AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(1): 672–693.
DOI: 10.3934/math.2025030
Received: 18 November 2024
Revised: 28 December 2024
Accepted: 30 December 2024
Published: 13 January 2025

Research article

Single wave solutions of the fractional Landau-Ginzburg-Higgs equation in space-time with accuracy via the beta derivative and mEDAM approach

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Abstract: The nonlinear wave behavior in the tropical and mid-latitude troposphere has been simulated using the space-time fractional Landau-Ginzburg-Higgs model. These waves are the consequence of interactions between equatorial and mid-latitude waves, fluid flow in dynamic systems, weak scattering, and extended linkages. The mEDAM method has been used to obtain new and extended closed-form solitary wave solutions of the previously published nonlinear fractional partial differential equation via the beta derivative. A wave transformation converts the fractional-order equation into an ordinary differential equation. Several soliton, single, kink, double, triple, anti-kink, and other soliton types are examples of known conventional wave shapes. The answers are displayed using the latest Python code, which enhances the usage of 2D and 3D plotlines, as well as contour plotlines, to emphasise the tangible utility of the solutions. The results of the study are clear, flexible, and easier to replicate.

Keywords: beta derivative; modified extended direct algebraic method; space-time fractional Landau-Ginzburg-Higgs equation; soliton solutions
Mathematics Subject Classification: 65F10, 65H10, 90C30, 90C33

1. Introduction

Fractional calculus, a field of mathematical analysis that examines the multiple ways to define real number powers or complex number powers of the differentiation operator, has gained increasing attention in modeling numerous applications. Because fractional differential equations can explain memory and the inherited characteristics of diverse physical processes, they are able to describe a variety of anomalous physical events more precisely than integer-order partial differential equations [1]. Readers can consult [2–6] for more pertinent literature regarding theoretical analysis and numerical techniques for fractional differential equations. Recently, it was discovered that fractional-order behavior can change over time and space. This subject has drawn the attention of numerous scientists in recent decades in an effort to comprehend the complexity and nonlinearity of natural occurrences. Recently, there has been a significant increase in the use of nonlinear fractional partial differential equations (NLFPDEs) in a variety of fields, including meteorology, solid-state physics, control theory, signal transmission, bio-genetics, chemical kinematics, optical fibers, plasma physics, geochemistry, electromagnetics, gas dynamics, elasticity, ecosystems, oceanic spectacles, fluid mechanics, system identifications, landscape evolution, signal and image processing, quantum mechanics, and other domains [7]. The generalized exponential rational function approach [8], the extended tanh-function approach [9, 10], the generalized Kudryashov method [11], the local fractional variational iteration transform method [12], the first integral method [13], the modified auxiliary expansion technique [14], the Hirota-bilinear method [15], the Riemann-Hilbert approach [16], the modified mapping method [17], the double $(G'/G, 1/G)$ method [18], and other helpful approaches have thus been studied and employed to find exact solutions of NLFPDEs. The methods include the exp-function technique, the improved fractional sub-equation approach [19], the bilinear method [20], the Adomian's decomposition scheme [21], the homotopy analysis method [22], the sine-Gordon expansion method [23], the differential transform method [24], the improved Bernoulli subequation function technique [25], the anomalous dispersion relations approach [26], the new generalized (G'/G)-expansion approach [27], and so forth. The enhanced Bernoulli sub-equation function (IBSEF) approach is the most effective. An alternative Bernoulli variable algorithm was proposed after investigating the wave energy associated with Bernoulli structures. Another team introduced a novel class of higher-order Bernoulli polynomials, which enabled several new applications for the Bernoulli model. Consequently, the Bernoulli sub-equation function method, which yields solutions of the NLFPDEs that are both hyperbolic and exponential in nature, was modified to create the IBSEFM [28]. The space–time fractional Landau-Ginzburg-Higgs (LGH) equation provides a major explanation for nonlinear waves in the mid-latitude and tropical troposphere. Weak dispersion and long-term linkages are characteristics of these waves, which originate from the interactions between Rossby waves at mid-latitudes and the equator [29]. Numerous disciplines, including condensed matter physics, high-energy physics, and cosmology, greatly benefit from the use of the NLGHE. Its implications have proven crucial in understanding the complex dynamics of superconductors, superfluids, and Bose-Einstein condensates, as well as providing insight into the early cosmos and the development of cosmic strings [30].

2. Definition and foundational arrangement

A broader understanding of differentiation is provided by the beta fractional derivative, which takes into account the behavior of complicated physical systems with memory and non-local effects that are not explained by integer-order derivatives. Viscoelasticity, control systems, anomalous diffusion, fractals, and long-range correlated systems are a few examples of applications. In terms of physics, the scaling behavior departure of a system from classical equations or the extent to which the past impacts the present are typically indicated by the parameter β.

Given a function g, let $g : (s, \infty) \to R$. The following is the equation for the β derivative [31, 32]:

$$0^A Dt^{\beta}(g(t)) = \lim_{\epsilon \to 0} \frac{g(t + \epsilon(t + \frac{1}{\Gamma(\beta)}))}{\epsilon}, \quad \forall t > s, \quad 0 < \beta \leq 1. \tag{2.1}$$

The provided expression is therefore referred to as beta-differentiable if and only if its limit exists. Notably, the interval mentioned has no bearing on the criteria given above. The definition given at the zero point in the context of function differentiability is different from zero.

Theorem 1. *The following relations can be met if f is not zero and g are two functions that are beta-differentiable, with beta lying in the interval* $\in (0, 1)$.

$$ {}_0^A D_x^{\beta}(c_1 g(x) + c_2 f(x)) = c_1 {}_0^A D_x^{\beta}(g(x)) + c_2 {}_0^A D_x^{\beta}(f(x)), \forall c_1, c_2 \in R \tag{2.2}$$

$$ {}_0^A D_x^{\beta}(c) = 0, \forall c \in R \tag{2.3}$$

$$ {}_0^A D_x^{\beta}(g(x) f(x)) = f(x) {}_0^A D_x^{\beta}(g(x), \tag{2.4}$$

$$ {}_0^A D_x^{\beta}(\frac{g(x)}{f(x)}) = \frac{f(x) {}_0^A D_x^{\beta}(g(x) - g(c) {}_0^A D_x^{\beta})}{f^2(x)}, \tag{2.5}$$

$$ {}_0^A D_x^{\beta}(fog(x)) = \left(x + \frac{1}{\Gamma(\beta)}\right)^{1-\beta} g'(x) f'(x). \tag{2.6}$$

3. MEDAM operational procedure

The purpose of this section is to give an overview of the EDAM. Examine the FPDE using the format given below [33, 34]:

$$E(w, D_t^{\alpha} w, D_{h_1}^{\beta} w, D_{h_2}^{\gamma} w, w D_{h_1}^{\beta} w, \ldots) = 0, \ \ 0 < \alpha, \beta, \gamma \leq 1, \tag{3.1}$$

where $w = w(t, h_1, h_2, h_3, \ldots, h_n)$ is the given. It is possible to solve (3.1) by following the steps below:

Step 1. In order to create a variable in the form $w(t, h_1, h_2, h_3, \ldots, h_n) = W(\eta)$, (3.1) must first be transformed. η represents a function of $t, h_1, h_2, h_3, \ldots, h_n$ and can take many different forms. (3.1) becomes a NODE with the following structure after this change:

$$F(W, W', WW', \ldots) = 0. \tag{3.2}$$

The primes in (3.2) present derivatives. Equation (3.2) may occasionally be integrated one or more times.

Step 2. Therefore, it is believed that the following closed-form solution to (3.2) exists:

$$V(\Omega) = \sum_{l=-M}^{M} s_j (\zeta(\Omega))^j. \tag{3.3}$$

In this case, parameters that need approximations are represented by s_j. Moreover, an alternate NODE of the following type is satisfied by $\zeta(\Omega)$:

$$\zeta'(\Omega) = \ln(\mho)(d + e\zeta(\Omega) + f(\zeta(\Omega))^2), \tag{3.4}$$

since $\mho \neq 1, 0$ and d, e and f are constants.

Step 3. We discover a positive integer M, which is expressed in (3.3), when we search for the homogeneous balance between the most dominating nonlinear element and the largest-order derivative in (3.2).

Step 4. After evaluating the equation that arises from integrating (3.2) with (3.3), a polynomial in $\zeta(\Omega)$ is obtained by merging all of the components in $\zeta(\Omega)$ in the same order. Setting the coefficients of the consequent polynomial to zero for all s_j and other parameters results in an algebraic equation system.

Step 5. Prior to that, Maple may be used to solve this system of nonlinear algebraic equations.

Step 6. The traveling wave solutions to (3.1) are obtained using (3.3) and its related solution $\zeta(\Omega)$ from (3.4), together with the unknown parameters. The following families of traveling wave solutions can be produced using the generic solution of Eq (10) [35–38].

Family 1. If $f \neq 0$ and $\Lambda < 0$ are present:

$$\zeta_1(\Omega) = -\frac{e}{2f} + \frac{\sqrt{-\Lambda}\tan_{\mho}\left(\frac{1}{2}\sqrt{-\Lambda}\Omega\right)}{2f},$$

$$\zeta_2(\Omega) = -\frac{e}{2f} - \frac{\sqrt{-\Lambda}\cot_{\mho}\left(\frac{1}{2}\sqrt{-\Lambda}\Omega\right)}{2f},$$

$$\zeta_3(\Omega) = -\frac{e}{2f} + \frac{\sqrt{-\Lambda}\left(\tan_{\mho}\left(\sqrt{-\Lambda}\Omega\right) + \sec_{\mho}\left(\sqrt{-\Delta}\Omega\right)\right)}{2f},$$

$$\zeta_4(\Omega) = -\frac{e}{2f} - \frac{\sqrt{-\Delta}\left(\cot_{\mho}\left(\sqrt{-\Lambda}\Omega\right) + \csc_{\mho}\left(\sqrt{-\Lambda}\Omega\right)\right)}{2f},$$

and

$$\zeta_5(\Omega) = -\frac{e}{2f} + \frac{\sqrt{-\Lambda}\left(\tan_{\mho}\left(\frac{1}{4}\sqrt{-\Lambda}\Omega\right) - \cot_{\mho}\left(\frac{1}{4}\sqrt{-\Lambda}\Omega\right)\right)}{4f}.$$

Family 2. If $f \neq 0$ and $\Lambda < 0$ are present:

$$\zeta_6(\Omega) = -\frac{e}{2f} - \frac{\sqrt{\Lambda}\tanh_{\mho}\left(\frac{1}{2}\sqrt{\Lambda}\Omega\right)}{2f},$$

$$\zeta_7(\Omega) = -\frac{e}{2f} - \frac{\sqrt{\Lambda}\coth_{\mho}\left(\frac{1}{2}\sqrt{\Lambda}\Omega\right)}{2f},$$

$$\zeta_8(\Omega) = -\frac{e}{2f} - \frac{\sqrt{\Lambda}\left(\tanh_{\mho}\left(\sqrt{\Lambda}\Omega\right) + i\,\mathrm{sech}_{\mho}\left(\sqrt{\Lambda}\Omega\right)\right)}{2f},$$

$$\zeta_9(\Omega) = -\frac{e}{2f} - \frac{\sqrt{\Lambda}\left(\coth_{\mho}\left(\sqrt{\Lambda}\Omega\right) + \mathrm{csch}_{\mho}\left(\sqrt{\Lambda}\Omega\right)\right)}{2f},$$

and

$$\zeta_{10}(\Omega) = -\frac{e}{2f} - \frac{\sqrt{\Lambda}\left(\tanh_{\mho}\left(\frac{1}{4}\sqrt{\Delta}\Omega\right) - \coth_{\mho}\left(\frac{1}{4}\sqrt{\Lambda}\Omega\right)\right)}{4f}.$$

Family 3. For $df > 0$ and $e = 0$, we have

$$\zeta_{11}(\Omega) = \sqrt{\frac{d}{f}}\tan_{\mho}\left(\sqrt{df}\Omega\right),$$

$$\mu_{12}(\Omega) = -\sqrt{\frac{d}{f}}\cot_{\mho}\left(\sqrt{df}\Omega\right),$$

$$\zeta_{13}(\Omega) = \sqrt{\frac{d}{f}}\left(\tan_{\mho}\left(2\sqrt{df}\eta\right) + \sec_{\mho}\left(2\sqrt{df}\Omega\right)\right),$$

$$\zeta_{14}(\Omega) = -\sqrt{\frac{d}{f}}\left(\cot_{\mho}\left(2\sqrt{df}\Omega\right) + \csc_{\mho}\left(2\sqrt{df}\Omega\right)\right),$$

and

$$\zeta_{15}(\Omega) = \frac{1}{2}\sqrt{\frac{d}{f}}\left(\tan_{\mho}\left(\frac{1}{2}\sqrt{df}\Omega\right) - \cot_{\mho}\left(\frac{1}{2}\sqrt{df}\Omega\right)\right).$$

Family 4. For $df < 0$ and $e = 0$, we have

$$\zeta_{16}(\Omega) = -\sqrt{-\frac{d}{f}}\tanh_{\mho}\left(\sqrt{-df}\Omega\right),$$

$$\zeta_{17}(\Omega) = -\sqrt{-\frac{d}{f}}\coth_{\mho}\left(\sqrt{-df}\Omega\right),$$

$$\zeta_{18}(\Omega) = -\sqrt{-\frac{d}{f}}\left(\tanh_{\mho}\left(2\sqrt{-df}\Omega\right) + i\,\mathrm{sech}_{\mho}\left(2\sqrt{-df}\Omega\right)\right),$$

$$\zeta_{19}(\Omega) = -\sqrt{-\frac{d}{f}}\left(\coth_{\mho}\left(2\sqrt{-df}\Omega\right) + \mathrm{csch}_{\mho}\left(2\sqrt{-df}\Omega\right)\right),$$

and

$$\zeta_{20}(\Omega) = -\frac{1}{2}\sqrt{-\frac{d}{f}}\left(\tanh_{\mho}\left(\frac{1}{2}\sqrt{-df}\Omega\right) + \coth_{\mho}\left(\frac{1}{2}\sqrt{-df}\Omega\right)\right).$$

Family 5. For $f = d$ and $e = 0$, we have

$$\zeta_{21}(\Omega) = \tan_{\mho}(d\Omega),$$

$$\zeta_{22}(\eta) = -\cot_{\mho}(d\Omega),$$

$$\mu_{23}(\Omega) = \tan_{\mho}(2d\Omega) + \sec_{\mho}(2d\Omega),$$

$$\zeta_{24}(\Omega) = -\cot_{\mho}(2\,d\Omega) + \csc_{\mho}(2\,d\Omega)\,,$$

and

$$\zeta_{25}(\Omega) = \frac{1}{2}\tan_{\mho}\left(\frac{1}{2}\,d\Omega\right) - \frac{1}{2}\,\cot_{\mho}\left(\frac{1}{2}\,a\Omega\right).$$

Family 6. For $f = -d$ and $e = 0$, we have

$$\zeta_{26}(\Omega) = -\tanh_{\mho}(d\Omega)\,,$$

$$\zeta_{27}(\Omega) = -\coth_{\mho}(d\Omega)\,,$$

$$\zeta_{28}(\Omega) = -\tanh_{\mho}(2\,d\Omega) + i\,\mathrm{sech}_{\mho}(2\,d\Omega)\,,$$

$$\zeta_{29}(\Omega) = -\coth_{\mho}(2\,d\Omega) + \mathrm{csch}_{\mho}(2\,d\Omega)\,,$$

and

$$\zeta_{30}(\Omega) = -\frac{1}{2}\tanh_{\mho}\left(\frac{1}{2}\,d\Omega\right) - \frac{1}{2}\,\coth_{\mho}\left(\frac{1}{2}\,d\Omega\right).$$

Family 7. For $\Lambda = 0$, we have

$$\zeta_{31}(\Omega) = -2\,\frac{d\,(e\Omega\ln\mho + 2)}{e^2\ln(\mho)\Omega}.$$

Family 8. For $f = 0$, $e = \varsigma$, and $d = n\varsigma$ (with $n \neq 0$), we have

$$\zeta_{32}(\Omega) = \mho^{\varsigma\Omega} - n.$$

Family 9. For $e = f = 0$, we have

$$\zeta_{33}(\Omega) = d\Omega\ln(\mho).$$

Family 10. For $e = d = 0$, we have

$$\zeta_{34}(\Omega) = -\frac{1}{f\Omega\ln(\mho)}.$$

Family 11. For $e \neq 0$, $f \neq 0$, and $d = 0$, we have

$$\zeta_{35}(\Omega) = -\frac{e}{f\,(\cosh_{\mho}(e\Omega) - \sinh_{\mho}(e\Omega) + 1)},$$

and

$$\zeta_{36}(\Omega) = -\frac{e\,(\cosh_{\mho}(e\Omega) + \sinh_{\mho}(e\Omega))}{f\,(\cosh_{\mho}(e\Omega) + \sinh_{\mho}(e\Omega) + 1)}.$$

Family 12. For $e = \varsigma$, $f = n\varsigma$ (with $n \neq 0$), and $d = 0$:

$$\zeta_{37}(\Omega) = \frac{\mho^{\varsigma\Omega}}{1 - n\mho^{\varsigma\Omega}}.$$

In the above solutions, $\Lambda = e^2 - 4df$. The generalized trigonometric and hyperbolic functions are expressed as below:

$$\sin_{\mho}(\Omega) = \frac{\mho^{i\Omega} - \mho^{-i\Omega}}{2i}, \quad \cos_{\mho}(\Omega) = \frac{\mho^{i\Omega} + \mho^{-i\Omega}}{2},$$

$$\sec_{\mho}(\Omega) = \frac{1}{\cos_{\mho}(\Omega)}, \quad \csc_{\mho}(\Omega) = \frac{1}{\sin_{\mho}(\Omega)},$$

$$\tan_{\mho}(\Omega) = \frac{\sin_{\mho}(\Omega)}{\cos_{\mho}(\Omega)}, \quad \cot_{\mho}(\Omega) = \frac{\cos_{\mho}(\Omega)}{\sin_{\mho}(\Omega)}.$$

Similarly,

$$\sinh_{\mho}(\Omega) = \frac{\mho^{\Omega} - \mho^{-\Omega}}{2}, \quad \cosh_{\mho}(\eta) = \frac{\mho^{\Omega} + \mho^{-\Omega}}{2},$$
$$\mathrm{sech}_{\mho}(\Omega) = \frac{1}{\cosh_{\mho}(\Omega)}, \quad \mathrm{csch}_{\mho}(\eta) = \frac{1}{\sinh_{\mho}(\Omega)},$$
$$\tanh_{\mho}(\Omega) = \frac{\sinh_{\mho}(\Omega)}{\cosh_{\mho}(\Omega)}, \quad \coth_{\mho}(\Omega) = \frac{\cosh_{\mho}(\Omega)}{\sinh_{\mho}(\Omega)}.$$

4. Space-time fractional Higgs-Landau-Ginzburg equation

A typical method for describing the behavior of the order parameter in superconductivity is the space-time fractional LGH equation. In order to derive precise answers for this problem, we have examined the subsequent format [34]:

$$D_t^{2\theta}\varpi - D_x^{2\theta}\varpi - m^2\varpi + q^2\varpi^3 = 0, 0 < \theta < 1, t > 0, \tag{4.1}$$

where $m, q \neq 0$ and θ is the fractional-order derivative.

To reduce (4.1) to an ordinary differential equation (ODE), assume:

$$\begin{cases} \varpi(x,t) = \varpi(\Xi), \\ \Xi = \frac{n}{\theta}\left(x + \frac{1}{\Gamma(\theta)}\right)^{\theta} + \frac{l}{\theta}\left(t + \frac{1}{\Gamma(\theta)}\right)^{\theta}. \end{cases} \tag{4.2}$$

By utilizing (4.2), it is possible to transform (4.1) into an integer-order nonlinear ordinary differential equation (NLODE) written as:

$$l^2\varpi'' - n^2\varpi'' - m^2\varpi + q^2\varpi^3 = 0. \tag{4.3}$$

Rewrite (4.3) as

$$(l^2 - n^2)\varpi'' - m^2\varpi + q^2\varpi^3 = 0, \tag{4.4}$$

and as the homogenous principle is used, we have the following equation:

$$\varpi(\Xi) = \sum_{i=-1}^{1} d_i(\varpi(\Xi))^i. \tag{4.5}$$

By inserting (4.5) into (4.4) and collecting all terms with the same orders of $\varpi(\Xi)$, an equation in $G(\Xi)$ is obtained. By reducing the coefficients of the formula to zero, it can be simplified into a system of nonlinear algebraic equations. Two types of solutions are obtained when using Maple to tackle the resulting problem:

Case 1.

$$\begin{aligned} &l = l, m = \frac{1}{2}\sqrt{2\,n^2\Omega^2 + 8\,l^2\lambda\gamma - 2\,l^2\Omega^2 - 8\,n^2\lambda\gamma}\,\ln(\Pi), n = n, \\ &d_{-1} = -2\,\frac{\ln(\Pi)\,\lambda\left(l^2 - n^2\right)}{q\sqrt{-2\,l^2 + 2\,n^2}}, d_0 = \frac{1}{2}\frac{\sqrt{-2\,l^2 + 2\,n^2}\Omega\,\ln(\Pi)}{q}, \\ &d_1 = 0. \end{aligned} \tag{4.6}$$

Case 2.

$$
\begin{aligned}
&l = l, m = \frac{1}{2}\sqrt{-8\,n^2\lambda\,\gamma + 2\,n^2\Omega^2 + 8\,l^2\lambda\,\gamma - 2\,l^2\Omega^2}\ln(\Pi), n = n,\\
&d_{-1} = 0, d_0 = -\frac{\ln(\Pi)\,\Omega\left(l^2 - n^2\right)}{q\sqrt{2\,n^2 - 2\,l^2}}, d_1 = \frac{\sqrt{2\,n^2 - 2\,l^2}\gamma\,\ln(\Pi)}{q}.
\end{aligned}
\tag{4.7}
$$

The following families of soliton solutions for (4.4) result from assuming scenario 1.
Cluster 1.1. Let $\Delta < 0$ and $\gamma \neq 0$, and we have

$$
\begin{aligned}
\varpi_{1,1}(x,t) = &\frac{-2\,\ln(\Pi)\,\lambda\left(l^2 - n^2\right)}{q\sqrt{-2\,l^2 + 2\,n^2}}\left(-\frac{1}{2}\frac{\Omega}{\gamma} + \frac{1}{2}\frac{\sqrt{-\Delta}\tan_{\Pi}\left(\frac{1}{2}\sqrt{-\Delta}\Xi\right)}{\gamma}\right)^{-1}\\
&+ \frac{1}{2}\frac{\sqrt{-2\,l^2 + 2\,n^2}\Omega\,\ln(\Pi)}{q},
\end{aligned}
\tag{4.8}
$$

$$
\begin{aligned}
\varpi_{1,2}(x,t) = &\frac{-2\,\ln(\Pi)\,\lambda\left(l^2 - n^2\right)}{q\sqrt{-2\,l^2 + 2\,n^2}}\left(-\frac{1}{2}\frac{\Omega}{\gamma} - \frac{1}{2}\frac{\sqrt{-\Delta}\cot_{\Pi}\left(\frac{1}{2}\sqrt{-\Delta}\Xi\right)}{\gamma}\right)^{-1}\\
&+ \frac{1}{2}\frac{\sqrt{-2\,l^2 + 2\,n^2}\Omega\,\ln(\Pi)}{q},
\end{aligned}
\tag{4.9}
$$

$$
\begin{aligned}
\varpi_{1,3}(x,t) = &\frac{-2\,\ln(\Pi)\,\lambda\left(l^2 - n^2\right)}{q\sqrt{-2\,l^2 + 2\,n^2}}\left(-\frac{1}{2}\frac{\Omega}{\gamma} + \frac{1}{2}\frac{\sqrt{-\Delta}\left(\tan_{\Pi}\left(\sqrt{-\Delta}\Xi\right) \pm \left(\sec_{\Pi}\left(\sqrt{-\Delta}\Xi\right)\right)\right)}{\gamma}\right)^{-1}\\
&+ \frac{1}{2}\frac{\sqrt{-2\,l^2 + 2\,n^2}\Omega\,\ln(\Pi)}{q},
\end{aligned}
\tag{4.10}
$$

$$
\begin{aligned}
\varpi_{1,4}(x,t) = &\frac{-2\,\ln(\Pi)\,\lambda\left(l^2 - n^2\right)}{q\sqrt{-2\,l^2 + 2\,n^2}}\left(-\frac{1}{2}\frac{\Omega}{\gamma} + \frac{1}{2}\frac{\sqrt{-\Delta}\left(\cot_{\Pi}\left(\sqrt{-\Delta}\Xi\right) \pm \left(\csc_{\Pi}\left(\sqrt{-\Delta}\Xi\right)\right)\right)}{\gamma}\right)^{-1}\\
&+ \frac{1}{2}\frac{\sqrt{-2\,l^2 + 2\,n^2}\Omega\,\ln(\Pi)}{q},
\end{aligned}
\tag{4.11}
$$

and

$$
\begin{aligned}
\varpi_{1,5}(x,t) = &\frac{-2\,\ln(\Pi)\,\lambda\left(l^2 - n^2\right)}{q\sqrt{-2\,l^2 + 2\,n^2}}\left(-\frac{1}{2}\frac{\Omega}{\gamma} + \frac{1}{4}\frac{\sqrt{-\Delta}\left(\tan_{\Pi}\left(\frac{1}{4}\sqrt{-\Delta}\Xi\right) - \cot_{\Pi}\left(\frac{1}{4}\sqrt{-\Delta}\Xi\right)\right)}{\gamma}\right)^{-1}\\
&+ \frac{1}{2}\frac{\sqrt{-2\,l^2 + 2\,n^2}\Omega\,\ln(\Pi)}{q}.
\end{aligned}
\tag{4.12}
$$

Cluster 1.2. Let $\Delta > 0$ and $\gamma \neq 0$, and we have

$$
\begin{aligned}
\varpi_{1,6}(x,t) = &\frac{-2\,\ln(\Pi)\,\lambda\left(l^2 - n^2\right)}{q\sqrt{-2\,l^2 + 2\,n^2}}\left(-\frac{1}{2}\frac{\Omega}{\gamma} - \frac{1}{2}\frac{\sqrt{\Delta}\tanh_{\Pi}\left(\frac{1}{2}\sqrt{-\Delta}\Xi\right)}{\gamma}\right)^{-1}\\
&+ \frac{1}{2}\frac{\sqrt{-2\,l^2 + 2\,n^2}\Omega\,\ln(\Pi)}{q},
\end{aligned}
\tag{4.13}
$$

$$\varpi_{1,7}(x,t) = \frac{-2\,\ln(\Pi)\,\lambda\,\left(l^2-n^2\right)}{q\sqrt{-2\,l^2+2\,n^2}}\left(-\frac{1}{2}\frac{\Omega}{\gamma}-\frac{1}{2}\frac{\sqrt{\Delta}\coth_{\Pi}\left(\frac{1}{2}\sqrt{-\Delta}\Xi\right)}{\gamma}\right)^{-1} + \frac{1}{2}\frac{\sqrt{-2\,l^2+2\,n^2}\Omega\,\ln(\Pi)}{q}, \tag{4.14}$$

$$\varpi_{1,8}(x,t) = \frac{-2\,\ln(\Pi)\,\lambda\,\left(l^2-n^2\right)}{q\sqrt{-2\,l^2+2\,n^2}}\left(-\frac{1}{2}\frac{\Omega}{\gamma}-\frac{1}{2}\frac{\sqrt{\Delta}\left(\tanh_{\Pi}\left(\sqrt{-\Delta}\Xi\right)\pm\left(\mathrm{sech}_{\Pi}\left(\sqrt{-\Delta}\Xi\right)\right)\right)}{\gamma}\right)^{-1} + \frac{1}{2}\frac{\sqrt{-2\,l^2+2\,n^2}\Omega\,\ln(\Pi)}{q}, \tag{4.15}$$

$$\varpi_{1,9}(x,t) = \frac{-2\,\ln(\Pi)\,\lambda\,\left(l^2-n^2\right)}{q\sqrt{-2\,l^2+2\,n^2}}\left(-\frac{1}{2}\frac{\Omega}{\gamma}-\frac{1}{2}\frac{\sqrt{\Delta}\left(\coth_{\Pi}\left(\sqrt{-\Delta}\Xi\right)\pm\left(\mathrm{sech}_{\Pi}\left(\sqrt{-\Delta}\Xi\right)\right)\right)}{\gamma}\right)^{-1} + \frac{1}{2}\frac{\sqrt{-2\,l^2+2\,n^2}\Omega\,\ln(\Pi)}{q}, \tag{4.16}$$

and

$$\varpi_{1,10}(x,t) = \frac{-2\,\ln(\Pi)\,\lambda\,\left(l^2-n^2\right)}{q\sqrt{-2\,l^2+2\,n^2}}\left(-\frac{1}{2}\frac{\Omega}{\gamma}-\frac{1}{4}\frac{\sqrt{\Delta}\left(\tanh_{\Pi}\left(\frac{1}{2}\sqrt{\Delta}\Xi\right)-\coth_{\Pi}\left(\frac{1}{4}\sqrt{\Delta}\Xi\right)\right)}{\gamma}\right)^{-1} + \frac{1}{2}\frac{\sqrt{-2\,l^2+2\,n^2}\Omega\,\ln(\Pi)}{q}. \tag{4.17}$$

Cluster 1.3. When $\gamma\lambda > 0$ and $\Omega = 0$, we have

$$\varpi_{1,11}(x,t) = -2\,\ln(\Pi)\,\lambda\,\left(l^2-n^2\right)\frac{1}{\sqrt{\frac{\lambda}{\gamma}}}\left(\tan_{\Pi}\left(\sqrt{\gamma\,\lambda}\Xi\right)\right)^{-1}\frac{1}{q\sqrt{-2\,l^2+2\,n^2}}, \tag{4.18}$$

$$\varpi_{1,12}(x,t) = 2\,\ln(\Pi)\,\lambda\,\left(l^2-n^2\right)\frac{1}{\sqrt{\frac{\lambda}{\gamma}}}\left(\cot_{\Pi}\left(\sqrt{\gamma\,\lambda}\Xi\right)\right)^{-1}\frac{1}{q\sqrt{-2\,l^2+2\,n^2}}, \tag{4.19}$$

$$\varpi_{1,13}(x,t) = -2\,\ln(\Pi)\,\lambda\,\left(l^2-n^2\right)\frac{1}{\sqrt{\frac{\lambda}{\gamma}}}\left(\tan_{\Pi}\left(2\sqrt{\gamma\,\lambda}\Xi\right)\pm\left(\sec_{\Pi}\left(2\sqrt{\gamma\,\lambda}\Xi\right)\right)\right)^{-1}\frac{1}{q\sqrt{-2\,l^2+2\,n^2}}, \tag{4.20}$$

$$\varpi_{1,14}(x,t) = 2\,\ln(\Pi)\,\lambda\,\left(l^2-n^2\right)\frac{1}{\sqrt{\frac{\lambda}{\gamma}}}\left(\cot_{\Pi}\left(2\sqrt{\gamma\,\lambda}\Xi\right)\pm\left(\csc_{\Pi}\left(2\sqrt{\gamma\,\lambda}\Xi\right)\right)\right)^{-1}\frac{1}{q\sqrt{-2\,l^2+2\,n^2}}, \tag{4.21}$$

and

$$\varpi_{1,15}(x,t) = -4\,\ln(\Pi)\,\lambda\,\left(l^2-n^2\right)\frac{1}{\sqrt{\frac{\lambda}{\gamma}}}\left(\tan_{\Pi}\left(\frac{1}{2}\sqrt{\gamma\,\lambda}\Xi\right)-\cot_{\Pi}\left(\frac{1}{2}\sqrt{\gamma\,\lambda}\Xi\right)\right)^{-1}\frac{1}{q\sqrt{-2\,l^2+2\,n^2}}. \tag{4.22}$$

Cluster 1.4. Let $\gamma\lambda < 0$ and $\Omega = 0$, and we have

$$\varpi_{1,16}(x,t) = 2\,\ln(\Pi)\,\lambda\left(l^2 - n^2\right)\frac{1}{\sqrt{-\frac{\lambda}{\gamma}}}\left(\tanh_\Pi\left(\sqrt{-\gamma\,\lambda}\Xi\right)\right)^{-1}\frac{1}{q\sqrt{-2\,l^2+2\,n^2}}, \tag{4.23}$$

$$\varpi_{1,17}(x,t) = 2\,\ln(\Pi)\,\lambda\left(l^2 - n^2\right)\frac{1}{\sqrt{-\frac{\lambda}{\gamma}}}\left(\coth_\Pi\left(\sqrt{-\gamma\,\lambda}\Xi\right)\right)^{-1}\frac{1}{q\sqrt{-2\,l^2+2\,n^2}}, \tag{4.24}$$

$$\varpi_{1,18}(x,t) = 2\,\ln(\Pi)\,\lambda\left(l^2 - n^2\right)\frac{1}{\sqrt{-\frac{\lambda}{\gamma}}}\left(\tanh_\Pi\left(2\,\sqrt{-\gamma\,\lambda}\Xi\right) \pm \left(i\mathrm{sech}_\Pi\left(2\,\sqrt{-\gamma\,\lambda}\Xi\right)\right)\right)^{-1}\frac{1}{q\sqrt{-2\,l^2+2\,n^2}}, \tag{4.25}$$

$$\varpi_{1,19}(x,t) = 2\,\ln(\Pi)\,\lambda\left(l^2 - n^2\right)\frac{1}{\sqrt{-\frac{\lambda}{\gamma}}}\left(\coth_\Pi\left(2\,\sqrt{-\gamma\,\lambda}\Xi\right) \pm \left(\mathrm{csch}_\Pi\left(2\,\sqrt{-\gamma\,\lambda}\Xi\right)\right)\right)^{-1}\frac{1}{q\sqrt{-2\,l^2+2\,n^2}}, \tag{4.26}$$

and

$$\varpi_{1,20}(x,t) = 4\,\ln(\Pi)\,\lambda\left(l^2 - n^2\right)\frac{1}{\sqrt{-\frac{\lambda}{\gamma}}}\left(\tanh_\Pi\left(\frac{1}{2}\,\sqrt{-\gamma\,\lambda}\Xi\right) + \coth_\Pi\left(\frac{1}{2}\,\sqrt{-\gamma\,\lambda}\Xi\right)\right)^{-1}\frac{1}{q\sqrt{-2\,l^2+2\,n^2}}. \tag{4.27}$$

Cluster 1.5. Suppose $\gamma = \lambda$ and $\Omega = 0$, and we have

$$\varpi_{1,21}(x,t) = -2\,\frac{\ln(\Pi)\,\lambda\left(l^2-n^2\right)}{q\sqrt{-2\,l^2+2\,n^2}\tan_\Pi(\lambda\,\Xi)}, \tag{4.28}$$

$$\varpi_{1,22}(x,t) = 2\,\frac{\ln(\Pi)\,\lambda\left(l^2-n^2\right)}{\cot_\Pi(\lambda\,\Xi)\,q\sqrt{-2\,l^2+2\,n^2}}, \tag{4.29}$$

$$\varpi_{1,23}(x,t) = -2\,\frac{\ln(\Pi)\,\lambda\left(l^2-n^2\right)}{q\sqrt{-2\,l^2+2\,n^2}\left(\tan_\Pi(2\,\lambda\,\Xi) \pm (\sec_\Pi(2\,\lambda\,\Xi))\right)}, \tag{4.30}$$

$$\varpi_{1,24}(x,t) = -2\,\frac{\ln(\Pi)\,\lambda\left(l^2-n^2\right)}{q\sqrt{-2\,l^2+2\,n^2}\left(-\cot_\Pi(2\,\lambda\,\Xi) \pm (\csc_\Pi(2\,\lambda\,\Xi))\right)}, \tag{4.31}$$

and

$$\varpi_{1,25}(x,t) = -2\,\frac{\ln(\Pi)\,\lambda\left(l^2-n^2\right)}{q\sqrt{-2\,l^2+2\,n^2}\left(\frac{1}{2}\tan_\Pi\left(\frac{1}{2}\lambda\,\Xi\right) - \frac{1}{2}\cot_\Pi\left(\frac{1}{2},\lambda\,\Xi\right)\right)}. \tag{4.32}$$

Cluster 1.6. When $\gamma = -\lambda$ and $\Omega = 0$, we have

$$\varpi_{1,26}(x,t) = 2\,\frac{\ln(\Pi)\,\lambda\left(l^2-n^2\right)}{\tanh_\Pi(\lambda\,\Xi)\,q\sqrt{-2\,l^2+2\,n^2}}, \tag{4.33}$$

$$\varpi_{1,27}(x,t) = 2\,\frac{\ln(\Pi)\,\lambda\left(l^2-n^2\right)}{\coth_\Pi(\lambda\,\Xi)\,q\,\sqrt{-2\,l^2+2\,n^2}}, \tag{4.34}$$

$$\varpi_{1,28}(x,t) = -2\,\frac{\ln(\Pi)\,\lambda\left(l^2-n^2\right)}{q\,\sqrt{-2\,l^2+2\,n^2}\,(-\tanh_\Pi(2\,\lambda\,\Xi)\pm(i\mathrm{sech}_\Pi(2\,\lambda\,\Xi)))}, \tag{4.35}$$

$$\varpi_{1,29}(x,t) = -2\,\frac{\ln(\Pi)\,\lambda\left(l^2-n^2\right)}{q\,\sqrt{-2\,l^2+2\,n^2}\,(-\coth_\Pi(2\,\lambda\,\Xi)\pm(\mathrm{csch}_\Pi(2\,\lambda\,\Xi)))}, \tag{4.36}$$

and

$$\varpi_{1,30}(x,t) = -2\,\frac{\ln(\Pi)\,\lambda\left(l^2-n^2\right)}{q\,\sqrt{-2\,l^2+2\,n^2}\left(-\frac{1}{2}\tanh_\Pi\left(\frac{1}{2}\lambda\,\Xi\right)-\frac{1}{2}\coth_\Pi\left(\frac{1}{2},\lambda\,\Xi\right)\right)}. \tag{4.37}$$

Cluster 1.7. When $\Delta = 0$, we have

$$\varpi_{1,31}(x,t) = \frac{(\ln(\Pi))^2\,\lambda\left(l^2-n^2\right)\Omega^2\Xi}{\lambda\,(\Omega\,\Xi\,\ln(\Pi)+2)\,q\,\sqrt{-2\,l^2+2\,n^2}} + \frac{1}{2}\,\frac{\sqrt{-2\,l^2+2\,n^2}\Omega\,\ln(\Pi)}{q}. \tag{4.38}$$

Cluster 1.8. When $\Omega := \rho, \lambda := \eta\rho(\eta \neq 0)$, and $\gamma := 0$,

$$\varpi_{1,32}(x,t) = -2\,\frac{\ln(\Pi)\,\eta\,\rho\left(l^2-n^2\right)}{q\,\sqrt{-2\,l^2+2\,n^2}\,(\Pi^{\rho\,\Xi}-\eta)} + \frac{1}{2}\,\frac{\sqrt{-2\,l^2+2\,n^2}\rho\,\ln(\Pi)}{q}. \tag{4.39}$$

Cluster 1.9. When $\Omega := 0$ and $\gamma := 0$, we have

$$\varpi_{1,33}(x,t) = -2\,\frac{\eta\left(l^2-n^2\right)}{\lambda\,\Xi\,q\,\sqrt{-2\,l^2+2\,n^2}}. \tag{4.40}$$

Cluster 1.10. When $\lambda := 0, \Omega \neq 0$, and $\gamma \neq 0$,

$$\begin{aligned}\varpi_{1,35}(x,t) = {} & 2\,\frac{\ln(\Pi)\,\lambda\left(l^2-n^2\right)\gamma\,(\cosh(\Omega\,\Xi)-\sinh(\Omega\,\Xi)+1)}{\Omega\,q\,\sqrt{-2\,l^2+2\,n^2}} \\ & + \frac{1}{2}\,\frac{\sqrt{-2\,l^2+2\,n^2}\Omega\,\ln(\Pi)}{q},\end{aligned} \tag{4.41}$$

$$\varpi_{1,36}(x,t) = 2\,\frac{\ln(\Pi)\,\lambda\left(l^2-n^2\right)\gamma}{\Omega\,q\,\sqrt{-2\,l^2+2\,n^2}} + \frac{1}{2}\,\frac{\sqrt{-2\,l^2+2\,n^2}\Omega\,\ln(\Pi)}{q}. \tag{4.42}$$

Cluster 1.11. When $\Omega := \rho, \gamma := \eta\rho$, and $\lambda := 0$, we have

$$\varpi_{1,37}(x,t) = \frac{1}{2}\,\frac{\sqrt{-2\,l^2+2\,n^2}\rho\,\ln(\Pi)}{q}. \tag{4.43}$$

Considering scenario number two, we obtain the subsequent sets of soliton solutions for Eq (4.4):

Household 2.1. When $\Delta < 0$ and $\gamma \neq 0$, we have

$$\varpi_{2,1}(x,t) = \frac{\sqrt{2\,n^2 - 2\,l^2}\gamma\,\ln(\Pi)}{q}\left(-\frac{1}{2}\frac{\Omega}{\gamma} + \frac{1}{2}\frac{\sqrt{-\Delta}\tan_{\Pi}\left(\frac{1}{2}\sqrt{-\Delta}\Xi\right)}{\gamma}\right) - \frac{\ln(\Pi)\,\Omega\left(l^2 - n^2\right)}{q\sqrt{2\,n^2 - 2\,l^2}}, \tag{4.44}$$

$$\varpi_{2,2}(x,t) = \frac{\sqrt{2\,n^2 - 2\,l^2}\gamma\,\ln(\Pi)}{q}\left(-\frac{1}{2}\frac{\Omega}{\gamma} - \frac{1}{2}\frac{\sqrt{-\Delta}\cot_{\Pi}\left(\frac{1}{2}\sqrt{-\Delta}\Xi\right)}{\gamma}\right) - \frac{\ln(\Pi)\,\Omega\left(l^2 - n^2\right)}{q\sqrt{2\,n^2 - 2\,l^2}}, \tag{4.45}$$

$$\varpi_{2,3}(x,t) = \frac{\sqrt{2\,n^2 - 2\,l^2}\gamma\,\ln(\Pi)}{q}\left(-\frac{1}{2}\frac{\Omega}{\gamma} + \frac{1}{2}\frac{\sqrt{-\Delta}\left(\tan_{\Pi}\left(\sqrt{-\Delta}\Xi\right) \pm \left(\sec_{\Pi}\left(\sqrt{-\Delta}\Xi\right)\right)\right)}{\gamma}\right) - \frac{\ln(\Pi)\,\Omega\left(l^2 - n^2\right)}{q\sqrt{2\,n^2 - 2\,l^2}}, \tag{4.46}$$

$$\varpi_{2,4}(x,t) = \frac{\sqrt{2\,n^2 - 2\,l^2}\gamma\,\ln(\Pi)}{q}\left(-\frac{1}{2}\frac{\Omega}{\gamma} + \frac{1}{2}\frac{\sqrt{-\Delta}\left(\cot_{\Pi}\left(\sqrt{-\Delta}\Xi\right) \pm \left(\csc_{\Pi}\left(\sqrt{-\Delta}\Xi\right)\right)\right)}{\gamma}\right) - \frac{\ln(\Pi)\,\Omega\left(l^2 - n^2\right)}{q\sqrt{2\,n^2 - 2\,l^2}}, \tag{4.47}$$

and

$$\varpi_{2,5}(x,t) = \frac{\sqrt{2\,n^2 - 2\,l^2}\gamma\,\ln(\Pi)}{q}\left(-\frac{1}{2}\frac{\Omega}{\gamma} + \frac{1}{4}\frac{\sqrt{-\Delta}\left(\tan_{\Pi}\left(\frac{1}{4}\sqrt{-\Delta}\Xi\right) - \cot_{\Pi}\left(\frac{1}{4}\sqrt{-\Delta}\Xi\right)\right)}{\gamma}\right) - \frac{\ln(\Pi)\,\Omega\left(l^2 - n^2\right)}{q\sqrt{2\,n^2 - 2\,l^2}}. \tag{4.48}$$

Household 2.2. When $\Delta > 0$ and $\gamma \neq 0$, we have

$$\varpi_{2,6}(x,t) = \frac{\sqrt{2\,n^2 - 2\,l^2}\gamma\,\ln(\Pi)}{q}\left(-\frac{1}{2}\frac{\Omega}{\gamma} - \frac{1}{2}\frac{\sqrt{\Delta}\tanh_{\Pi}\left(\frac{1}{2}\sqrt{-\Delta}\Xi\right)}{\gamma}\right) - \frac{\ln(\Pi)\,\Omega\left(l^2 - n^2\right)}{q\sqrt{2\,n^2 - 2\,l^2}}, \tag{4.49}$$

$$\varpi_{2,7}(x,t) = \frac{\sqrt{2\,n^2 - 2\,l^2}\gamma\,\ln(\Pi)}{q}\left(-\frac{1}{2}\frac{\Omega}{\gamma} - \frac{1}{2}\frac{\sqrt{\Delta}\coth_{\Pi}\left(\frac{1}{2}\sqrt{-\Delta}\Xi\right)}{\gamma}\right) - \frac{\ln(\Pi)\,\Omega\left(l^2 - n^2\right)}{q\sqrt{2\,n^2 - 2\,l^2}}, \tag{4.50}$$

$$\varpi_{2,8}(x,t) = \frac{\sqrt{2\,n^2-2\,l^2}\gamma\,\ln(\Pi)}{q}\left(-\frac{1}{2}\frac{\Omega}{\gamma}-\frac{1}{2}\frac{\sqrt{\Delta}\left(\tanh_\Pi\left(\sqrt{-\Delta}\Xi\right)\pm\left(\operatorname{sech}_\Pi\left(\sqrt{-\Delta}\Xi\right)\right)\right)}{\gamma}\right) - \frac{\ln(\Pi)\,\Omega\left(l^2-n^2\right)}{q\sqrt{2\,n^2-2\,l^2}}, \tag{4.51}$$

$$\varpi_{2,9}(x,t) = \frac{\sqrt{2\,n^2-2\,l^2}\gamma\,\ln(\Pi)}{q}\left(-\frac{1}{2}\frac{\Omega}{\gamma}-\frac{1}{2}\frac{\sqrt{\Delta}\left(\coth_\Pi\left(\sqrt{-\Delta}\Xi\right)\pm\left(\operatorname{sech}_\Pi\left(\sqrt{-\Delta}\Xi\right)\right)\right)}{\gamma}\right) - \frac{\ln(\Pi)\,\Omega\left(l^2-n^2\right)}{q\sqrt{2\,n^2-2\,l^2}}, \tag{4.52}$$

and

$$\varpi_{2,10}(x,t) = \frac{\sqrt{2\,n^2-2\,l^2}\gamma\,\ln(\Pi)}{q}\left(-\frac{1}{2}\frac{\Omega}{\gamma}-\frac{1}{4}\frac{\sqrt{\Delta}\left(\tanh_\Pi\left(\frac{1}{4}\sqrt{\Delta}\Xi\right)-\coth_\Pi\left(\frac{1}{4}\sqrt{\Delta}\Xi\right)\right)}{\gamma}\right) - \frac{\ln(\Pi)\,\Omega\left(l^2-n^2\right)}{q\sqrt{2\,n^2-2\,l^2}}. \tag{4.53}$$

Household 2.3. When $\gamma\lambda > 0$ and $\Omega = 0$, we have

$$\varpi_{2,11}(x,t) = \sqrt{2\,n^2-2\,l^2}\gamma\,\ln(\Pi)\sqrt{\frac{\lambda}{\gamma}}\tan_\Pi\left(\sqrt{\gamma\,\lambda}\Xi\right)q^{-1}, \tag{4.54}$$

$$\varpi_{2,12}(x,t) = -\frac{\sqrt{2\,n^2-2\,l^2}\gamma\,\ln(\Pi)}{q}\sqrt{\frac{\lambda}{\gamma}}\cot_\Pi\left(\sqrt{\gamma\,\lambda}\Xi\right), \tag{4.55}$$

$$\varpi_{2,13}(x,t) = \frac{\sqrt{2\,n^2-2\,l^2}\gamma\,\ln(\Pi)}{q}\sqrt{\frac{\lambda}{\gamma}}\left(\tan_\Pi\left(2\sqrt{\gamma\,\lambda}\Xi\right)\pm\left(\sec_\Pi\left(2\sqrt{\gamma\,\lambda}\Xi\right)\right)\right), \tag{4.56}$$

$$\varpi_{2,14}(x,t) = -\frac{\sqrt{2\,n^2-2\,l^2}\gamma\,\ln(\Pi)}{q}\sqrt{\frac{\lambda}{\gamma}}\left(\cot_\Pi\left(2\sqrt{\gamma\,\lambda}\Xi\right)\pm\left(\csc_\Pi\left(2\sqrt{\gamma\,\lambda}\Xi\right)\right)\right), \tag{4.57}$$

and

$$\varpi_{2,15}(x,t) = \frac{\sqrt{2\,n^2-2\,l^2}\gamma\,\ln(\Pi)}{q}\sqrt{\frac{\lambda}{\gamma}}\left(\tan_\Pi\left(\frac{1}{2}\sqrt{\gamma\,\lambda}\Xi\right)-\cot_\Pi\left(\frac{1}{2}\sqrt{\gamma\,\lambda}\Xi\right)\right). \tag{4.58}$$

Household 2.4. When $\gamma\lambda < 0$ and $\Omega = 0$, we have

$$\varpi_{2,16}(x,t) = -\frac{\sqrt{2\,n^2-2\,l^2}\gamma\,\ln(\Pi)}{q}\sqrt{-\frac{\lambda}{\gamma}}\tanh_\Pi\left(\sqrt{-\gamma\,\lambda}\Xi\right), \tag{4.59}$$

$$\varpi_{2,17}(x,t) = -\frac{\sqrt{2\,n^2-2\,l^2}\gamma\,\ln(\Pi)}{q}\sqrt{-\frac{\lambda}{\gamma}}\coth_\Pi\left(\sqrt{-\gamma\,\lambda}\Xi\right), \tag{4.60}$$

$$\varpi_{2,18}(x,t) = -\frac{\sqrt{2\,n^2 - 2\,l^2}\gamma\,\ln(\Pi)}{q}\sqrt{-\frac{\lambda}{\gamma}}\left(\tanh_\Pi\left(2\,\sqrt{-\gamma\,\lambda}\Xi\right) \pm \left(i\mathrm{sech}_\Pi\left(2\,\sqrt{-\gamma\,\lambda}\Xi\right)\right)\right), \tag{4.61}$$

$$\varpi_{2,19}(x,t) = -\frac{\sqrt{2\,n^2 - 2\,l^2}\gamma\,\ln(\Pi)}{q}\sqrt{-\frac{\lambda}{\gamma}}\left(\coth_\Pi\left(2\,\sqrt{-\gamma\,\lambda}\Xi\right) \pm \left(\mathrm{csch}_\Pi\left(2\,\sqrt{-\gamma\,\lambda}\Xi\right)\right)\right), \tag{4.62}$$

and

$$\varpi_{2,20}(x,t) = -\frac{\sqrt{2\,n^2 - 2\,l^2}\gamma\,\ln(\Pi)}{q}\sqrt{-\frac{\lambda}{\gamma}}\left(\tanh_\Pi\left(\frac{1}{2}\,\sqrt{-\gamma\,\lambda}\Xi\right) + \coth_\Pi\left(\frac{1}{2}\,\sqrt{-\gamma\,\lambda}\Xi\right)\right). \tag{4.63}$$

Household 2.5. When $\lambda = \gamma$ and $\Omega = 0$, we have

$$\varpi_{2,21}(x,t) = \frac{\sqrt{2\,n^2 - 2\,l^2}\lambda\,\ln(\Pi)\tan_\Pi(\lambda\,\Xi)}{q}, \tag{4.64}$$

$$\varpi_{2,22}(x,t) = -\frac{\sqrt{2\,n^2 - 2\,l^2}\lambda\,\ln(\Pi)\cot_\Pi(\lambda\,\Xi)}{q}, \tag{4.65}$$

$$\Pi_{2,23}(x,t) = \frac{\sqrt{2\,n^2 - 2\,l^2}\lambda\,\ln(\Pi)}{q\,(\tan_\Pi(2\,\lambda\,\Xi) \pm (\sec_\Pi(2\,\lambda\,\Xi)))}, \tag{4.66}$$

$$\varpi_{2,24}(x,t) = \frac{\sqrt{2\,n^2 - 2\,l^2}\lambda\,\ln(\Pi)\,(-\cot_\Pi(2\,\lambda\,\Xi) \pm (\csc_\Pi(2\,\lambda\,\Xi)))}{q}, \tag{4.67}$$

and

$$\varpi_{2,25}(x,t) = \frac{\sqrt{2\,n^2 - 2\,l^2}\lambda\,\ln(\Pi)\left(\frac{1}{2}\tan_\Pi\left(\frac{1}{2}\lambda\,\Xi\right) - \frac{1}{2}\cot_\Pi\left(\frac{1}{2}\lambda\,\Xi\right)\right)}{q}. \tag{4.68}$$

Household 2.6. When $\gamma = -\lambda$ and $\Omega = 0$, we have

$$\varpi_{2,26}(x,t) = -\frac{\sqrt{2\,n^2 - 2\,l^2}\lambda\,\ln(\Pi)\tanh_\Pi(\lambda\,\Xi)}{q}, \tag{4.69}$$

$$\varpi_{2,27}(x,t) = -\frac{\sqrt{2\,n^2 - 2\,l^2}\lambda\,\ln(\Pi)\coth_\Pi(\lambda\,\Xi)}{q}, \tag{4.70}$$

$$\varpi_{2,28}(x,t) = \frac{\sqrt{2\,n^2 - 2\,l^2}\lambda\,\ln(\Pi)\,(-\tanh_\Pi(2\,\lambda\,\Xi) \pm (i\mathrm{sech}_\Pi(2\,\lambda\,\Xi)))}{q}, \tag{4.71}$$

$$\varpi_{2,29}(x,t) = \frac{\sqrt{2\,n^2 - 2\,l^2}\lambda\,\ln(\Pi)\,(-\coth_\Pi(2\,\lambda\,\Xi) \pm (\mathrm{csch}_\Pi(2\,\lambda\,\Xi)))}{q}, \tag{4.72}$$

and

$$\varpi_{2,30}(x,t) = \frac{\sqrt{2\,n^2 - 2\,l^2}\lambda\,\ln(\Pi)\left(-\frac{1}{2}\tanh_\Pi\left(\frac{1}{2}\lambda\,\Xi\right) - \frac{1}{2}\coth_\Pi\left(\frac{1}{2}\lambda\,\Xi\right)\right)}{q}. \tag{4.73}$$

Household 2.7. When $\Delta = 0$,

$$\varpi_{2,31}(x,t) = -2\,\frac{\sqrt{2\,n^2 - 2\,l^2}\lambda^2\,(\Omega\,\Xi\,\ln(\Pi) + 2)}{\Omega^2\Xi\,q} - \frac{\ln(\Pi)\,\Omega\,\left(l^2 - n^2\right)}{q\,\sqrt{2\,n^2 - 2\,l^2}}. \tag{4.74}$$

Household 2.8. When $\Omega := \rho, \lambda := \eta\rho(\eta \neq 0)$, and $\gamma := 0$, we have

$$\varpi_{2,32}(x,t) = \frac{\sqrt{-2\,l^2 + 2\,n^2}\eta\,\rho\,\ln(\Pi)\left(\Pi^{\rho\,\Xi} - \eta\right)}{q} - \frac{\ln(\Pi)\,\rho\,\left(l^2 - n^2\right)}{q\,\sqrt{-2\,l^2 + 2\,n^2}}. \tag{4.75}$$

Household 2.9. When $\Omega := 0$ and $\gamma := 0$,

$$\varpi_{2,33}(x,t) = \frac{\sqrt{2\,n^2 - 2\,l^2}\lambda^2\,(\ln(\Pi))^2\,\Xi}{q}. \tag{4.76}$$

Household 2.10. When $\lambda := 0, \Omega \neq 0$, and $\gamma \neq 0$,

$$\varpi_{2,34}(x,t) = -\frac{\ln(\Pi)\,\Omega\,\left(l^2 - n^2\right)}{q\,\sqrt{2\,n^2 - 2\,l^2}}, \tag{4.77}$$

and

$$\varpi_{2,35}(x,t) = -\frac{\ln(\Pi)\,\Omega\,\left(l^2 - n^2\right)}{q\,\sqrt{2\,n^2 - 2\,l^2}}. \tag{4.78}$$

Household 2.11. When $\Omega := \rho, \gamma := \eta\rho$, and $\lambda := 0$, we have

$$\varpi_{2,36}(x,t) = -\frac{\ln(\Pi)\,\rho\,\left(l^2 - n^2\right)}{q\,\sqrt{2\,n^2 - 2\,l^2}}. \tag{4.79}$$

5. Discussion and graphics

This groundbreaking study uses the modified extended differential algebraic method (mEDAM) to analyze the space-time linear Ginzburg-Landau heat equation (STLGHE), offering new insights on kink soliton structures and 3D wave patterns. Our findings demonstrate the great dependability and efficiency of mEDAM in capturing coherent structures that differentiate between various reaction-diffusion system phases. The obtained results have significantly improved our understanding of temporal evolution processes and related physical phenomena, including phase transitions, pattern formation, nonlinear wave propagation, and thermodynamic processes. Notably, the resulting soliton solutions exhibit a broad range of kink patterns, characterized by abrupt and narrow oscillations that rapidly transition between stable states. The detection of these kink waves is essential to understanding the behavior of physical processes in STLGHE-powered systems, including optical communications and condensed matter physics. To provide a deeper understanding of the obtained results, we include a physical interpretation of the graphical figures.

Figure 1 illustrates the evolution of the kink soliton solution over time, demonstrating the rapid transition between stable states. Figure 2 shows the profile of the kink soliton solution, highlighting the abrupt and narrow oscillations that characterize these structures. Figure 3 presents the wave pattern of the kink soliton solution, illustrating the complex and dynamic behavior of these structures. Figure 4 demonstrates the effect of varying the free parameter values on the kink soliton solution, highlighting the flexibility and adaptability of the mEDAM method. Figure 5 shows the temporal evolution of the kink soliton solution for different initial conditions, demonstrating the robustness and stability of the obtained solutions. Figure 6 illustrates the spatial distribution of the kink soliton solution, highlighting the localized and coherent nature of these structures. Figure 7 presents the energy density of the kink soliton solution, demonstrating the efficient and effective transfer of energy between different regions of the system. Figure 8 shows the phase portrait of the kink soliton solution, illustrating the complex and dynamic behavior of these structures in the phase space.

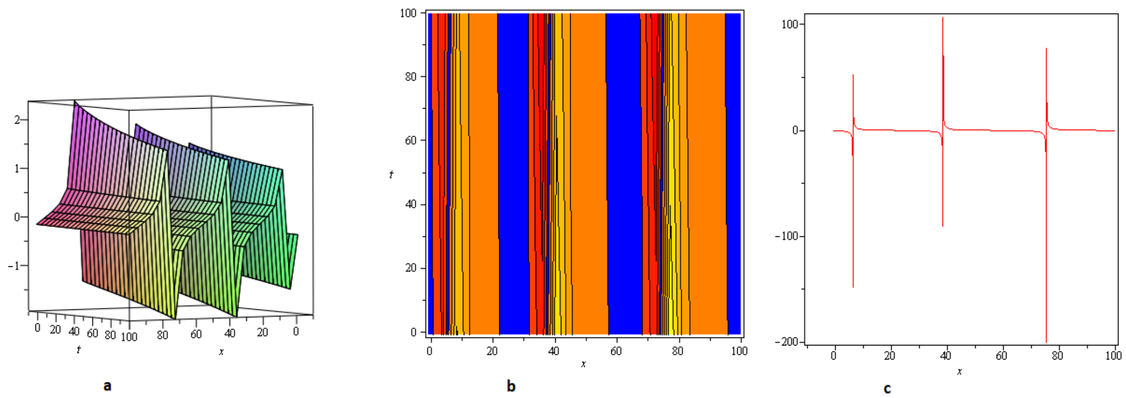

Figure 1. The three-dimensional and contour soliton solutions stated in (4.8) are graphed for $\Omega := 0.4671, \gamma := 0.21, \lambda := 1.0, \Pi := e, q := 0.65, n := 0.9, l := 0.72$, and $\theta := 0.861$.

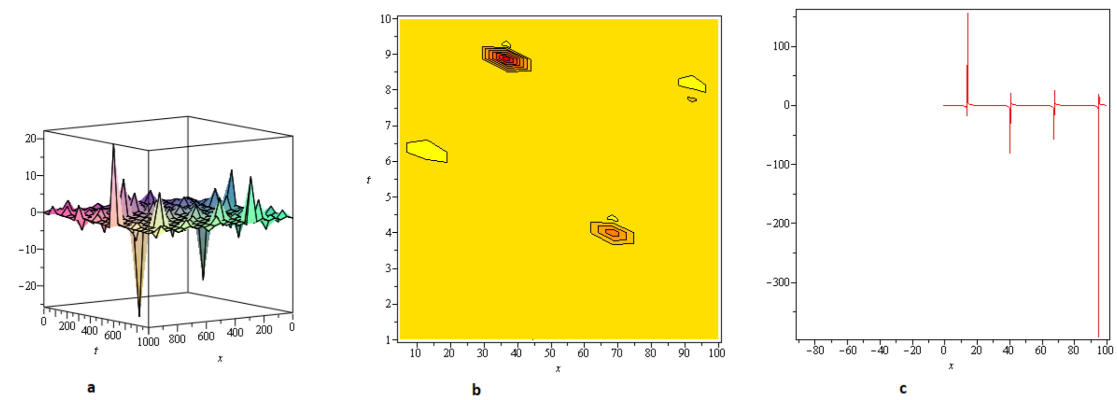

Figure 2. The three-dimensional and contour soliton solutions stated in (4.9) are graphed for $\Omega := 0.4871, \gamma := 0.21, \lambda := 1.0, \Pi := e, q := 0.36, n := 0.9, l := 0.792$, and $\theta := 0.961$.

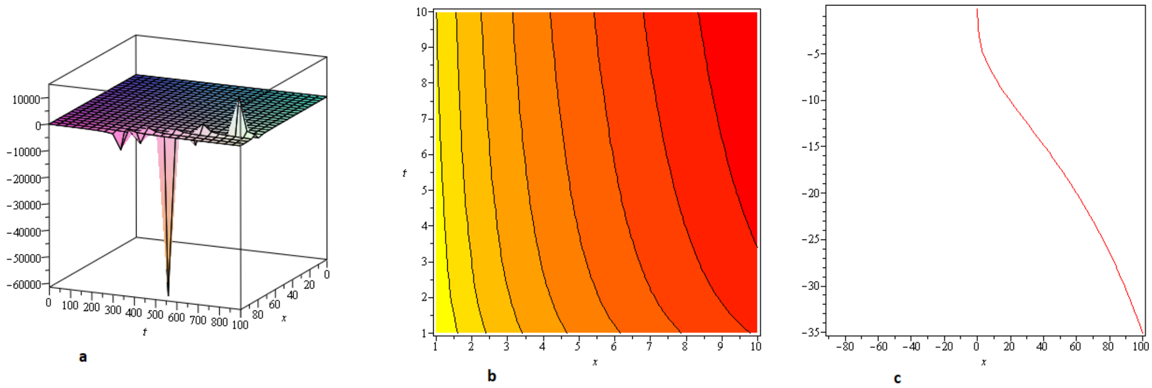

Figure 3. The three-dimensional and contour soliton solutions stated in (4.23) are graphed for $\Omega := 0, \gamma := 11, \lambda := 11., \Pi := e, q := 0.36, n := .519, l := 0.92$, and $\theta := 0.961$.

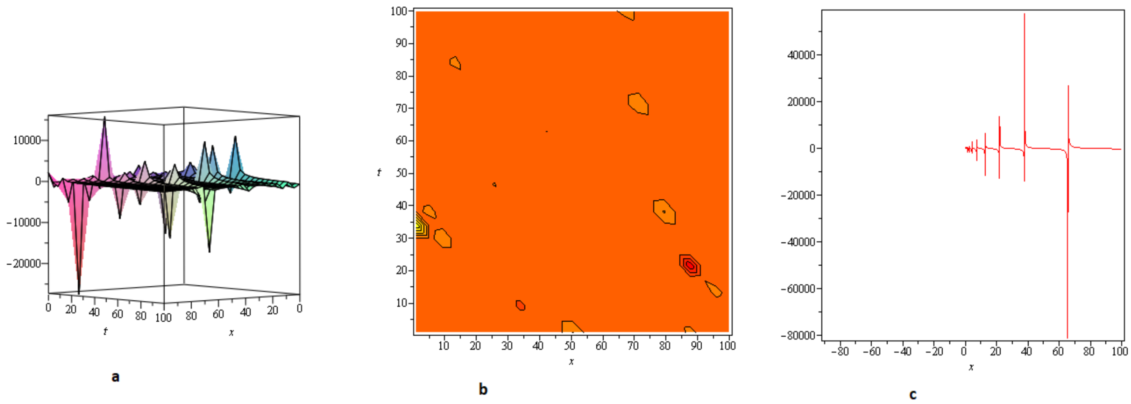

Figure 4. The three-dimensional and contour soliton solutions stated in (4.44) are graphed for $\Omega := 0, \gamma := 11, \lambda := 11., \Pi := e, q := 0.365, n := .519, l := 0.92$, and $\theta := 0.1$.

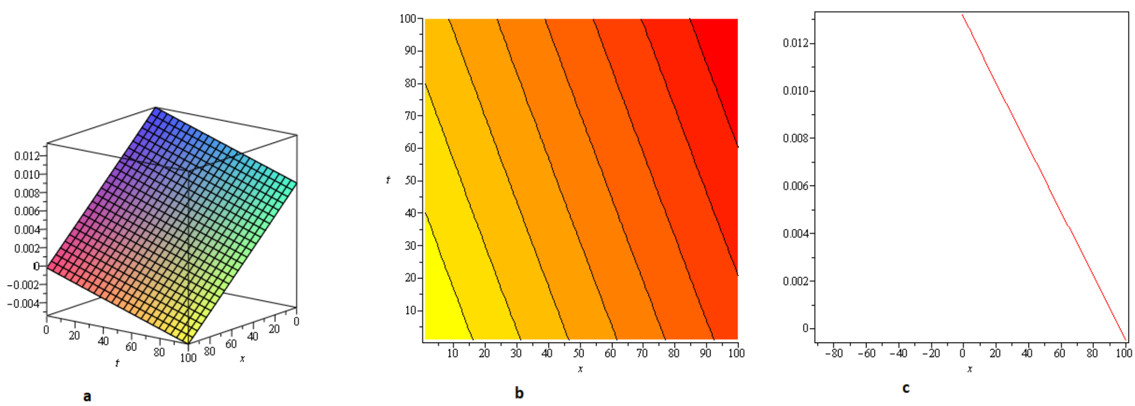

Figure 5. The three-dimensional and contour soliton solutions stated in (4.49) are graphed for $\Omega := 2, r := 1, \lambda := 1, \Pi := e, q := 0.5, n := 0.519, l := 0.2e - 2$, and $\theta := 0.99$.

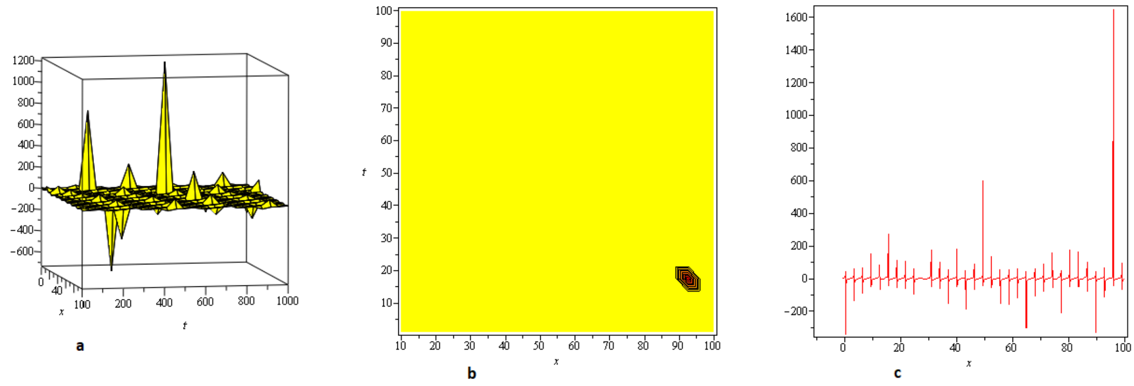

Figure 6. The three-dimensional and contour soliton solutions stated in (4.55) are graphed for $\Omega := 2, \gamma := 11, \lambda := 11, \Pi := e, q := 0.5, n := 0.96519, l := 0.2$, and $\theta := 0.99$.

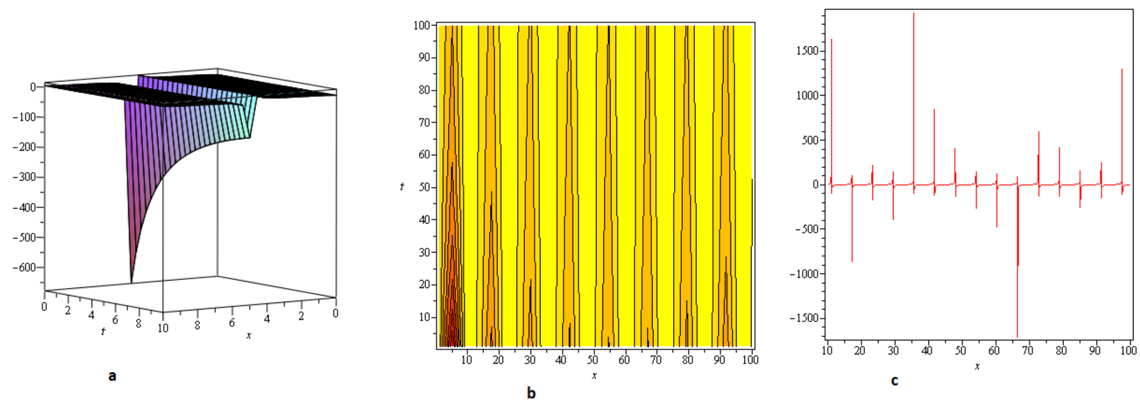

Figure 7. The three-dimensional and contour soliton solutions stated in (4.66) are graphed for $\Omega := 0.5, \gamma := 1.1, \lambda := 1, \Pi := e, q := 0.25, n := 0.519, l := 0.2$, and $\theta := 0.99$.

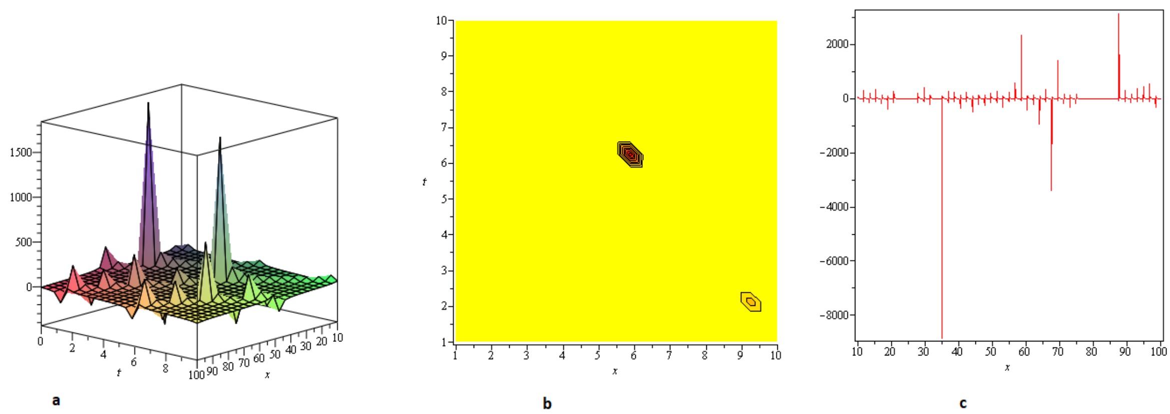

Figure 8. The three-dimensional and contour soliton solutions stated in (4.70) are graphed for $\Omega := 0, r := -1.1, \lambda := -11, \Pi := e, q := 0.25, n := 0.519, l := 0.2$, and $\theta := 0.99$.

By providing a detailed physical interpretation of the graphical figures, we aim to enhance the understanding and appreciation of the obtained results, and to facilitate further research and applications in this exciting field.

6. Conclusions

This study employs the beta derivative and the mEDAM technique to derive exact solitary wave solutions for the space-time fractional Landau-Ginzburg-Higgs problem. By varying the free parameter values, our solutions exhibit diverse shapes, including hyperbolic and exponential trigonometric functions. Notably, single soliton, double soliton, kink, anti-kink, multiple soliton, and various other solution types are obtained by adjusting the free parameters. To illustrate the physical mechanisms, Python software is utilized to generate contour, 2D, 3D, and specific plots. These solutions have far-reaching implications, as they can be applied to investigate the propagation of shallow water waves, fluid dynamics in dynamic systems, and weakly scattering nonlinear waves in the mid-latitude troposphere resulting from interactions between the equator and mid-latitude Rossby waves. The validated approach provides a plethora of solutions to nonlinear partial differential equations (NLFPDEs) that arise in engineering, nonlinear science, and applied mathematics. This method is demonstrated to be simple, reliable, consistent, and effective. In conclusion, this study contributes significantly to the understanding of soliton interactions and their applications. Given the recent experimental observations of soliton interactions in various fields, it is expected that the structures presented in this study will be experimentally observed in the near future. The experimental verification of these structures will not only deepen our understanding of soliton interactions but also pave the way for potential applications in fields such as optics, condensed matter physics, and fluid dynamics.

Author contributions

Ikram Ullah, Muhammad Bilal, Javed Iqbal, Hasan Bulut and Funda Turk: Conceptualization, Methodology, Resources, Writing-original draft, Formal analysis, Investigation, Validation, Visualization, Software. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This research is funded by the Department of Mathematics, Faculty of Science, Bartın University, Bartın, Turkey, and the School of Mathematics and Statistics, Central South University, Changsha, Hunan, China.

Conflict of interest

The authors have no conflicts of interest to declare.

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