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Research article

The reverse order laws for $\{1,2,3M\}$- and $\{1,2,4N\}$- inverse of three matrix products

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Abstract: The reverse order laws for weighted generalized inverses often appear in linear algebra problems of several applied fields, having attracted considerable attention. In this paper, by using the maximal and minimal ranks of the generalized Schur complement, we obtained some necessary and sufficient conditions for the reverse order laws

$$A_3\{1,2,3M_3\}A_2\{1,2,3M_2\}A_1\{1,2,3M_1\} \subseteq (A_1A_2A_3)\{1,2,3M_1\}$$

and

$$A_3\{1,2,4N_4\}A_2\{1,2,4N_3\}A_1\{1,2,4N_2\} \subseteq (A_1A_2A_3)\{1,2,4N_4\}.$$

Keywords: reverse order law; weighted generalized inverses; generalized Schur complement; maximal and minimal ranks; matrix product
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1. Introduction

At first, we will provide the following explanations of some of the notations for the convenience of readers.

Definition 1.1. $C^{m\times n}$: the set of $m \times n$ complex matrices,
I_k: k-order identity matrix,
$O_{m\times n}$: zero matrix of order $m \times n$,
$R(A)$, $N(A)$: the range space and the null space of A, respectively,
$r(A)$, A^*: the rank and the conjugate transpose of A, respectively.

Definition 1.2. [1, 2] Let $A \in C^{m\times n}$, and let $M \in C^{m\times m}$, $N \in C^{n\times n}$ be two positive definite Hermitian matrices. X is the weighted Moore-Penrose inverse of A when it satisfies

$$(1)\ AXA = A,\ \ (2)\ XAX = X,\ \ (3M)\ (MAX)^* = MAX,\ \ (4N)\ (NXA)^* = NXA, \tag{1.1}$$

where X is denoted by $X = A^{(1,2,3M,4N)} = A^{\dagger}_{M,N}$.

For a given matrix $A \in C^{m\times n}$, the sets $A\{1,2,3M\}$- and $A\{1,2,4N\}-$ inverses of A are
$A\{1,2,3M\} = \{X \in C^{n\times m} \mid AXA = A,\ XAX = X,\ (MAX)^* = MAX\}$,
$A\{1,2,4N\} = \{X \in C^{n\times m} \mid AXA = A,\ XAX = X,\ (NXA)^* = NXA\}$;
more relevant theories can be found in [1,3].

The reverse order law for weighted generalized inverses is a key tool in the study of the weighted least squares problem, the weighted perturbation theory, optimization problems, and other related topics [4–6].

The reverse order laws for generalized inverses of matrix products are a class of interesting problems that are fundamental in the theory of generalized inverses [7–10]. In 1966, Greville [7] first gave an equivalent condition for the so-called reverse order law $B^{\dagger}A^{\dagger} = (AB)^{\dagger}$. Since then, many authors have studied this problem [11–15]. On studying the reverse order for any $\{i, j, \cdots, k\}$-inverse of matrix products, one important relations problem is: under what conditions

$$A_3\{i, j, \cdots, k\}A_2\{i, j, \cdots, k\}A_1\{i, j, \cdots, k\} \subseteq (A_1A_2A_3)\{i, j, \cdots, k\}$$

holds, where $\{(i), (j), \cdots, (k)\} \subseteq \{(1), (2), (3M), (4N)\}$.

In [16], some necessary and sufficient conditions were presented for the first times for several types of reverse order laws to hold for weighted generalized inverses. Since then, reverse order laws for weighted generalized inverses of matrix products have attracted considerable attention and some interesting results have been derived [17–20].

The purpose of this paper is to show some equivalent conditions for the following so-called reverse order laws

$$A_3\{1,2,3M_3\}A_2\{1,2,3M_2\}A_1\{1,2,3M_1\} \subseteq (A_1A_2A_3)\{1,2,3M_1\}$$

and

$$A_3\{1,2,4N_4\}A_2\{1,2,4N_3\}A_1\{1,2,4N_2\} \subseteq (A_1A_2A_3)\{1,2,4N_4\},$$

where $A_i \in C^{m_i\times m_{i+1}}, i = 1,2,3,\ M_i \in C^{m_i\times m_i}, i = 1,2,3,\ N_i \in C^{m_i\times m_i}, and\ \ i = 2,3,4$ are six positive definite Hermitian matrices.

The following lemmas are essential to the rest of this paper.

Lemma 1.1. *[3] Let L, M be two complementary subspaces of C^m, and let $P_{L,M}$ be the projector on L along M, then*

$$P_{L,M}A = A \Longleftrightarrow R(A) \subseteq L, \tag{1.2}$$

$$AP_{L,M} = A \Longleftrightarrow N(A) \supseteq M. \tag{1.3}$$

Lemma 1.2. *[1,3] Let $A \in C^{m\times n}$, $X \in C^{n\times m}$, and let M, N be two positive definite Hermitian matrices of order m and n, respectively, then*

$$X \in A\{1,2,3M\} \Longleftrightarrow A^*MAX = A^*M \ \ and\ \ r(X) = r(A), \tag{1.4}$$

$$X \in A\{1,2,4N\} \Longleftrightarrow XAN^{-1}A^* = N^{-1}A^* \ \ and \ \ r(X) = r(A), \tag{1.5}$$

$$X \in A\{1,2,4N\} \Longleftrightarrow X^* \in A^*\{1,2,3N^{-1}\}. \tag{1.6}$$

Lemma 1.3. *[21] Let $A \in C^{m\times n}$, $B \in C^{m\times k}$, $C \in C^{l\times n}$, $D \in C^{l\times k}$, and let $M \in C^{m\times m}$, $N \in C^{n\times n}$ be two positive definite Hermitian matrices, then*

$$\max_{A^{(1,2,3M)}} r(D - CA^{(1,2,3M)}B) = \min\left\{ r\begin{pmatrix} A^*MA & A^*MB \\ C & D \end{pmatrix} - r(A),\ \ r\begin{pmatrix} A^*MB \\ D \end{pmatrix} \right\}, \tag{1.7}$$

$$\min_{A^{(1,2,3M)}} r(D - CA^{(1,2,3M)}B) = r\begin{pmatrix} A^*MA & A^*MB \\ C & D \end{pmatrix} + r\begin{pmatrix} A^*MB \\ D \end{pmatrix} - r\begin{pmatrix} A & O \\ O & A^*MB \\ C & D \end{pmatrix}, \tag{1.8}$$

$$\max_{A^{(1,2,4N)}} r(D - CA^{(1,2,4N)}B) = \min\left\{ r\begin{pmatrix} CN^{-1}A^*, & D \end{pmatrix}, r\begin{pmatrix} AN^{-1}A^* & B \\ CN^{-1}A^* & D \end{pmatrix} - r(A) \right\}, \tag{1.9}$$

$$\min_{A^{(1,2,4N)}} r(D - CA^{(1,2,4N)}B) = r(CN^{-1}A^*, \quad D) + r\begin{pmatrix} AN^{-1}A^* & B \\ CN^{-1}A^* & D \end{pmatrix} - r\begin{pmatrix} A & O & B \\ O & CN^{-1}A^* & D \end{pmatrix}. \tag{1.10}$$

Lemma 1.4. *[22] Let $A \in C^{m\times n}$, $B \in C^{m\times k}$ and $C \in C^{p\times n}$, then*

$$r\begin{pmatrix} A, & B \end{pmatrix} = r(A) + r(E_A B) = r(B) + r(E_B A), \tag{1.11}$$

$$r\begin{pmatrix} A \\ C \end{pmatrix} = r(A) + r(CF_A) = r(C) + r(AF_C), \tag{1.12}$$

$$r\begin{pmatrix} A \\ C \end{pmatrix} \le r(A) + r(C), \quad r\begin{pmatrix} A, & B \end{pmatrix} \le r(A) + r(B), \tag{1.13}$$

where the projectors $E_A = I_m - AA^\dagger$, $E_B = I_m - BB^\dagger$, $F_A = I_n - A^\dagger A$, $F_C = I_n - C^\dagger C$.

2. Main results

In this section, we will present some necessary and sufficient conditions for the reverse order laws for the weighted generalized inverses $\{1,2,3M\}-$ and $\{1,2,4N\}-$ of three matrix products. The following theorem is the main result in this section.

Theorem 2.1. *Let* $A_i \in C^{m_i \times m_{i+1}}$, $A_i^{(1,2,3M_i)} \in A_i\{1,2,3M_i\}$ *and* $i \in \{1,2,3\}$. *Let* $M_i \in C^{m_i \times m_i}$, $i \in \{1,2,3\}$ *be three positive definite Hermitian matrices, then*

$$\begin{aligned} & A_3\{1,2,3M_3\}A_2\{1,2,3M_2\}A_1\{1,2,3M_1\} \subseteq (A_1A_2A_3)\{1,2,3M_1\} \\ \Longleftrightarrow \quad & r\begin{pmatrix} A_3^* & O \\ O & A_2^* \\ A_3^*A_2^*A_1^*M_1A_1A_2M_3^{-1} & A_3^*A_2^*A_1^*M_1A_1M_2^{-1} \end{pmatrix} = r(A_2) + r(A_3) \ \text{ and} \\ & r(A_1A_2A_3) = \min\{r(A_1), r(A_2), r(A_3)\} = \sum_{i=1}^{3} r(A_i) - r\begin{pmatrix} A_2^* & O \\ O & A_3^* \\ A_1M_2^{-1} & A_1A_2M_3^{-1} \end{pmatrix}. \end{aligned} \tag{2.1}$$

Proof. According to the formula (1.4) in Lemma 1.2, for any $A_i^{(1,2,3M_i)} \in A_i\{1,2,3M_i\}$, $i \in \{1,2,3\}$, we can reach the conclusion that

$$A_3\{1,2,3M_3\}A_2\{1,2,3M_2\}A_1\{1,2,3M_1\} \subseteq (A_1A_2A_3)\{1,2,3M_1\}$$

holds if and only if

$$A_3^*A_2^*A_1^*M_1A_1A_2A_3A_3^{(1,2,3M_3)}A_2^{(1,2,3M_2)}A_1^{(1,2,3M_1)} = A_3^*A_2^*A_1^*M_1 \ \text{ and}$$

$$r(A_3^{(1,2,3M_3)}A_2^{(1,2,3M_2)}A_1^{(1,2,3M_1)}) = r(A_1A_2A_3)$$

holds, which are respectively equivalent to the following two identities

$$\max_{A_3^{(1,2,3M_3)},A_2^{(1,2,3M_2)},A_1^{(1,2,3M_1)}} r(A_3^*A_2^*A_1^*M_1 - A_3^*A_2^*A_1^*M_1A_1A_2A_3A_3^{(1,2,3M_3)}A_2^{(1,2,3M_2)}A_1^{(1,2,3M_1)}) = 0 \tag{2.2}$$

and

$$\begin{aligned} & \max_{A_3^{(1,2,3M_3)},A_2^{(1,2,3M_2)},A_1^{(1,2,3M_1)}} r(A_3^{(1,2,3M_3)}A_2^{(1,2,3M_2)}A_1^{(1,2,3M_1)}) \\ = & \min_{A_3^{(1,2,3M_3)},A_2^{(1,2,3M_2)},A_1^{(1,2,3M_1)}} r(A_3^{(1,2,3M_3)}A_2^{(1,2,3M_2)}A_1^{(1,2,3M_1)}) \\ = & \ r(A_1A_2A_3). \end{aligned} \tag{2.3}$$

Using the formula (1.7) in Lemma 1.3 with $A = A_1$, $B = I_{m_1}$, $D = A_3^*A_2^*A_1^*M_1$ and $C = A_3^*A_2^*A_1^*M_1A_1A_2A_3A_3^{(1,2,3M_3)}A_2^{(1,2,3M_2)}$, we have

$$\max_{A_1^{(1,2,3M_1)}} r(A_3^*A_2^*A_1^*M_1 - A_3^*A_2^*A_1^*M_1A_1A_2A_3A_3^{(1,2,3M_3)}A_2^{(1,2,3M_2)}A_1^{(1,2,3M_1)})$$

$$
\begin{aligned}
&= \min\left\{r\begin{pmatrix} A_1^*M_1A_1 & A_1^*M_1 \\ A_3^*A_2^*A_1^*M_1A_1A_2A_3A_3^{(1,2,3M_3)}A_2^{(1,2,3M_2)} & A_3^*A_2^*A_1^*M_1 \end{pmatrix} - r(A_1),\right. \\
&\qquad \left. r\begin{pmatrix} A_1^*M_1 \\ A_3^*A_2^*A_1^*M_1 \end{pmatrix}\right\} \\
&= \min\left\{r(A_3^*A_2^*A_1^*M_1A_1 - A_3^*A_2^*A_1^*M_1A_1A_2A_3A_3^{(1,2,3M_3)}A_2^{(1,2,3M_2)}),\ \ r(A_1)\right\} \\
&= r(A_3^*A_2^*A_1^*M_1A_1 - A_3^*A_2^*A_1^*M_1A_1A_2A_3A_3^{(1,2,3M_3)}A_2^{(1,2,3M_2)}). \qquad (2.4)
\end{aligned}
$$

From (2.4) and again by formula (1.7) in Lemma 1.3 with $A = A_2$, $B = I_{m_2}$, $D = A_3^*A_2^*A_1^*M_1A_1$, and $C = A_3^*A_2^*A_1^*M_1A_1A_2A_3A_3^{(1,2,3M_3)}$, we have

$$
\begin{aligned}
&\max_{A_2^{(1,2,3M_2)},A_1^{(1,2,3M_1)}} r(A_3^*A_2^*A_1^*M_1 - A_3^*A_2^*A_1^*M_1A_1A_2A_3A_3^{(1,2,3M_3)}A_2^{(1,2,3M_2)}A_1^{(1,2,3M_1)}) \\
&= \max_{A_2^{(1,2,3M_2)}} r(A_3^*A_2^*A_1^*M_1A_1 - A_3^*A_2^*A_1^*M_1A_1A_2A_3A_3^{(1,2,3M_3)}A_2^{(1,2,3M_2)}) \\
&= \min\left\{r\begin{pmatrix} A_2^*M_2A_2 & A_2^*M_2 \\ A_3^*A_2^*A_1^*M_1A_1A_2A_3A_3^{(1,2,3M_3)} & A_3^*A_2^*A_1^*M_1A_1 \end{pmatrix} - r(A_2),\ \ r\begin{pmatrix} A_2^*M_2 \\ A_3^*A_2^*A_1^*M_1A_1 \end{pmatrix}\right\}. \qquad (2.5)
\end{aligned}
$$

According to the formulas (1.12) and (1.13) of Lemma 1.4, we have

$$
\begin{aligned}
&\min\left\{r\begin{pmatrix} A_2^*M_2A_2 & A_2^*M_2 \\ A_3^*A_2^*A_1^*M_1A_1A_2A_3A_3^{(1,2,3M_3)} & A_3^*A_2^*A_1^*M_1A_1 \end{pmatrix} - r(A_2),\ \ r\begin{pmatrix} A_2^*M_2 \\ A_3^*A_2^*A_1^*M_1A_1 \end{pmatrix}\right\} \\
&= \min\left\{r\begin{pmatrix} O & A_2^*M_2 \\ A_3^*A_2^*A_1^*M_1A_1A_2A_3A_3^{(1,2,3M_3)} - A_3^*A_2^*A_1^*M_1A_1A_2 & A_3^*A_2^*A_1^*M_1A_1 \end{pmatrix} - r(A_2), r\begin{pmatrix} A_2^*M_2 \\ A_3^*A_2^*A_1^*M_1A_1 \end{pmatrix}\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
&r\begin{pmatrix} O & A_2^*M_2 \\ A_3^*A_2^*A_1^*M_1A_1A_2A_3A_3^{(1,2,3M_3)} - A_3^*A_2^*A_1^*M_1A_1A_2 & A_3^*A_2^*A_1^*M_1A_1 \end{pmatrix} - r(A_2) \\
&\leq r\left(A_3^*A_2^*A_1^*M_1A_1A_2A_3A_3^{(1,2,3M_3)} - A_3^*A_2^*A_1^*M_1A_1A_2\right) + r\begin{pmatrix} A_2^*M_2 \\ A_3^*A_2^*A_1^*M_1A_1 \end{pmatrix} - r(A_2) \\
&\leq r\left(A_3^*A_2^*A_1^*M_1A_1A_2(A_3A_3^{(1,2,3M_3)} - I_{m_3})\right) + r\begin{pmatrix} A_2^*M_2 \\ A_3^*A_2^*A_1^*M_1A_1 \end{pmatrix} - r(A_2) \\
&\leq r(A_2) + r\begin{pmatrix} A_2^*M_2 \\ A_3^*A_2^*A_1^*M_1A_1 \end{pmatrix} - r(A_2) \\
&= r\begin{pmatrix} A_2^*M_2 \\ A_3^*A_2^*A_1^*M_1A_1 \end{pmatrix} \qquad (2.6)
\end{aligned}
$$

and

$$
r\begin{pmatrix} O & A_2^*M_2 \\ A_3^*A_2^*A_1^*M_1A_1A_2A_3A_3^{(1,2,3M_3)} - A_3^*A_2^*A_1^*M_1A_1A_2 & A_3^*A_2^*A_1^*M_1A_1 \end{pmatrix} - r(A_2)
$$

$$= r\begin{pmatrix} O & A_2^* \\ A_3^*A_2^*A_1^*M_1A_1A_2A_3A_3^{(1,2,3M_3)} - A_3^*A_2^*A_1^*M_1A_1A_2 & A_3^*A_2^*A_1^*M_1A_1M_2^{-1} \end{pmatrix} - r(A_2)$$
$$= r\left(A_3^*A_2^*A_1^*M_1A_1A_2A_3A_3^{(1,2,3M_3)} - A_3^*A_2^*A_1^*M_1A_1A_2,\ \ A_3^*A_2^*A_1^*M_1A_1M_2^{-1}F_{A_2^*}\right). \tag{2.7}$$

Combining (2.5), (2.6), and (2.7), we have

$$\max_{A_2^{(1,2,3M_2)},A_1^{(1,2,3M_1)}} r(A_3^*A_2^*A_1^*M_1 - A_3^*A_2^*A_1^*M_1A_1A_2A_3A_3^{(1,2,3M_3)}A_2^{(1,2,3M_2)}A_1^{(1,2,3M_1)})$$
$$= \min\left\{ r\begin{pmatrix} A_2^*M_2A_2 & A_2^*M_2 \\ A_3^*A_2^*A_1^*M_1A_1A_2A_3A_3^{(1,2,3M_3)} & A_3^*A_2^*A_1^*M_1A_1 \end{pmatrix} - r(A_2),\ \ r\begin{pmatrix} A_2^*M_2 \\ A_3^*A_2^*A_1^*M_1A_1 \end{pmatrix} \right\}$$
$$= r\begin{pmatrix} O & A_2^*M_2 \\ A_3^*A_2^*A_1^*M_1A_1A_2A_3A_3^{(1,2,3M_3)} - A_3^*A_2^*A_1^*M_1A_1A_2 & A_3^*A_2^*A_1^*M_1A_1 \end{pmatrix} - r(A_2)$$
$$= r\left(A_3^*A_2^*A_1^*M_1A_1A_2A_3A_3^{(1,2,3M_3)} - A_3^*A_2^*A_1^*M_1A_1A_2,\ \ A_3^*A_2^*A_1^*M_1A_1M_2^{-1}F_{A_2^*}\right). \tag{2.8}$$

From (2.8) and again by formula (1.7) in Lemma 1.3 with $A = A_3$, $B = (I_{m_3}, O)$, $C = A_3^*A_2^*A_1^*M_1A_1A_2A_3$, and $D = (A_3^*A_2^*A_1^*M_1A_1A_2,\ \ A_3^*A_2^*A_1^*M_1A_1M_2^{-1}F_{A_2^*})$, we have

$$r\left(A_3^*A_2^*A_1^*M_1A_1A_2A_3A_3^{(1,2,3M_3)} - A_3^*A_2^*A_1^*M_1A_1A_2,\ \ A_3^*A_2^*A_1^*M_1A_1M_2^{-1}F_{A_2^*}\right)$$
$$= r\left((A_3^*A_2^*A_1^*M_1A_1A_2,\ \ A_3^*A_2^*A_1^*M_1A_1M_2^{-1}F_{A_2^*}) - A_3^*A_2^*A_1^*M_1A_1A_2A_3A_3^{(1,2,3M_3)}(I_{m_3}, O)\right)$$

and

$$\max_{A_3^{(1,2,3M_3)},A_2^{(1,2,3M_2)},A_1^{(1,2,3M_1)}} r(A_3^*A_2^*A_1^*M_1 - A_3^*A_2^*A_1^*M_1A_1A_2A_3A_3^{(1,2,3M_3)}A_2^{(1,2,3M_2)}A_1^{(1,2,3M_1)})$$
$$= \max_{A_3^{(1,2,3M_3)}} r\left(A_3^*A_2^*A_1^*M_1A_1A_2A_3A_3^{(1,2,3M_3)} - A_3^*A_2^*A_1^*M_1A_1A_2,\ \ A_3^*A_2^*A_1^*M_1A_1M_2^{-1}F_{A_2^*}\right)$$
$$= \min\left\{ r\begin{pmatrix} A_3^*M_3A_3 & A_3^*M_3 & O \\ A_3^*A_2^*A_1^*M_1A_1A_2A_3 & A_3^*A_2^*A_1^*M_1A_1A_2 & A_3^*A_2^*A_1^*M_1A_1M_2^{-1}F_{A_2^*} \end{pmatrix} - r(A_3), \right.$$
$$\left. r\begin{pmatrix} A_3^*M_3 & O \\ A_3^*A_2^*A_1^*M_1A_1A_2 & A_3^*A_2^*A_1^*M_1A_1M_2^{-1}F_{A_2^*} \end{pmatrix} \right\}$$
$$= \min\left\{ r\begin{pmatrix} O & A_3^*M_3 & O \\ O & A_3^*A_2^*A_1^*M_1A_1A_2 & A_3^*A_2^*A_1^*M_1A_1M_2^{-1}F_{A_2^*} \end{pmatrix} - r(A_3), \right.$$
$$\left. r\begin{pmatrix} A_3^*M_3 & O \\ A_3^*A_2^*A_1^*M_1A_1A_2 & A_3^*A_2^*A_1^*M_1A_1M_2^{-1}F_{A_2^*} \end{pmatrix} \right\}$$
$$= r\begin{pmatrix} A_3^*M_3 & O \\ A_3^*A_2^*A_1^*M_1A_1A_2 & A_3^*A_2^*A_1^*M_1A_1M_2^{-1}F_{A_2^*} \end{pmatrix} - r(A_3)$$
$$= r\left(A_3^*A_2^*A_1^*M_1A_1A_2M_3^{-1}F_{A_3^*},\ \ A_3^*A_2^*A_1^*M_1A_1M_2^{-1}F_{A_2^*}\right), \tag{2.9}$$

According to (2.9) and formula (1.12) of the Lemma 1.4, we have

$$\max_{A_3^{(1,2,3M_3)},A_2^{(1,2,3M_2)},A_1^{(1,2,3M_1)}} r(A_3^*A_2^*A_1^*M_1 - A_3^*A_2^*A_1^*M_1A_1A_2A_3A_3^{(1,2,3M_3)}A_2^{(1,2,3M_2)}A_1^{(1,2,3M_1)})$$

$$= r\left(A_3^*A_2^*A_1^*M_1A_1A_2M_3^{-1}F_{A_3^*},\ \ A_3^*A_2^*A_1^*M_1A_1M_2^{-1}F_{A_2^*}\right)$$

$$= r\begin{pmatrix} A_3^* & O \\ O & A_2^* \\ A_3^*A_2^*A_1^*M_1A_1A_2M_3^{-1} & A_3^*A_2^*A_1^*M_1A_1M_2^{-1} \end{pmatrix} - r(A_2) - r(A_3). \tag{2.10}$$

Combining (2.2) and (2.10), we have

$$\max_{A_3^{(1,2,3M_3)},A_2^{(1,2,3M_2)},A_1^{(1,2,3M_1)}} r(A_3^*A_2^*A_1^*M_1 - A_3^*A_2^*A_1^*M_1A_1A_2A_3A_3^{(1,2,3M_3)}A_2^{(1,2,3M_2)}A_1^{(1,2,3M_1)}) = 0$$

if and only if

$$r\begin{pmatrix} A_3^* & O \\ O & A_2^* \\ A_3^*A_2^*A_1^*M_1A_1A_2M_3^{-1} & A_3^*A_2^*A_1^*M_1A_1M_2^{-1} \end{pmatrix} = r(A_2) + r(A_3). \tag{2.11}$$

In the rest of the section, we will find the equivalent conditions of (2.3). By Lemma 1.3 (1.7) with $A = A_1$, $B = I_{m_1}$, $C = A_3^{(1,2,3M_3)}A_2^{(1,2,3M_2)}$, and $D = O$, we have

$$\begin{aligned} &\max_{A_1^{(1,2,3M_1)}} r(A_3^{(1,2,3M_3)}A_2^{(1,2,3M_2)}A_1^{(1,2,3M_1)}) \\ =\ & \min\left\{ r\begin{pmatrix} A_1^*M_1A_1 & A_1^*M_1 \\ A_3^{(1,2,3M_3)}A_2^{(1,2,3M_2)} & O \end{pmatrix} - r(A_1),\ \ r\begin{pmatrix} A_1^*M_1 \\ O \end{pmatrix} \right\} \\ =\ & \min\left\{ r\left(A_3^{(1,2,3M_3)}A_2^{(1,2,3M_2)}\right), r(A_1) \right\}. \end{aligned} \tag{2.12}$$

Form (2.12) and using Lemma 1.3 (1.7) with $A = A_2$, $B = I_{m_2}$, $C = A_3^{(1,2,3M_3)}$, and $D = O$, we have

$$\begin{aligned} &\max_{A_2^{(1,2,3M_2)},A_1^{(1,2,3M_1)}} r(A_3^{(1,2,3M_3)}A_2^{(1,2,3M_2)}A_1^{(1,2,3M_1)}) \\ =\ & \min\left\{ \left(\max_{A_2^{(1,2,3M_2)}} r\left(A_3^{(1,2,3M_3)}A_2^{(1,2,3M_2)}\right) \right), r(A_1) \right\}. \end{aligned} \tag{2.13}$$

From (2.13) and formula (1.7) of Lemma 1.3, we have

$$\begin{aligned} &\min\left\{ \left(\max_{A_2^{(1,2,3M_2)}} r\left(A_3^{(1,2,3M_3)}A_2^{(1,2,3M_2)}\right) \right), r(A_1) \right\} \\ =\ & \min\left\{ \left(\min\left\{ r\begin{pmatrix} A_2^*M_2A_2 & A_2^*M_2 \\ A_3^{(1,2,3M_3)} & O \end{pmatrix} - r(A_2),\ \ r\begin{pmatrix} A_2^*M_2 \\ O \end{pmatrix} \right\} \right), r(A_1) \right\} \\ =\ & \min\left\{ r(A_3^{(1,2,3M_3)}), r(A_2), r(A_1) \right\}. \end{aligned} \tag{2.14}$$

Combining (2.13) and (2.14), we have

$$\max_{A_2^{(1,2,3M_2)},A_1^{(1,2,3M_1)}} r(A_3^{(1,2,3M_3)}A_2^{(1,2,3M_2)}A_1^{(1,2,3M_1)})$$

$$= \min\left\{r(A_3^{(1,2,3M_3)}), r(A_2), r(A_1)\right\}. \tag{2.15}$$

Since $r(A_3^{(1,2,3M_3)}) = r(A_3)$, then

$$\begin{aligned}
&\max_{A_3^{(1,2,3M_3)}, A_2^{(1,2,3M_2)}, A_1^{(1,2,3M_1)}} r(A_3^{(1,2,3M_3)} A_2^{(1,2,3M_2)} A_1^{(1,2,3M_1)}) \\
=\ & \min\left\{r(A_3^{(1,2,3M_3)}), r(A_2), r(A_1)\right\} \\
=\ & \min\left\{r(A_3), r(A_2), r(A_1)\right\}.
\end{aligned} \tag{2.16}$$

On the other hand, according to (1.8) of Lemma 1.3 with $A = A_3$, $B = A_2^{(1,2,3M_2)} A_1^{(1,2,3M_1)}$, $C = I_{m_4}$, and $D = O$, we have

$$\begin{aligned}
&\min_{A_3^{(1,2,3M_3)}} r(A_3^{(1,2,3M_3)} A_2^{(1,2,3M_2)} A_1^{(1,2,3M_1)}) \\
=\ & r\begin{pmatrix} A_3^* M_3 A_3 & A_3^* M_3 A_2^{(1,2,3M_2)} A_1^{(1,2,3M_1)} \\ I_{m_4} & O \end{pmatrix} + r\begin{pmatrix} A_3^* M_3 A_2^{(1,2,3M_2)} A_1^{(1,2,3M_1)} \\ O \end{pmatrix} \\
& - r\begin{pmatrix} A_3 & O \\ O & A_3^* M_3 A_2^{(1,2,3M_2)} A_1^{(1,2,3M_1)} \\ I_{m_4} & O \end{pmatrix} \\
=\ & r(A_3^* M_3 A_2^{(1,2,3M_2)} A_1^{(1,2,3M_1)}).
\end{aligned} \tag{2.17}$$

From (2.17) and again by formula (1.8) in Lemma 1.3 with $A = A_2$, $B = A_1^{(1,2,3M_1)}$, $C = A_3^* M_3$, and $D = O$, we have

$$\begin{aligned}
&\min_{A_2^{(1,2,3M_2)},\, A_3^{(1,2,3M_3)}} r(A_3^{(1,2,3M_3)} A_2^{(1,2,3M_2)} A_1^{(1,2,3M_1)}) \\
=\ & \min_{A_2^{(1,2,3M_2)}} r(A_3^* M_3 A_2^{(1,2,3M_2)} A_1^{(1,2,3M_1)}) \\
=\ & r\begin{pmatrix} A_2^* M_2 A_2 & A_2^* M_2 A_1^{(1,2,3M_1)} \\ A_3^* M_3 & O \end{pmatrix} + r\begin{pmatrix} A_2^* M_2 A_1^{(1,2,3M_1)} \\ O \end{pmatrix} - r\begin{pmatrix} A_2 & O \\ O & A_2^* M_2 A_1^{(1,2,3M_1)} \\ A_3^* M_3 & O \end{pmatrix} \\
=\ & r\left(A_2^* M_2 A_2 M_3^{-1} F_{A_3^*},\ A_2^* M_2 A_1^{(1,2,3M_1)}\right) + r(A_3) - r\begin{pmatrix} A_2 \\ A_3^* M_3 \end{pmatrix}.
\end{aligned} \tag{2.18}$$

By formula (1.12) of Lemma 1.4, we have

$$\begin{aligned}
& r\begin{pmatrix} A_2^* M_2 A_2 & A_2^* M_2 A_1^{(1,2,3M_1)} \\ A_3^* M_3 & O \end{pmatrix} = r\begin{pmatrix} A_3^* M_3 & O \\ A_2^* M_2 A_2 & A_2^* M_2 A_1^{(1,2,3M_1)} \end{pmatrix} \\
=\ & r\begin{pmatrix} A_3^* & O \\ A_2^* M_2 A_2 M_3^{-1} & A_2^* M_2 A_1^{(1,2,3M_1)} \end{pmatrix} \\
=\ & r\left(A_2^* M_2 A_2 M_3^{-1} F_{A_3^*},\ A_2^* M_2 A_1^{(1,2,3M_1)}\right) + r(A_3)
\end{aligned} \tag{2.19}$$

and

$$r\begin{pmatrix} A_2 & O \\ O & A_2^*M_2A_1^{(1,2,3M_1)} \\ A_3^*M_3 & O \end{pmatrix} = r\begin{pmatrix} A_2 \\ A_3^*M_3 \end{pmatrix} + r(A_2^*M_2A_1^{(1,2,3M_1)}). \tag{2.20}$$

Combining Eqs (2.19) and (2.20), we obtain Eq (2.18).

Using Eq (2.18) and formula (1.8) from Lemma 1.3 with ($A = A_1$), ($B = (O, \ -I_{m_1})$), ($C = A_2^*M_2$), and ($D = (A_2^*M_2A_2M_3^{-1}F_{A_3^*}, \ O)$), we derive

$$\begin{aligned}
&\min_{A_1^{(1,2,3M_1)}, A_2^{(1,2,3M_2)}, A_3^{(1,2,3M_3)}} r(A_3^{(1,2,3M_3)}A_2^{(1,2,3M_2)}A_1^{(1,2,3M_1)}) \\
= &\left(\min_{A_1^{(1,2,3M_1)}} r\left(A_2^*M_2A_2M_3^{-1}F_{A_3^*}, \ A_2^*M_2A_1^{(1,2,3M_1)}\right)\right) + r(A_3) - r\begin{pmatrix} A_2 \\ A_3^*M_3 \end{pmatrix} \\
= &\left(\min_{A_1^{(1,2,3M_1)}} r\left[(A_2^*M_2A_2M_3^{-1}F_{A_3^*}, O) - A_2^*M_2A_1^{(1,2,3M_1)}(O, -I_{m_1})\right]\right) + r(A_3) - r\begin{pmatrix} A_2 \\ A_3^*M_3 \end{pmatrix} \\
= &\ r\begin{pmatrix} A_1^*M_1A_1 & O & -A_1^*M_1 \\ A_2^*M_2 & A_2^*M_2A_2M_3^{-1}F_{A_3^*} & O \end{pmatrix} + r\begin{pmatrix} O & -A_1^*M_1 \\ A_2^*M_2A_2M_3^{-1}F_{A_3^*} & O \end{pmatrix} \\
&- r\begin{pmatrix} A_1 & O & O \\ O & O & -A_1^*M_1 \\ A_2^*M_2 & A_2^*M_2A_2M_3^{-1}F_{A_3^*} & O \end{pmatrix} + r(A_3) - r\begin{pmatrix} A_2 \\ A_3^*M_3 \end{pmatrix} \\
= &\ r\begin{pmatrix} A_1^*M_1A_1 & A_1^*M_1A_1A_2M_3^{-1}F_{A_3^*} & A_1^*M_1 \\ A_2^*M_2 & O & O \end{pmatrix} + r(A_1) + r(A_2^*M_2A_2M_3^{-1}F_{A_3^*}) - r(A_1) \\
&- r\begin{pmatrix} A_1 & O \\ A_2^*M_2 & A_2^*M_2A_2M_3^{-1}F_{A_3^*} \end{pmatrix} + r(A_3) - r\begin{pmatrix} A_2 \\ A_3^*M_3 \end{pmatrix},
\end{aligned} \tag{2.21}$$

where

$$\begin{aligned}
&r\begin{pmatrix} A_1^*M_1A_1 & A_1^*M_1A_1A_2M_3^{-1}F_{A_3^*} & A_1^*M_1 \\ A_2^*M_2 & O & O \end{pmatrix} \\
= &\ r\begin{pmatrix} A_1^*M_1A_1M_2^{-1} & A_1^*M_1A_1A_2M_3^{-1}F_{A_3^*} & A_1^*M_1 \\ A_2^* & O & O \end{pmatrix} \\
= &\ r(A_2) + r(A_1^*M_1, A_1^*M_1A_1M_2^{-1}F_{A_2^*}, A_1^*M_1A_1A_2M_3^{-1}F_{A_3^*})
\end{aligned} \tag{2.22}$$

and

$$\begin{aligned}
&r(A_2^*M_2A_2M_3^{-1}F_{A_3^*}) \\
= &\ r\left((M_2^{1/2}A_2)^*(M_2^{1/2}A_2)M_3^{-1}F_{A_3^*}\right) \\
\leq &\ r((M_2^{1/2}A_2)M_3^{-1}F_{A_3^*}) \\
= &\ r\left((M_2^{1/2}A_2)(M_2^{1/2}A_2)^{\dagger}(M_2^{1/2}A_2)M_3^{-1}F_{A_3^*}\right) \\
= &\ r\left((M_2^{1/2}A_2)((M_2^{1/2}A_2)^*M_2^{1/2}A_2)^{\dagger}(M_2^{1/2}A_2)^*(M_2^{1/2}A_2)M_3^{-1}F_{A_3^*}\right)
\end{aligned}$$

$$\begin{aligned} &\le \left((M_2^{1/2}A_2)^*(M_2^{1/2}A_2)M_3^{-1}F_{A_3^*}\right) \\ &= r(A_2^*M_2A_2M_3^{-1}F_{A_3^*}). \end{aligned} \tag{2.23}$$

By the formula (2.23), we have

$$r(A_2^*M_2A_2M_3^{-1}F_{A_3^*}) = r((M_2^{1/2}A_2)M_3^{-1}F_{A_3^*}) = r(A_2M_3^{-1}F_{A_3^*}). \tag{2.24}$$

By the formula (1.12) of Lemma 1.4, we have

$$r\begin{pmatrix} A_2 \\ A_3^*M_3 \end{pmatrix} = r\begin{pmatrix} A_2M_3^{-1} \\ A_3^* \end{pmatrix} = r(A_3) + r(A_2M_3^{-1}F_{A_3^*}). \tag{2.25}$$

Combining (2.21), (2.22), (2.24), and (2.25), we have

$$\begin{aligned} &\min_{A_1^{(1,2,3M_1)},A_2^{(1,2,3M_2)},A_3^{(1,2,3M_3)}} r(A_3^{(1,2,3M_3)}A_2^{(1,2,3M_2)}A_1^{(1,2,3M_1)}) \\ &= r(A_2) + r(A_1^*M_1, A_1^*M_1A_1M_2^{-1}F_{A_2^*}, A_1^*M_1A_1A_2M_3^{-1}F_{A_3^*}) - r\begin{pmatrix} A_1 & O \\ A_2^*M_2 & A_2^*M_2A_2M_3^{-1}F_{A_3^*} \end{pmatrix} \\ &= r(A_1) + r(A_2) - r\begin{pmatrix} A_1 & O \\ A_2^*M_2 & A_2^*M_2A_2M_3^{-1}F_{A_3^*} \end{pmatrix} \\ &= r(A_1) + r(A_2) - r\begin{pmatrix} A_1 & A_1A_2M_3^{-1}F_{A_3^*} \\ A_2^*M_2 & O \end{pmatrix} \\ &= r(A_1) - r\left(A_1M_2^{-1}F_{A_2^*},\ A_1A_2M_3^{-1}F_{A_3^*}\right) \\ &= \sum_{i=1}^{3} r(A_i) - r\begin{pmatrix} A_2^* & O \\ O & A_3^* \\ A_1M_2^{-1} & A_1A_2M_3^{-1} \end{pmatrix}. \end{aligned} \tag{2.26}$$

According to (2.2), (2.3), (2.11), (2.16), and (2.26), we have proved Theorem 2.1. □

From Lemma 1.1, Lemma 1.4, and Theorem 2.1, we immediately obtain the following corollary.

Corollary 2.1. *Let $A_i \in C^{m_i \times m_{i+1}}$, $A_i^{(1,2,3M_i)} \in A_i\{1,2,3M_i\}$, where $i \in \{1,2,3\}$. Let $M_i \in C^{m_i \times m_i}$, $i \in \{1,2,3\}$ be three positive definite Hermitian matrices. Then the following statements are equivalent:*

(1)

$$A_3\{1,2,3M_3\}A_2\{1,2,3M_2\}A_1\{1,2,3M_1\} \subseteq (A_1A_2A_3)\{1,2,3M_1\}; \tag{2.27}$$

(2)

$$r\begin{pmatrix} A_3^*M_3 & O \\ O & A_2^*M_2 \\ A_3^*A_2^*A_1^*M_1A_1A_2 & A_3^*A_2^*A_1^*M_1A_1 \end{pmatrix} = r(A_2) + r(A_3) \tag{2.28}$$

and

$$r(A_1A_2A_3) = \min\{r(A_1), r(A_2), r(A_3)\} = \sum_{i=1}^{3} r(A_i) - r\begin{pmatrix} A_2^*M_2 & O \\ O & A_3^*M_3 \\ A_1 & A_1A_2 \end{pmatrix}; \tag{2.29}$$

(3)

$$R(A_2^*A_1^*M_1A_1A_2A_3) \subseteq R(M_3A_3), \tag{2.30}$$

$$R(A_1^*M_1A_1A_2A_3) \subseteq R(M_2A_2) \tag{2.31}$$

and

$$r(A_1A_2A_3) = \min\{r(A_1), r(A_2), r(A_3)\} = \sum_{i=1}^{3} r(A_i) - r\begin{pmatrix} A_2^*M_2 & O \\ O & A_3^*M_3 \\ A_1 & A_1A_2 \end{pmatrix};$$

(4)

$$A_3(A_3)^{\dagger}_{M_3,I_{m_4}} M_3^{-1}A_2^*A_1^*M_1A_1A_2A_3 = M_3^{-1}A_2^*A_1^*M_1A_1A_2A_3, \tag{2.32}$$

$$A_2(A_2)^{\dagger}_{M_2,I_{m_3}} M_2^{-1}A_1^*M_1A_1A_2A_3 = M_2^{-1}A_1^*M_1A_1A_2A_3 \tag{2.33}$$

and

$$r(A_1A_2A_3) = \min\{r(A_1), r(A_2), r(A_3)\} = \sum_{i=1}^{3} r(A_i) - r\begin{pmatrix} A_2^*M_2 & O \\ O & A_3^*M_3 \\ A_1 & A_1A_2 \end{pmatrix}.$$

Proof. According to Theorem 2.1, we have (1) $\Leftrightarrow$ (2). Now, we will prove (3) $\Rightarrow$ (2). By (2.30) and (2.31), we have $R(M_3^{-1}A_2^*A_1^*M_1A_1A_2A_3) \subseteq R(A_3)$ and $R(M_2^{-1}A_1^*M_1A_1A_2A_3) \subseteq R(A_2)$. Using Lemma 1.1, we have

$$\begin{aligned}
&R(A_2^*A_1^*M_1A_1A_2A_3) \subseteq R(M_3A_3) \text{ and } R(A_1^*M_1A_1A_2A_3) \subseteq R(M_2A_2) \Rightarrow \\
&R(M_3^{-1}A_2^*A_1^*M_1A_1A_2A_3) \subseteq R(A_3) \text{ and } R(M_2^{-1}A_1^*M_1A_1A_2A_3) \subseteq R(A_2) \Rightarrow \\
&r\begin{pmatrix} A_3 & O & M_3^{-1}A_2^*A_1^*M_1A_1A_2A_3 \\ O & A_2 & M_2^{-1}A_1^*M_1A_1A_2A_3 \end{pmatrix} = r(A_2) + r(A_3) \Rightarrow \\
&r\begin{pmatrix} A_3^*M_3 & O \\ O & A_2^*M_2 \\ A_3^*A_2^*A_1^*M_1A_1A_2 & A_3^*A_2^*A_1^*M_1A_1 \end{pmatrix} = r(A_2) + r(A_3).
\end{aligned}$$

Next, we will prove (2) $\Rightarrow$ (3), that is, (2.28) $\Rightarrow$ (2.30) and (2.31). According to (2.28) and the formula (1.11) of Lemma 1.4, we have

$$\begin{aligned}
&r\begin{pmatrix} A_3^*M_3 & O \\ O & A_2^*M_2 \\ A_3^*A_2^*A_1^*M_1A_1A_2 & A_3^*A_2^*A_1^*M_1A_1 \end{pmatrix} = r(A_2) + r(A_3) \\
\Longleftrightarrow\ &r\begin{pmatrix} A_3 & O & M_3^{-1}A_2^*A_1^*M_1A_1A_2A_3 \\ O & A_2 & M_2^{-1}A_1^*M_1A_1A_2A_3 \end{pmatrix} = r(A_2) + r(A_3)
\end{aligned}$$

$$\iff r\begin{pmatrix} E_{A_3}M_3^{-1}A_2^*A_1^*M_1A_1A_2A_3 \\ E_{A_2}M_2^{-1}A_1^*M_1A_1A_2A_3 \end{pmatrix} + r(A_2) + r(A_3) = r(A_2) + r(A_3)$$

$$\iff r\begin{pmatrix} E_{A_3}M_3^{-1}A_2^*A_1^*M_1A_1A_2A_3 \\ E_{A_2}M_2^{-1}A_1^*M_1A_1A_2A_3 \end{pmatrix} = 0$$

and

$$A_3A_3^{\dagger}M_3^{-1}A_2^*A_1^*M_1A_1A_2A_3 = M_3^{-1}A_2^*A_1^*M_1A_1A_2A_3$$

and

$$A_2A_2^{\dagger}M_2^{-1}A_1^*M_1A_1A_2A_3 = M_2^{-1}A_1^*M_1A_1A_2A_3.$$

According to (1.2) of Lemma 1.1, we have (2) $\Rightarrow$ (3).

Finally, we will prove (3) $\iff$ (4), that is (2.30) $\Leftrightarrow$ (2.32) and (2.31) $\Leftrightarrow$ (2.33). According to (1.2) of Lemma 1.1, we have

$$R(A_2^*A_1^*M_1A_1A_2A_3) \subseteq R(M_3A_3) \Leftrightarrow R(M_3^{-1}A_2^*A_1^*M_1A_1A_2A_3) \subseteq R(A_3)$$

and

$$\begin{aligned} A_3(A_3)^{\dagger}_{M_3,I_{m_4}}M_3^{-1}A_2^*A_1^*M_1A_1A_2A_3 &= P_{R(A_3(A_3)^{\dagger}_{M_3,I_{m_4}}),N(A_3(A_3)^{\dagger}_{M_3,I_{m_4}})}M_3^{-1}A_2^*A_1^*M_1A_1A_2A_3 \\ &= P_{R(A_3),N(A_3(A_3)^{\dagger}_{M_3,I_{m_4}})}M_3^{-1}A_2^*A_1^*M_1A_1A_2A_3 \\ &= M_3^{-1}A_2^*A_1^*M_1A_1A_2A_3 \end{aligned}$$

and

$$R(A_1^*M_1A_1A_2A_3) \subseteq R(M_2A_2) \Leftrightarrow R(M_2^{-1}A_1^*M_1A_1A_2A_3) \subseteq R(A_2)$$

$$\begin{aligned} A_2(A_2)^{\dagger}_{M_2,I_{m_3}}M_2^{-1}A_1^*M_1A_1A_2A_3 &= P_{R(A_2(A_2)^{\dagger}_{M_2,I_{m_3}}),N(A_2(A_2)^{\dagger}_{M_2,I_{m_3}})}M_2^{-1}A_1^*M_1A_1A_2A_3 \\ &= P_{R(A_2),N(A_2(A_2)^{\dagger}_{M_2,I_{m_3}})}M_2^{-1}A_1^*M_1A_1A_2A_3 \\ &= M_2^{-1}A_1^*M_1A_1A_2A_3. \end{aligned}$$

So we have that (1), (2), (3), and (4) are equivalent. □

From Lemma 1.2, we have $X \in A\{1,2,4N\}$ if and only if $X^* \in A^*\{1,2,3N^{-1}\}$. Thus according to the results obtained in Theorem 2.1 and Corollary 2.1, we obtain the following theorem and corollary without any proof.

Theorem 2.2. *Let $A_i \in C^{m_i\times m_{i+1}}$, $A_i^{(1,2,4N_{i+1})} \in A_i\{1,2,4N_{i+1}\}$, where $i \in \{1,2,3\}$. Let $N_i \in C^{m_i\times m_i}$, $i \in \{1,2,3,4\}$ be four positive definite Hermitian matrices. Then*

$$\begin{aligned} &A_3\{1,2,4N_4\}A_2\{1,2,4N_3\}A_1\{1,2,4N_2\} \subseteq (A_1A_2A_3)\{1,2,4N_4\} \\ \iff\ & r\begin{pmatrix} A_1^* & O & N_2A_2A_3N_4^{-1}A_3^*A_2^*A_1^* \\ O & A_2^* & N_3A_3N_4^{-1}A_3^*A_2^*A_1^* \end{pmatrix} = r(A_1) + r(A_2) \end{aligned}$$

and

$$r(A_1A_2A_3) = \min\{r(A_1), r(A_2), r(A_3)\} = \sum_{i=1}^{3} r(A_i) - r\begin{pmatrix} A_2^* & O & N_3A_3 \\ O & A_1^* & N_2A_2A_3 \end{pmatrix}.$$

From Lemma 1.1, Lemma 1.4, and Theorem 2.2, we immediately obtain the corollary as follows:

Corollary 2.2. *Let* $A_i \in C^{m_i\times m_{i+1}}$, $A_i^{(1,2,4N_{i+1})} \in A_i\{1,2,4N_{i+1}\}$, *where* $i \in \{1,2,3\}$. *Let* $N_i \in C^{m_i\times m_i}$, $i \in \{1,2,3,4\}$ *be four positive definite Hermitian matrices. Then the following statements are equivalent:*

(1)

$$A_3\{1,2,4N_4\}A_2\{1,2,4N_3\}A_1\{1,2,4N_2\} \subseteq (A_1A_2A_3)\{1,2,4N_4\},$$

(2)

$$r\begin{pmatrix} N_2^{-1}A_1^* & O & A_2A_3N_4^{-1}A_3^*A_2^*A_1^* \\ O & N_3^{-1}A_2^* & A_3N_4^{-1}A_3^*A_2^*A_1^* \end{pmatrix} = r(A_1) + r(A_2)$$

and

$$r(A_1A_2A_3) = \min\{r(A_1), r(A_2), r(A_3)\} = \sum_{i=1}^{3} r(A_i) - r\begin{pmatrix} N_3^{-1}A_2^* & O & A_3 \\ O & N_2^{-1}A_1^* & A_2A_3 \end{pmatrix}.$$

(3)

$$R(A_2A_3N_4^{-1}A_3^*A_2^*A_1^*) \subseteq R(N_2^{-1}A_1^*),$$

$$R(A_3N_4^{-1}A_3^*A_2^*A_1^*) \subseteq R(N_3^{-1}A_2^*)$$

and

$$r(A_1A_2A_3) = \min\{r(A_1), r(A_2), r(A_3)\} = \sum_{i=1}^{3} r(A_i) - r\begin{pmatrix} N_3^{-1}A_2^* & O & A_3 \\ O & N_2^{-1}A_1^* & A_2A_3 \end{pmatrix}.$$

(4)

$$(A_1)^{\dagger}_{I_{m_1},N_2}A_1A_2A_3N_4^{-1}A_3^*A_2^*A_1^* = A_2A_3N_4^{-1}A_3^*A_2^*A_1^*,$$

$$(A_2)^{\dagger}_{I_{m_2},N_3}A_3A_3N_4^{-1}A_3^*A_2^*A_1^* = A_3N_4^{-1}A_3^*A_2^*A_1^*$$

and

$$r(A_1A_2A_3) = \min\{r(A_1), r(A_2), r(A_3)\} = \sum_{i=1}^{3} r(A_i) - r\begin{pmatrix} N_3^{-1}A_2^* & O & A_3 \\ O & N_2^{-1}A_1^* & A_2A_3 \end{pmatrix}.$$

3. Conclusions

The reverse order law for the inverses $\{1,2,3M\}$- and $\{1,2,4N\}$- of matrix products has been studied in this article by using the ranks of the generalized Schur complement. The work performed in this paper is a useful tool for many algorithms for the computation of the weighted least squares technique of matrix equations.

Author contributions

Baifeng Qiu: Resources; Yingying Qin: Resources; Zhiping Xiong: Conceptualization, writing-review and editing. All authors have read and agree to the published version of the manuscript..

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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