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Research article

A Diophantine approximation problem with unlike powers of primes

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Abstract: Let λ_1, λ_2, λ_3, and λ_4 be non-zero real numbers, not all negative. Suppose that λ_1/λ_3 is irrational and algebraic, $\delta > 0$, and the set $\mathcal{V}$ is a well-spaced sequence. In this paper, we prove that, for any $\varepsilon > 0$, the number of $v \in \mathcal{V}$ with $v \leq X$ such that the inequality

$$\left|\lambda_1 p_1 + \lambda_2 {p_2}^2 + \lambda_3 {p_3}^3 + \lambda_4 {p_4}^4 - v\right| < v^{-\delta}$$

has no solution in primes p_1, p_2, p_3, p_4 that does not exceed $O(X^{1-\frac{83}{144}+2\delta+2\varepsilon})$.

Keywords: Davenport-Heilbronn method; Diophantine inequality; primes; exceptional set
Mathematics Subject Classification: 11D75, 11P32, 11P55

1. Introduction

In 1953, Prachar [1] demonstrated that any sufficiently large odd integer N can be expressed as

$$N = p_1 + p_2^2 + p_3^3 + p_4^4 + p_5^5,$$

where p_1, p_2, p_3, p_4, and p_5 are prime numbers. Subsequently, as a result of [2, Theorem 1], Ren and Tsang reached the same conclusion as Prachar. It is significant to explore the analogous formulation for Diophantine inequalities. In 1973, Vaughan [3] first proved that for any real η, there are infinitely many solutions in primes p_j to the inequality

$$|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < (\max p_j)^{-\xi+\varepsilon} \tag{1.1}$$

with $\xi = 1/10$. The exponent was subsequently improved several times, and the best result up to now is due to K. Matomäki [4] with $\xi = 2/9$. Moreover, it is worth mentioning that Diophantine inequality (1.1) has also been resolved for special prime numbers. The solvability of (1.1) with primes p_1, p_2, and p_3, where $p_i + 2$ are almost-primes, is discussed in [5]. Similarly, the cases where $p_3 = x^2 + y^2 + 1$ or $p_i = [n^c]$ (Piatetski-Shapiro primes) are addressed in [6, 7], respectively.

Then for related results for Diophantine inequalities with unlike powers of primes, we consider the following examples. Let λ_1, λ_2, λ_3, λ_4, and λ_5 be non-zero real numbers, not all negative, and suppose that λ_1/λ_2 is irrational. In 2016, Ge and Li [8] demonstrated that for any given real numbers η and σ with $0 < \sigma < \frac{1}{720}$, there exist infinitely many solutions in prime numbers p_j to the inequality

$$\left|\lambda_1 p_1 + \lambda_2 {p_2}^2 + \lambda_3 {p_3}^3 + \lambda_4 {p_4}^4 + \lambda_5 {p_5}^5 + \eta\right| < (\max_{1\le j\le 5} p_j)^{-\sigma}. \tag{1.2}$$

In 2017, Mu [9] made further improvements by extending the range of the exponent to $\sigma \le \frac{1}{180}$ using the method outlined in Languasco and Zaccagnini [10]. Subsequently, Liu [11] optimized the result further, obtaining $\sigma \le \frac{5}{288}$. Inspired by the work of Wang and Yao [12], Mu and Qu [13] combined the sieve methods from Harman [14] and Harman and Kumchev [15] to refine Liu's result and prove that (1.2) holds for $\sigma \le \frac{5}{252}$. Very recently, this result was improved by Zhu [16], who obtained $\sigma \le \frac{1}{48}$. In 2022, Zhu [17] employed an innovative approach to estimate the correlation integral over the minor arcs, leading to a more robust estimation that $\sigma \le \frac{19}{756}$.

Building upon the advancements made in the study of Diophantine inequalities, subsequent research has explored different variations and forms of these inequalities. For clarity in discussion, we first introduce the definition of a well-spaced sequence. A set of positive real numbers $\mathcal{V}$ is called a well-space sequence if there exists a constant $c > 0$ such that

$$u, v \in \mathcal{V}, u \ne v \Rightarrow |u - v| > c.$$

On this basis, in 2018, Ge and Zhao [18] dropped the linear prime variable in (1.2) and considered the exceptional set for the inequality

$$\left|\lambda_2 {p_2}^2 + \lambda_3 {p_3}^3 + \lambda_4 {p_4}^4 + \lambda_5 {p_5}^5 - v\right| < v^{-\delta}, \tag{1.3}$$

where λ_2, λ_3, λ_4, and λ_5 are non-zero real numbers, not all negative. Let $\mathcal{D}$ be a well-spaced sequence, and suppose $\delta > 0$, and λ_1/λ_2 is algebraic and irrational. Denote by $E(\mathcal{D}, X, \delta)$ the number of $v \in \mathcal{D}$ with $v \le X$ such that the inequality (1.3) has no solution in primes p_2, p_3, p_4, and p_5. Recently, Ge and Zhao [18] demonstrated that

$$E(\mathcal{D}, X, \delta) \ll X^{1-\frac{1}{72}+2\delta+\varepsilon}.$$

Subsequently, Mu and Gao [19], as well as Liu and Liu [20], further improved upon the result of Ge and Zhao.

In this paper, we drop the final prime variable in (1.2) and examine the exceptional set for the inequality

$$\left|\lambda_1 p_1 + \lambda_2 {p_2}^2 + \lambda_3 {p_3}^3 + \lambda_4 {p_4}^4 - v\right| < v^{-\delta}. \tag{1.4}$$

Drawing on techniques from [18, 20], we establish the following results.

Theorem 1.1. Let λ_1, λ_2, λ_3, and λ_4 be non-zero real numbers, not all negative. Suppose that λ_1/λ_3 is irrational and algebraic, $\delta > 0$, and the set $\mathcal{V}$ is a well-spaced sequence. Denote by $E(\mathcal{V}, X, \delta)$ the number of $v \in \mathcal{V}$ with $v \le X$ such that the inequality

$$\left|\lambda_1 p_1 + \lambda_2 {p_2}^2 + \lambda_3 {p_3}^3 + \lambda_4 {p_4}^4 - v\right| < v^{-\delta}$$

has no solution in primes p_1, p_2, p_3, p_4. Then, for any $\varepsilon > 0$, we have

$$E(\mathcal{V}, X, \delta) \ll X^{1-\frac{83}{144}+2\delta+2\varepsilon}. \tag{1.5}$$

Theorem 1.2. Let λ_1, λ_2, λ_3, and λ_4 be non-zero real numbers, not all negative. Suppose that λ_1/λ_3 is irrational and algebraic, $\delta > 0$, and the set $\mathcal{V}$ is a well-spaced sequence. Then there is a sequence $X_j \to \infty$ such that

$$E(\mathcal{V}, X_j, \delta) \ll {X_j}^{1-\frac{83}{144}+2\delta+2\varepsilon} \tag{1.6}$$

for any $\varepsilon > 0$. Moreover, if the convergent denominators q_j for λ_1/λ_3 satisfy $q_{j+1}^{1-w} \ll q_j$ for $w \in [0\ ,\ 1)$, then, for all $X \ge 1$ and any $\varepsilon > 0$,

$$E(\mathcal{V}, X, \delta) \ll X^{\frac{1}{2}-\frac{11}{12}\chi+2\delta+2\varepsilon}, \tag{1.7}$$

with

$$\chi = \min\left(\tfrac{41-90\omega}{492(1-\omega)}, \tfrac{1}{12}\right). \tag{1.8}$$

Remark. Theorem 1.1 follows directly from Theorem 1.2. Since λ_1/λ_3 is algebraic, we can set $w = \varepsilon$, and thus $\chi = \frac{1}{12}$. Therefore, we focus on proving Theorem 1.2 in the following.

Notation. Throughout this paper, the letter p denotes a prime number, with or without subscripts. The non-zero real numbers λ_1, λ_2, λ_3, λ_4, and δ are given constants. Implicit constants in the O, $\ll$ and $\gg$ notations usually depend at most on λ_1, λ_2, λ_3, and λ_4. The letter ε represents an arbitrarily small positive constant, but not necessarily the same each time. We write $e(x) = \exp(2\pi i x)$.

2. Outline of the method

Let X be a sufficiently large positive number. Suppose that $I_k = \left[(\eta X)^{1/k}, (X)^{1/k}\right]$ for $k = 1, 2, 3, 4$ and $0 < \tau < 1$. Define

$$K_\tau(\alpha) = \begin{cases} \left(\frac{\sin \pi\tau\alpha}{\pi\alpha}\right)^2 & \alpha \neq 0; \\ \tau^2 & \alpha = 0, \end{cases} \tag{2.1}$$

and

$$S_k(\alpha) = \sum_{p \in I_k} (\log p) e(\alpha p^k), \quad U_k(\alpha) = \sum_{n \in I_k} e(\alpha n^k), \tag{2.2}$$

for $k = 1, 2, 3, 4$. Then, it is simple to obtain

$$K_\tau(\alpha) \ll \min(\tau^2, |\alpha|^{-2}), \quad \int_{-\infty}^{+\infty} K_\tau(\alpha) e(\alpha x) d\alpha = \max(0, \tau - |x|). \tag{2.3}$$

For convenience, Φ is defined as an arbitrary measurable subset of $\mathbb{R}$, and then

$$\mathcal{I}(v,X;\Phi)=\int_{\Phi}S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)K_\tau(\alpha)e(-v\alpha)d\alpha.$$

By (2.3), we have

$$\begin{aligned}\mathcal{I}(v,X;\mathbb{R})&=\int_{-\infty}^{+\infty}S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)K_\tau(\alpha)e(-v\alpha)d\alpha\\&=\sum_{p_k\in I_k}(\log p_1)\cdots(\log p_4)\int_{-\infty}^{+\infty}e\Big((\lambda_1p_1+\lambda_2p_2{}^2+\lambda_3p_3{}^3+\lambda_4p_4{}^4-v)\alpha\Big)K_\tau(\alpha)d\alpha\\&=\sum_{p_k\in I_k}(\log p_1)\cdots(\log p_4)\max\Big(0,\tau-\big|\lambda_1p_1+\lambda_2p_2{}^2+\lambda_3p_3{}^3+\lambda_4p_4{}^4-v\big|\Big)\qquad(2.4)\\&\ll(\log X)^4\sum_{p_k\in I_k}\max\Big(0,\tau-\big|\lambda_1p_1+\lambda_2p_2{}^2+\lambda_3p_3{}^3+\lambda_4p_4{}^4-v\big|\Big).\end{aligned}$$

Thus

$$0\le\mathcal{I}(v,X;\mathbb{R})\le\tau(\log X)^4\mathcal{N}(v,X),$$

where $\mathcal{N}(v,X)$ counts the number of solutions to the inequality

$$\big|\lambda_1p_1+\lambda_2p_2{}^2+\lambda_3p_3{}^3+\lambda_4p_4{}^4-v\big|<\tau,\quad p_k\in I_k(k=1,2,3,4).$$

To estimate (2.4), we divide the region of integration into three parts: The major arc $\mathcal{M}$, the minor arc m, and the trivial arc t, defined as

$$\mathcal{M}=\{\alpha:|\alpha|\le\phi\}\cup\{\alpha:\phi<|\alpha|\le\xi\},m=\{\alpha:\xi<|\alpha|<\gamma\},t=\{\alpha:|\alpha|\ge\gamma\},$$

and then we can write

$$\mathcal{I}(v,X;\mathbb{R})=\mathcal{I}(v,X;\mathcal{M})+\mathcal{I}(v,X;m)+\mathcal{I}(v,X;t),\qquad(2.5)$$

where $\phi=X^{\frac{5}{24}-1-\varepsilon},\xi=X^{-\frac{7}{9}-\varepsilon},\gamma=\tau^{-2}X^{\frac{7}{12}+2\varepsilon}$.

It is sufficient to establish a positive lower bound for $\mathcal{I}(v,X;\mathbb{R})$. We employ a standard dyadic argument, focusing on those values of v that satisfy $\frac{1}{2}X\le v\le X$. We will estimate the three terms on the right-hand side of (2.5) in Sections 4–6. Additionally, Sections 3 and 6 will present preliminary lemmas and conclude the proof of Theorem 1.2.

3. Preliminary lemmas

Lemma 3.1. Suppose that $X\ge Z_1\ge X^{\frac{4}{5}+2\varepsilon}$ and $|S_1(\lambda_1\alpha)|>Z_1$. Then, there exist two coprime integers a and q such that

$$1\le q\ll\left(\frac{X^{1+\varepsilon}}{Z_1}\right)^2,|q\lambda_1\alpha-a|\ll\left(\frac{X^{1+\varepsilon}}{Z_1}\right)^2X^{-1}.$$

Proof. This is extracted from [13, Lemma 4.1].

Lemma 3.2. Suppose that α is a real number, and there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with

$$(q, a) = 1, 1 \leq q \leq P^{\frac{3}{2}}, |q\alpha - a| < P^{-\frac{3}{2}}.$$

Then one has

$$\sum_{P<p\leq 2P} (\log p)e(\alpha p^3) \ll P^{\frac{11}{12}+\varepsilon} + \frac{P^{1+\varepsilon}}{q^{\frac{1}{2}}\left(1 + P^3 \left|\alpha - \frac{a}{q}\right|\right)^{\frac{1}{2}}}.$$

Proof. See [18, Lemma 2.3].

Lemma 3.3. Let $k \geq 1$ be a real number. For any fixed real number $A \geq 6$, we have

$$\int_{|\alpha|\leq X^{-1+\frac{5}{6k}-\varepsilon}} |S_k(\alpha) - U_k(\alpha)|^2 d\alpha \ll X^{\frac{2}{k}-1}(\log X)^{-A}.$$

Proof. This follows from [21, Corollary 2.3].

Lemma 3.4. Let k be a positive integer, we have

$$\int_{|\alpha|\leq X^{-1+\frac{2}{3k}-\varepsilon}} |S_k(\lambda\alpha)|^2 d\alpha \ll X^{\frac{2}{k}-1}.$$

Proof. This lemma corresponds to [13, Lemma 3.3] or [19, Lemma 3.1].

Lemma 3.5. Suppose $X^{\frac{1}{8}-1} \leq |\alpha| \leq X^{-\frac{1}{8}}$, we have

$$|S_4(\lambda_4\alpha)| \ll X^{\frac{1}{4}-\frac{1}{512}+\varepsilon}.$$

Proof. This result is derived from [22, Theorem 1] by taking $q = [|\lambda_4\alpha|^{-1}]$ and $a = 1$. It can also be found in [20].

Lemma 3.6. For $k = 1, 2, 3, 4$, and a positive integer m such that $1 \leq m \leq k$, we have

$$\int_{-\infty}^{+\infty} |S_k(\alpha)|^{2^m} K_\tau(\alpha)d\alpha \ll \tau X^{(2^m-m)/k+\varepsilon}, \int_{-1}^{1} |S_k(\alpha)|^{2^m} d\alpha \ll X^{(2^m-m)/k+\varepsilon}.$$

Proof. These results are derived from Hua's Lemma, with further details outlined in [3, Lemma 2.5].

Lemma 3.7. Suppose that α is a real number, and there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with

$$(q, a) = 1, 1 \leq q \leq P^{\frac{47}{21}}, |q\alpha - a| \leq P^{-\frac{47}{21}}.$$

Then one has

$$\sum_{p\leq P} (\log p)e(\alpha p^4) \ll P^{\frac{23}{24}+\varepsilon} + \frac{P^{1+\varepsilon}}{q^{\frac{1}{2}}\left(1 + P^4 \left|\alpha - \frac{a}{q}\right|\right)^{\frac{1}{2}}}.$$

Proof. This result is derived from [23, Theorem 1] by taking $k = 4$.

Lemma 3.8. Let $m(3) = \frac{1}{36}$, $m(4) = \frac{1}{96}$, λ and μ be nonzero constants. For $i = 3, 4$, suppose that $X^{\frac{1}{i}-m(i)+\varepsilon} \leq Z_i \leq X^{\frac{1}{i}}$. We define

$$\rho(Z_i) = \left\{\alpha \in \mathbb{R} : Z_i \leq |S_i(\lambda_i\alpha)| \leq X^{\frac{1}{i}}\right\}.$$

i) For $\alpha \in \rho(Z_i)$, there exist coprime integers a_i, q_i satisfying

$$1 \le q_i \ll \left(\frac{X^{\frac{1}{i}+\varepsilon}}{Z_i}\right)^2, |\alpha q_i \lambda_i - a_i| \ll \left(\frac{X^{\frac{1}{i}+\varepsilon}}{Z_i}\right)^2 X^{-1}.$$

ii) For $l \ge 4$, we have

$$\int_{\rho(Z_i)} |S_i(\lambda\alpha)|^2 |S_l(\mu\alpha)|^2 K_\tau(\alpha)\, d\alpha \ll \tau(X^{\frac{2}{i}+\frac{2}{l}-1+\varepsilon} + X^{\frac{4}{i}+\frac{1}{l}-1+\varepsilon} Z_i^{-2}).$$

Proof. For $i = 3$, let $P = X^{\frac{1}{3}}$, $Q = P^{\frac{3}{2}}$. By Dirichlet's approximation theorem, there exist two coprime integers a_3, q_3 with $1 \le q_3 \le Q$ and $|\alpha\lambda_3 q_3 - a_3| \le Q^{-1}$. Then it follows from Lemma 3.2 and the hypothesis $Z_3 \ge X^{\frac{1}{3}-m(3)+\varepsilon}$, we have

$$X^{\frac{1}{3}-m(3)+\varepsilon} \le Z_3 \le |S_3(\lambda_3\alpha)| \le X^{\frac{1}{3}-m(3)+\varepsilon} + \frac{X^{\frac{1}{3}+\varepsilon}}{(q_3 + X|\alpha\lambda_3 q_3 - a_3|)^{\frac{1}{2}}}. \tag{3.1}$$

Thus, we have

$$1 \le q_3 \ll \left(\frac{X^{\frac{1}{3}+\varepsilon}}{Z_3}\right)^2, |\alpha q_3 \lambda_3 - a_3| \ll \left(\frac{X^{\frac{1}{3}+\varepsilon}}{Z_3}\right)^2 X^{-1}.$$

For $i = 4$, let $P = X^{\frac{1}{4}}$, $Q = P^{\frac{47}{21}}$. By Dirichlet's approximation theorem, there exist two coprime integers a_4, q_4 with $1 \le q_4 \le Q$ and $|\alpha\lambda_4 q_4 - a_4| \le Q^{-1}$. Then it follows from Lemma 3.7 and the hypothesis $Z_4 \ge X^{\frac{1}{4}-m(4)+\varepsilon}$, we have

$$X^{\frac{1}{4}-m(4)+\varepsilon} \le Z_4 \le |S_4(\lambda_4\alpha)| \le X^{\frac{1}{4}-m(4)+\varepsilon} + \frac{X^{\frac{1}{4}+\varepsilon}}{(q_4 + X|\alpha\lambda_4 q_4 - a_4|)^{\frac{1}{2}}}. \tag{3.2}$$

Thus, we have

$$1 \le q_4 \ll \left(\frac{X^{\frac{1}{4}+\varepsilon}}{Z_4}\right)^2, |\alpha q_4 \lambda_4 - a_4| \ll \left(\frac{X^{\frac{1}{4}+\varepsilon}}{Z_4}\right)^2 X^{-1},$$

which completes the proof of Lemma 3.8 i).

Next, we give the proof of Lemma 3.8 ii). Using Brüdern's method from [24], the result can be established for the case when $\lambda = 1$. To simplify the analysis, we assume that $\lambda = 1$. Define $Q_i = X^{\frac{2}{i}+\varepsilon} Z_i^{-2}$ and

$$\rho_i'(q_i, a_i) = \left\{\alpha : \left|\alpha - \frac{a_i}{q_i}\right| \le \frac{Q_i}{q_i X}\right\}, i = 3, 4.$$

Let $V_i(\alpha)$ be a function of period 1. Further, define the function $V_i(\alpha)$ for $\alpha \in \left(Q_i X^{-1}, 1 + Q_i X^{-1}\right]$ as

$$V_i(\alpha) = \begin{cases} (q_i + X|\alpha q_i - a_i|)^{-1}, & \alpha \in \rho_i'; \\ 0, & \alpha \in \left(Q_i X^{-1}, 1 + Q_i X^{-1}\right] \backslash \rho_i', \end{cases}$$

where ρ_i' is the union of all intervals $\rho_i'(q_i, a_i)$ such that $1 \leq a_i \leq q_i \leq Q_i$ and $(a_i, q_i) = 1$. Let $\rho_i^* = \rho_i' + \mathbb{Z}$ be the union of all such intervals, then

$$\rho(Z_i) \subseteq \rho_i^*. \tag{3.3}$$

For $\alpha \in \rho(Z_i)$, by combining (3.1)–(3.3) , we obtain

$$|S_i(\lambda\alpha)| \ll X^{\frac{1}{i}+\varepsilon} V_i(\alpha)^{\frac{1}{2}}.$$

Next, we express

$$|S_l(\mu\alpha)|^2 = \sum_v e(\alpha v)\psi(v),$$

where $l \geq 4$,

$$\psi(v) = \sum_{\substack{p_3, p_4 \in I_l \\ \mu\left(p_3^l - p_4^l\right) = v.}} (\log p_3)(\log p_4),$$

We can deduce from [24, Lemma3] that

$$\begin{aligned}\int_{\rho(Z_i)} |S_i(\lambda\alpha)|^2 |S_l(\mu\alpha)|^2 K_\tau(\alpha)\,d\alpha &\ll X^{\frac{2}{i}+\varepsilon} \int_{\rho_i^*} V_i(\alpha) \sum_v (\psi(v)\,e(\alpha v))\,K_\tau(\alpha)\,d\alpha \\ &\ll \tau X^{\frac{2}{i}-1+\varepsilon}(1+\tau)^{1+\varepsilon}\left(\sum_v \psi(v) + Q_i \sum_{|v|\leq\tau} \psi(v)\right). \end{aligned} \tag{3.4}$$

We easily derive

$$\sum_v \psi(v) = |S_l(0)|^2 \ll X^{\frac{2}{l}}. \tag{3.5}$$

Since $\tau \to 0$ as $X \to \infty$, by the Prime Number Theorem, we have

$$\sum_{|v|\leq\tau} \psi(v) = \sum_{p\in I_l} (\log p)^2 \ll X^{\frac{1}{l}+\varepsilon}. \tag{3.6}$$

Lemma 3.8 ii) can be readily proven by substituting (3.5) and (3.6) into (3.4).

4. The major arc

In this section, we refine our analysis by dividing the major arc into two distinct regions, $\mathcal{M}_1$ and $\mathcal{M}_2$, which are defined as follows:

$$\mathcal{M}_1 = \left\{\alpha : |\alpha| \leq X^{\frac{5}{24}-1-\varepsilon}\right\}, \mathcal{M}_2 = \left\{\alpha : X^{\frac{5}{24}-1-\varepsilon} < |\alpha| \leq X^{-\frac{7}{9}-\varepsilon}\right\}.$$

4.1. The region $\mathcal{M}_1$

In this subsection, we establish a lower bound for the integral over the $\mathcal{M}_1$. For k=1, 2, 3, 4, define

$$F_k(\alpha) = \int_{I_k} e(\alpha u^k)du.$$

Applying the first derivative estimate for trigonometric integrals (see [25, Lemma 4.2]), we derive the bound

$$F_k(\alpha) \ll X^{\frac{1}{k}-1}\min(X, |\alpha|^{-1}). \tag{4.1}$$

Lemma 4.1. We have

$$\mathcal{I}(v, X; \mathcal{M}_1) \gg \tau^2 X^{\frac{13}{12}}.$$

Proof. It can be easily demonstrated that

$$\begin{aligned}
&\int_{\mathcal{M}_1} S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)K_\tau(\alpha)e(-v\alpha)d\alpha \\
&= \int_{\mathcal{M}_1} F_1(\lambda_1\alpha)F_2(\lambda_2\alpha)F_3(\lambda_3\alpha)F_4(\lambda_4\alpha)K_\tau(\alpha)e(-v\alpha)d\alpha \\
&+ \int_{\mathcal{M}_1} (S_1(\lambda_1\alpha) - F_1(\lambda_1\alpha))F_2(\lambda_2\alpha)F_3(\lambda_3\alpha)F_4(\lambda_4\alpha)K_\tau(\alpha)e(-v\alpha)d\alpha \\
&+ \int_{\mathcal{M}_1} S_1(\lambda_1\alpha)(S_2(\lambda_2\alpha) - F_2(\lambda_2\alpha))F_3(\lambda_3\alpha)F_4(\lambda_4\alpha)K_\tau(\alpha)e(-v\alpha)d\alpha \\
&+ \int_{\mathcal{M}_1} S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)(S_3(\lambda_3\alpha) - F_3(\lambda_3\alpha))F_4(\lambda_4\alpha)K_\tau(\alpha)e(-v\alpha)d\alpha \\
&+ \int_{\mathcal{M}_1} S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)(S_4(\lambda_4\alpha) - F_4(\lambda_4\alpha))K_\tau(\alpha)e(-v\alpha)d\alpha \\
&= J_0 + J_1 + J_2 + J_3 + J_4,
\end{aligned}$$

where it is shown that $J_0 \gg \tau^2 X^{\frac{13}{12}}$ and $J_k = o(\tau^2 X^{\frac{13}{12}})$ for k=1, 2, 3, 4.

The analysis begins by establishing a lower bound for J_0,

$$\begin{aligned}
J_0 = &\int_{\mathbb{R}} F_1(\lambda_1\alpha)F_2(\lambda_2\alpha)F_3(\lambda_3\alpha)F_4(\lambda_4\alpha)K_\tau(\alpha)e(-v\alpha)d\alpha \\
&+ O\left(\int_{|\alpha|>X^{\frac{5}{24}-1-\varepsilon}} |F_1(\lambda_1\alpha)F_2(\lambda_2\alpha)F_3(\lambda_3\alpha)F_4(\lambda_4\alpha)|K_\tau(\alpha)d\alpha\right).
\end{aligned} \tag{4.2}$$

Utilizing Eqs (2.3) and (4.1), we deduce that the error term in (4.2) satisfies

$$\ll \tau^2 X^{-\frac{23}{12}} \int_{|\alpha|>X^{\frac{5}{24}-1-\varepsilon}} \frac{1}{|\alpha|^4} d\alpha \ll \tau^2 X^{-\frac{23}{12}+\frac{19}{8}+3\varepsilon} = o(\tau^2 X^{\frac{13}{12}}). \tag{4.3}$$

For brevity, let $y_j = \lambda_j x_j{}^j$ for j=1, 2, 3, 4. By changing the order of integration and substituting variables, we note that the domains of integration for y_1, y_2, y_3, and y_4 are $\left[\lambda_j\eta X, \lambda_j X\right]$. Moreover, the integral satisfies $\left|\sum_{j=1}^{4} y_j - v\right| < \tau$. Considering $\left|\sum_{j=1}^{4} y_j - v\right| < \frac{\tau}{2}$ and the bounds on y_4, we derive the lower bound for the final integral. Therefore

$$\begin{aligned}
&\int_{\mathbb{R}} F_1(\lambda_1\alpha)F_2(\lambda_2\alpha)F_3(\lambda_3\alpha)F_4(\lambda_4\alpha)K_\tau(\alpha)e(-v\alpha)d\alpha \\
&= \frac{1}{24\lambda_1{\lambda_2}^{1/2}{\lambda_3}^{1/3}{\lambda_4}^{1/4}} \int\int\int\int {y_1}^0{y_2}^{-\frac{1}{2}}{y_3}^{-\frac{2}{3}}{y_4}^{-\frac{3}{4}} \int_{\mathbb{R}} e\left(\left(\sum_{j=1}^{4} y_j - v\right)\alpha\right)K_\tau(\alpha)d\alpha dy_1dy_2dy_3dy_4 \\
&\gg \tau X^{-\frac{3}{4}} \int_{\lambda_1\eta X}^{\lambda_1 X}\int_{\lambda_2\eta X}^{\lambda_2 X}\int_{\lambda_3\eta X}^{\lambda_3 X}\int_{-\sum_{j=1}^{3} y_j+v-\frac{\tau}{2}}^{-\sum_{j=1}^{3} y_j+v+\frac{\tau}{2}} {y_1}^0{y_2}^{-\frac{1}{2}}{y_3}^{-\frac{2}{3}}dy_1dy_2dy_3dy_4 \gg \tau^2X^{\frac{13}{12}},
\end{aligned}$$

which means that

$$\int_{\mathbb{R}} F_1(\lambda_1\alpha)F_2(\lambda_2\alpha)F_3(\lambda_3\alpha)F_4(\lambda_4\alpha)K_\tau(\alpha)e(-v\alpha)d\alpha \gg \tau^2X^{\frac{13}{12}}. \tag{4.4}$$

Thus, combining (4.2)–(4.4), we have

$$J_0 \gg \tau^2X^{\frac{13}{12}}. \tag{4.5}$$

Then we turn to dealing with J_1, J_2, J_3, and J_4. According to Euler's summation formula, we have

$$|U_k(\alpha) - F_k(\alpha)| \ll 1 + |\alpha| X, k = 1, 2, 3, 4. \tag{4.6}$$

Using (2.3), we estimate J_1 as follows:

$$\begin{aligned}
J_1 &\ll \tau^2 \int_{\mathcal{M}_1} |S_1(\lambda_1\alpha) - F_1(\lambda_1\alpha)|\,|F_2(\lambda_2\alpha)F_3(\lambda_3\alpha)F_4(\lambda_4\alpha)|\,d\alpha \\
&\ll \tau^2 \int_{\mathcal{M}_1} |S_1(\lambda_1\alpha) - U_1(\lambda_1\alpha)|\,|F_2(\lambda_2\alpha)F_3(\lambda_3\alpha)F_4(\lambda_4\alpha)|\,d\alpha \\
&\quad + \tau^2 \int_{\mathcal{M}_1} |U_1(\lambda_1\alpha) - F_1(\lambda_1\alpha)|\,|F_2(\lambda_2\alpha)F_3(\lambda_3\alpha)F_4(\lambda_4\alpha)|\,d\alpha \\
&=: \tau^2(A_1 + B_1).
\end{aligned} \tag{4.7}$$

Utilizing Cauchy's inequality along with Lemma 3.3 and (4.1), we derive

$$\begin{aligned}
A_1 &\ll X^{\frac{7}{12}} \left(\int_{\mathcal{M}_1} |S_1(\lambda_1\alpha) - U_1(\lambda_1\alpha)|^2 d\alpha\right)^{\frac{1}{2}} \left(\int_{\mathcal{M}_1} |F_2(\lambda_2\alpha)|^2 d\alpha\right)^{\frac{1}{2}} \\
&\ll X^{\frac{13}{12}}(\log X)^{-\frac{A}{2}} \left(\int_0^{\frac{1}{X}} X d\alpha + \int_{\frac{1}{X}}^{X^{-1+\frac{5}{24}-\varepsilon}} X^{-1}|\alpha|^{-2}d\alpha\right)^{\frac{1}{2}} \\
&\ll X^{\frac{13}{12}}(\log X)^{-\frac{A}{2}}.
\end{aligned} \tag{4.8}$$

Similarly, from (4.1) and (4.6), we have

$$
\begin{aligned}
B_1 &\ll \int_0^{\frac{1}{X}} |F_2(\lambda_2\alpha)F_3(\lambda_3\alpha)F_4(\lambda_4\alpha)|\,d\alpha + X\int_{\frac{1}{X}}^{X^{-1+\frac{5}{24}-\varepsilon}} |\alpha|\,|F_2(\lambda_2\alpha)F_3(\lambda_3\alpha)F_4(\lambda_4\alpha)|d\alpha \\
&\ll X^{\frac{1}{12}} + X\int_{\frac{1}{X}}^{X^{-1+\frac{5}{24}-\varepsilon}} |\alpha|^{-2}X^{-\frac{23}{12}}d\alpha \qquad (4.9)\\
&\ll X^{\frac{1}{12}}.
\end{aligned}
$$

Hence, by combining (4.7)–(4.9), it follows that

$$J_1 = o(\tau^2 X^{\frac{13}{12}}). \qquad (4.10)$$

Following analogous reasoning for J_2 by (2.3), we have

$$
\begin{aligned}
J_2 &\ll \tau^2 \int_{\mathcal{M}_1} |S_1(\lambda_1\alpha)|\,|S_2(\lambda_2\alpha) - F_2(\lambda_2\alpha)||F_3(\lambda_3\alpha)F_4(\lambda_4\alpha)|d\alpha \\
&\ll \tau^2 \int_{\mathcal{M}_1} |S_1(\lambda_1\alpha)|\,|S_2(\lambda_2\alpha) - U_2(\lambda_2\alpha)||F_3(\lambda_3\alpha)F_4(\lambda_4\alpha)|d\alpha \qquad (4.11)\\
&\quad + \tau^2 \int_{\mathcal{M}_1} |S_1(\lambda_1\alpha)|\,|U_2(\lambda_2\alpha) - F_2(\lambda_2\alpha)|\,|F_3(\lambda_3\alpha)F_4(\lambda_4\alpha)|d\alpha \\
&=: \tau^2(A_2 + B_2).
\end{aligned}
$$

Applying Cauchy's inequality along with Lemmas 3.3 and 3.4, it can be further derived that

$$
\begin{aligned}
A_2 &\ll X^{\frac{7}{12}}\left(\int_{|\alpha|\le X^{-1+\frac{2}{3}-\varepsilon}} |S_1(\lambda_1\alpha)|^2 d\alpha\right)^{\frac{1}{2}}\left(\int_{\mathcal{M}_1} |S_2(\lambda_2\alpha) - U_2(\lambda_2\alpha)|^2 d\alpha\right)^{\frac{1}{2}} \\
&\ll X^{\frac{13}{12}}(\log X)^{-\frac{A}{2}}. \qquad (4.12)
\end{aligned}
$$

Similarly, using the estimates derived from (4.1) and (4.6), along with the results from Lemma 3.4, we have

$$
\begin{aligned}
B_2 &\ll \int_0^{\frac{1}{X}} |S_1(\lambda_1\alpha)F_3(\lambda_3\alpha)F_4(\lambda_4\alpha)|d\alpha + X\int_{\frac{1}{X}}^{X^{-1+\frac{5}{24}-\varepsilon}} |\alpha|\,|S_1(\lambda_1\alpha)F_3(\lambda_3\alpha)F_4(\lambda_4\alpha)|d\alpha \\
&\ll X^{\frac{7}{12}} + X^{-\frac{5}{12}}\int_{\frac{1}{X}}^{X^{-1+\frac{5}{24}-\varepsilon}} |\alpha|^{-1}\,|S_1(\lambda_1\alpha)|d\alpha \qquad (4.13)\\
&\ll X^{\frac{7}{12}} + X^{-\frac{5}{12}}\left(\int_{\frac{1}{X}}^{X^{-1+\frac{5}{24}-\varepsilon}} |\alpha|^{-2}d\alpha\right)^{\frac{1}{2}}\left(\int_{|\alpha|\le X^{-1+\frac{2}{3}-\varepsilon}} |S_1(\lambda_1\alpha)|^2 d\alpha\right)^{\frac{1}{2}} \\
&\ll X^{\frac{7}{12}}.
\end{aligned}
$$

Therefore, combining (4.11)–(4.13) results in the conclusion that

$$J_2 = o(\tau^2 X^{\frac{13}{12}}). \qquad (4.14)$$

Similar arguments for J_3 and J_4, utilizing Lemmas 3.3 and 3.4, we obtain

$$J_j = o(\tau^2 X^{\frac{13}{12}}),\ j = 3, 4. \qquad (4.15)$$

Hence, by combining (4.5), (4.10), (4.14), and (4.15), the proof of Lemma 4.1 is complete.

4.2. The region $\mathcal{M}_2$

In this subsection, we derive an upper bound for the integral over the region $\mathcal{M}_2$ to quantify its contribution to the overall integral.

Lemma 4.2. We conclude that

$$\mathcal{I}(v, X; \mathcal{M}_2) = o(\tau^2 X^{\frac{13}{12}}). \tag{4.16}$$

Proof. By applying Cauchy's inequality, along with the results from (2.3) and Lemmas 3.4 and 3.5, we derive the following:

$$\begin{aligned}\mathcal{I}(v, X\,; \mathcal{M}_2) &= \int_{\mathcal{M}_2} S_1(\lambda_1\alpha) S_2(\lambda_2\alpha) S_3(\lambda_3\alpha) S_4(\lambda_4\alpha) K_\tau(\alpha) e(-v\alpha) d\alpha \\ &\ll \tau^2 X \max_{\alpha\in\mathcal{M}_2} |S_4(\lambda_4\alpha)| \left(\int_{\mathcal{M}_2} |S_2(\lambda_2\alpha)|^2 \, d\alpha\right)^{\frac{1}{2}} \left(\int_{\mathcal{M}_2} |S_3(\lambda_3\alpha)|^2 \, d\alpha\right)^{\frac{1}{2}} \\ &\ll \tau^2 X^{\frac{5}{4}-\frac{1}{512}+\varepsilon} \left(\int_{\mathcal{M}_2} |S_2(\lambda_2\alpha)|^2 \, d\alpha\right)^{\frac{1}{2}} \left(\int_{\mathcal{M}_2} |S_3(\lambda_3\alpha)|^2 \, d\alpha\right)^{\frac{1}{2}} \\ &\ll \tau^2 X^{\frac{13}{12}-\frac{1}{512}+\varepsilon} = o\left(\tau^2 X^{\frac{13}{12}}\right).\end{aligned}$$

Subsequently, combining the results from Lemmas 4.1 and 4.2, we conclude the integral estimation over the major arc $\mathcal{M}$, to obtain the following lemma.

Lemma 4.3. We have

$$\mathcal{I}(v, X; \mathcal{M}) \gg \tau^2 X^{\frac{13}{12}}.$$

5. The trivial arc

In this section, we demonstrate

$$\mathcal{I}(v, X; t) = o(\tau^2 X^{\frac{13}{12}}). \tag{5.1}$$

Utilizing Cauchy's inequality and the trivial bounds of $S_1(\lambda_1\alpha)$, $S_4(\lambda_4\alpha)$, we obtain

$$\begin{aligned}\mathcal{I}(v, X; t) &= \int_t S_1(\lambda_1\alpha) S_2(\lambda_2\alpha) S_3(\lambda_3\alpha) S_4(\lambda_4\alpha) K_\tau(\alpha) e(-v\alpha) d\alpha \\ &\ll X^{1+\frac{1}{4}} \left(\int_t |S_2(\lambda_2\alpha)|^2 K_\tau(\alpha) d\alpha\right)^{\frac{1}{2}} \left(\int_t |S_3(\lambda_3\alpha)|^2 K_\tau(\alpha) d\alpha\right)^{\frac{1}{2}}.\end{aligned} \tag{5.2}$$

By the periodicity of $S_2(\lambda_2\alpha)$, along with (2.3) and Lemma 3.6, we obtain

$$\begin{aligned}\int_t |S_2(\lambda_2\alpha)|^2 K_\tau(\alpha)\, \mathrm{d}\alpha &\ll \int_\gamma^\infty |S_2(\lambda_2\alpha)|^2 \frac{1}{|\alpha|^2} \mathrm{d}\alpha \ll \int_{|\lambda_2|\gamma}^\infty |S_2(\alpha)|^2 \frac{1}{|\alpha|^2} \mathrm{d}\alpha \\ &\ll \sum_{m=\left[|\lambda_2|\gamma\right]}^\infty \int_m^{m+1} |S_2(\alpha)|^2 \frac{1}{|\alpha|^2} \mathrm{d}\alpha \ll \sum_{m=\left[|\lambda_2|\gamma\right]}^\infty \int_m^{m+1} |S_2(\alpha)|^2 \frac{1}{m^2} \mathrm{d}\alpha\end{aligned} \tag{5.3}$$

$$\ll \int_0^1 |S_2(\alpha)|^2 \mathrm{d}\alpha \sum_{m=[|\lambda_2|\gamma]}^{\infty} \frac{1}{m^2} \ll X^{\frac{1}{2}+\varepsilon}\gamma^{-1}.$$

Similarly, we can conclude that

$$\int_t |S_3(\lambda_3\alpha)|^2 K_\tau(\alpha)\,\mathrm{d}\alpha \ll X^{\frac{1}{3}+\varepsilon}\gamma^{-1}. \tag{5.4}$$

By combining (5.2)–(5.4), we have

$$\int_t S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)K_\tau(\alpha)e(-v\alpha)d\alpha \ll X^{\frac{20}{12}+\varepsilon}\gamma^{-1} \ll X^{\frac{20}{12}+\varepsilon}\tau^2 X^{-\frac{7}{12}-2\varepsilon} \ll \tau^2 X^{\frac{13}{12}-\varepsilon}.$$

Thus, (5.1) follows directly.

6. The minor arc and the proof of Theorem 1.2

This section provides a precise estimation of the integral over the minor arc and proves Theorem 1.2. We define $m = \hat{m} \cup m^*$, with $\rho = \frac{1}{12}$ and $\tau = X^{-\delta}$.

Let $\mathcal{E} = \mathcal{E}(\mathcal{V}, X, \delta)$ denote the set of elements v in $\mathcal{V}$ for which the inequality (1.4) has no solution in the prime variables p_j for $j = 1,2,3,4$. Thus, we have $E(\mathcal{V}, X, \delta) = |\mathcal{E}(\mathcal{V}, X, \delta)|$. By selecting an appropriate complex number ϑ_v such that $|\vartheta_v| = 1$, we can reformulate the integral as follows:

$$\begin{aligned} E(\mathcal{V}, X, \delta)\,\tau^2 X^{\frac{13}{12}} &\ll \sum_{v\in\mathcal{E}} \left| \int_m S_1(\lambda_1\alpha) S_2(\lambda_2\alpha) S_3(\lambda_3\alpha) S_4(\lambda_4\alpha) K_\tau(\alpha)\, e(-v\alpha)\, d\alpha \right| \\ &= \sum_{v\in\mathcal{E}} \vartheta_v \int_m S_1(\lambda_1\alpha) S_2(\lambda_2\alpha) S_3(\lambda_3\alpha) S_4(\lambda_4\alpha) K_\tau(\alpha)\, e(-v\alpha)\, d\alpha \\ &= \int_m S_1(\lambda_1\alpha) S_2(\lambda_2\alpha) S_3(\lambda_3\alpha) S_4(\lambda_4\alpha) T(\alpha) K_\tau(\alpha)\, d\alpha. \end{aligned} \tag{6.1}$$

In this case, we have

$$T(\alpha) = \sum_{v\in\mathcal{E}} \vartheta_v e(-v\alpha).$$

By applying Cauchy's inequality, we can bound the expression as follows:

$$E(\mathcal{V}, X, \delta)\,\tau^2 X^{\frac{13}{12}} \ll \left(\int_m |S_2(\lambda_2\alpha) T(\alpha)|^2 K_\tau(\alpha)\, d\alpha \right)^{\frac{1}{2}} \times \left(\int_m |S_1(\lambda_1\alpha) S_3(\lambda_3\alpha) S_4(\lambda_4\alpha)|^2 K_\tau(\alpha)\, d\alpha \right)^{\frac{1}{2}}. \tag{6.2}$$

Next, we define $\hat{m} = m_1 \cup m_2 \cup m_3$ and $m^* = m \backslash \hat{m}$, with $X = q^{\frac{90}{49}}$, where

$$\begin{aligned} m_1 &= \left\{ \alpha : |S_3(\lambda_3\alpha)| < X^{\frac{1}{3}-\frac{1}{3}\rho+\varepsilon}, |S_4(\lambda_4\alpha)| < X^{\frac{1}{4}-\frac{1}{8}\rho+\varepsilon} \right\}, \\ m_2 &= \left\{ \alpha : |S_1(\lambda_1\alpha)| < X^{1-\frac{12}{5}\rho+\varepsilon}, |S_3(\lambda_3\alpha)| \geq X^{\frac{1}{3}-\frac{1}{3}\rho+\varepsilon} \right\}, \\ m_3 &= \left\{ \alpha : |S_1(\lambda_1\alpha)| < X^{1-\frac{12}{5}\rho+\varepsilon}, |S_4(\lambda_4\alpha)| \geq X^{\frac{1}{4}-\frac{1}{8}\rho+\varepsilon} \right\}. \end{aligned} \tag{6.3}$$

Lemma 6.1. We have

$$\int_{m} |S_2(\lambda_2\alpha) T(\alpha)|^2 K_\tau(\alpha)\, d\alpha \ll \tau\Big(E(\mathcal{V}, X, \delta) X^{\frac{1}{2}+\varepsilon} + (E(\mathcal{V}, X, \delta))^2 X^{\varepsilon}\Big). \tag{6.4}$$

Proof. According to (2.3), we have

$$\begin{aligned}
&\int_{R} |S_2(\lambda_2\alpha) T(\alpha)|^2 K_\tau(\alpha)\, d\alpha \\
&= \sum_{v_1, v_2 \in \mathcal{E}} \vartheta_{v_1}\vartheta_{v_2} \sum_{\eta X \le p_1{}^2, p_2{}^2 \le X} (\log p_1)(\log p_2) \int_{R} e\left[\Big(\lambda_2(p_1{}^2 - p_2{}^2) - (v_1 - v_2)\Big)\alpha\right] K_\tau(\alpha)\, d\alpha \\
&= \sum_{v_1, v_2 \in \mathcal{E}} \vartheta_{v_1}\vartheta_{v_2} \sum_{\eta X \le p_1{}^2, p_2{}^2 \le X} (\log p_1)(\log p_2) \max(0, \tau - \left|\lambda_2(p_1{}^2 - p_2{}^2) - (v_1 - v_2)\right|) \\
&\ll (\log X)^2 \mathfrak{L}(X),
\end{aligned}$$

where $\mathfrak{L}(X)$ represents the number of solutions to the inequality

$$\left|\lambda_2(p_1{}^2 - p_2{}^2) - (v_1 - v_2)\right| < \tau$$

with $v_1, v_2 \in \mathcal{E}$ and $\eta X \le p_1{}^2, p_2{}^2 \le X$.

Since X is sufficiently large and $\tau = X^{-\delta}$. When $v_1 = v_2$, there must exist the case that $p_1 = p_2$. Under this condition, we can deduce that

$$\mathfrak{L}(X) \ll \tau E(\mathcal{V}, X, \delta) X^{\frac{1}{2}}.$$

When $v_1 \ne v_2$, there exists at most one integer n such that $n \ll X$ and the inequality $|\lambda_2 n - (v_1 - v_2)| < \tau$ holds. For any integer n , the number of solutions to $n = p_1{}^2 - p_2{}^2$ is bounded by X^{ε}. Consequently, we obtain

$$\mathfrak{L}(X) \ll \tau (E(\mathcal{V}, X, \delta))^2 X^{\varepsilon}.$$

Combining both cases, we obtain

$$\mathfrak{L}(X) \ll \tau\Big(E(\mathcal{V}, X, \delta) X^{\frac{1}{2}} + (E(\mathcal{V}, X, \delta))^2 X^{\varepsilon}\Big).$$

Thus, Lemma 6.1 is established.

Lemma 6.2. We have

$$\int_{\hat{m}} |S_1(\lambda_1\alpha) S_3(\lambda_3\alpha) S_4(\lambda_4\alpha)|^2 K_\tau(\alpha)\, d\alpha \ll \tau X^{\frac{13}{6} - \frac{11}{12}\rho + \varepsilon}. \tag{6.5}$$

Proof. Utilizing Lemmas 3.6 and 3.8 ii) along with the trivial bounds for $S_3(\lambda_3\alpha)$ and $S_4(\lambda_4\alpha)$, we analyze the integrals over m_1, m_2, and m_3 separately.

For the integral over the interval m_1, we obtain

$$\begin{aligned}
&\int_{m_1} |S_1(\lambda_1\alpha) S_3(\lambda_3\alpha) S_4(\lambda_4\alpha)|^2 K_\tau(\alpha)\, d\alpha \\
&\ll (X^{\frac{1}{3} - \frac{1}{36} + \varepsilon})^2 (X^{\frac{1}{4} - \frac{1}{96} + \varepsilon})^2 \left(\int_{m_1} |S_1(\lambda_1\alpha)|^2 K_\tau(\alpha)\, d\alpha\right) \ll \tau X^{\frac{13}{6} - \frac{11}{12}\rho + \varepsilon}.
\end{aligned}$$

For the integral over the interval m_2, we obtain

$$\begin{aligned}&\int_{m_2} |S_1(\lambda_1\alpha) S_3(\lambda_3\alpha) S_4(\lambda_4\alpha)|^2 K_\tau(\alpha)\, d\alpha \\ &\ll \left(X^{1-\frac{1}{5}+\varepsilon}\right)^2 \left(\int_{m_2} |S_3(\lambda_3\alpha)|^2 |S_4(\lambda_4\alpha)|^2 K_\tau(\alpha)\, d\alpha\right) \\ &\ll \tau X^{\frac{13}{6}-\frac{24}{5}\rho+\varepsilon}.\end{aligned}$$

For the integral over the interval m_3, we obtain

$$\begin{aligned}&\int_{m_3} |S_1(\lambda_1\alpha) S_3(\lambda_3\alpha) S_4(\lambda_4\alpha)|^2 K_\tau(\alpha)\, d\alpha \\ &\ll \left(X^{1-\frac{1}{5}+\varepsilon}\right)^2 \left(\int_R |S_3(\lambda_3\alpha)|^4 K_\tau(\alpha)\, d\alpha\right)^{\frac{1}{2}} \left(\int_{m_3} |S_4(\lambda_4\alpha)|^2 |S_4(\lambda_4\alpha)|^2 K_\tau(\alpha)\, d\alpha\right)^{\frac{1}{2}} \\ &\ll \tau X^{\frac{13}{6}-\frac{14}{5}\rho+\varepsilon}.\end{aligned}$$

By combining these estimates, Lemma 6.2 follows immediately.
Lemma 6.3. We have

$$\int_{m^*} |S_1(\lambda_1\alpha) S_3(\lambda_3\alpha) S_4(\lambda_4\alpha)|^2 K_\tau(\alpha)\, d\alpha \ll \tau X^{\frac{13}{6}-\frac{98}{15}\rho+5\varepsilon}. \tag{6.6}$$

Proof. Employing Harman's method as outlined in [14], we partition the region m^* into disjoint sets $S(Z_1, Z_3, y)$ defined as

$$S(Z_1, Z_3, y) = \{\alpha \in m^* : Z_1 \leq |S_1(\lambda_1\alpha)| < 2Z_1, Z_3 \leq |S_3(\lambda_3\alpha)| < 2Z_3, y \leq |\alpha| \leq 2y\},$$

where $Z_1 = X^{1-\frac{12}{5}\rho+2\varepsilon}2^{t_1}$, $Z_3 = X^{\frac{1}{3}-\frac{1}{3}\rho+2\varepsilon}2^{t_2}$, $y = \xi 2^{t_3}$ for some positive integers t_1, t_2, and t_3.

By invoking Lemmas 3.1 and 3.8 i), we obtain integers a_1, q_1 and a_3, q_3 such that $(a_1, q_1) = 1$ and $(a_3, q_3) = 1$ satisfying

$$1 \leq q_1 \ll \left(\frac{X^{1+\varepsilon}}{Z_1}\right)^2, |q_1\lambda_1\alpha - a_1| \ll X^{-1}\left(\frac{X^{1+\varepsilon}}{Z_1}\right)^2, \tag{6.7}$$

$$1 \leq q_3 \ll \left(\frac{X^{\frac{1}{3}+\varepsilon}}{Z_3}\right)^2, |q_3\lambda_3\alpha - a_3| \ll X^{-1}\left(\frac{X^{\frac{1}{3}+\varepsilon}}{Z_3}\right)^2. \tag{6.8}$$

Note that $a_1 a_3 \neq 0$.

We further dissect $S(Z_1, Z_3, y)$ into subsets $S(Z_1, Z_3, y, Q_1, Q_3)$ with α satisfying $|\alpha| \geq y = \xi 2^{t_3} \geq \xi = X^{-\frac{7}{9}-\varepsilon}$ and $\left|\frac{a_i}{\lambda_i\alpha}\right| \ll q_i$ for $i = 1, 3$, where

$$Q_1 \leq q_1 < 2Q_1, Q_3 \leq q_3 < 2Q_3, Q_1 \ll \left(\frac{X^{1+\varepsilon}}{Z_1}\right)^2, Q_3 \ll \left(\frac{X^{\frac{1}{3}+\varepsilon}}{Z_3}\right)^2.$$

Then, we have

$$\begin{aligned}\left|a_3q_1\frac{\lambda_1}{\lambda_3}-a_1q_3\right| &= \left|\frac{a_1\left(a_3-q_3\lambda_3\alpha\right)+a_3\left(q_1\lambda_1\alpha-a_1\right)}{\lambda_3\alpha}\right| \\ &\ll Q_1X^{-1}\left(\frac{X^{\frac{1}{3}+\varepsilon}}{Z_3}\right)^2+Q_3X^{-1}\left(\frac{X^{1+\varepsilon}}{Z_1}\right)^2 \\ &\ll X^{-1+\frac{82}{15}\rho-4\varepsilon} \ll X^{-\frac{49}{90}-4\varepsilon}.\end{aligned} \tag{6.9}$$

Assuming that $|a_3q_1|$ takes only R distinct values. By the pigeonhole principle, we have $R \ll \frac{yQ_1Q_3}{q}$. Due to bounds on the divisor function, each value of $|a_3q_1|$ corresponds to significantly fewer than X^{ε} pairs a_3, q_1. For fixed a_3 and q_1, the value of $|a_1q_3|$ is the integral part of $a_3q_1\frac{\lambda_1}{\lambda_3}$; thus there are significantly fewer than X^{ε} pairs a_1, q_3. Consequently, by (6.7) and (6.8), the length of $S\left(Z_1, Z_3, y, Q_1, Q_3\right)$ is

$$\begin{aligned}&\ll RX^{\varepsilon}\min\left(\frac{1}{Q_1X}\left(\frac{X^{1+\varepsilon}}{Z_1}\right)^2, \frac{1}{Q_3X}\left(\frac{X^{\frac{1}{3}+\varepsilon}}{Z_3}\right)^2\right) \\ &\ll \frac{X^{\frac{101}{180}+\varepsilon}y}{qZ_1Z_3}.\end{aligned}$$

Evaluating the integral over $S\left(Z_1, Z_3, y, Q_1, Q_3\right)$, we obtain

$$\begin{aligned}&\int_{S(Z_1,Z_3,y,Q_1,Q_3)}\left|S_1\left(\lambda_1\alpha\right)S_3\left(\lambda_3\alpha\right)S_4\left(\lambda_4\alpha\right)\right|^2K_\tau\left(\alpha\right)d\alpha \\ &\ll \min\left(\tau^2, y^{-2}\right)Z_1{}^2Z_3^2X^{\frac{2}{4}}\int_{S(Z_1,Z_3,y,Q_1,Q_3)}d\alpha \ll \tau y^{-1}Z_1^2Z_3^2X^{\frac{1}{2}}\frac{X^{\frac{101}{180}+\varepsilon}y}{qZ_1Z_3} \\ &\ll \tau y^{-1}Z_1Z_3X^{\frac{1}{2}}\frac{X^{\frac{101}{180}+\varepsilon}y}{q} \ll \frac{\tau X^{\frac{431}{180}-\frac{41}{15}\rho+5\varepsilon}}{q} \ll \frac{\tau X^{\frac{13}{6}+5\varepsilon}}{q} \ll \tau X^{\frac{73}{45}+5\varepsilon}.\end{aligned}$$

Finally, summing over all possible values of Z_1, Z_3, y, Q_1, Q_3, we have

$$\int_{m^*}\left|S_1\left(\lambda_1\alpha\right)S_3\left(\lambda_3\alpha\right)S_4\left(\lambda_4\alpha\right)\right|^2K_\tau\left(\alpha\right)d\alpha \ll \tau X^{\frac{73}{45}+6\varepsilon}.$$

Combining Lemmas 6.2 and 6.3, we arrive at Lemma 6.4.
Lemma 6.4. We have

$$\int_{m}\left|S_1\left(\lambda_1\alpha\right)S_3\left(\lambda_3\alpha\right)S_4\left(\lambda_4\alpha\right)\right|^2K_\tau\left(\alpha\right)d\alpha \ll \tau X^{\frac{13}{6}-\frac{11}{12}\rho+\varepsilon}.$$

Proof of Theorem 1.2. We now proceed to prove the first part of Theorem 1.2. Substituting (6.4) and Lemma 6.4 into (6.2), we have

$$\begin{aligned}E\left(\mathcal{V}, X, \delta\right)\tau^2X^{\frac{13}{12}} &\ll \left(\tau\left(E\left(\mathcal{V}, X, \delta\right)X^{\frac{1}{2}+\varepsilon}+\left(E\left(\mathcal{V}, X, \delta\right)\right)^2X^{\varepsilon}\right)\right)^{\frac{1}{2}}\left(\tau X^{\frac{13}{6}-\frac{11}{144}+\varepsilon}\right)^{\frac{1}{2}} \\ &\ll \tau E(\mathcal{V}, X, \delta)^{\frac{1}{2}}X^{\frac{16}{12}-\frac{11}{288}+\varepsilon}+\tau E\left(\mathcal{V}, X, \delta\right)X^{\frac{13}{12}-\frac{11}{288}+\varepsilon}.\end{aligned}$$

Due to $0<\delta<\frac{11}{288}$, there is $\tau X^{\frac{13}{12}-\frac{11}{288}+\varepsilon}=o\left(\tau^2 X^{\frac{13}{12}}\right)$, we obtain

$$E(\mathcal{V},X,\delta)\,\tau^2 X^{\frac{13}{12}} \ll \tau E(\mathcal{V},X,\delta)^{\frac{1}{2}} X^{\frac{16}{12}-\frac{11}{288}+\varepsilon}.$$

Thus, we find

$$E(\mathcal{V},X,\delta) \ll \tau^{-2} X^{\frac{1}{2}-\frac{11}{144}+2\varepsilon} \ll X^{\frac{61}{144}+2\delta+2\varepsilon}.$$

Since λ_1/λ_3 is irrational, there exist infinitely many values of q that can be selected with a sequence $X_j\to\infty$ such that

$$E\left(\mathcal{V},X_j,\delta\right) \ll X_j^{\frac{61}{144}+2\delta+2\varepsilon}.$$

This completes the proof of the first part of Theorem 1.2.

Next, we prove the second part of Theorem 1.2. By the proof methods from Lemmas 6.2 and 6.3, we observe that replacing ρ with χ (as defined in Theorem 1.2) is sufficient, resulting in the following conditions:

$$X^{(1-\omega)\left(1-\frac{82}{15}\chi\right)} \ll q \ll X^{\left(1-\frac{82}{15}\chi\right)}.$$

Substituting and simplifying, we obtain

$$E(\mathcal{V},X,\delta)\,\tau^2 X^{\frac{13}{12}} \ll \tau(E(\mathcal{V},X,\delta))^{\frac{1}{2}} X^{\frac{16}{12}-\frac{11}{24}\chi+\varepsilon} + \tau E(\mathcal{V},X,\delta)\, X^{\frac{13}{12}-\frac{11}{24}\chi+\varepsilon}.$$

Given the conditions of the theorem, specifically $0<\delta<\frac{11}{24}\chi$, we have the asymptotic relation $\tau X^{\frac{13}{12}-\frac{11}{24}\chi+\varepsilon}=o\left(\tau^2 X^{\frac{13}{12}}\right)$. From this, we can deduce that

$$E(\mathcal{V},X,\delta) \ll \tau^{-2} X^{\frac{1}{2}-\frac{11}{12}\chi+2\varepsilon} \ll X^{\frac{1}{2}-\frac{11}{12}\chi+2\delta+2\varepsilon}.$$

Thus, the second part of Theorem 1.2 is proved.

7. Conclusions

In this paper, we prove that, for any $\varepsilon>0$, the number of $v\in\mathcal{V}$ with $v\le X$ such that the inequality

$$\left|\lambda_1 p_1+\lambda_2 {p_2}^2+\lambda_3 {p_3}^3+\lambda_4 {p_4}^4-v\right|<v^{-\delta}$$

has no solution in primes p_1, p_2, p_3, p_4 that does not exceed $O(X^{1-\frac{83}{144}+2\delta+2\varepsilon})$.

Author contributions

Xinyan Li: Writing-review and editing, writing-original draft, validation, resources, methodology, formal analysis, conceptualization; Wenxu Ge: Writing-review and editing, resources, methodology, supervision, validation, formal analysis. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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