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# Some new oscillation results for second-order differential equations with neutral term

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**Abstract:** In this paper, we study the oscillatory behavior of second-order differential equations. Using the comparison method, we obtain new oscillation criteria that improve the relevant results in the literature. Additionally, an example is given to illustrate the importance of the obtained oscillation criteria.

**Keywords:** oscillation criteria; comparison method; neutral; second-order
**Mathematics Subject Classification:** 34C10, 34K11

## 1. Introduction

In this paper, we will study the oscillation behavior of the following differential equations (DEs):

$$\mathcal{P}''(s) + q(s)\, x(\beta(s)) = 0. \tag{1.1}$$

We assume the following:

(I) $\mathcal{P}(s) = x(s) + \mu(s)\, x(\alpha(s))$, $\mu \in C([s_0,\infty),[0,1))$, $\alpha \in C^1([s_0,\infty),(0,\infty))$, $\alpha(s) \le s$, $\alpha'(s) > 0$ and $\lim_{s\to\infty} \alpha(s) = \infty$;

(II) $\beta \in C^1([s_0,\infty),(0,\infty))$, $\beta(s) \le s$, $\beta'(s) > 0$, and $\lim_{s\to\infty} \beta(s) = \infty$ and

(III) $q \in C^1([s_0,\infty))$ and $q(s) > 0$.

By a solution of (1.1), we mean a function $x \in C[s_u,\infty)$, with $s_u := \min\{\beta(s_b), \alpha(s_b)\}$ for some $s_b \ge s_0$, which has the property $\mathcal{P}'(s) \in C^1([s_u,\infty),\mathbb{R})$, and satisfies (1.1) on $[s_b,\infty)$. We only consider those nontrivial solutions $x$ of (1.1), which are defined on some half-line $[s_b,\infty)$.

As is customary, a solution $x$ of (1.1) is said to be oscillatory if it has arbitrarily large zeros; otherwise, it is said to be nonoscillatory. The equation itself is called oscillatory if all its solutions oscillate.

Recently, there has been a lot of interest in the oscillatory properties of solutions to diverse types of functional DEs. The oscillation of neutral DEs was a special focus of several authors. We direct the reader to the articles [1–3], monographs [4–6], and the references listed within. The fact that neutral DEs appear in several practical issues in the natural sciences, control, and engineering [7] explains one of the primary causes of this interest.

For second-order equations, Agarwal et al. [8], Kusano et al. [9], Sun and Meng [10], and Dzurina and Stavroulakis [11] derived oscillation criteria for the DE

$$\left(\lambda(s)\left|x'(s)\right|^{\psi-1}x'(s)\right)'+q(s)\left|x(\beta(s))\right|^{\psi-1}x(\beta(s))=0,$$

where $\lambda(s)\in C^1([s_0,\infty),\mathbb{R})$, $\lambda(s)>0$, and $\psi>0$. Baculikova [12] established some oscillatory properties of the DE

$$(\lambda(s)x'(s))'+q(s)x(\beta(s))=0.$$

Their results complement and improve on the results of [13, 14].

We briefly discuss the relevant findings that inspired our work in the remaining portion of this section.

Some oscillation criteria for the functional DE of neutral type

$$\left(\lambda(s)\left|(x(s)+\mu(s)x(\alpha(s)))'\right|^{\psi-1}(x(s)+\mu(s)x(\alpha(s)))'\right)'+q(s)\left|x(\beta(s))\right|^{\phi-1}x(\beta(s))=0, \qquad (1.2)$$

where $\lambda(s)\in C^1([s_0,\infty),\mathbb{R})$, $\lambda(s)>0$, $\lambda'(s)\geq 0$, $\psi>0$, and $\phi>0$ are established by Wu et al. [15]. They proved that the DE (1.2) is oscillatory if

$$\int_{s_0}^{\infty}\left[\rho(s)q(s)(1-\mu(\beta(s)))^{\phi}-\frac{(\rho'(s))^{\kappa+1}\lambda(\varrho(s))}{(\kappa+1)^{\kappa+1}(m\rho(s)\beta'(s))^{\kappa}}\right]\mathrm{d}s=\infty, \qquad (1.3)$$

where $\kappa=\min\{\psi,\phi\}$,

$$\varrho(s)=\begin{cases}\beta(s), & \psi\leq\phi,\\ s, & \psi>\phi,\end{cases}$$

and

$$m=\begin{cases}1, & \psi=\phi,\\ 0<m\leq 1, & \psi\neq\phi.\end{cases}$$

Using the Riccati substitution technique and comparing with first-order delay equations, Moaaz [16] derived oscillation criteria for the DE

$$\left(\lambda(s)\left((x(s)+\mu(s)x(\alpha(s)))'\right)^{\psi}\right)'+f(s,x(\beta(s)))=0, \qquad (1.4)$$

where $|f(s,x)|\geq q(s)|x|^{\phi}$, $\psi$ and $\phi$ are quotients of odd positive integers, $\lambda(s)\in C([s_0,\infty),\mathbb{R})$, $\lambda(s)>0$, and $\int^{\infty}\lambda^{-1/\psi}(\varsigma)\,\mathrm{d}\varsigma=\infty$. They proved that the DE (1.4) is oscillatory if $\psi\geq\phi$ and

$$\liminf_{s\to\infty}\int_{\beta(s)}^{s}\mathcal{I}(v)\left(\widehat{\Theta}(\beta(v))\right)^{\phi}\mathrm{d}v>\frac{1}{\mathrm{e}}, \qquad (1.5)$$

where

$$\mathcal{I}(s) = q(s)(1-\mu(\beta(s)))^{\phi},$$

$$\eta(s,\varsigma) = \int_{\varsigma}^{s} \lambda^{-1/\psi}(u)\,\mathrm{d}u,$$

$$\widehat{\Theta}(s) = \eta(s,s_1) + \frac{C^{\phi-\psi}}{\psi}\int_{s_1}^{s} \eta(\nu,s_1)\,\eta(\beta(\nu),s_1)\,\mathcal{I}(\nu)\,\mathrm{d}\nu,$$

and $C$ is a positive real constant.

Comparing the DE with either first-order delay DEs or inequalities, whose oscillatory behavior is known beforehand, is one of the fundamental methods in the oscillation theory for acquiring the criteria. Using the comparison method, this study aims to develop some oscillation criteria for the (1.1) solutions. The results obtained here improve some of the established results in the literature.

## 2. Main results

We begin by mentioning an important lemma that we will need to prove our results.

**Lemma 2.1.** *[17] Let $x$ be a positive solution of (1.1). Then,*

$$\mathcal{P}(s) > 0,\ \mathcal{P}'(s) > 0,\ \text{ and } \mathcal{P}''(s) \le 0, \tag{2.1}$$

*for all sufficiently large $s$.*

**Theorem 2.1.** *If*

$$\int_{s_0}^{\infty} \beta(\varsigma)\,q(\varsigma)(1-\mu(\beta(\varsigma)))\,\mathrm{d}\varsigma = \infty, \tag{2.2}$$

*and*

$$\liminf_{s\to\infty} \int_{\beta(s)}^{s} \frac{1}{u^2}\int_0^{u} \varsigma q(\varsigma)\beta(\varsigma)(1-\mu(\beta(\varsigma)))\,\mathrm{d}\varsigma\mathrm{d}u > \frac{1}{\mathrm{e}}, \tag{2.3}$$

*then (1.1) is oscillatory.*

*Proof.* Let (1.1) have a positive solution. It is clear that

$$-\left(s^2\left(\frac{\mathcal{P}(s)}{s}\right)'\right)' = -\left(s\mathcal{P}'(s) - \mathcal{P}(s)\right)' = sq(s)\,x(\beta(s)). \tag{2.4}$$

Integrating (2.4) from $s_0$ to $\infty$, we obtain

$$-s^2\left(\frac{\mathcal{P}(s)}{s}\right)' = \mathcal{P}(s_0) - s_0\mathcal{P}'(s_0) + \int_{s_0}^{s} \varsigma q(\varsigma)\,x(\beta(\varsigma))\,\mathrm{d}\varsigma = k + \int_{s_0}^{s} \varsigma q(\varsigma)\,x(\beta(\varsigma))\,\mathrm{d}\varsigma, \tag{2.5}$$

where $k = \mathcal{P}(s_0) - s_0\mathcal{P}'(s_0)$. Since $\mathcal{P}(s) = x(s) + \mu(s)\,x(\alpha(s))$, we obtain

$$x(s) = \mathcal{P}(s) - \mu(s)\,x(\alpha(s)) \ge \mathcal{P}(s) - \mu(s)\,\mathcal{P}(\alpha(s)) \ge (1-\mu(s))\,\mathcal{P}(s). \tag{2.6}$$

Substituting (2.6) in (2.5), we find

$$-s^2\left(\frac{\mathcal{P}(s)}{s}\right)' \ge k + \int_{s_0}^{s} \varsigma q(\varsigma)(1-\mu(\beta(\varsigma)))\,\mathcal{P}(\beta(\varsigma))\,\mathrm{d}\varsigma, \tag{2.7}$$

since $\mathcal{P}'(s) > 0$, and using (2.2), we see that

$$-s^2\left(\frac{\mathcal{P}(s)}{s}\right)' \geq k + \mathcal{P}(\beta(s_0))\int_{s_0}^{s} \varsigma q(\varsigma)(1-\mu(\beta(\varsigma)))\,\mathrm{d}\varsigma \to \infty \text{ as } s \to \infty. \tag{2.8}$$

Hence

$$\frac{\mathcal{P}(s)}{s} \text{ is decreasing,} \tag{2.9}$$

which also implies that $k > 0$ for a large enough $s_0$.
Therefore, we expect

$$\lim_{s\to\infty} \frac{\mathcal{P}(s)}{s} = n = 0. \tag{2.10}$$

If $\lim_{s\to\infty} \mathcal{P}(s)/s = n > 0$. From (2.7), we get

$$z'(s) + \frac{1}{s^2}\int_{s_0}^{s} \varsigma q(\varsigma)(1-\mu(\beta(\varsigma)))\beta(\varsigma)z(\beta(\varsigma))\,\mathrm{d}\varsigma \leq -\frac{k}{s^2} < 0. \tag{2.11}$$

Integrating (2.11) from $s_0$ to $\infty$, we obtain

$$\begin{aligned} z(s_0) - n &\geq n\int_{s_0}^{s} \frac{1}{u^2}\int_{s_0}^{u} \varsigma q(\varsigma)(1-\mu(\beta(\varsigma)))\beta(\varsigma)z(\beta(\varsigma))\,\mathrm{d}\varsigma\mathrm{d}u \\ &= n\int_{s_0}^{\infty} \beta(\varsigma)q(\varsigma)(1-\mu(\beta(\varsigma)))\,\mathrm{d}\varsigma, \end{aligned}$$

which contradicts (2.2); therefore, we find that $n = 0$.
Now, from (2.11), we find

$$\begin{aligned} 0 &\geq z'(s) + \frac{1}{s^2}\left(z(\beta(s))\int_{s_0}^{s} \varsigma q(\varsigma)(1-\mu(\beta(\varsigma)))\beta(\varsigma)\,\mathrm{d}\varsigma + k\right) \\ &= z'(s) + \frac{1}{s^2}\left(z(\beta(s))\int_{0}^{s} \varsigma q(\varsigma)(1-\mu(\beta(\varsigma)))\beta(\varsigma)\,\mathrm{d}\varsigma + k - z(\beta(s))\int_{0}^{s_0} \varsigma q(\varsigma)(1-\mu(\beta(\varsigma)))\beta(\varsigma)\,\mathrm{d}\varsigma\right). \end{aligned}$$

Since $z'(s) < 0$ and $\lim_{s\to\infty} z(s) = 0$, we find that

$$z'(s) + \left(\frac{1}{s^2}\int_{0}^{s} \varsigma q(\varsigma)(1-\mu(\beta(\varsigma)))\beta(\varsigma)\,\mathrm{d}\varsigma\right)z(\beta(s)) \leq 0 \tag{2.12}$$

has a positive solution $z(s)$, which contradicts (2.3). End of proof. □

**Lemma 2.2.** *Let (2.2) hold. Then*

$$z(s)\psi_{\varrho}(s) \text{ is decreasing,} \tag{2.13}$$

*where*

$$\psi_{\varrho}(s) = \mathrm{e}^{\phi_{\varrho}(s)},\ \phi_{\varrho}'(s) = \theta_{\varrho}(s) \text{ and } \theta_{\varrho}(s) = \frac{1}{s^2}\int_{0}^{s} \varsigma q(\varsigma)(1-\mu(\beta(\varsigma)))\beta(\varsigma)\,\mathrm{d}\varsigma.$$

*Proof.* Let (1.1) have a positive solution. From (2.12), we get

$$z'(s)+\theta_{\varrho}(s)z(s)\leq 0. \tag{2.14}$$

Therefore,

$$\left(z(s)\psi_{\varrho}(s)\right)'=z'(s)\mathrm{e}^{\phi_{\varrho}(s)}+z(s)\mathrm{e}^{\phi_{\varrho}(s)}\theta_{\varrho}(s).$$

Using (2.14), we see that

$$\left(z(s)\psi_{\varrho}(s)\right)'\leq 0.$$

End of proof. □

**Theorem 2.2.** *Let (2.2) hold. If*

$$\liminf_{s\to\infty}\int_{\beta(s)}^{s}\frac{\psi_{\varrho}(\beta(u))}{u^{2}}\int_{s_0}^{u}\frac{\varsigma q(\varsigma)(1-\mu(\beta(\varsigma)))\beta(\varsigma)}{\psi_{\varrho}(\beta(\varsigma))}\mathrm{d}\varsigma\mathrm{d}u>\frac{1}{\mathrm{e}}, \tag{2.15}$$

*then (1.1) is oscillatory, where* $\psi_{\varrho}(s)$ *is defined as in Lemma 2.2.*

*Proof.* Let (1.1) have a positive solution. From (2.11), we find

$$0\geq z'(s)+\frac{1}{s^{2}}\int_{s_0}^{s}\frac{\varsigma q(\varsigma)(1-\mu(\beta(\varsigma)))\beta(\varsigma)\psi_{\varrho}(\beta(\varsigma))}{\psi_{\varrho}(\beta(\varsigma))}z(\beta(\varsigma))\,\mathrm{d}\varsigma.$$

Using (2.13), we have

$$z'(s)+\left(\frac{\psi_{\varrho}(\beta(s))}{s^{2}}\int_{s_0}^{s}\frac{\varsigma q(\varsigma)(1-\mu(\beta(\varsigma)))\beta(\varsigma)}{\psi_{\varrho}(\beta(\varsigma))}\mathrm{d}\varsigma\right)z(\beta(s))\leq 0 \tag{2.16}$$

which a positive solution $z(s)$, and contradicts (2.15). End of proof. □

**Corollary 2.1.** *Let (2.2) hold and* $\lim_{s\to\infty}\psi_{\varrho}(\beta(s))/s=0$. *If*

$$\liminf_{s\to\infty}\int_{\beta(s)}^{s}\frac{\psi_{\varrho}(\beta(u))}{u^{2}}\int_{0}^{u}\frac{\varsigma q(\varsigma)(1-\mu(\beta(\varsigma)))\beta(\varsigma)}{\psi_{\varrho}(\beta(\varsigma))}\mathrm{d}\varsigma\mathrm{d}u>\frac{1}{\mathrm{e}}, \tag{2.17}$$

*then (1.1) is oscillatory, where* $\psi_{\varrho}(s)$ *is defined as in Lemma 2.2.*

*Proof.* This is similar to the proof of Theorem 2.1, and thus we omit it. □

**Lemma 2.3.** *Let (2.2) hold. Then*

$$x(\beta(s))\psi_2(s)\ \text{ is increasing,} \tag{2.18}$$

*where*

$$\psi_2(s)=q(s)\mathrm{e}^{-\phi_2(s)},\ \phi_2'(s)=\theta_2(s),$$

*and*

$$\begin{aligned}\theta_2(s) &= \frac{(q(s)(1-\mu(\beta(s))))'}{q(s)}+\frac{\beta'(s)(1-\mu(\beta(s)))}{\beta(s)}\int_{\beta(s)}^{s}q(\varsigma)(1-\mu(\beta(\varsigma)))\beta(\varsigma)\,\mathrm{d}\varsigma\\ &\quad+\beta'(s)(1-\mu(\beta(s)))\int_{s}^{\infty}q(\varsigma)(1-\mu(\beta(\varsigma)))\,\mathrm{d}\varsigma.\end{aligned}$$

*Proof.* Let (1.1) have a positive solution. From (1.1) and (2.6), we obtain

$$\mathcal{P}''(s) + q(s)(1-\mu(\beta(s)))\mathcal{P}(\beta(s)) \le 0. \tag{2.19}$$

Differentiating (2.19), we find

$$\mathcal{P}'''(s) + (q(s)(1-\mu(\beta(s))))'\mathcal{P}(\beta(s)) + q(s)(1-\mu(\beta(s)))\beta'(s)\mathcal{P}'(\beta(s)) \le 0. \tag{2.20}$$

Since $\mathcal{P}(s) \ge x(s)$, we see that (2.20) becomes

$$\mathcal{P}'''(s) + (q(s)(1-\mu(\beta(s))))' x(\beta(s)) + q(s)(1-\mu(\beta(s)))\beta'(s)\mathcal{P}'(\beta(s)) \le 0. \tag{2.21}$$

Integrating (2.19) from $\beta(s)$ to $\infty$, we have

$$\begin{aligned}\mathcal{P}'(\beta(s)) &\ge \int_{\beta(s)}^{s} q(\varsigma)(1-\mu(\beta(\varsigma)))\beta(\varsigma)\frac{\mathcal{P}(\beta(\varsigma))}{\beta(\varsigma)}\mathrm{d}\varsigma + \int_{s}^{\infty} q(\varsigma)(1-\mu(\beta(\varsigma)))\mathcal{P}(\beta(\varsigma))\,\mathrm{d}\varsigma \\ &\ge \frac{\mathcal{P}(\beta(s))}{\beta(s)}\int_{\beta(s)}^{s} q(\varsigma)(1-\mu(\beta(\varsigma)))\beta(\varsigma)\,\mathrm{d}\varsigma + \mathcal{P}(\beta(s))\int_{s}^{\infty} q(\varsigma)(1-\mu(\beta(\varsigma)))\,\mathrm{d}\varsigma,\end{aligned} \tag{2.22}$$

where we used (2.1) and (2.9). Since $\mathcal{P}(s) \ge x(s)$, we see that (2.22) becomes

$$\mathcal{P}'(\beta(s)) \ge \frac{x(\beta(s))}{\beta(s)}\int_{\beta(s)}^{s} q(\varsigma)(1-\mu(\beta(\varsigma)))\beta(\varsigma)\,\mathrm{d}\varsigma + x(\beta(s))\int_{s}^{\infty} q(\varsigma)(1-\mu(\beta(\varsigma)))\,\mathrm{d}\varsigma. \tag{2.23}$$

By using (2.21) and (2.23), we obtain

$$0 \ge \mathcal{P}'''(s) + q(s)\,x(\beta(s))\,\theta_2(s).$$

From (1.1) and the above inequality, we see that

$$0 \ge \mathcal{P}'''(s) - \mathcal{P}''(s)\,\theta_2(s).$$

Consequently,

$$\left(\mathrm{e}^{-\phi_2(s)}\mathcal{P}''(s)\right)' = \mathrm{e}^{-\phi_2(s)}(-\theta_2(s))\mathcal{P}''(s) + \mathrm{e}^{-\phi_2(s)}\mathcal{P}'''(s) = \mathrm{e}^{-\phi_2(s)}\left[-\theta_2(s)\mathcal{P}''(s) + \mathcal{P}'''(s)\right] \le 0;$$

hence, $\mathrm{e}^{-\phi_2(s)}\mathcal{P}''(s)$ is decreasing. In addition, we see that

$$(\psi_2(s)\,x(\beta(s)))' = \left(q(s)\,\mathrm{e}^{-\phi_2(s)}\left(-\frac{\mathcal{P}''(s)}{q(s)}\right)\right)' = \left(\mathrm{e}^{-\phi_2(s)}(-\mathcal{P}''(s))\right)' \ge 0.$$

End of proof. □

**Lemma 2.4.** *Let (2.2) hold. Then,*

$$\mathcal{P}(s) \ge s\int_{s}^{\infty} q(\varsigma)\,x(\beta(\varsigma))\,\mathrm{d}\varsigma + \int_{s_0}^{s} \varsigma q(\varsigma)\,x(\beta(\varsigma))\,\mathrm{d}\varsigma. \tag{2.24}$$

*Proof.* The proof is similar to that of [18, Lemma 3.2]. Therefore, it has been omitted. □

**Theorem 2.3.** *Let (2.2) hold. If*

$$\limsup_{s\to\infty}\Bigg[(1-\mu(\beta(s)))\Bigg(\beta(s)\psi_2(s)\int_s^\infty\frac{q(\varsigma)}{\psi_2(\varsigma)}\mathrm{d}\varsigma+\psi_\varrho(\beta(s))\int_{\beta(s)}^s\frac{q(\varsigma)(1-\mu(\beta(\varsigma)))\beta(\varsigma)}{\psi_\varrho(\beta(\varsigma))}\mathrm{d}\varsigma$$
$$+\frac{\psi_\varrho(\beta(s))}{\beta(s)}\int_{s_0}^{\beta(s)}\varsigma q(\varsigma)(1-\mu(\beta(\varsigma)))\frac{\beta(\varsigma)}{\psi_\varrho(\beta(\varsigma))}\mathrm{d}\varsigma\Bigg)\Bigg]>1, \tag{2.25}$$

*then (1.1) is oscillatory, where $\psi_\varrho(s)$ and $\psi_2(s)$ are defined as in Lemmas 2.2 and 2.3, respectively.*

*Proof.* Let (1.1) have a positive solution. From (2.24), we have

$$\begin{aligned}\mathcal{P}(\beta(s))&\geq\beta(s)\int_{\beta(s)}^\infty q(\varsigma)x(\beta(\varsigma))\mathrm{d}\varsigma+\int_{s_0}^{\beta(s)}\varsigma q(\varsigma)x(\beta(\varsigma))\mathrm{d}\varsigma\\&\geq\beta(s)\int_s^\infty q(\varsigma)x(\beta(\varsigma))\mathrm{d}\varsigma+\beta(s)\int_{\beta(s)}^s q(\varsigma)x(\beta(\varsigma))\mathrm{d}\varsigma+\int_{s_0}^{\beta(s)}\varsigma q(\varsigma)x(\beta(\varsigma))\mathrm{d}\varsigma.\end{aligned} \tag{2.26}$$

Using (2.6) in (2.26), we get

$$\begin{aligned}\mathcal{P}(\beta(s))\ \geq\ &\beta(s)\int_s^\infty q(\varsigma)x(\beta(\varsigma))\mathrm{d}\varsigma+\beta(s)\int_{\beta(s)}^s q(\varsigma)(1-\mu(\beta(\varsigma)))\mathcal{P}(\beta(\varsigma))\mathrm{d}\varsigma\\&+\int_{s_0}^{\beta(s)}\varsigma q(\varsigma)(1-\mu(\beta(\varsigma)))\mathcal{P}(\beta(\varsigma))\mathrm{d}\varsigma.\end{aligned}$$

Since $\beta(s)\leq s,\beta'(s)>0$, and using (2.18) and (2.13), we find

$$\begin{aligned}\mathcal{P}(\beta(s))\ \geq\ &x(\beta(s))\Bigg(\beta(s)\psi_2(s)\int_s^\infty\frac{q(\varsigma)}{\psi_2(\varsigma)}\mathrm{d}\varsigma+\psi_\varrho(\beta(s))\int_{\beta(s)}^s\frac{q(\varsigma)(1-\mu(\beta(\varsigma)))\beta(\varsigma)}{\psi_\varrho(\beta(\varsigma))}\mathrm{d}\varsigma\\&+\frac{\psi_\varrho(\beta(s))}{\beta(s)}\int_{s_0}^{\beta(s)}\frac{\varsigma q(\varsigma)(1-\mu(\beta(\varsigma)))\beta(\varsigma)}{\psi_\varrho(\beta(\varsigma))}\mathrm{d}\varsigma\Bigg).\end{aligned}$$

Using (2.6), we get

$$\begin{aligned}x(\beta(s))\ \geq\ &x(\beta(s))(1-\mu(\beta(s)))\Bigg(\beta(s)\psi_2(s)\int_s^\infty\frac{q(\varsigma)}{\psi_2(\varsigma)}\mathrm{d}\varsigma+\psi_\varrho(\beta(s))\int_{\beta(s)}^s\frac{(1-\mu(\beta(\varsigma)))\beta(\varsigma)}{q^{-1}(\varsigma)\psi_\varrho(\beta(\varsigma))}\mathrm{d}\varsigma\\&+\frac{\psi_\varrho(\beta(s))}{\beta(s)}\int_{s_0}^{\beta(s)}\frac{\varsigma q(\varsigma)(1-\mu(\beta(\varsigma)))\beta(\varsigma)}{\psi_\varrho(\beta(\varsigma))}\mathrm{d}\varsigma\Bigg),\end{aligned}$$

which contradicts (2.26). End of proof. □

**Example 2.1.** *Consider the following equation:*

$$(x(s)+\mu_0x(\delta s))''+\frac{q_0}{s^2}x(\gamma s)=0, \tag{2.27}$$

*where $q(s)=q_0/s^2$, $q_0>0$, $\mu(s)=\mu_0$, $\mu_0\in[0,1)$, $\beta(s)=\gamma s$, and $\alpha(s)=\delta s$. Now, we see that the condition (2.2) is satisfied,*

$$\theta_\varrho(s)=\frac{1}{s}q_0(1-\mu_0)\gamma,\ \phi_\varrho(s)=q_0(1-\mu_0)\gamma\ln s,\ \psi_\varrho(s)=s^{q_0(1-\mu_0)\gamma},$$

$$\theta_2(s) = \frac{-(1-\mu_0)2 + \gamma q_0 (1-\mu_0)^2 \ln\frac{1}{\gamma} + \gamma q_0 (1-\mu_0)^2}{s},$$

$$\phi_2(s) = \left(-(1-\mu_0)2 + \gamma q_0 (1-\mu_0)^2 \ln\frac{1}{\gamma} + \gamma q_0 (1-\mu_0)^2\right)\ln s,$$

*and*

$$\psi_2(s) = q_0 s^{(1-\mu_0)2-\gamma q_0(1-\mu_0)^2 \ln\frac{1}{\gamma} - \gamma q_0 (1-\mu_0)^2 - 2}.$$

*Thus, we find* $\lim_{s\to\infty} \psi_\varrho(\beta(s))/s = 0.$

*From Corollary 2.1, we find that (2.27) is oscillatory if*

$$\frac{q_0(1-\mu_0)\gamma}{1-q_0(1-\mu_0)\gamma} > \frac{1}{\mathrm{e}\ln\frac{1}{\gamma}}. \tag{2.28}$$

*From Theorem 2.3, we find that (2.27) is oscillatory if*

$$1 < (1-\mu_0)\left(\frac{\gamma q_0}{(1-\mu_0)2 - \gamma q_0(1-\mu_0)^2\ln\frac{1}{\gamma} - \gamma q_0(1-\mu_0)^2 - 1} + \left(\gamma^{-q_0(1-\mu_0)\gamma} - 1\right) + \frac{(1-\mu_0)q_0\gamma^{1-q_0(1-\mu_0)\gamma}}{1-q_0(1-\mu_0)\gamma}\right). \tag{2.29}$$

*Figure 1 shows some numerical oscillatory solutions to (2.27).*

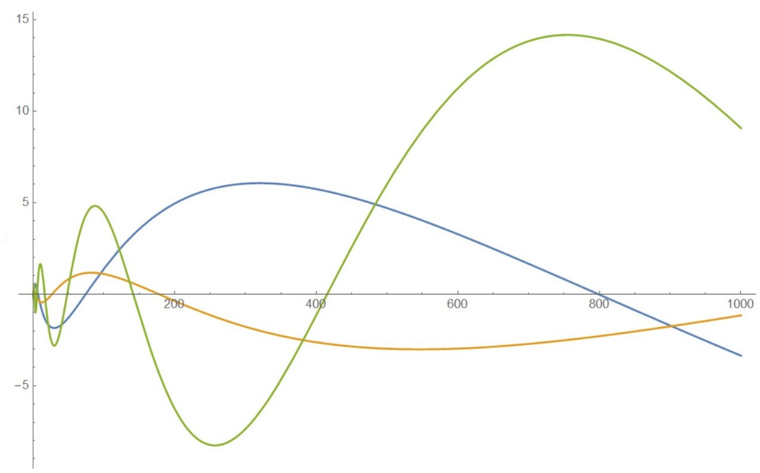


**Figure 1.** Some numerical oscillatory solutions to Eq (2.27).

**Remark 2.1.** *Let*

$$\left(x(s) + \frac{1}{10}x\left(\frac{1}{2}s\right)\right)'' + \frac{(1.05)}{s^2}x\left(\frac{1}{5}s\right) = 0 \tag{2.30}$$

*be a special case of the Eq (2.27), where* $q_0 = (1.05)$, $\mu_0 = 1/10$, $\delta = 1/2$, *and* $\gamma = 1/5$.

*Let* $\rho(s) = s$. *By applying Theorem 1 in [15], we find that the condition (1.3) is not satisfied, where*

$$(-0.305)\int_{s_0}^{\infty} \frac{1}{s}\mathrm{d}s \neq \infty.$$

*Thus, Theorem 1 in [15] fails to study the oscillation of Eq (2.30).*

*By applying Corollary 2.2 in [16], we find that the condition (1.5) is not satisfied, where*

$$0.361\,67 \ngtr \frac{1}{\mathrm{e}}.$$

*Thus, Corollary 2.2 in [16] fails to study the oscillation of Eq (2.30).*

*Now, we see that the condition (2.28) becomes*

$$\frac{(1.05)\left(1-\frac{1}{10}\right)\frac{1}{5}}{1-(1.05)\left(1-\frac{1}{10}\right)\frac{1}{5}} > \frac{1}{\mathrm{e}\ln 5}.$$

*By using Corollary 2.1, we find that (2.30) is oscillatory.*

*Additionally, condition (2.29) becomes*

$$1 < \left(\frac{9}{10}\right)\left(\frac{\frac{(1.05)}{5}}{\left(\frac{9}{10}\right)2-\frac{(1.05)}{5}\left(\frac{9}{10}\right)^2\ln 5-\frac{(1.05)}{5}\left(\frac{9}{10}\right)^2-1}+\left(\left(\frac{1}{5}\right)^{-\frac{(1.05)}{5}\left(\frac{9}{10}\right)}-1\right)+\frac{\left(\frac{9}{10}\right)(1.05)\left(\frac{1}{5}\right)^{1-\frac{(1.05)}{5}\left(\frac{9}{10}\right)}}{1-\frac{(1.05)}{5}\left(\frac{9}{10}\right)}\right).$$

*By using Theorem 2.3, we find that (2.30) is oscillatory.*

*Hence, and through the above, we find that the criteria we obtained produces results for the oscillation of Eq (2.30), while previous studies failed to study the oscillation of Eq (2.30).*

## 3. Conclusions

In this paper, the oscillatory behavior of (1.1) was studied. We succeeded in establishing new monotonic properties for the positive solutions of (1.1); from them, we obtained new oscillation criteria for (1.1). In addition, we provided an example and compared the results we obtained with some previous studies to show that the results we obtained improved these studies. As future work, we will try to extend our proposed results to third-order DEs.

## Author contributions

Abdullah Mohammed Alomair and Ali Muhib: Conceptualization, Methodology, Validation, Writing-original draft, Writing-review & editing. The authors contributed equally to this work.

## Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Conflict of interest

The authors declare no conflicts of interest.

## References

1. S. R. Grace, J. R. Graef, E. Tunc, Oscillatory behaviour of third order nonlinear differential equations with a nonlinear nonpositive neutral term, *J. Taibah Univ. Sci.*, **13** (2019), 704–710. https://doi.org/10.1080/16583655.2019.1622847
2. O. Moaaz, A. Muhib, S. Owyed, E. E. Mahmoud, A. Abdelnaser, Second-order neutral differential equations: improved criteria for testing the oscillation, *J. Math.*, **2021** (2021), 6665103. https://doi.org/10.1155/2021/6665103
3. A. Muhib, O. Moaaz, C. Cesarano, S. Askar, E. M. Elabbasy, Neutral differential equations of fourth-order: new asymptotic properties of solutions, *Axioms*, **11** (2022), 52. https://doi.org/10.3390/axioms11020052
4. R. P. Agarwal, M. Bohner, W. T. Li, *Nonoscillation and oscillation: theory for functional differential equations*, Boca Raton: CRC Press, 2004. https://doi.org/10.1201/9780203025741
5. L. Erbe, Q. Kong, B. G. Zhang, *Oscillation theory for functional differential equations*, New York: Routledge, 1995. https://doi.org/10.1201/9780203744727
6. I. Gyori, G. Ladas, *Oscillation theory of delay differential equations with applications*, Oxford: Oxford University Press, 1991. https://doi.org/10.1093/oso/9780198535829.001.0001
7. J. K. Hale, *Theory of functional differential equations*, New York: Springer, 1977. https://doi.org/10.1007/978-1-4612-9892-2
8. R. Agarwal, S. Shieh, C. Yeh, Oscillation criteria for second-order retarded differential equations, *Math. Comput. Model.*, **26** (1997), 1–11. https://doi.org/10.1016/S0895-7177(97)00141-6
9. T. Kusano, Y. Naito, Oscillation and nonoscillation criteria for second order quasilinear differential equations, *Acta Math. Hung.*, **76** (1997), 81–99. https://doi.org/10.1007/bf02907054
10. Y. G. Sun, F. W. Meng, Note on the paper of Dzurina and Stavroulakis, *Appl. Math. Comput.*, **174** (2006), 1634–1641. https://doi.org/10.1016/j.amc.2005.07.008
11. J. Dzurina, I. P. Stavroulakis, Oscillation criteria for second-order delay differential equations, *Appl. Math. Comput.*, **140** (2003), 445–453. https://doi.org/10.1016/S0096-3003(02)00243-6
12. B. Baculikova, Oscillatory behavior of the second order noncanonical differential equations, *Electron. J. Qual. Theory Differ. Equ.*, **2019** (2019), 1–11. https://doi.org/10.14232/ejqtde.2019.1.89
13. B. Baculikova, Oscillation of second-order nonlinear noncanonical differential equations with deviating argument, *Appl. Math. Lett.*, **91** (2019), 68–75. https://doi.org/10.1016/j.aml.2018.11.021
14. J. Dzurina, I. Jadlovska, A note on oscillation of second-order delay differential equations, *Appl. Math. Lett.*, **69** (2017), 126–132. https://doi.org/10.1016/j.aml.2017.02.003
15. Y. Wu, Y. Yu, J. Zhang, J. Xiao, Oscillation criteria for second order Emden-Fowler functional differential equations of neutral type, *J. Inequal. Appl.*, **2016** (2016), 328. https://doi.org/10.1186/s13660-016-1268-9
16. O. Moaaz, New criteria for oscillation of nonlinear neutral differential equations, *Adv. Differ. Equ.*, **2019** (2019), 484. https://doi.org/10.1186/s13662-019-2418-4

17. B. Baculikova, J. Dzurina, Oscillation theorems for second-order nonlinear neutral differential equations, *Comput. Math. Appl.*, **62** (2011), 4472–4478. https://doi.org/10.1016/j.camwa.2011.10.024

18. R. Koplatadze, G. Kvinkadze, I. P. Stavroulakis, Properties A and B of n-th order linear differential equations with deviating argument, *Gorgian Math. J.*, **6** (1999), 553–566. https://doi.org/10.1023/A:1022962129926