1-\eta}$, and has the following properties:

$$b^\eta_0 = 1, \quad \sum_{k=0}^{n-1} b^\eta_k = n^{1-\eta}. \tag{3.2}$$

The truncation error is denoted as R^n, and we have the following result from [50]:

$$|R^n| \le C(\Delta t)^{2-\eta},$$

where C is a constant independent of Δt.

From this, we express:

$$\begin{aligned}
P_{\alpha,\alpha_1,\cdots,\alpha_l}(D_t)u(x,t) =& \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}\Bigg(\Big(1 + \sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i(\Delta t)^{\alpha-\alpha_i}\Big)u(x,t_n) \\
&+ \sum_{k=1}^{n-1}\Big(b^\alpha_{n-k} - b^\alpha_{n-k-1} + \sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i(b^{\alpha_i}_{n-k} - b^{\alpha_i}_{n-k-1})(\Delta t)^{\alpha-\alpha_i}\Big)u(x,t_k) \\
&- \Big(b^\alpha_{n-1} + \sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i b^{\alpha_i}_{n-1}(\Delta t)^{\alpha-\alpha_i}\Big)u(x,t_0)\Bigg) + R^n_{\alpha,\alpha_1,\cdots,\alpha_l}.
\end{aligned} \tag{3.3}$$

To reformulate Eq (1.1), we introduce the following first-order system:

$$p = u_x, \quad P_{\alpha,\alpha_1,\cdots,\alpha_l}(D_t)u(x,t) + \rho u(x,t) = p_x + f(x,t). \tag{3.4}$$

Let $u_h^n, p_h^n \in V_h^k$ denote the approximate solutions for $u(\cdot, t_n)$ and $p(\cdot, t_n)$, respectively, and let $f^n(x) = f(x, t_n)$. The fully discrete local discontinuous Galerkin (LDG) scheme is formulated as follows:

Find $u_h^n, p_h^n \in V_h^k$ such that for all test functions $v, \xi \in V_h^k$,

$$\begin{aligned}
&(\beta_0 + \rho)\int_\Omega u_h^n v\,dx + \beta_1\left(\int_\Omega p_h^n v_x\,dx - \sum_{j=1}^{N}\left((\widehat{p_h^n}v^-)_{j+\frac{1}{2}} - (\widehat{p_h^n}v^+)_{j-\frac{1}{2}}\right)\right)\\
&= \sum_{k=1}^{n-1}\left(b_{n-k-1}^{\alpha} - b_{n-k}^{\alpha} + \sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i(b_{n-k-1}^{\alpha_i} - b_{n-k}^{\alpha_i})(\Delta t)^{\alpha-\alpha_i}\right)\int_\Omega u_h^k v\,dx\\
&+ \left(b_{n-1}^{\alpha} + \sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i b_{n-1}^{\alpha_i}(\Delta t)^{\alpha-\alpha_i}\right)\int_\Omega u_h^0 v\,dx + \beta\int_\Omega f^n v\,dx,\\
&\int_\Omega p_h^n\xi\,dx + \int_\Omega u_h^n\xi_x\,dx - \sum_{j=1}^{N}\left((\widehat{u_h^n}\xi^-)_{j+\frac{1}{2}} - (\widehat{u_h^n}\xi^+)_{j-\frac{1}{2}}\right) = 0,
\end{aligned} \tag{3.5}$$

where

$$\beta_0 = 1 + \sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i(\Delta t)^{\alpha-\alpha_i}$$

and

$$\beta_1 = \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}.$$

The initial condition u_h^0 is determined by the L^2 projection of $u(\cdot, 0)$, given by:

$$\int_\Omega u_h^0 v\,dx = \int_\Omega \mathcal{P}u(x,0)v\,dx = \int_\Omega u_0(x)v\,dx, \quad \forall v \in V_h^k.$$

The terms marked with a "hat" in (3.5) are numerical fluxes, representing single-valued functions at cell boundaries, which are designed to ensure stability. For this system, we choose the fluxes as:

$$\widehat{u_h^n} = \delta(u_h^n)^- + (1-\delta)(u_h^n)^+, \quad \widehat{p_h^n} = (1-\delta)(p_h^n)^- + \delta(p_h^n)^+, \tag{3.6}$$

with a given parameter $\delta \neq \frac{1}{2}$. Choosing $\delta = 0$ or $\delta = 1$ results in purely alternating numerical fluxes.

4. Stability

To streamline the notation and without any loss of generality, we focus on the numerical analysis for the case where $f = 0$.

Theorem 4.1. *Under the assumption of periodic or compactly supported boundary conditions, the fully discrete LDG scheme given in (3.5) is unconditionally stable. Specifically, there exists a constant* $C > 0$*, depending on* u *and* T*, such that*

$$\|u_h^n\| \le \|u_h^0\|, \quad n = 1, 2, \ldots, M.$$

Proof. We begin by selecting the test functions $v = u_h^n$ and $\xi = \beta_1 p_h^n$ in the scheme (3.5). Using the chosen numerical fluxes as specified in (3.6), we derive:

$$\begin{aligned}
&(\beta_0+\rho)\|u_h^n\|^2 + \beta_1\|p_h^n\|^2 + \beta_1\sum_{j=1}^{N}(\Psi(u_h^n,p_h^n)_{j+\frac{1}{2}} - \Psi(u_h^n,p_h^n)_{j-\frac{1}{2}} + \Theta(u_h^n,p_h^n)_{j-\frac{1}{2}})\\
&\quad= \sum_{k=1}^{n-1}(b_{n-k-1}^{\alpha} - b_{n-k}^{\alpha} + \sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i(b_{n-k-1}^{\alpha_i} - b_{n-k}^{\alpha_i})(\Delta t)^{\alpha-\alpha_i})\int_\Omega u_h^k u_h^n dx\\
&\qquad+ (b_{n-1}^{\alpha} + \sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i b_{n-1}^{\alpha_i}(\Delta t)^{\alpha-\alpha_i})\int_\Omega u_h^0 u_h^n dx\\
&\quad\le \sum_{k=1}^{n-1}(b_{n-k-1}^{\alpha} - b_{n-k}^{\alpha} + \sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i(b_{n-k-1}^{\alpha_i} - b_{n-k}^{\alpha_i})(\Delta t)^{\alpha-\alpha_i})\|u_h^k\|\|u_h^n\|\\
&\qquad+ (b_{n-1}^{\alpha} + \sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i b_{n-1}^{\alpha_i}(\Delta t)^{\alpha-\alpha_i})\|u_h^0\|\|u_h^n\|,
\end{aligned} \tag{4.1}$$

where the terms $\Psi(u_h^n, p_h^n)$ and $\Theta(u_h^n, p_h^n)$ are defined as:

$$\begin{aligned}
\Psi(u_h^n,p_h^n) &= (p_h^n)^-(u_h^n)^- - \widehat{p_h^n}(u_h^n)^- - \widehat{u_h^n}(p_h^n)^-,\\
\Theta(u_h^n,p_h^n) &= (p_h^n)^-(u_h^n)^- - (p_h^n)^+(u_h^n)^+ - \widehat{p_h^n}(u_h^n)^- + \widehat{p_h^n}(u_h^n)^+ - \widehat{u_h^n}(p_h^n)^- + \widehat{u_h^n}(p_h^n)^+.
\end{aligned}$$

Through straightforward computation, we observe that

$$\Theta(u_h^n, p_h^n) = 0.$$

From Eq (4.1), we can simplify further to obtain:

$$\begin{aligned}
(\beta_0+\rho)\|u_h^n\| \le& \sum_{k=1}^{n-1}(b_{n-k-1}^{\alpha} - b_{n-k}^{\alpha} + \sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i(b_{n-k-1}^{\alpha_i} - b_{n-k}^{\alpha_i})(\Delta t)^{\alpha-\alpha_i})\|u_h^k\|\\
&+ (b_{n-1}^{\alpha} + \sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i b_{n-1}^{\alpha_i}(\Delta t)^{\alpha-\alpha_i})\|u_h^0\|.
\end{aligned} \tag{4.2}$$

We proceed by induction to confirm the theorem. For $n = 1$, Eq (4.2) simplifies to:

$$(\beta_0+\rho)\|u_h^1\| \le b_0^{\alpha} + \sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i b_0^{\alpha_i}(\Delta t)^{\alpha-\alpha_i})\|u_h^0\|.$$

Since $b_0^{\alpha} = b_0^{\alpha_i} = 1$, it follows that:

$$\|u_h^1\| \le \|u_h^0\|. \tag{4.3}$$

Assume that for $m = 1, 2, \ldots, n-1$, the inequality

$$\|u_h^m\| \le \|u_h^0\|$$

holds. Substituting this into (4.2), we deduce:

$$(\beta_0+\rho)\|u_h^n\| \le \beta_0\|u_h^0\| \le (\beta_0+\rho)\|u_h^0\|. \tag{4.4}$$

Thus,

$$\|u_h^n\| \le \|u_h^0\|.$$

This completes the proof of stability. □

Lemma 4.1. *[51] For each $t \in (0,T]$, if $u_{tt} \in L^2(D)$, the following holds:*

$$\|R^n_{\alpha,\alpha_1,\cdots,\alpha_l}\| \le C(\Delta t)^{2-\alpha}.$$

We also provide the following lemma for analysis:

Lemma 4.2. *[51] If $\psi^n \ge 0$ for $n = 1,2,\ldots,N$ and $\psi^0 = 0$, then under the inequality:*

$$\begin{aligned}\beta_0\psi^n \le& \sum_{k=1}^{n-1}\left(b^\alpha_{n-k-1} - b^\alpha_{n-k} + \sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i(b^{\alpha_i}_{n-k-1} - b^{\alpha_i}_{n-k})(\Delta t)^{\alpha-\alpha_i}\right)\psi^k\\ &+\left(b^\alpha_{n-1} + \sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i b^{\alpha_i}_{n-1}(\Delta t)^{\alpha-\alpha_i}\right)\psi^0 + \chi,\end{aligned}$$

we can conclude:

$$\psi^n \le C(\Delta t)^\alpha\chi,$$

where C is a positive constant independent of h and Δt.

Theorem 4.2. *Let $u(x,t_n)$ denote the exact solution of problem (1.1), assumed to be sufficiently smooth such that $u \in H^{m+1}$ with $0 \le m \le k+1$. Let u_h^n represent the numerical solution obtained from the fully discrete LDG scheme (3.5). Then, the following error estimate holds:*

$$\|u(x,t_n) - u_h^n\| \le C(h^{k+1} + (\Delta t)^{2-\alpha}), n = 1,\cdots,M, \tag{4.5}$$

where C is a constant depending on u, T, and α.

Proof. Let us define

$$\begin{aligned} e_u^n &= u(x,t_n) - u_h^n = \mathcal{P}_\delta e_u^n - (\mathcal{P}_\delta u(x,t_n) - u(x,t_n)),\\ e_p^n &= p(x,t_n) - p_h^n = \mathcal{P}_{1-\delta} e_p^n - (\mathcal{P}_{1-\delta} p(x,t_n) - p(x,t_n)).\end{aligned} \tag{4.6}$$

Incorporating the fluxes from (3.6), by applying (4.6), the resulting error equation can be expressed as follows:

$$\begin{aligned}&(\beta_0+\rho)\int_\Omega \mathcal{P}_\delta e_u^n v dx + \beta_1(\int_\Omega \mathcal{P}_{1-\delta}e_p^n v_x dx - \sum_{j=1}^{N}(((\mathcal{P}_{1-\delta}e_p^n)^+ v^-)_{j+\frac{1}{2}} - ((\mathcal{P}_{1-\delta}e_p^n)^+ v^+)_{j-\frac{1}{2}}))\\ &+\int_\Omega \mathcal{P}_{1-\delta}e_p^n \xi dx + \int_\Omega \mathcal{P}_\delta e_u^n \xi_x dx - \sum_{j=1}^{N}(((\mathcal{P}_\delta e_u^n)^- \xi^-)_{j+\frac{1}{2}} - ((\mathcal{P}_\delta e_u^n)^- \xi^+)_{j-\frac{1}{2}})\end{aligned}$$

$$
\begin{aligned}
&=\sum_{k=1}^{n-1}(b^{\alpha}_{n-k-1}-b^{\alpha}_{n-k}+\sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i(b^{\alpha_i}_{n-k-1}-b^{\alpha_i}_{n-k})(\Delta t)^{\alpha-\alpha_i})\int_{\Omega}\mathcal{P}_{\delta}e^{k}_{u}vdx\\
&+(b^{\alpha}_{n-1}+\sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_ib^{\alpha_i}_{n-1}(\Delta t)^{\alpha-\alpha_i})\int_{\Omega}\mathcal{P}_{\delta}e^{0}_{u}vdx-\beta_1\int_{\Omega}R_nvdx\\
&+(\beta_0+\rho)\int_{\Omega}(\mathcal{P}_{\delta}u(x,t_n)-u(x,t_n))vdx+\beta_1(\int_{\Omega}(\mathcal{P}_{1-\delta}p(x,t_n)-p(x,t_n))v_xdx\\
&-\sum_{j=1}^{N}(((\mathcal{P}_{1-\delta}p(x,t_n)-p(x,t_n))^{+}v^{-})_{j+\frac{1}{2}}-((\mathcal{P}_{1-\delta}p(x,t_n)-p(x,t_n))^{+}v^{+})_{j-\frac{1}{2}}))\\
&+\int_{\Omega}(\mathcal{P}_{1-\delta}p(x,t_n)-p(x,t_n))\xi dx+\int_{\Omega}(\mathcal{P}_{\delta}u(x,t_n)-u(x,t_n))\xi_xdx\\
&-\sum_{j=1}^{N}(((\mathcal{P}_{\delta}u(x,t_n)-u(x,t_n))^{-}\xi^{-})_{j+\frac{1}{2}}-((\mathcal{P}_{\delta}u(x,t_n)-u(x,t_n))^{-}\xi^{+})_{j-\frac{1}{2}})\\
&-\sum_{k=1}^{n-1}(b^{\alpha}_{n-k-1}-b^{\alpha}_{n-k}+\sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i(b^{\alpha_i}_{n-k-1}-b^{\alpha_i}_{n-k})(\Delta t)^{\alpha-\alpha_i})\int_{\Omega}(\mathcal{P}_{\delta}u(x,t_k)-u(x,t_k))vdx\\
&-(b^{\alpha}_{n-1}+\sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_ib^{\alpha_i}_{n-1}(\Delta t)^{\alpha-\alpha_i})\int_{\Omega}(\mathcal{P}_{\delta}u(x,t_0)-u(x,t_0))vdx.
\end{aligned}
\tag{4.7}
$$

By substituting the test functions $v = \mathcal{P}_{\delta}e^{n}_{u}$ and $\xi = \beta_1\mathcal{P}_{1-\delta}e^{n}_{p}$ into (4.7) and utilizing the properties (2.1), the following equality is derived:

$$
\begin{aligned}
&(\beta_0+\rho)\|\mathcal{P}_{\delta}e^{n}_{u}\|^2dx+\beta_1\|\mathcal{P}_{1-\delta}e^{n}_{p}\|^2dx\\
&=\sum_{k=1}^{n-1}(b^{\alpha}_{n-k-1}-b^{\alpha}_{n-k}+\sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i(b^{\alpha_i}_{n-k-1}-b^{\alpha_i}_{n-k})(\Delta t)^{\alpha-\alpha_i})\int_{\Omega}\mathcal{P}_{\delta}e^{k}_{u}\mathcal{P}_{\delta}e^{n}_{u}dx\\
&+(b^{\alpha}_{n-1}+\sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_ib^{\alpha_i}_{n-1}(\Delta t)^{\alpha-\alpha_i})\int_{\Omega}\mathcal{P}_{\delta}e^{0}_{u}\mathcal{P}_{\delta}e^{n}_{u}dx-\beta_1\int_{\Omega}R_n\mathcal{P}_{\delta}e^{n}_{u}dx\\
&+\beta_1\int_{\Omega}(\mathcal{P}_{1-\delta}p(x,t_n)-p(x,t_n))\mathcal{P}_{1-\delta}e^{n}_{p}dx+\mathbb{H}.
\end{aligned}
\tag{4.8}
$$

Define

$$
\begin{aligned}
\mathbb{H}&=(\beta_0+\rho)\int_{\Omega}(\mathcal{P}_{\delta}u(x,t_n)-u(x,t_n))\mathcal{P}_{\delta}e^{n}_{u}dx\\
&-\sum_{k=1}^{n-1}(b^{\alpha}_{n-k-1}-b^{\alpha}_{n-k}+\sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i(b^{\alpha_i}_{n-k-1}-b^{\alpha_i}_{n-k})(\Delta t)^{\alpha-\alpha_i})\int_{\Omega}(\mathcal{P}_{\delta}u(x,t_k)-u(x,t_k))\mathcal{P}_{\delta}e^{n}_{u}dx\\
&-(b^{\alpha}_{n-1}+\sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_ib^{\alpha_i}_{n-1}(\Delta t)^{\alpha-\alpha_i})\int_{\Omega}(\mathcal{P}_{\delta}u(x,t_0)-u(x,t_0))\mathcal{P}_{\delta}e^{n}_{u}dx.
\end{aligned}
$$

After performing some manual calculations, we arrive at the result:

$$
\mathbb{H}=-\Delta t\sum_{k=0}^{n-1}(b^{\alpha}_{k}+\sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_ib^{\alpha_i}_{k}(\Delta t)^{\alpha-\alpha_i}\int_{\Omega}\partial_t(\mathcal{P}_{\delta}u(x,t_{n-k})-u(x,t_{n-k}))\mathcal{P}_{\delta}e^{n}_{u}dx),
$$

where

$$\partial_t \varphi(x,t_k) = \frac{\varphi(x,t_k) - \varphi(x,t_{k-1})}{\Delta t}.$$

We know that

$$\|\partial_t(\mathcal{P}_\delta u(x,t_{n-k}) - u(x,t_{n-k}))\| \le Ch^{k+1}.$$

By using the property (3.2), we have

$$\sum_{k=0}^{n-1}(b_k^{\alpha} + \sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i b_k^{\alpha_i}(\Delta t)^{\alpha-\alpha_i}) = n^{1-\alpha} + \sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i n^{1-\alpha_i}(\Delta t)^{\alpha-\alpha_i}),$$

and then we get

$$\begin{aligned} |\mathbb{H}| &\le Ch^{k+1}(n^{1-\alpha}\Delta t + \sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i n^{1-\alpha_i}\Delta t)\|\mathcal{P}_\delta e_u^n\| \\ &\le Ch^{k+1}(\Delta t)^{\alpha}(T^{1-\alpha} + \sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i T^{1-\alpha_i})\|\mathcal{P}_\delta e_u^n\|. \end{aligned} \tag{4.9}$$

Notice

$$ab \le \frac{1}{2\beta_0}a^2 + \frac{\beta_0}{2}b^2,$$

and applying the Cauchy-Schwarz inequality and Lemma 4.1, we obtain

$$\begin{aligned} (\beta_0+\rho)\|\mathcal{P}_\delta e_u^n\| \le& \sum_{k=1}^{n-1}(b_{n-k-1}^{\alpha} - b_{n-k}^{\alpha} + \sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i(b_{n-k-1}^{\alpha_i} - b_{n-k}^{\alpha_i})(\Delta t)^{\alpha-\alpha_i})\|\mathcal{P}_\delta e_u^k\| \\ &+ (b_{n-1}^{\alpha} + \sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i b_{n-1}^{\alpha_i}(\Delta t)^{\alpha-\alpha_i})\|\mathcal{P}_\delta e_u^0\| + C(\Delta t)^2 \\ &+ Ch^{k+1}(\Delta t)^{\alpha} + Ch^{k+1}(\Delta t)^{\alpha}(T^{1-\alpha} + \sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i T^{1-\alpha_i}). \end{aligned} \tag{4.10}$$

By applying Lemma 4.2, we can conclude the result:

$$\|\mathcal{P}_\delta e_u^n\| \le C(h^{k+1} + (\Delta t)^{2-\alpha}).$$

Therefore, Theorem 4.2 follows from the triangle inequality and the interpolation property (2.1). □

5. Numerical examples

In this section, we will perform numerical experiments to illustrate the efficiency and numerical accuracy of the proposed fully discrete local discontinuous Galerkin method.

Example 5.1. *Consider the original fractional diffusion equation (1.1) in* $\Omega = [0,1]$*, and take* $l = 2$*:*

$$P_{\alpha,\alpha_1,\alpha_2}(D_t)u(x,t) - \frac{\partial^2 u(x,t)}{\partial x^2} + \rho u(x,t) = f(x,t), \qquad x \in (a,b), t \in (0,T]. \tag{5.1}$$

For the numerical experiments, we consider the periodic boundary condition and take the following initial condition:

$$u(x,0) = \sin 2\pi x, \quad x \in [0,1].$$

Then the exact solution is

$$u(x,t) = (t+1)^2 \sin 2\pi x.$$

First, the numerical convergence orders of the scheme (3.5) in space for computing this example are tested. With the fixed and sufficiently small step sizes $\Delta t = 1/10,000$*, and the varying* $h = 1/5, 1/10, 1/15, 1/20$*, respectively, the numerical errors and convergence orders in* L^2*-norm and* L^∞*-norm for different values of fractional order are recorded in Tables 1 and 2. From these tables, we can see that the errors attain* $(k+1)$*-th order of accuracy for piecewise* P^k *polynomials.*

We then test the time convergence rate using the presented scheme. For $\alpha = 0.7$*,* $\alpha_1 = 0.4$*, and* $\alpha_2 = 0.5$*, the numerical errors and convergence orders in the* L^2*-norm and* L^∞*-norm are presented in Table 3. As observed, the accuracy order is* $2 - \alpha$*, which is consistent with the theoretical analysis.*

Table 1. Spatial accuracy test using piecewise P^k polynomials when $\alpha = 0.8, \alpha_1 = 0.3, \alpha_2 = 0.7, \Delta t = \frac{1}{10,000}$, and $\delta = 0.8$.

	N	L^2-error	order	L^∞-error	order
	5	5.456655447655487e-02	-	3.684655468765437e-01	-
P^1	10	1.545439256065108e-02	1.82	1.036363224572394e-01	1.83
	15	7.418682089199397e-03	1.81	4.934748111840253e-02	1.83
	20	4.197087875971981e-03	1.98	2.898199831180425e-02	1.85
	5	5.516456123131424e-03	-	6.511456361527128e-02	-
P^2	10	7.545778986029944e-04	2.87	9.480139901367918e-03	2.78
	15	2.454320602848064e-04	2.77	2.901538048669126e-03	2.92
	20	1.084189710075339e-04	2.84	1.267081962360050e-03	2.88

Table 2. Spatial accuracy test using piecewise P^k polynomials when $\alpha = 0.9, \alpha_1 = 0.4, \alpha_2 = 0.5, \Delta t = \frac{1}{10,000}$, and $\delta = 0.3$.

	N	L^2-error	order	L^∞-error	order
	5	4.684541651565467e-02	-	1.989864651665547e-01	-
P^1	10	1.373548542522102e-02	1.77	6.082221675704989e-02	1.71
	15	6.866381974648757e-03	1.71	3.028205047660128e-02	1.72
	20	3.998003174324391e-03	1.88	1.799062146393744e-02	1.81
	5	4.984516546165546e-03	-	5.984145715685175e-02	-
P^2	10	7.009840790419134e-04	2.83	8.474177810702038e-03	2.82
	15	2.298567129699161e-04	2.75	2.657520182860610e-03	2.86
	20	9.951429447030076e-05	2.91	1.180726564442627e-03	2.82

Table 3. Temporal errors and convergence rates for piecewise P^2 basis functions when $\alpha = 0.7, \alpha_1 = 0.4, \alpha_2 = 0.5, N = 600, \delta = 0.6$, and $T = 1$.

M	L^2-error	order	L^∞-error	order
5	2.614564766548656e-03	-	1.925546155361646e-03	-
10	1.114634573373263e-03	1.23	8.266036876295551e-04	1.22
20	4.495553965778067e-04	1.31	3.403916859859762e-04	1.28
40	1.877091403979780e-04	1.26	1.411467392104509e-04	1.27

Example 5.2. *Now we consider the following time fractional, reaction-diffusion equation with initial singularities:*

$$P_{\alpha,\alpha_1}(D_t)u(x,t) - \frac{\partial^2 u(x,t)}{\partial x^2} + \rho u(x,t) = f(x,t), \quad x \in (0,1),\ t \in (0,T], \tag{5.2}$$

where $P_{\alpha,\alpha_1}(D_t)$ is the multi-term Caputo fractional derivative operator defined as:

$$P_{\alpha,\alpha_1}(D_t)u(x,t) = D_t^{\alpha}u(x,t) + \alpha_1 D_t^{\alpha_1}u(x,t),$$

with $0 < \alpha, \alpha_1 < 1$, and $\rho > 0$ is the reaction coefficient.

Let the exact solution be:

$$u(x,t) = t^{-\beta}x^2(1-x)^2, \quad \beta > 0.$$

Then the initial and boundary conditions are

$$u(0,t) = u(1,t) = 0, \quad t > 0,$$

and the source term $f(x,t)$ is

$$f(x,t) = \left[\frac{\Gamma(1-\beta)}{\Gamma(1-\beta-\alpha)}t^{-\beta-\alpha} + \alpha_1\frac{\Gamma(1-\beta)}{\Gamma(1-\beta-\alpha_1)}t^{-\beta-\alpha_1}\right]x^2(1-x)^2$$

$$-2t^{-\beta}(2-6x+6x^2) + \rho t^{-\beta}x^2(1-x)^2.$$

We solve the problem using the proposed numerical scheme, which combines the local discontinuous Galerkin method for spatial discretization and the finite difference method for temporal discretization. The computational parameters are set as follows: the temporal step size is $\Delta t = \frac{1}{10{,}000}$, the reaction coefficient is $\rho = 1$, the fractional orders are $\alpha = 0.8$ and $\alpha_1 = 0.6$, and the singularity parameter is $\beta = 0.5$.

The numerical solution is compared with the exact solution, and the results are presented in Table 4. The numerical method effectively handles the singularity at $t = 0$ introduced by the initial condition, while the observed convergence rates are consistent with theoretical predictions, demonstrating the robustness and accuracy of the proposed approach for multi-term fractional equations.

Table 4. Accuracy test using piecewise P^k polynomials when $\alpha = 0.8, \alpha_1 = 0.6, \Delta t = \frac{1}{10{,}000}, \delta = 0.3$, and $\beta = 0.5$.

	N	L^2-error	order	L^∞-error	order
P^1	5	7.759837594594854e-02	-	1.937598459345438e-01	-
	10	2.687056130819109e-02	1.53	5.881555343372386e-02	1.72
	15	1.398845698040959e-02	1.61	2.976177799048298e-02	1.68
	20	8.430971770294258e-03	1.76	1.788616813492260e-02	1.77
P^2	5	8.723894482394492e-03	-	3.652198385546779e-02	-
	10	1.510443605795659e-03	2.53	5.543100745658434e-03	2.72
	15	4.756028262196357e-04	2.85	1.810259757096130e-03	2.76
	20	2.156070219803091e-04	2.75	8.277659034441626e-04	2.72

6. Conslusions

This paper presented a numerical method for solving the multi-term fractional reaction-diffusion equation, which involves logarithmic decay of solutions as time progresses. The proposed method employed the discontinuous Galerkin method with generalized alternating numerical fluxes for spatial discretization and the finite difference method for temporal discretization. The authors rigorously established the unconditional stability of the scheme through mathematical induction. To validate its accuracy and efficiency, a series of comprehensive numerical experiments were conducted, demonstrating the potential of the proposed method for practical applications in modeling complex diffusion processes. In the future we plan to extend the proposed method to high-dimensional problems, where the complexity of the equations and computational cost increase significantly.

Author contributions

Leqiang Zou: Methodology, Stability analysis, Error estimate, Investigation, Writing-original draft; Yanzi Zhang: Methodology, Supervision, Writing-original draft. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare that there are no competing interests.

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