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Research article

Metric and strong metric dimension in TI-power graphs of finite groups

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Abstract: Given a finite group G with the identity e, the TI-power graph (trivial intersection power graph) of G is an undirected graph with vertex set G, in which two distinct vertices x and y are adjacent if $\langle x\rangle \cap \langle y\rangle = \{e\}$. In this paper, we obtain closed formulas for the metric and strong metric dimensions of the TI-power graph of a finite group. As applications, we compute the metric and strong metric dimensions of the TI-power graph of a cyclic group, a dihedral group, a generalized quaternion group, and a semi-dihedral group.

Keywords: TI-power graph; metric dimension; strong metric dimension; finite group
Mathematics Subject Classification: 05C25

1. Introduction

Every graph in this paper is assumed to be a *simple graph*, that is, an undirected graph with no loops and multiple edges. Given a graph Γ, we always use $V(\Gamma)$ and $E(\Gamma)$ to denote the vertex set and edge set of Γ, respectively. The *complement* of Γ, denoted by $\overline{\Gamma}$, is a graph with the same vertices as Γ and with an edge between vertices x and y if and only if there is no edge between x and y in Γ. In general, we use K_n to denote the complete graph of size n. A subset of $V(\Gamma)$ is called a *clique* of Γ provided that every two distinct vertices in this subset are adjacent in Γ. The maximum cardinality of a clique in Γ is called the *clique number* of Γ and is denoted by $\omega(\Gamma)$. Let $x, y, z \in V(\Gamma)$. The *distance* between x and y in Γ, denoted by $d_\Gamma(x, y)$ ($d(x, y)$ for simplicity), is the length of a shortest path from x to y in Γ. The *neighborhood* of x in Γ, denoted by $N_\Gamma(x)$, is the set $\{y \in V(\Gamma) : d(y, x) = 1\}$. In Γ, the *closed neighborhood* of x, denoted by $N_\Gamma[x]$, is the set $N_\Gamma(x) \cup \{x\}$. If the situation is unambiguous, we denote $N_\Gamma(x)$ and $N_\Gamma[x]$ simply by $N(x)$ and $N[x]$, respectively. In Γ, we say vertex z *resolves* vertices x and y if $d(x, z) \neq d(y, z)$. Let S be a subset of $V(\Gamma)$. If any two distinct vertices in Γ can be resolved by some element in S, then S is said to be a *resolving set* of Γ. The minimum cardinality of a resolving set of Γ

is called the *metric dimension* of Γ and is denoted by $\dim(\Gamma)$. Also, if there is some shortest path either from z to x containing y or from z to y containing x, then we say z *strongly resolves* these two vertices x, y of Γ. If every two distinct vertices of Γ can be strongly resolved by some vertex in S, then S is said to be a *strong resolving set* of Γ. The *strong metric dimension* of Γ, denoted by $\mathrm{sdim}(\Gamma)$, is the smallest cardinality of a strong resolving set of Γ.

The metric dimension of a graph enables robotics engineers to design a moving robot which may measure its current location in some network of navigating agents. This concept was independently introduced by Slater and Harary et al. in [6, 21], respectively. Sebő and Tannier introduced the strong metric dimension of a graph in [22], and they also introduced some applications of this parameter in combinatorial searching.

Graphs associated with some algebraic structures have been actively investigated, such as, the famous Cayley graphs which have a long history. Moreover, graphs from algebraic structures have valuable applications (cf. [8]): for example, Cayley graphs as classifiers for data mining [9]. Let G be a group with the identity e. In 2000, Kelarev and Quinn [10] introduced the *power graph* of a group, denoted by $\overrightarrow{\mathcal{P}}(G)$, which is a directed graph with vertex set G, and for distinct $x, y \in G$, there is an arc from x to y if and only if y is a power of x. In 2009, Chakrabarty et al. [3] introduced the *undirected power graph* of G, denoted by $\mathcal{P}(G)$, as the underlying graph of $\overrightarrow{\mathcal{P}}(G)$. Namely, $\mathcal{P}(G)$ has vertex set G, and between two distinct vertices have an edge if and only if one is a power of the other. Afterward, for convenience, the term "power graph" refers to an undirected power graph defined as above. In 2013 and 2021, Abawajy et al. [1] and Kumar et al. [11] respectively published two surveys with power graphs that contain a large number of results on Hamiltonian, clique number, planarity, automorphism group, chromatic number, spectrum, independent number, connectivity, and so on.

Note that in $\mathcal{P}(G)$, if two vertices a and b are adjacent, then we must have $\langle a\rangle \cap \langle b\rangle = \langle a\rangle$ or $\langle b\rangle$. Motivated by the fact, Bera [2] first introduced the *intersection power graph* of G, denoted by $\mathcal{P}_I(G)$, as an undirected graph with vertex set G, and two distinct vertices a and b are adjacent if either one of x, y is e, or $\langle x\rangle \cap \langle y\rangle \neq \{e\}$. Clearly, $\mathcal{P}(G)$ must be a spanning subgraph of $\mathcal{P}_I(G)$. In [2], Bera studied some basic properties of an intersection power graph. In [14], the authors classified all finite groups whose intersection power graph is toroidal or projective-planar. Ma et al. [17] obtained necessary and sufficient conditions when $\mathcal{P}_I(G)-\{e\}$ admits a perfect code, where $\mathcal{P}_I(G)-\{e\}$ is the subgraph obtained by deleting e from $\mathcal{P}_I(G)$. Recently, Ma and Fu [16] studied the metric and strong metric dimensions of an intersection power graph.

The *TI-power graph* (trivial intersection power graph) defined on G, denoted by $\mathcal{N}(G)$, is a simple graph with vertex set G where two distinct vertices a and b are adjacent if $\langle a\rangle \cap \langle b\rangle = \{e\}$. By the definition of a TI-power graph, we see that if we only consider the set $G \setminus \{e\}$, then the induced subgraph of $\mathcal{N}(G)$ by $G \setminus \{e\}$ is equal to the complement of the induced subgraph of $\mathcal{P}_I(G)$ by $G \setminus \{e\}$. In [13], Li, the second and third authors introduced the TI-power graph of a group and classified all finite groups whose TI-power graph is claw-free, $K_{1,4}$-free, C_4-free, or P_4-free.

In [5], it was noted that determining the metric dimension of a graph is an NP-hard problem. Also, it was showed in [12, 19] that the problem of computing strong metric dimension of a graph is NP-hard, which suggests one should obtain closed formulas for the metric and strong metric dimension of specific families of graphs or bounding the value of this invariant as tight as possible. Nikandish et al. [18] studied the metric and strong metric dimensions of the co-zero-divisor graph defined on a commutative ring. Zhai et al. [23] investigated the metric and strong metric dimensions of the commuting graph of a finite group. Ma and Fu [16] studied the metric and strong metric dimensions of the intersection power graph of a finite group.

In this paper, we continue the study of TI-power graphs of finite groups. We will obtain closed formulas for the metric and strong metric dimensions of the TI-power graph of a finite group. As applications, we compute the metric and strong metric dimensions of the TI-power graph of a cyclic group, a dihedral group, a generalized quaternion group, and a semi-dihedral group.

2. Preliminaries

Every group considered in our paper is assumed to be finite. In general, G always denotes a finite group with the identity e. Given an element g of G, $\langle g\rangle$ denotes the cyclic group with generator g, and the *order* $o(g)$ of g is the smallest positive integer m such that $g^m = e$. In a group G, an element $x \in G$ is called an *involution* if $o(x) = 2$, and an involution u is called an *maximal involution* if $\langle u\rangle \subseteq \langle w\rangle$ implies $w = u$, where $w \in G$. In general, $\mathbb{Z}_n$ denotes the cyclic group of order n. As usual, the k-fold direct product of $\mathbb{Z}_n$ is denoted by $\mathbb{Z}_n^k$.

Given an integer $n \geq 3$, we use D_{2n} to denote the *dihedral group* with $2n$ elements. It is widely known that D_{2n} has a presentation as follows:

$$D_{2n} = \langle x, y : x^n = y^2 = e, yx = x^{-1}y\rangle. \tag{2.1}$$

Note that for every $n \geq 3$, D_{2n} is non-abelian.

Given an integer $n \geq 2$, the *generalized quaternion group* with $4n$ elements is denoted by Q_{4n} and possesses a presentation as follows:

$$Q_{4n} = \langle x, y : x^n = y^2, x^{2n} = y^4 = e, xy = yx^{-1}\rangle. \tag{2.2}$$

Q_{4n} is also called a *dicyclic group* in the literature. Note that Q_{4n} is an extension of $\mathbb{Z}_2$ by $\mathbb{Z}_{2n}$. Also, for every $n \geq 2$, Q_{4n} is non-abelian. One can verify that Q_{4n} has a unique involution, which is y^2.

Let G be a group. Now we define the relation $\approx$ on G by the following rule:

$$x \approx y \Leftrightarrow N(x) = N(y) \text{ in } \mathcal{N}(G), \text{ where } x, y \in G.$$

It is readily seen that $\approx$ is an equivalence relation on G. The equivalence $\approx$-class containing the element $g \in G$ is denoted by $\widetilde{g}$. For any $g \in G$, write

$$[g] := \{x \in G : \langle x\rangle = \langle g\rangle\}.$$

Namely, $[g]$ is the set of the generators of $\langle g\rangle$. Denote by $\pi(G)$ the set of all prime divisors of $|G|$. For $x \in G$, we denote $\pi(\langle x\rangle)$ simply by $\pi(x)$.

Lemma 2.1. *For any group G, the following hold:*

(a) *For distinct $x, y \in G$, if $N(x) = N(y)$, then $\{x, y\} \notin E(\mathcal{N}(G))$. In particular, $\widetilde{g}$ is an independent set of $\mathcal{N}(G)$ for any $g \in G$;*
(b) *For any $g \in G$, $[g] \subseteq \widetilde{g}$;*

(c) *For any $g \in G$, then*

$$\widetilde{g} = \{x \in G : \pi(g) = \pi(x) = \pi(\langle g\rangle \cap \langle x\rangle)\}. \tag{2.3}$$

In particular, if $a \in \langle g\rangle$, $\pi(g) = \pi(a)$ and $o(a)$ is a product of distinct primes, then $a \in \widetilde{g}$.

Proof. It is easy to get (a) and (b). We only prove (c). Let $x \in \widetilde{g}$. Now $\pi(\langle g\rangle \cap \langle x\rangle) \subseteq \pi(x)$ and $\pi(\langle g\rangle \cap \langle x\rangle) \subseteq \pi(g)$. Suppose for a contradiction that $\pi(\langle g\rangle \cap \langle x\rangle) \subsetneq \pi(x)$. Thus, we may assume that there exists $a \in \langle x\rangle$ such that $o(a) = p$ and $a \notin \langle g\rangle$, where p is a prime. It follows that $a \in N(g)$, but $a \notin N(g)$, a contradiction. As a result, $\pi(\langle g\rangle \cap \langle x\rangle) = \pi(x)$. Similarly, we also have $\pi(\langle g\rangle \cap \langle x\rangle) = \pi(g)$, and thus (2.3) holds, as desired. □

Lemma 2.2. *For distinct $x, y \in G$, $N[x] = N[y]$ if and only if each of x and y is either e or a maximal involution. In particular, if x is a maximal involution, then $N[x] = G$.*

Proof. Suppose first that $N[x] = N[y]$. Then, x and y are adjacent in $\mathcal{N}(G)$. If $o(x) \geq 3$, then $x^{-1} \in N[y]$, so $x^{-1} \in N[x]$, which is impossible. As a consequence, $o(x) \leq 2$. Similarly, we also have $o(y) \leq 2$. Suppose that $x \neq e$. Then we claim that x is a maximal involution. For the sake of contradiction, suppose that there exists $a \in G \setminus \{x\}$ such that $x \in \langle a\rangle$. If $y = e$, then $N[y] = N[x] = G$, a contradiction as $a \notin N[x]$. It follows that y is an involution. Therefore, $a \in N[y]$, which is also impossible as $a \notin N[x]$. Thus, each of x and y is either e or a maximal involution.

Note that $N[x] = G$ if x is e or a maximal involution. Thus, the converse is clear. □

For two vertices $x, y \in \mathcal{N}(G)$, we now define the binary relation $\equiv$ on G as follows:

$$x \equiv y \Leftrightarrow N[x] = N[y] \text{ or } N(x) = N(y) \text{ in } \mathcal{N}(G).$$

In fact, it is easy to see that $x \equiv y \Leftrightarrow N(x) \setminus \{y\}$ is equivalent to $N(y) \setminus \{x\}$. Hernando et al. [7] showed that $\equiv$ is an equivalence relation. We use $\overline{g}$ to denote the $\equiv$-class having $g \in G$. Write

$$\overline{G} := \{\overline{g} : g \in G\}.$$

Recall the following elementary result.

Theorem 2.3. ([20, Theorem 5.4.10 (ii)]) *For a prime p, a p-group possessing a unique subgroup of order p is either a cyclic group or a generalized quaternion group.*

In the following, we use Ψ to denote the set of finite groups G satisfying the following two conditions:

(I) G is a 2-group;
(II) G has at least one maximal involution and has precisely one non-maximal involution.

If $G \in \Psi$, then G is called a Ψ-*group*. Remark that if $G \in \Psi$ and $a \in G$ with $o(a) \geq 4$, then the non-maximal unique involution must belong to $\langle a\rangle$.

Example 2.4. *For Ψ-groups, we have the following:*

(i) *If G is abelian, then G is a Ψ-group if and only if $G \cong \mathbb{Z}_2^k \times \mathbb{Z}_{2^t}$ for some integers $k \geq 1, t \geq 2$;*
(ii) *For any $k \geq 1$ and $t \geq 3$, we have $\mathbb{Z}_2^k \times Q_{2^t} \in \Psi$;*
(iii) *For any $t \geq 2$, the dihedral group $D_{2\times 2^t} \in \Psi$;*

(iv) *The modular group of order* 16*, denoted by* M_{16}*, has a presentation:*

$$M_{16} = \langle y, x : y^8 = x^2 = e, xyx = y^5 \rangle.$$

Then $M_{16} = \langle y \rangle \cup \{yx, y^2x, \ldots, y^7x, x\}$*,* M_{16} *has precisely three involutions* x, y^4, y^4x*, and* $o(yx) = o(y^3x) = o(y^5x) = o(y^7x) = 8$*. Thus,* M_{16} *has only two cyclic subgroups of order* 4*, that is* $\langle y^2 \rangle$ *and* $\langle y^2x \rangle$*. Since* $\langle y^2 \rangle \cap \langle y^2x \rangle = \{e, y^4\}$*, we have* $M_{16} \in \Psi$*.*

For any group G, let $\mathcal{I}_G$ denote the set of all maximal involutions of G. Note that $\mathcal{I}_G$ may be an empty set. For example, $\mathcal{I}_{S_3} = \{(1,2),(1,3),(2,3)\}$, $\mathcal{I}_{\mathbb{Z}_6} = \emptyset$ and $\mathcal{I}_{Q_{4n}} = \emptyset$ for any positive integer $n \geq 2$.

Lemma 2.5. *For any group G, the following hold:*

(a) $|\overline{G}| = 1$ *if and only if* $G \cong \mathbb{Z}_2^m$ *for some positive integer* $m \geq 1$*, which in turn is true if and only if* $\mathcal{N}(G)$ *is complete;*
(b) $|\overline{G}| = 2$ *if and only if G is isomorphic to one of the following:*

$$Q_{4\cdot 2^t}, \ \mathbb{Z}_{p^l}, \ D_{2\cdot q^n}, \ a\ \Psi\text{-group}, \tag{2.4}$$

where $t \geq 1$*, p is a prime, q is an odd prime,* $l \geq 2$ *if* $p = 2$ *and* $l \geq 1$ *if* $p > 2$*, and* $n \geq 1$*;*
(c) *Every* $\equiv$*-class is one of* $\{e\} \cup \mathcal{I}_G$ *and* $\approx$*-classes.*

Proof. (a) It is clear that $\mathcal{N}(G)$ is complete if and only if $G \cong \mathbb{Z}_2^m$ where $m \geq 1$, and if $\mathcal{N}(G)$ is complete, then $|\overline{G}| = 1$. Now suppose that $|\overline{G}| = 1$. Then $\overline{e} = G$. For any $x \in G \setminus \{e\}$, we have that $x \in \overline{e}$, which implies that $N[x] = N[e]$ since e and x are adjacent in $\mathcal{N}(G)$. In view of Lemma 2.2, we have that x is a maximal involution of G. Thus, G is an elementary abelian 2-group, as desired.

(b) If G is one group in (2.4), then combining the definition of a Ψ-group, we know easily that $|\overline{G}| = 2$.

Conversely, suppose that $|\overline{G}| = 2$. Note that by (a), we see that G is not an elementary abelian 2-group, which implies that G has an element x of order at least 3. Clearly, $|G| \geq 3$ and $\overline{G} = \{\overline{e}, \overline{x}\}$. We consider two cases:

Case 1. $|\overline{e}| = 1$.

Note that by Lemma 2.2, G has no maximal involutions. For distinct $a, b \in G \setminus \{e\}$, we have $a, b \in \overline{x}$, so Lemma 2.2 implies that $N(a) = N(b)$. Thus, by Lemma 2.1, we have that a and b are nonadjacent in $\mathcal{N}(G)$. This means that G is a p-group for some prime p. This also means that G has a unique subgroup of order p. Note that if $G \not\cong \mathbb{Z}_2$, it follows from Theorem 2.3 that $G \cong Q_{4\cdot 2^t}$ or $\mathbb{Z}_{p^l}$, as desired.

Case 2. $|\overline{e}| > 1$.

By Lemmas 2.1 and 2.2, we have that $G \setminus \overline{e}$ is an independent set of $\mathcal{N}(G)$. Suppose that there exists an involution in $G \setminus \overline{e}$. Then G is a 2-group, has maximal involutions and has only one non-maximal involution, say u. Let $a \in G \setminus \overline{e}$ with maximum order. Then $o(a) \geq 4$, so $u \in \langle a \rangle$ as $\{u, a\} \notin E(\mathcal{N}(G))$. It follows that in this case, $G \in \Psi$, as desired.

In the following, suppose that there is no involution in $G \setminus \overline{e}$. Then we may assume that $|G| = 2^m q^n$, where q is an odd prime and $m, n \geq 1$. Let Q be a Sylow q-subgroup of G. Clearly, G has a unique subgroup of order q, say $\langle a \rangle$, and so $Q \cong \mathbb{Z}_{q^n}$ by Theorem 2.3. We next prove that G has a unique Sylow q-subgroup Q. Suppose for a contradiction that Q' is also a Sylow q-subgroup of G. Then $a \in Q'$ and

Q' is cyclic. It follows that $Q, Q' \subseteq C_G(a)$, and so $|C_G(a)|$ is even because $C_G(a)$ is a subgroup of G. This means that G has a non-maximal involution which does not belong to $\overline{e}$, a contradiction since this non-maximal involution is adjacent to a. This means that Q is normal in G. Let now P be a Sylow 2-subgroup of G. Note that G can not have elements of order 4. This implies that $P \cong \mathbb{Z}_2^m$ for some $m \geq 1$.

Let $Q = \langle b \rangle$. In the following we prove $m = 1$. Suppose for a contradiction that $m \geq 2$. Choosing distinct $x, y \in P \setminus \{e\}$, we have that $xb, yb \notin Q$. It follows that both xb and yb are maximal involutions. Thus,

$$xbx = b^{-1}, \;\; yby = b^{-1}.$$

Since xy is an involution, we have

$$(xy)b(xy) = x(yby)x = xb^{-1}x = b.$$

As a consequentce, xy and b commute, and so $o(xyb) = 2q^n$, a contradiction since G has no non-maximal involutions. We conclude that $m = 1$. Now we may assume that $P = \langle x \rangle$, where x is an involution. It follows that

$$G = \langle x, b : x^2 = b^{q^n} = e, xbx = b^{-1} \rangle \cong D_{2q^n},$$

as desired.

(c) The result follows from Lemma 2.2 and the definition of $\equiv$. □

Lemma 2.6. *Let x be an involution of G. If there exists $y \in G$ such that $x \in \langle y \rangle$ and $o(y) = 2^k$ for some $k \geq 2$, then $x \equiv y$.*

Proof. By (2.3) of Lemma 2.1(c) and the definition of $\equiv$, it is easy to obtain the required result. □

Lemma 2.7. *Let $x \in G$. Then, $|\overline{x}| = 1$ if and only if one of the following occurs:*

(i) *$x = e$ and $\mathcal{I}_G = \emptyset$;*
(ii) *x is a non-maximal involution, and if $\langle x \rangle \subsetneq \langle g \rangle$ for some $g \in G$, then $4 \nmid o(g)$.*

Proof. Suppose first that $|\overline{x}| = 1$. Note that $\widetilde{x} \subseteq \overline{x}$. By Lemma 2.1(b), we deduce $o(x) \leq 2$. If $x = e$, then by Lemma 2.2, it must be $\mathcal{I}_G = \emptyset$, so (i) follows. In the following, let $o(x) = 2$. Then Lemma 2.2 implies that x is a non-maximal involution. Suppose for a contradiction that there exists $g \in G$ such that $\langle x \rangle \subsetneq \langle g \rangle$ and $4 \mid o(g)$. Let $a \in \langle g \rangle$ with $o(a) = 4$. Then Lemma 2.6 implies that $x \equiv a$, so $a \in \overline{x}$, a contradiction. Thus, (ii) follows, as desired.

Conversely, it is clear that if (i) occurs, then by Lemma 2.2, $|\overline{x}| = 1$. Now suppose that (ii) occurs. Note that x is a non-maximal involution. Suppose for a contradiction that $y \in \overline{x}$. It follows from Lemma 2.5(c) that $N(x) = N(y)$. Then by Lemma 2.1(a), we have $\langle x \rangle \cap \langle y \rangle = \{e, x\}$. On the other hand, Lemma 2.1(c) also implies that $o(y)$ is a power of 2, which implies that $x \in \langle y \rangle$ and $4 \mid o(y)$, a contradiction. We conclude that $|\overline{x}| = 1$, as desired. □

Lemma 2.8. *If $|\overline{G}| \geq 3$, then there exist at least two distinct $\equiv$-classes having size at least 2.*

Proof. By Lemma 2.5(a), G has an element x with $o(x) \geq 3$. Then $|\overline{x}| \geq 2$. If G has a maximal involution, then $|\overline{e}| \geq 2$ by Lemma 2.2, as desired. Thus, in the following, we may assume that G has no maximal involutions. Note that $|\overline{a}| \geq 2$ if $o(a)$ is an odd prime. If $|G|$ has two distinct odd prime divisors, then considering these elements of prime order, it is easy to see that the desired result follows. Thus, now, we may assume that $|G| \mid 2^m q^n$, where $m, n \geq 1$ and q is an odd prime.

Assume that G is a 2-group. By Theorem 2.3 and Lemma 2.5(b), G cannot have a unique involution. Thus, G has at least 2 involutions, say u and v. Note that both u and v are not maximal. It follows that there exist $a, b \in G$ such that $o(a) \geq 4$, $o(b) \geq 4$, $u \in \langle a \rangle$, and $v \in \langle b \rangle$. Now Lemma 2.6 implies that $|\overline{u}| \geq 2$ and $|\overline{v}| \geq 2$, as desired. Similarly, if G is a q-group, then we can also get the required result.

Finally, we assume that $|G| = 2^m q^n$ for $m, n \geq 1$. Clearly, if G has two distinct subgroups of order q, then the required result follows. Therefore, in the following, we assume that G has a unique subgroup of order q, $\langle w \rangle$. Let z be an involution of G. If $|\overline{z}| \geq 2$, then the required result follows. Thus, we assume that $|\overline{z}| = 1$. Note that z is non-maximal. It follows from Lemma 2.7 that there exists $g \in G$ such that $z \in \langle g \rangle$ and $o(g) = 2q^l$ for some $l \geq 1$. As a result, we have that $w \in \langle g \rangle$, so $z \in N(w)$ and $z \in N(g)$. In view of Lemma 2.5(c), we have that $\overline{w} \neq \overline{g}$, $|\overline{w}| \geq 2$, and $|\overline{g}| \geq 2$, as desired. □

Recall that G is a finite group. Suppose that there exists an involution $u \in G$ such that $|\overline{u}| = 1$. Clearly, u is non-maximal. We call u an *exceptional involution* if the following two conditions hold:

(i) There exists $a \in G$ such that $o(a)$ is an odd integer and $\langle u, a \rangle \cong \mathbb{Z}_{2 \cdot o(a)}$;
(ii) There is no $b \in G \setminus \overline{e}$ such that $o(b)$ is odd, $\langle b \rangle \cap \langle a \rangle = \{e\}$ and $\langle u, b \rangle \cong \mathbb{Z}_{2 \cdot o(b)}$.

Denote by $\mathfrak{E}_G$ the set of all exceptional involutions of G. Note that $\mathfrak{E}_G$ may be $\emptyset$.

Example 2.9. *Suppose that $G = \mathbb{Z}_2 \times \mathbb{Z}_6$. Then G has three involutions, that is, $(0, 3)$, $(1, 3)$ and $(1, 0)$, and every involution of G is non-maximal. Notice that*

$$\langle (0,3), (0,2) \rangle \cong \langle (1,3), (0,2) \rangle \cong \langle (1,0), (0,2) \rangle \cong \mathbb{Z}_6,$$

and G has a unique subgroup of order 3. It is easy to see that $\mathfrak{E}_G = \{(0,3), (1,3), (1,0)\}$.

In $\mathcal{N}(G)$, for $a, b \in G$, define

$$R\{a, b\} := \{x \in G : d(a, x) \neq d(b, x)\}$$

as the set of vertices resolving a and b in $\mathcal{N}(G)$. Note that $\mathcal{N}(G)$ has diameter of at most 2 and $N[e] = G$.

Lemma 2.10. *Let $u \in \mathfrak{E}_G$. Then there exist precisely two distinct $\overline{x}, \overline{y} \in \overline{G}$ such that*

$$u \in \langle x \rangle, u \notin \langle y \rangle, y \in \langle x \rangle, R\{x, y\} = \{x, y, u\}, \pi(\langle x \rangle \cap \langle y \rangle) = \pi(y) = \pi(x) \setminus \{2\}. \tag{2.5}$$

In particular, for any $x' \in \overline{x}$ and $y' \in \overline{y}$, $R\{x', y'\} = \{x', y', u\}$.

Proof. By the definition of an exceptional involution, there exists $a \in G$ such that $o(a)$ is an odd integer and $\langle u, a \rangle \cong \mathbb{Z}_{2 \cdot o(a)}$, and there is no $b \in G \setminus \overline{e}$ such that $o(b)$ is odd, $\langle b \rangle \cap \langle a \rangle = \{e\}$ and $\langle u, b \rangle \cong \mathbb{Z}_{2 \cdot o(b)}$. Now let $x = ua$ and $y = a$. Then $u \in \langle x \rangle$ and $u \notin \langle y \rangle$. Now Lemmas 2.5(c) and 2.1(c) imply that $\overline{x} \neq \overline{y}$. Also, it is easy to see that $R\{x, y\} = \{x, y, u\}$ and $\pi(\langle x \rangle \cap \langle y \rangle) = \pi(y) = \pi(x) \setminus \{2\}$. Moreover, using the definition of an exceptional involution again, we see that such $\overline{x}$ and $\overline{y}$ are uniquely determined by u, as desired. □

Example 2.11. *Let $G = \langle a \rangle \cong \mathbb{Z}_{90}$. By Lemma 2.1(c), the unique involution a^{45} is an exceptional involution. Also, let $x = a$ and $y = a^2$. Then $\overline{x}$ and $\overline{y}$ are uniquely determined by a^{45} and satisfy* (2.5), *where*

$$\overline{x} = \{g \in G : o(g) = 90 \text{ or } 30\}, \quad \overline{y} = \{g \in G : o(g) = 45 \text{ or } 15\}.$$

Let $u \in \mathfrak{E}_G$. By Lemma 2.10, we denote by $\overline{x_u}$ and $\overline{y_u}$ the two $\equiv$-classes determined uniquely by u, where $u \in \langle x_u \rangle$. If $\mathfrak{E}_G \neq \emptyset$, then we define a binary relation $\curlywedge$ on $\mathfrak{E}_G$ as follows:

$$u \curlywedge v \Leftrightarrow \overline{y_u} = \overline{y_v}, \quad u, v \in \mathfrak{E}_G.$$

Clearly, $\curlywedge$ is an equivalence relation over $\mathfrak{E}_G$. We use $\widehat{u}$ to denote the $\curlywedge$-class containing element $u \in \mathfrak{E}_G$.

Now, by the definitions of an exceptional involution and the equivalence relation $\curlywedge$, we have the following result.

Lemma 2.12. *Suppose that $u \in \mathfrak{E}_G$ and $\widehat{u} = \{u_1, u_2, \ldots, u_t\}$, where $u = u_1$ and $t \geq 2$. Then for each two distinct indices $1 \leq i, j \leq t$, if $a \in \overline{x_{u_i}}$ and $b \in \overline{x_{u_j}}$, then*

$$R\{a, b\} = \{u_i, u_j, a, b\}.$$

3. Metric dimension

In this section, we will give an explicit formula for the metric dimension of the TI-power graph of a finite group. The main result of this section is the following theorem.

Theorem 3.1. *Let G be a finite group. Then*

$$\dim(\mathcal{N}(G)) = |G| - |\overline{G}| + |\mathfrak{E}_G|. \tag{3.1}$$

We will prove a few lemmas before giving the proof of Theorem 3.1.

Lemma 3.2. *If S is a resolving set of $\mathcal{N}(G)$ and $a \in G$, then $|S \cap \overline{a}| \geq |\overline{a}| - 1$.*

Proof. If $|\overline{a}| = 1$, then clearly, the required result holds. In the following, we assume that $|\overline{a}| \geq 2$. Suppose, by contradiction, that $|S \cap \overline{a}| < |\overline{a}| - 1$. Then there exist two distinct $x, y \in \overline{a}$ such that $x, y \in G \setminus S$. Combining $\overline{x} = \overline{y}$ and [7, Lemma 2.3], we have that $d(x, z) = d(z, y)$ for any $z \in G \setminus \{x, y\}$. This means that no such elements of S can resolve x and y, a contradiction. □

Proposition 3.3. *Let $|\overline{G}| = r \geq 3$. Then* $\dim(\mathcal{N}(G)) \geq |G| - |\overline{G}| + |\mathfrak{E}_G|$.

Proof. Suppose that S is a resolving set of $\mathcal{N}(G)$ with size $\dim(\mathcal{N}(G))$. If $\mathfrak{E}_G = \emptyset$, then by Lemma 3.2, we have

$$\dim(\mathcal{N}(G)) = |S| = \sum_{\overline{a} \in \overline{G}} |S \cap \overline{a}| \geq \sum_{\overline{a} \in \overline{G}} (|\overline{a}| - 1) = |G| - |\overline{G}|,$$

as desired. In the following, suppose that $\mathfrak{E}_G \neq \emptyset$. Let

$$\mathfrak{E}_G = \dot{\bigcup_{1 \leq i \leq k}} \widehat{w_i}$$

and let

$$\widehat{w_i} = \{w_{i1}, w_{i2}, \ldots, w_{it_i}\},\ \mathcal{F}_i := \{\overline{w_{i1}}, \ldots, \overline{w_{it_i}}, \overline{x_{w_{i1}}}, \ldots, \overline{x_{w_{it_i}}}, \overline{y_{w_i}}\}.$$

Then, Lemmas 3.2, 2.10 and 2.12 imply that

$$|S \cap (\bigcup_{\overline{w} \in \mathcal{F}_i} \overline{w})| \geq \sum_{\overline{w} \in \mathcal{F}_i} (|\overline{w}| - 1) + t_i. \tag{3.2}$$

Now write

$$\mathcal{W} := \bigcup_{i=1}^{k} \mathcal{F}_i.$$

Then we have

$$\bigcup_{w \in \mathfrak{E}_G} \{\overline{x_w}, \overline{y_w}, \overline{w}\} = \mathcal{W}.$$

Thus, by (3.2) and Lemma 3.2, we deduce

$$\begin{aligned}
\dim(\mathcal{N}(G)) =& |S| = \sum_{\overline{w} \in \mathcal{W}} |S \cap \overline{w}| + \sum_{\overline{w} \in \overline{G} \setminus \mathcal{W}} |S \cap \overline{w}| \\
\geq& \sum_{i=1}^{k} |S \cap (\bigcup_{\overline{w} \in \mathcal{F}_i} \overline{w})| + \sum_{\overline{w} \in \overline{G} \setminus \mathcal{W}} (|\overline{w}| - 1) \\
\geq& \sum_{i=1}^{k} \sum_{\overline{w} \in \mathcal{F}_i} (|\overline{w}| - 1) + \sum_{i=1}^{k} t_i + \sum_{\overline{w} \in \overline{G} \setminus \mathcal{W}} (|\overline{w}| - 1) \\
=& \sum_{\overline{w} \in \mathcal{W}} (|\overline{w}| - 1) + \sum_{\overline{w} \in \overline{G} \setminus \mathcal{W}} (|\overline{w}| - 1) + \sum_{i=1}^{k} t_i \\
=& \sum_{\overline{w} \in \overline{G}} (|\overline{w}| - 1) + \sum_{i=1}^{k} t_i \\
=& |G| - |\overline{G}| + |\mathfrak{E}_G|,
\end{aligned}$$

as desired. □

Proposition 3.4. *Let $|\overline{G}| = r \geq 3$ and let $\{x_1, x_2, \ldots, x_r\}$ be a system of representatives for the $\equiv$-classes of G. Then*

$$S := (G \setminus \{x_1, x_2, \ldots, x_r\}) \cup \mathfrak{E}_G$$

is a resolving set of $\mathcal{N}(G)$.

Proof. By Lemma 2.8, we have that $|S| \geq 2$. Now it suffices to show that for distinct indices $1 \leq i, j \leq r$, if $x_i, x_j \notin \mathfrak{E}_G$, then there exists a vertex in S such that it resolves x_i and x_j. In the following, let $a, b \in \{x_1, x_2, \ldots, x_r\} \setminus \mathfrak{E}_G$ with $a \neq b$. Then $\overline{a} \neq \overline{b}$.

Case 1. $|\overline{a}| \geq 2$ and $|\overline{b}| \geq 2$.

Suppose that one of $\overline{a}$ and $\overline{b}$ contains e, and without loss of generality, let $e \in \overline{a}$. Then by Lemmas 2.5(c) and 2.1(a), $\overline{b}$ must be an independent set. Let $b' \in \overline{b}$. It follows that $d(b, b') = 2$ and

$d(a, b') = 1$, which implies that $b' \in R\{a, b\} \cap S$, as desired. In the following, suppose that $e \notin \overline{a} \cup \overline{b}$. Then $\overline{a} = \widetilde{a}$ and $\overline{b} = \widetilde{b}$ by Lemma 2.5(c). Note that $a \approx b$. It follows from Lemma 2.1(c) that there exists a subgroup $\langle u \rangle$ of prime order such that $\langle u \rangle$ is contained in one of $\langle a \rangle$ and $\langle b \rangle$, and is not contained in the other. Without loss of generality, we assume that $\langle u \rangle \subseteq \langle a \rangle$ and $\langle u \rangle \nsubseteq \langle b \rangle$. If $o(u) \geq 3$, then one of u and u^{-1} belongs to S, say $u \in S$, which implies that $u \in R\{a, b\}$, as desired.

Thus, in the following, we assume that u is an involution, and no such subgroup $\langle v \rangle$ of odd prime order such that $\langle v \rangle$ is contained in one of $\langle a \rangle$ and $\langle b \rangle$, and is not contained in the other. Note that u is not maximal. If $|\overline{u}| \geq 2$, then from Lemma 2.1(c), it follows that there exists an element x of order 4 such that $u \in \langle x \rangle$, which implies that one of x and x^{-1} belongs to $S \cap R\{a, b\}$, as desired.

Finally, we may assume that $|\overline{u}| = 1$. Note that in this case, $a = ua'$ for some element a' of odd order. Also, let $b = wb'$, where $o(w)$ is a power of 2 and $o(b')$ is odd. Then by our hypotheses, we have

$$\pi(\langle a' \rangle \cap \langle b' \rangle) = \pi(a') = \pi(b').$$

If there exists an element $c \in G \setminus \overline{e}$ such that $o(c)$ is odd, $\langle c \rangle \cap \langle a' \rangle = \{e\}$ and $\langle u, c \rangle \cong \mathbb{Z}_{2 \cdot o(c)}$, then one of uc and $(uc)^{-1}$ belongs to $S \cap R\{a, b\}$, as desired. Otherwise, $u \in \mathfrak{E}_G$, and so $u \in S$. It is clear that $u \in R\{a, b\}$, as required.

Case 2. One of $\overline{a}$ and $\overline{b}$ has size 1.

Without loss of generality, let $|\overline{a}| = 1$. It follows from Lemma 2.7 that $a = e$ or a is an involution.

Subcase 2.1. $a = e$.

Then, Lemmas 2.7, 2.2, and 2.5 imply that $\overline{b} = \widetilde{b}$. Note that $N[e] = G$ in $\mathcal{N}(G)$. If $|\overline{b}| \geq 2$, then taking $b' \in \overline{b} \setminus \{b\}$, we have that $b' \in S$, so by Lemma 2.1(a) we see that $b' \in S \cap R\{a, b\}$, as desired. In the following, we may assume that $|\overline{b}| = 1$. Then b is an involution. Clearly, in this case, b is a non-maximal involution. It follows from Lemma 2.7 that there exists an element $x \in G$ such that $\langle b \rangle \subsetneq \langle x \rangle$ and $4 \nmid o(x)$. Now one of x and x^{-1} must belong to $S \cap R\{a, b\}$, as desired.

Subcase 2.1. a is an involution.

Then by Lemma 2.7, there exists an element $y \in G$ such that $\langle a \rangle \subsetneq \langle y \rangle$ and $4 \nmid o(y)$. Suppose first that $|\overline{b}| = 1$. If $b = e$, then it is similar to Subcase 2.1, and the required result holds. Now by Lemma 2.7, let b be also an involution. Then, $\langle y \rangle \cap \langle b \rangle = \{e\}$; therefore, one of y and y^{-1} must belong to $S \cap R\{a, b\}$, as desired.

Next, we assume that $|\overline{b}| \geq 2$. If one of $b = e$ and $o(b) = 2$ occurs, then one of y and y^{-1} must belong to $S \cap R\{a, b\}$ since $\langle y \rangle \cap \langle b \rangle = \{e\}$, as desired. As a result, we may assume that $o(b) \geq 3$. Note that $b^{-1} \in S$. If there exists $w \in \langle b^{-1} \rangle$ such that $o(w)$ is an odd prime, then one of w and w^{-1} must belong to $S \cap R\{a, b\}$, as desired. We now may assume that $o(b^{-1})$ is a power of 2. Since $|\overline{a}| = 1$, we know that $a \notin \langle b^{-1} \rangle$ by Lemma 2.7. This means that $b^{-1} \in R\{a, b\}$, as desired. □

For two graphs Γ_1 and Γ_2 with $V(\Gamma_1) \cap V(\Gamma_2) = \emptyset$, the *sum* of Γ_1 and Γ_2, denoted by $\Gamma_1 \vee \Gamma_2$, is the graph with vertex set $V(\Gamma_1) \cup V(\Gamma_2)$, and its edge set is the union of $E(\Gamma_1)$, $E(\Gamma_2)$ and the set of all edges between vertices from the two different graphs Γ_1 and Γ_2.

We are now ready to prove our main theorems.

Proof of Theorem 3.1. If $|\overline{G}| \geq 3$, then by Propositions 3.3 and 3.4, we see that (3.1) holds. In the following, we only need to consider $|\overline{G}| \leq 2$. Suppose that $|\overline{G}| = 1$. Then Lemma 2.5(a) implies that G is an elementary abelian 2-group and $\mathcal{N}(G)$ is complete. Thus, $\mathfrak{E}_G = \emptyset$, and it follows from [4, Theorem 3] that $\dim(\mathcal{N}(G)) = |G| - 1$, which implies that (3.1) holds, as desired.

Suppose that $|\overline{G}| = 2$. Then Lemma 2.5(b) implies that G is isomorphic to one group in (2.4). If $G \cong Q_{4\cdot 2^t}$ or $\mathbb{Z}_{p^l}$ where $t \geq 1$, p is a prime, $l \geq 2$ if $p = 2$, and $l \geq 1$ if $p > 2$, then $\mathcal{N}(G)$ is a star and $\mathfrak{E}_G = \emptyset$ by Lemma 2.7, so it follows from [4, Theorem 4] that $\dim(\mathcal{N}(G)) = |G| - 2$, as desired. Assume that $G \cong D_{2\cdot q^n}$ or a Ψ-group where q is an odd prime and $n \geq 1$. Then by the definition of a Ψ-group, one can easily obtain that $\mathfrak{E}_G = \emptyset$ and

$$\mathcal{N}(G) \cong K_m \vee \overline{K_{|G|-m}},$$

where m is the number of all maximal involution. Let $\overline{G} = \{\overline{e}, \overline{a}\}$. Since $|\overline{a}| \geq 2$, we may choose $a' \in \overline{a} \setminus \{a\}$. As a result, $a' \in R\{e, a\}$. It follows that $\dim(\mathcal{N}(G)) \leq |G| - 2$. Also, by Lemma 3.2, $\dim(\mathcal{N}(G)) \geq |G| - 2$, so (3.1) holds, as desired. □

Corollary 3.5. *If G is a finite group with odd order, then* $\dim(\mathcal{N}(G)) = |G| - |\overline{G}|$.

4. Strong metric dimension

Let G be a group. We define a binary relation $\sim$ on G as follows:

$$x \sim y \Leftrightarrow N[x] = N[y] \text{ in } \mathcal{N}(G),\ x, y \in G.$$

Observe that $\sim$ is an equivalence relation. Let $\mathcal{U}(G)$ be a complete set of distinct representative elements for the equivalence relation $\sim$. Next, we define the *reduced graph* of $\mathcal{N}(G)$, denoted simply by $\mathcal{R}(G)$, as a simple graph with vertex set $\mathcal{U}(G)$, where two distinct vertices are adjacent if and only if they are adjacent in $\mathcal{N}(G)$. It is clear that for distinct $\sim$-classes $\mathcal{U}_1$ and $\mathcal{U}_2$, if there exist a vertex in $\mathcal{U}_1$ and a vertex in $\mathcal{U}_2$ which are adjacent in $\mathcal{N}(G)$, then every vertex in $\mathcal{U}_1$ and every vertex in $\mathcal{U}_2$ are adjacent in $\mathcal{N}(G)$. Thus, $\mathcal{R}(G)$ does not depend on the choice of representatives.

Ma et al. [15] characterized the strong metric dimension of a graph with diameter two by the reduced graph of this graph. If $\mathcal{N}(G)$ is complete, then clearly, $|\mathcal{U}(G)| = 1$ and $\mathrm{sdim}(\mathcal{N}(G)) = |G| - 1$, which also implies $\omega(\mathcal{R}(G)) = 1$. Otherwise, $\mathcal{N}(G)$ has diameter two. Thus, by [15, Theorem 2.2], we have the following result.

Theorem 4.1. *Let G be a group of order n. Then* $\mathrm{sdim}(\mathcal{N}(G)) = n - \omega(\mathcal{R}(G))$.

Given a finite group G, write

$$\mathcal{Q}(G) := \{\langle u\rangle \subseteq G : \ o(u) \text{ is a prime and if } o(u) = 2 \text{ then } u \text{ is not maximal}\}.$$

Namely, $\mathcal{Q}(G)$ is the set of all cyclic subgroups of prime order other than the cyclic subgroup generated by some maximal involution.

Proposition 4.2. $\omega(\mathcal{R}(G)) = |\mathcal{Q}(G)| + 1$.

Proof. By Lemma 2.2, we may assume that $\mathcal{U}(G) = G \setminus \mathcal{I}_G$. Write

$$\mathcal{Q}(G) = \{\langle u_1\rangle, \langle u_2\rangle, \dots, \langle u_t\rangle\}.$$

Note that $u_i \in \mathcal{U}(G)$ for all $1 \le i \le t$. Clearly,

$$C := \{e, u_1, u_2, \dots, u_t\}$$

is a clique of $\mathcal{R}(G)$, so $\omega(\mathcal{R}(G)) \ge t + 1$. Now let $S := \{x_1, x_2, \dots, x_k\}$ be a clique of $\mathcal{R}(G)$ with $k = \omega(\mathcal{R}(G))$. In the following, we will prove $k \le t + 1$.

We first claim that $o(x_i)$ is a prime power for any $1 \le i \le k$. Suppose for a contradiction that there exists $j \in \{1, 2, \dots, k\}$ such that $o(x_j)$ has two distinct prime divisors p, q. Without loss of generality, let $j = 1$. Take now $u, v \in \langle x_1 \rangle$ with $o(u) = p$ and $o(v) = q$. Let

$$S' := \{u, v, x_2, \dots, x_k\}.$$

If $|\langle u\rangle \cap \langle x_i\rangle| \neq 1$ for all $2 \le i \le k$, then $u \in \langle x_1\rangle \cap \langle x_i\rangle$, which is impossible since S is a clique. Similarly, we also have $|\langle v\rangle \cap \langle x_i\rangle| = 1$ for all $2 \le i \le k$. It follows that S' is a clique, but $|S'| > |S| = \omega(\mathcal{R}(G))$, a contradiction. So, our claim is valid.

For all $1 \le i \le k$, if $\langle x_i\rangle \neq \{e\}$, then take $y_i \in \langle x_i\rangle$ with $o(y_i)$ as a prime; if $\langle x_i\rangle = \{e\}$, then take $x_i = e$. Note that at most one of S is e. Then $\langle y_i\rangle \neq \langle y_j\rangle$ for distinct $1 \le i, j \le k$, since S is a clique. Write

$$T := \{y_1, y_2, \dots, y_k\}.$$

Then it must be that T is also a clique. If y_l is a maximal involution for some $1 \le l \le k$, then x_l is also a maximal involution, so $x_l \notin \mathcal{U}(G)$, a contradiction. By the definition of $\mathcal{Q}(G)$, it follows that $k \le t + 1$, as desired. □

Now Theorem 4.1 and Proposition 4.2 imply our main theorem as follows.

Theorem 4.3. *Let G be a group of order n. Then* $\mathrm{sdim}(\mathcal{N}(G)) = n - |\mathcal{Q}(G)| - 1$.

It is clear that by Theorem 2.5, $\mathrm{sdim}(\mathcal{N}(G)) = |G| - 1$ if and only if G is an elementary abelian 2-group. Combining Theorems 2.3 and 4.3, we conclude this section by the following corollary, which classifies all finite group G with $\mathrm{sdim}(\mathcal{N}(G)) = |G| - 2$.

Corollary 4.4. *Let G be a group of order n. Then* $\mathrm{sdim}(\mathcal{N}(G)) = n - 2$ *if and only if G is isomorphic to one of the following:*

(i) $Q_{4\cdot 2^t}$, *where* $t \ge 1$*;*
(ii) $\mathbb{Z}_{2^m}$, *where* $m \ge 2$*;*
(iii) $\mathbb{Z}_{q^l}$, *where q is an odd prime and* $l \ge 1$.

5. Examples

In this section, using Theorems 3.1 and 4.3, we will compute $\dim(\mathcal{N}(G))$ and $\mathrm{sdim}(\mathcal{N}(G))$ if G is a cyclic group, a dihedral group, a generalized quaternion group and a semi-dihedral group.

For a positive integer n, let

$$n = p_1^{r_1} p_2^{r_2} \dots p_t^{r_t} \tag{5.1}$$

be its canonical factorization, where $p_1 < p_2 < \cdots < p_t$ are pair-wise distinct primes and $r_i \ge 1$ for all $1 \le i \le t$.

Lemma 5.1. *Let $G = \mathbb{Z}_n$, where n is a positive integer as presented in* (5.1)*. Then, G has an exceptional involution if and only if $t \geq 2$, $p_1 = 2$ and $r_1 = 1$.*

Proof. Suppose that G has an exceptional involution u. Then $|\overline{u}| = 1$. Moreover, by Lemma 2.1, G has no elements of order 4, which implies that $p_1 = 2$ and $r_1 = 1$. Since u is not maximal, we deduce $t \geq 2$, as desired. For the converse, assume that $t \geq 2$, $p_1 = 2$, and $r_1 = 1$. Let u be the unique involution of G. Clearly, $|\overline{u}| = 1$. Also, an element a with $o(a) = p_2^{r_2} \dots p_t^{r_t}$ satisfies the conditions (i) and (ii) in the definition of an exceptional involution, as desired. □

Theorem 5.2. *Let n be a positive integer as presented in* (5.1)*. Then,*

$$\dim(\mathcal{N}(\mathbb{Z}_n)) = \begin{cases} 1, & \text{if } p_1 = 2, r_1 = 1, t = 1; \\ n - 2^t + 1, & \text{if } t \geq 2,\ p_1 = 2 \text{ and } r_1 = 1; \\ n - 2^t, & \text{otherwise,} \end{cases}$$

and

$$\operatorname{sdim}(\mathcal{N}(\mathbb{Z}_n)) = \begin{cases} 1, & \text{if } p_1 = 2, r_1 = 1, t = 1; \\ n - t - 1, & \text{otherwise.} \end{cases}$$

Proof. We first compute $\dim(\mathcal{N}(\mathbb{Z}_n))$. If $p_1 = 2, r_1 = 1, t = 1$, then $n = 2$ and $\mathcal{N}(\mathbb{Z}_2)$ is complete, so it follows from [4, Theorem 3] that $\dim(\mathcal{N}(\mathbb{Z}_n)) = 1$, as desired. In the following, let $n \geq 3$. Then $\mathbb{Z}_n$ has no maximal involutions. From Lemma 2.5(c), it follows that every $\equiv$-class is a $\approx$-class. By Lemma 2.1 and (2.3), it is easy to see that if $x \in \mathbb{Z}_n$, then

$$\widetilde{x} = \{g \in \mathbb{Z}_n : \pi(g) = \pi(x)\}.$$

As a result, we conclude that $|\overline{\mathbb{Z}_n}| = 2^t$. Now Lemma 5.1 and Theorem 3.1 imply the required result.

For $\operatorname{sdim}(\mathcal{N}(\mathbb{Z}_n))$, if $n = 2$, then $\mathcal{N}(\mathbb{Z}_n)$ is complete; otherwise, $\mathbb{Z}_n$ has no maximal involutions and so $|Q(\mathbb{Z}_n)| = t$, as desired. □

Theorem 5.3. *Let D_{2n} be the dihedral group as presented in* (2.1)*, where $n \geq 3$ is a positive integer as presented in* (5.1)*. Then,*

$$\dim(\mathcal{N}(D_{2n})) = \begin{cases} 2n - 2^t + 1, & \text{if } t \geq 2,\ p_1 = 2 \text{ and } r_1 = 1; \\ 2n - 2^t, & \text{otherwise,} \end{cases}$$

and $\operatorname{sdim}(\mathcal{N}(D_{2n})) = 2n - t - 1$.

Proof. It is easy to verify that

$$D_{2n} = \langle x \rangle \cup \{xy, x^2y, \dots, x^{n-1}y, y\}, \quad \mathcal{I}_{D_{2n}} = \{xy, x^2y, \dots, x^{n-1}y, y\}. \tag{5.2}$$

It follows from Lemma 2.5(c) that $\overline{e} = \{e, xy, x^2y, \dots, x^{n-1}y, y\}$, hence $|\overline{D_{2n}}| = |\overline{\langle x \rangle}|$. Note that $\langle x \rangle \cong \mathbb{Z}_n$. By Lemma 5.1, and Theorems 3.1 and 5.2, we can obtain $\dim(\mathcal{N}(D_{2n}))$. Now in view of (5.2), we have that $|Q(D_{2n})| = |Q(\mathbb{Z}_n)| = t$. Thus, it follows from Theorem 4.3 that $\operatorname{sdim}(\mathcal{N}(D_{2n})) = 2n - t - 1$. □

Theorem 5.4. *Let $2n = p_1^{r_1} p_2^{r_2} \dots p_t^{r_t}$ be a positive integer of at least 4, where $p_1, p_2, \dots, p_t$ are pairwise distinct primes with $2 = p_1 < p_2 < \dots < p_t$, and $r_i \geq 1$ for all $1 \leq i \leq t$. Let Q_{4n} be the generalized quaternion group as presented in* (2.2)*. Then,* $\dim(\mathcal{N}(Q_{4n})) = 4n - 2^t$ *and* $\operatorname{sdim}(\mathcal{N}(Q_{4n})) = 4n - t - 1$.

Proof. Note that Q_{4n} has a unique involution $y^2 = x^n$ which is not maximal and $\langle x\rangle \cong \mathbb{Z}_{2n}$. It is easy to verify that

$$Q_{4n} = \langle x\rangle \cup \{x^i y : 1 \le i \le 2n\}, \;\; o(x^i y) = 4 \text{ for all } 1 \le i \le 2n. \tag{5.3}$$

Thus, $x^n = (x^i y)^2$ for all $1 \le i \le 2n$. It follows from Lemma 2.5(c) that

$$\overline{x^n} = \{g \in \langle x\rangle : o(g) \text{ is a power of } 2\} \cup \{x^i y : 1 \le i \le 2n\}.$$

Therefore, we have $|\overline{Q_{4n}}| = |\overline{\langle x\rangle}|$ and Q_{4n} has no exceptional involutions. It follows from Theorems 3.1 and 5.2 that $\dim(\mathcal{N}(Q_{4n})) = 4n - 2^t$. Moreover, by (5.3), we have that $|\mathcal{Q}(Q_{4n})| = |\mathcal{Q}(\mathbb{Z}_{2n})| = t$. As a result, Theorem 4.3 implies that $\mathrm{sdim}(\mathcal{N}(Q_{4n})) = 4n - t - 1$. □

We conclude the paper by the following example, which determines the metric and strong metric dimensions of the TI-power graph of a semi-dihedral group.

Given a positive integer $n \ge 2$, the *semi-dihedral group* of order $8n$, denoted by SD_{8n}, is a group having the presentation as follows:

$$SD_{8n} = \langle a, b : a^{4n} = b^2 = e, bab = a^{2n-1}\rangle. \tag{5.4}$$

We remark that $SD_{8n} = \langle a\rangle \cup \{ab, a^2b, a^3b, \dots, a^{4n}b\}$, $o(a^ib) = 2$ for any even i, and for any odd j, $o(a^jb) = 4$, and $(a^jb)^2 = a^{2n}$. It follows that SD_{8n} has no exceptional involutions. Moreover, we have

$$\overline{e} = \{e\} \cup \{a^ib : \; 2 \le i \le 4n \text{ and } i \text{ is even}\} \tag{5.5}$$

and

$$\overline{a^{2n}} = \{a^jb : \; 1 \le j < 4n \text{ and } j \text{ is odd}\} \cup \{g \in \langle a\rangle : o(g) \text{ is a power of } 2\}.$$

Thus, it is easy to see that $|\overline{SD_{8n}}| = |\overline{\mathbb{Z}_{4n}}|$. Furthermore, by (5.5), we have that $|\mathcal{Q}(SD_{8n})| = |\mathcal{Q}(\mathbb{Z}_{4n})|$. Now, by Theorems 3.1, 5.2 and 4.3, we have the following result.

Theorem 5.5. *Let* $4n = p_1^{r_1} p_2^{r_2} \dots p_t^{r_t}$ *be a positive integer of at least* 8*, where* $p_1, p_2, \dots, p_t$ *are pairwise distinct primes with* $2 = p_1 < p_2 < \cdots < p_t$*,* $r_1 \ge 2$*, and* $r_i \ge 1$ *for all* $2 \le i \le t$*. Let* SD_{8n} *be the semi-dihedral group as presented in* (5.4)*. Then,* $\dim(\mathcal{N}(SD_{8n})) = 8n - 2^t$ *and* $\mathrm{sdim}(\mathcal{N}(SD_{8n})) = 8n - t - 1$.

6. Conclusions

For a finite group G with the identity e, the *TI-power graph* (trivial intersection power graph) of G, denoted by $\mathcal{N}(G)$, is an undirected graph with vertex set G, in which two distinct vertices x and y are adjacent if $\langle x\rangle \cap \langle y\rangle = \{e\}$. In a paper to appear in Open Mathematics, Li, the second and third authors introduced the TI-power graph of a group and classified all finite groups whose TI-power graph is claw-free, $K_{1,4}$-free, C_4-free, or P_4-free.

In 2018, Bera [2] introduced the *intersection power graph* of G, denoted by $\mathcal{P}_I(G)$, as an undirected graph with vertex set G, and two distinct vertices a and b are adjacent if either one of x, y is e, or $\langle x\rangle \cap \langle y\rangle \ne \{e\}$. Thus, according to the definition of a TI-power graph, we see that if we only consider the set $G \setminus \{e\}$, then the induced subgraph of $\mathcal{N}(G)$ by $G \setminus \{e\}$ is equal to the complement of the induced subgraph of $\mathcal{P}_I(G)$ by $G \setminus \{e\}$.

This paper continued the study of TI-power graphs of finite groups. Specifically, we obtained closed formulas for the metric and strong metric dimensions of the TI-power graph of a finite group. As applications, we computed the metric and strong metric dimensions of the TI-power graph of a cyclic group, a dihedral group, a generalized quaternion group, and a semi-dihedral group.

Author contributions

Chunqiang Cui: Conceptualization, Writing-original draft; Jin Chen: Formal analysis, Writing-original draft; Shixun Lin: Editing, Writing-original draft. All the contributors have perused and consented to the publishable draft of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors state no conflict of interests.

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