AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(1): 634–653.
DOI: 10.3934/math.2025028
Received: 28 November 2024
Revised: 19 December 2024
Accepted: 02 January 2025
Published: 13 January 2025

Research article

Exploring the q-analogue of Fibonacci sequence spaces associated with c and c_0

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Abstract: We have proposed a q-analogue $c(\mathcal{F}(q))$ and $c_0(\mathcal{F}(q))$ of Fibonacci sequence spaces, where $\mathcal{F}(q) = (f^q_{km})$ denotes a q-Fibonacci matrix defined in the following manner:

$$f^q_{km} = \begin{cases} q^{m+1}\frac{f_{m+1}(q)}{f_{k+3}(q)-1}, & \text{if } 0 \leq m \leq k, \\ 0, & \text{if } m > k, \end{cases}$$

for all $k, m \in \mathbb{Z}^+_0$, where $(f_k(q))$ denotes a sequence of q-Fibonacci numbers. We developed a Schauder basis and determined several important duals (α-, β-, γ-) of the aforesaid constructed spaces $c(\mathcal{F}(q))$ and $c_0(\mathcal{F}(q))$. Additionally, we examined certain characterization results for the matrix class $(\mathfrak{U}, \mathfrak{V})$, where $\mathfrak{U} \in \{c(\mathcal{F}(q)), c_0(\mathcal{F}(q))\}$ and $\mathfrak{V} \in \{\ell_\infty, c, c_0, \ell_1\}$. Essential conditions for the compactness of the matrix operators on the space $c_0(\mathcal{F}(q))$ via the Hausdorff measure of noncompactness (Hmnc) were presented.

Keywords: q-Fibonacci sequence spaces; duals; matrix mappings; compactness
Mathematics Subject Classification: 40C05, 46A45, 46B45, 47B37

1. Introduction and preliminaries

Let $\mathbb{C}$ denotes the complex field and $\mathbb{Z}^+_0 = \mathbb{Z}^+ \cup \{0\}$, where $\mathbb{Z}^+$ represents the set of all positive integers. In the entirety of this article, by ω, we mean the set

$$\{v = (v_m) : v_m \in \mathbb{C} \text{ for all } m \in \mathbb{Z}^+_0\}.$$

The addition and scalar multiplication operations of sequences in ω are defined by

$$(u_m) + (v_m) = (u + v)_m \text{ and } \lambda(u_m) = (\lambda u_m)$$

for all $u = (u_m), v = (v_m) \in \omega$, and $\lambda \in \mathbb{C}$. Under these operations, the set ω forms a linear space. Any linear subspace of ω is known as a sequence space. In the literature, several types of sequence spaces have been witnessed. Among these, the spaces ℓ_p of absolutely p-summable, c of convergent, c_0 of null, and ℓ_∞ of bounded sequences are very frequently utilized by the researchers in this domain, and are sometimes referred to as classical sequence spaces.

A BK-space is associated with the combined concept of completeness and coordinate-wise continuity. More specifically, a BK-space is a Banach space (complete space endowed with a norm) under which coordinate functionals are continuous. Some prominent examples of BK-spaces are ℓ_p $(1 \leq p < \infty)$ due to norm

$$\|u\|_{\ell_p} = \left(\sum_{k=0}^{\infty} |u_k|^p\right)^{1/p}$$

and $\mathfrak{U} \in \{\ell_\infty, c, c_0\}$ due to norm

$$\|u\|_{\ell_\infty} = \sup_{k \in \mathbb{Z}_0^+} |u_k|.$$

Consider $\Theta = (\theta_{km})$ to be an arbitrary infinite matrix having entries that are either complex or real. Denote by $\Theta_k = (\theta_{km})_{m=0}^{\infty}$ the k^{th} row of the matrix Θ. For any $u = (u_m) \in \omega$, the sequence

$$\Theta u = \{(\Theta u)_k\} = \left(\sum_{m=0}^{\infty} \theta_{km} u_m\right)_{k \in \mathbb{Z}_0^+}$$

is called a Θ-transform of $u = (u_m)$, assuming the sum in the last equality is finite for every $k \in \mathbb{Z}_0^+$.

Suppose $\mathfrak{U}, \mathfrak{V} \subset \omega$. Then, an infinite matrix Θ is said to correspond a matrix mapping from $\mathfrak{U}$ to $\mathfrak{V}$ if for all $u \in \mathfrak{U}$, $\Theta u \in \mathfrak{V}$. Let $(\mathfrak{U}, \mathfrak{V})$ denote the set of all matrices that maps from $\mathfrak{U}$ to $\mathfrak{V}$. Given a matrix Θ, it is known that the domain

$$\mathfrak{U}_\Theta = \{u = (u_m) \in \omega : \Theta u \in \mathfrak{U}\}$$

of the matrix Θ in the space $\mathfrak{U}$ is itself a sequence space. When Θ is a triangular matrix and $\mathfrak{U}$ is a BK-space, $\mathfrak{U}_\Theta$ inherits certain properties from $\mathfrak{U}$, such as being a BK-space itself. The norm on $\mathfrak{U}_\Theta$ is defined as

$$\|u\|_{\mathfrak{U}_\Theta} = \|\Theta u\|_{\mathfrak{U}}.$$

For a deeper understanding and specific examples of trianglular matrices in classical sequence spaces, one should consult the monographs [5, 21] that provide detailed explanations and examples illustrating the behavior of such matrices in various sequence spaces.

1.1. Some preliminaries from q-calculus

The concept of q-analogue indeed provides a powerful framework for generalizing classical mathematical concepts by introducing a new parameter q. The versatility of q-analogue theory lies in its ability to extend classical concepts while maintaining a connection to the original theory, thus allowing for deeper insights and novel applications across various mathematical domains. As q approaches 1^-, the q-analogue is reduced to the original expression, preserving the classical results.

Although Euler laid some foundational work in this area, it was Jackson [11] who made significant contributions by formalizing q-analogues and developing the concepts of q-differentiation and

q-integration. The acceptance of q-analogue theory by the mathematical community has led to its widespread application in various branches of mathematics. In hypergeometric functions, combinatorics, algebra, approximation theory, integro-differential equations, special functions, and more, q-analogues find numerous applications. Recently, q-theory has also been utilized in the study of summability as well as sequence spaces, as indicated in [1, 8, 22, 31].

We proceed to discuss certain fundamental concepts in q-theory:

Definition 1.1. For a whole number $[z]_q$, the q-integer is given by

$$[z]_q = \begin{cases} \sum_{m=0}^{z-1} q^m & , \quad z \in \mathbb{Z}^+, \\ 0 & , \quad z = 0. \end{cases}$$

This definition ensures that in the limit as $q \to 1^-$, the q-integer $[z]_q$ converges to the ordinary integer z.

Definition 1.2. The notation $\binom{k}{m}_q$ is defined by

$$\binom{k}{m}_q = \begin{cases} \dfrac{[k]_q!}{[k-m]_q![m]_q!} & , \quad k \geq m, \\ 0 & , \quad k < m. \end{cases}$$

This represents the q-analogue of the standard binomial coefficient $\binom{k}{m}$. Note that $[m]_q! = \prod_{k=1}^{m}[k]_q$ denotes the q-analogue of the factorial $m!$.

Indeed, the equalities $\binom{0}{0}_q = \binom{k}{0}_q = \binom{k}{k}_q = 1$ and $\binom{k}{k-m}_q = \binom{k}{m}_q$ suffice for the q-binomial coefficient $\binom{k}{m}_q$. For further understanding of q-theory, we recommend consulting the monograph [13].

We now shift our focus to specific sequence spaces constructed utilizing the q-analogue of special matrices. Table 1 may be consulted for this purpose. The q-matrices $C(q) = (c^q_{km})$, $E(q) = (e^q{}_{km})$, $\nabla^2(q) = (\delta^{2;q}_{km})$, $\nabla^n(q) = (\delta^{n;q}_{km})$, $C(q) = (\tilde{c}_{km}(q))$, $\mathcal{P}(q) = (p^q_{km})$, $\hat{F}(q) = \left(\hat{f}_{km}(q)\right)$, and $\mathcal{F}(q) = (f^q_{km})$ listed below will aid in the interpretation of the results presented in Table 1:

$$c^q_{km} = \begin{cases} \frac{q^m}{[k+1]_q} & (0 \leq m \leq k), \\ 0 & (m > k), \end{cases}$$

$$e^q_{km}(\alpha,\beta) = \begin{cases} \frac{1}{(\alpha+\beta)^k_q}\binom{k}{m}_q q^{\binom{m}{2}}\alpha^m\beta^{k-m} & (0 \leq m \leq k), \\ 0 & (m > k), \end{cases}$$

$$\delta^{2;q}_{km} = \begin{cases} (-1)^{k-m}q^{\binom{k-m}{2}}\binom{2}{k-m} & (0 \leq m \leq k), \\ 0 & (m > k), \end{cases}$$

$$\delta^{n;q}_{km} = \begin{cases} (-1)^{k-m}q^{\binom{k-m}{2}}\binom{n}{k-m} & (\max\{0, k-n\} \leq m \leq k), \\ 0 & (0 \leq m \leq \max\{k-n\} \text{ or } m > k), \end{cases}$$

$$\tilde{c}_{km}(q) = \begin{cases} q^m \dfrac{c_m(q)c_{k-m}(q)}{c_{k+1}(q)} & , \quad 0 \leq m \leq k, \\ 0 & , \quad m > k, \end{cases}$$

$$p^q_{km} = \begin{cases} \binom{k}{m}_q & , \quad 0 \le m \le k, \\ 0 & , \quad m > k, \end{cases}$$

$$\hat{f}_{km}(q) = \begin{cases} -\frac{f_{k+1}(q)-1}{q^k f_k(q)} & (m = k-1), \\ \frac{f_{k+2}(q)-1}{q^k f_k(q)} & (m = k), \\ 0 & \text{otherwise}, \end{cases} \tag{1.1}$$

$$f^q_{km} = \begin{cases} q^{m+1} \frac{f_{m+1}(q)}{f_{k+3}(q)-1} & , \quad 0 \le m \le k, \\ 0 & , \quad m > k. \end{cases} \tag{1.2}$$

Table 1. Domain of special q-matrices.

$\mathfrak{U}$	Θ	$\mathfrak{U}_\Theta$	References
$\ell_p, c_0, c, \ell_\infty$	$C(q)$	$(\ell_p)_{C(q)}, (c_0)_{C(q)}, c_{C(q)}, (\ell_\infty)_{C(q)}$	[8, 31]
$\ell_p, c_0, c, \ell_\infty$	$E(q)$	$(\ell_p)_{E(q)}, (c_0)_{E(q)}, c_{E(q)}, (\ell_\infty)_{E(q)}$	[29, 32]
$\ell_p, c_0, c, \ell_\infty$	∇^2_q	$(\ell_p)_{\nabla^2_q}, (c_0)_{\nabla^2_q}, c_{\nabla^2_q}, (\ell_\infty)_{\nabla^2_q}$	[2, 33]
$\ell_p, c_0, c, \ell_\infty$	∇^m_q	$(\ell_p)_{\nabla^m_q}, (c_0)_{\nabla^m_q}, c_{\nabla^m_q}, (\ell_\infty)_{\nabla^m_q}$	[9, 30]
$\ell_p, c_0, c, \ell_\infty$	$C(q)$	$(\ell_p)_{C(q)}, (c_0)_{C(q)}, c_{C(q)}, (\ell_\infty)_{C(q)}$	[27, 34]
$\ell_p, c_0, c, \ell_\infty$	$\mathcal{P}(q)$	$(\ell_p)_{\mathcal{P}(q)}, (c_0)_{\mathcal{P}(q)}, c_{\mathcal{P}(q)}, (\ell_\infty)_{\mathcal{P}(q)}$	[28]
$\ell_p, c_0, c, \ell_\infty$	$\hat{F}(q)$	$(\ell_p)_{\hat{F}(q)}, (c_0)_{\hat{F}(q)}, c_{\hat{F}(q)}, (\ell_\infty)_{\hat{F}(q)}$	[4]
c_0, c	$\mathcal{F}(q)$	$(c_0)_{\mathcal{F}(q)}, c_{\mathcal{F}(q)}$	[35]

Here, $(c_m(q))_{m\in\mathbb{Z}^+_0}$ and $(f_m(q))_{m\in\mathbb{Z}^+_0}$ denote q-Catalan and q-Fibonacci sequences, respectively.

1.2. Fibonacci numbers and associated sequence spaces

The sequence of natural numbers

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

represents the Fibonacci sequence, often known as Nature's numbers. These numbers are prevalent in various natural phenomena, including the arrangement of sunflower seeds, pinecone bracts, tree branch patterns, pineapple scales, and fern shapes. Their diverse applications span engineering, architecture, mathematics, and the natural sciences.

Let f_m signify the m^{th} Fibonacci number. These numbers follow a linear recurrence relation:

$$f_m = f_{m-1} + f_{m-2}, \quad \text{with} \quad f_0 = 0 \text{ and } f_1 = 1.$$

We proceed to explore several well-established properties associated with Fibonacci numbers (refer to [16]):

$$\lim_{m\to\infty} \frac{f_{m+1}}{f_m} = \frac{1+\sqrt{5}}{2} \text{ (Golden Ratio)};$$

$$\sum_{m=0}^{k} f_m = f_{k+2} - 1, \; k \in \mathbb{Z}^+_0; \tag{1.3}$$

$$\sum_{m=0}^{\infty}\frac{1}{f_m} \text{ converges;}$$
$$f_{m-1}f_{m+1}-f_m^2=(-1)^{m+1},\ m\in\mathbb{Z}^+.$$

Fibonacci numbers, known for their fascinating properties, have also been applied in the fields of sequence spaces and summability. Despite numerous studies on Fibonacci numbers within these areas, we will briefly discuss some pioneering research:

Define the matrix $\hat{F}=(\hat{f}_{km})_{k,m\in\mathbb{Z}_0^+}$ in the following manner:

$$\hat{f}_{km}=\begin{cases}-\frac{f_{k+1}}{f_k} & \text{if } m=k-1,\\ \frac{f_k}{f_{k+1}} & \text{if } m=k,\\ 0 & \text{otherwise.}\end{cases}$$

The domains $\ell_p(\hat{F})=(\ell_p)_{\hat{F}}$ and $\ell_\infty(\hat{F})=(\ell_\infty)_{\hat{F}}$ have been explored by Kara [14], and $c_0(\hat{F})=(c_0)_{\hat{F}}$ and $c(\hat{F})=c_{\hat{F}}$ by Başarır et al. [6].

Kara and Başarır [15] defined the Fibonacci space $\mathfrak{U}(\tilde{F})=\mathfrak{U}_{\tilde{F}}$ for $\mathfrak{U}\in\{\ell_p,c_0,c,\ell_\infty\}$ using a regular Fibonacci matrix $\tilde{F}=(\tilde{f}_{rm})_{r,m\in\mathbb{Z}_0^+}$ given by the following expression:

$$\tilde{f}_{rm}=\begin{cases}\frac{f_m^2}{f_rf_{r+1}} & \text{if } 0\le m\le r,\\ 0 & \text{if } m>r.\end{cases}$$

Debnath and Saha [7] proposed an alternative regular Fibonacci matrix $\mathcal{F}=(f_{km})_{k,m\in\mathbb{Z}_0^+}$ defined by

$$f_{rm}=\begin{cases}\frac{f_m}{f_{r+2}-1} & \text{if } 0\le m\le r,\\ 0 & \text{if } m>r.\end{cases}$$

They constructed the spaces $c_0(\mathcal{F})=(c_0)_{\mathcal{F}}$ and $c(\mathcal{F})=c_{\mathcal{F}}$. Following this, Ercan [10] extended the work by developing Fibonacci spaces $\ell_p(\mathcal{F})=(\ell_p)_{\mathcal{F}}$ for $0\le p<\infty$ and $\ell_\infty(\mathcal{F})=(\ell_\infty)_{\mathcal{F}}$.

Let us shift our attention to the q-analogue $f(q)=(f_m(q))_{m\in\mathbb{Z}_0^+}$ of the Fibonacci sequence $f=(f_m)$. The q-Fibonacci numbers are given, as shown in [3, 24], in the following manner:

$$f_r(q)=\begin{cases}0 & (r=0),\\ 1 & (r=1),\\ f_{r-1}(q)+q^{r-2}f_{r-2}(q) & (r>1).\end{cases}$$

In simpler terms:

$$\begin{aligned}&f_0(q)=0,\\&f_1(q)=1,\\&f_2(q)=1,\\&f_3(q)=1+q,\\&f_4(q)=1+q+q^2,\end{aligned}$$

$f_5(q) = 1 + q + q^2 + q^3 + q^4,$
$f_6(q) = 1 + q + q^2 + q^3 + 2q^4 + q^5 + q^6,$
and so forth.

Additionally, as q tends to 1^-, $f_m(q)$ converges to f_m for all $m \in \mathbb{Z}_0^+$. Numerous researchers have dedicated efforts to investigating the q-analogues of the interesting relations displayed by Fibonacci numbers. A prominent example is the q-analogue of the property (1.3), as discussed in [3, Theorem 2]:

$$f_{k+2}(q) - 1 = \sum_{m=1}^{k} q^m f_m(q),$$

which is the same as writing

$$f_{k+3}(q) - 1 = \sum_{m=0}^{k} q^{m+1} f_{m+1}(q). \tag{1.4}$$

Recently, Atabey et al. [4] introduced the q-Fibonacci difference sequence space $\mathfrak{U}(\hat{F}(q))$, where $\mathfrak{U}$ is any one of the spaces in $\{\ell_p, c_0, c, \ell_\infty\}$. Here, $\hat{F}(q) = \left(\hat{f}_{km}(q)\right)_{k,m\in\mathbb{Z}_0^+}$ represents the double band q-Fibonacci difference matrix, defined as in (1.1).

More recently, Yaying et al. [35] utilized the relation (1.4) to construct a q-analogue $\mathcal{F}(q)$, defined as in (1.2), of the Fibonacci matrix $\mathcal{F}$. Using $\mathcal{F}(q)$, they developed q-Fibonacci sequence spaces $\ell_p(\mathcal{F}(q))$ and $\ell_\infty(\mathcal{F}(q))$, defined as the domain of $\mathcal{F}(q)$ in ℓ_p and ℓ_∞ (classical spaces).

This paper naturally extends the research conducted in [35], thereby extending the investigation to the spaces c_0 and c. Specifically, our aim is to introduce q-Fibonacci spaces $c_0(\mathcal{F}(q))$ and $c(\mathcal{F}(q))$, and explore the various intriguing properties that emerge from these newly defined spaces.

2. The domains $c_0(\mathcal{F}(q))$ and $c(\mathcal{F}(q))$

The sequence $v = (v_k)_{k\in\mathbb{Z}_0^+}$, defined by the relation

$$v_k = (\mathcal{F}(q)u)_k = \sum_{l=0}^{k} q^{l+1} \frac{f_{l+1}(q)}{f_{k+3}(q) - 1} u_l, \tag{2.1}$$

is called the $\mathcal{F}(q)$-transform of $u = (u_k)_{k\in\mathbb{Z}_0^+}$.

Next, we introduce the spaces $c(\mathcal{F}(q))$ and $c_0(\mathcal{F}(q))$ in the following manner:

$$c(\mathcal{F}(q)) := \{u = (u_m) \in \omega : v = \mathcal{F}(q)u \in c\},$$
$$c_0(\mathcal{F}(q)) := \{u = (u_m) \in \omega : v = \mathcal{F}(q)u \in c_0\}.$$

Alternatively, the aforementioned spaces are re-expressed as:

$$c(\mathcal{F}(q)) = c_{\mathcal{F}(q)} \quad \text{and} \quad c_0(\mathcal{F}(q)) = (c_0)_{\mathcal{F}(q)}.$$

In other words, $c(\mathcal{F}(q))$ and $c_0(\mathcal{F}(q))$ are considered the domains of the q-Fibonacci matrix in c and c_0, respectively. Indeed, as q approaches 1^-, these domains are reduced to $c(\mathcal{F})$ and $c_0(\mathcal{F})$, respectively, a topic studied by Debnath and Saha [7].

Lemma 2.1. [35, Lemma 2.1] *The inverse* $\mathcal{G}(q) = \{\mathcal{F}(q)\}^{-1} = (g^q_{km})_{k,m\in\mathbb{Z}_0^+}$ *of the q-Fibonacci matrix* $\mathcal{F}(q)$ *is expressed as follows:*

$$g^q_{km} = \begin{cases} (-1)^{k-m}\frac{f_{m+3}(q)-1}{q^{k+1}f_{k+1}(q)}, & \text{if } k-1 \le m \le k, \\ 0, & \text{if } m > k. \end{cases}$$

The above lemma allows us to define the $\{\mathcal{F}(q)\}^{-1}$-transform, or $\mathcal{G}(q)$-transform, of the sequence $v = (v_k)$ in the following manner:

$$u_k = \sum_{m=k-1}^{k} (-1)^{k-m}\frac{f_{m+3}(q)-1}{q^{k+1}f_{k+1}(q)}v_m. \tag{2.2}$$

Indeed, the Eqs (2.1) and (2.2) imply each other, and are thus equivalent.

Theorem 2.2. *Associated with a bounded norm*

$$\|u\|_{c(\mathcal{F}(q))} = \|u\|_{c_0(\mathcal{F}(q))} = \sup_{m\in\mathbb{Z}_0^+}\left|\sum_{l=0}^{m} q^{l+1}\frac{f_{m+1}(q)}{f_{m+3}(q)-1}u_l\right|,$$

the spaces $c(\mathcal{F}(q))$ *and* $c_0(\mathcal{F}(q))$ *form BK-spaces.*

Proof. This can be routinely verified. □

Remark 2.3. *It can be noted that as q tends to* 1^-, Theorem 2.2 *yields* Theorem 2.1 *of Debnath and Saha* [7].

Theorem 2.4. *For* $\mathfrak{U} \in \{c, c_0\}$, *it holds that* $\mathfrak{U}(\mathcal{F}(q)) \cong \mathfrak{U}$.

Proof. Let $\mathfrak{U} \in \{c, c_0\}$. Consider a mapping $\mathcal{M}$ defined in the following manner:

$$\begin{aligned} \mathcal{M} \quad &: \quad \mathfrak{U}(\mathcal{F}(q)) \to \mathfrak{U}, \\ u \quad &\mapsto \quad \mathcal{M}u = v = \mathcal{F}(q)u. \end{aligned}$$

It is evident that $\mathcal{F}(q)$ acts as the matrix representation of the operator $\mathcal{M}$. Since $\mathcal{F}(q)$ is triangular, it can be deduced that $\mathcal{M}$ forms a linear bijection and preserves the norm. The result follows straightforwardly. □

Definition 2.5. A normed linear space $\mathfrak{U}$ having norm $\|\cdot\|$ possesses a Schauder basis $b = (b_m)$ if ∃ is a unique sequence of real numbers $c = (c_m)$ for each $u = (u_m) \in \mathfrak{U}$ such that

$$\lim_{k\to\infty}\left\|u - \sum_{m=0}^{k} c_m b_m\right\| = 0.$$

Suppose Θ is a triangle. It follows that the sequence space/matrix domain $\mathfrak{U}_\Theta$ possesses a Schauder basis iff $\mathfrak{U}$ has a basis (refer to [12, Theorem 2.3]). The following result readily emerges from this observation:

Theorem 2.6. *Define the sequence* $\{s^{(m)}(q)\}_{m\in\mathbb{Z}_0^+} \subset c_0(\mathcal{F}(q))$ *by*

$$s_k^{(m)}(q) = \begin{cases} (-1)^{k-m}\frac{f_{m+3}(q)-1}{q^{k+1}f_{k+1}(q)}, & \text{if } k-1 \leq m \leq k, \\ 0, & \text{otherwise.} \end{cases}$$

The following assertions hold true:

(1) The sequence

$$\left\{b^{(k)}(q)\right\}_{k\in\mathbb{Z}_0^+}$$

forms a basis of $c_0(\mathcal{F}(q))$, *and each* $u \in c_0(\mathcal{F}(q))$ *is uniquely expressed as*

$$u = \sum_{k=0}^{\infty} v_k s^{(k)}(q),$$

where $v_k = (\mathcal{F}(q)u)_k$ *for each* $k \in \mathbb{Z}_0^+$.

(2) The set

$$\left\{e, b^{(k)}(q)\right\}$$

forms a basis of $c(\mathcal{F}(q))$, *and every* $u \in c(\mathcal{F}(q))$ *is uniquely determined as*

$$u = le + \sum_{k=0}^{\infty}(v_k - l)s^{(k)}(q),$$

where $l = \lim_{k\to\infty} v_k = \lim_{k\to\infty}(\mathcal{F}(q)u)_k$.

3. The duals $\{\mathfrak{U}(\mathcal{F}(q))\}^{\lambda}$, $\mathfrak{U} \in \{c, c_0\}$, and $\lambda \in \{\alpha, \beta, \gamma\}$

Let $\mathfrak{U}$ and $\mathfrak{V}$ denote any two sequence spaces. We define the set $\mathcal{M}(\mathfrak{U}, \mathfrak{V})$ in the following manner:

$$\mathcal{M}(\mathfrak{U}, \mathfrak{V}) = \{c = (c_m) \in \omega : cu = (c_m u_m) \in \mathfrak{V} \text{ for all } u \in \mathfrak{U}\}.$$

In the cases where $\mathfrak{V} = \ell_1$, cs (the space of all convergent series), and bs (the space of all bounded series), the set $\mathcal{M}(\mathfrak{U}, \mathfrak{V})$ is referred to as the α-dual, β-dual, and γ-dual of $\mathfrak{U}$, denoted respectively by $\mathfrak{U}^{\alpha}$, $\mathfrak{U}^{\beta}$, and $\mathfrak{U}^{\gamma}$.

Now we focus on certain established results crucial for examining the duals of the new spaces. The symbol $\mathfrak{Z}$ denotes the family of all finite subsets of $\mathbb{Z}_0^+$.

Lemma 3.1. [26] *Let* $\Theta = (\theta_{km})$ *be an infinite matrix. Then, we have the following results:*

(i) $\Theta \in (c_0, \ell_1) = (c, \ell_1)$ *iff*

$$\sup_{\mathcal{K}\in\mathfrak{Z}} \sum_{m=0}^{\infty} \left| \sum_{k\in\mathcal{K}} \theta_{km} \right| < \infty. \tag{3.1}$$

(ii) $\Theta \in (c_0, c)$ *iff*

$$\exists \xi_m \in \mathbb{C} \ni \lim_{k\to\infty} \theta_{km} = \xi_m \text{ for each } m \in \mathbb{Z}_0^+, \tag{3.2}$$

$$\sup_{k\in\mathbb{Z}_0^+} \sum_{m=0}^{\infty} |\theta_{km}| < \infty. \tag{3.3}$$

(iii) $\Theta \in (c, c)$ *iff* (3.2) *and* (3.3) *hold, and*

$$\exists \zeta \in \mathbb{C} \ni \lim_{k\to\infty} \sum_{m=0}^{\infty} \theta_{km} = \zeta. \tag{3.4}$$

(iv) $\Theta \in (c_0, \ell_\infty) = (c, \ell_\infty)$ *iff* (3.3) *holds.*

(v) $\Theta \in (\ell_\infty, c)$ *iff* (3.2) *holds, and*

$$\sum_{m=0}^{\infty} |\theta_{km}| \text{ converges uniformly in } k. \tag{3.5}$$

Theorem 3.2. *Let $c = (c_m) \in \omega$. Define the matrix $\Lambda(q) = (\lambda^q_{km})_{k,m\in\mathbb{Z}_0^+}$ and the set $\nu(q)$ as follows:*

$$\lambda^q_{km} = \begin{cases} (-1)^{k-m} \frac{f_{m+3}(q)-1}{q^{k+1} f_{k+1}(q)} c_k & , \quad k-1 \leq m \leq k, \\ 0 & , \quad \text{otherwise}, \end{cases}$$

$$\nu(q) = \left\{ c = (c_k) \in \omega : \sup_{\mathcal{K}\in\mathfrak{Z}} \sum_{m=0}^{\infty} \left| \sum_{k\in\mathcal{K}} \lambda^q_{km} \right| < \infty \right\}.$$

Then, it holds that

$$\{c(\mathcal{F}(q))\}^\alpha = \{c_0(\mathcal{F}(q))\}^\alpha = \nu(q).$$

Proof. Let $\mathfrak{U} \in \{c, c_0\}$. By utilizing the matrix $\Lambda(q) = (\lambda^q_{km})$ and the sequence $u = (u_k)$ (see Eq (2.2)), we have

$$c_k u_k = \sum_{m=k-1}^{k} (-1)^{k-m} \frac{f_{m+3}(q) - 1}{q^{k+1} f_{k+1}(q)} v_m c_k = (\Lambda(q)v)_k$$

for each $k \in \mathbb{Z}_0^+$. Keeping in mind this equality, it is observed that $cu = (c_m u_m) \in \ell_1$ whenever $u \in \mathfrak{U}(\mathcal{F}(q))$ iff $\Lambda(q)v \in \ell_1$ whenever $v \in \mathfrak{U}$. Therefore, $c = (c_m) \in \{\mathfrak{U}(\mathcal{F}(q))\}^\alpha$ iff $\Lambda(q) \in (\mathfrak{U}, \ell_1)$. By substituting c and c_0 for $\mathfrak{U}$ and applying Lemma 3.1(i), we obtain the required fact that

$$\{c(\mathcal{F}(q))\}^\alpha = \{c_0(\mathcal{F}(q))\}^\alpha = \nu(q).$$

This ends the proof. □

Theorem 3.3. *Let $d = (d_m) \in \omega$. Define the matrix $\Omega(q) = (\omega^q_{km})_{k,m\in\mathbb{Z}_0^+}$ by*

$$\omega^q_{km} = (f_{m+3}(q) - 1) \left\{ \frac{d_m}{q^{m+1} f_{m+1}(q)} - \frac{d_{m+1}}{q^{m+2} f_{m+2}(q)} \right\}.$$

Then, it holds that

(i) $d = (d_k) \in \{c(\mathcal{F}(q))\}^\beta$ *iff* $\Omega(q) \in (c, c)$ *and*

$$\left\{ \frac{f_{k+3}(q) - 1}{q^{k+1} f_{k+1}(q)} d_k \right\} \in c_0. \tag{3.6}$$

(ii) $d = (d_k) \in \{c_0(\mathcal{F}(q))\}^\beta$ *iff* $\Omega(q) \in (c_0, c)$, *and* (3.6) *is satisfied.*

Proof. We focus on the proof of the Beta-dual of the space $c(\mathcal{F}(q))$.

Assume that $d = (d_m) \in \{c(\mathcal{F}(q))\}^\beta$. By the definition of the Beta-dual, the series $\sum_{m=0}^{\infty} d_m u_m$ converges for any $u = (u_m) \in c(\mathcal{F}(q))$. Using Abel's partial summation on the r-th partial sum of the infinite series $\sum_{m=0}^{\infty} d_m u_m$, we obtain the following equality:

$$\begin{aligned}\sum_{m=0}^{r} d_m u_m &= \sum_{m=0}^{r}\left\{\sum_{l=m-1}^{m}(-1)^{m-l}\frac{f_{l+3}(q)-1}{q^{l+1}f_{l+1}(q)}v_l\right\}d_m \\ &= \sum_{m=0}^{r-1}(f_{m+3}(q)-1)\left\{\frac{d_m}{q^{m+1}f_{m+1}(q)}-\frac{d_{m+1}}{q^{m+2}f_{m+2}(q)}\right\}v_m+\frac{f_{r+3}(q)-1}{q^{r+1}f_{r+1}(q)}v_r d_r \qquad (3.7)\end{aligned}$$

for all $r \in \mathbb{Z}_0^+$. By hypothesis, the series $\sum_{m=0}^{\infty} d_m u_m$ is convergent. Taking the limit as $r \to \infty$ in (3.7), we observe that the series

$$\sum_{m=0}^{\infty}(f_{m+3}(q)-1)\left\{\frac{d_m}{q^{m+1}f_{m+1}(q)}-\frac{d_{m+1}}{q^{m+2}f_{m+2}(q)}\right\}v_m$$

is convergent and

$$\left\{\frac{f_{r+3}(q)-1}{q^{r+1}f_{r+1}(q)}v_r d_r\right\} \in c_0.$$

Considering that $c(\mathcal{F}(q)) \cong c$, which implies $v = (v_m) \in c$, the above condition is satisfied by

$$\left\{\frac{f_{r+3}(q)-1}{q^{r+1}f_{r+1}(q)}d_r\right\} \in c_0.$$

Therefore, we obtain

$$\sum_{m=0}^{\infty} d_m u_m = \sum_{m=0}^{\infty}(f_{m+3}(q)-1)\left\{\frac{d_m}{q^{m+1}f_{m+1}(q)}-\frac{d_{m+1}}{q^{m+2}f_{m+2}(q)}\right\}v_m = (\Omega(q)v)_k \qquad (3.8)$$

for each $k \in \mathbb{Z}_0^+$. Thus, $\Omega(q) \in (c, c)$. Alternatively, the matrix $\Omega(q)$ satisfies the conditions (3.2)–(3.4) of Lemma 3.1 (iii). This completes the necessary part of the proof.

Conversely, assume that $\Omega(q) \in (c, c)$ and the condition (3.6) is satisfied. Using (3.7), we derive (3.8). Since $\Omega(q) \in (c, c)$, the series $\sum_{m=0}^{\infty} d_m u_m$ converges for all $u = (u_m) \in c(\mathcal{F}(q))$. This implies that $d = (d_m) \in \{c(\mathcal{F}(q))\}^\beta$. Thus, the conditions are sufficient.

A similar proof may be given for the Beta-dual of the space $c_0(\mathcal{F}(q))$ except that Lemma 3.1(iii) is replaced by Lemma 3.1(ii). We skip the detailed proof to avoid redundant statements. □

Theorem 3.4. *Let $d = (d_m) \in \omega$. Then, it holds that $d = (d_m) \in \{c(\mathcal{F}(q))\}^\beta = \{c_0(\mathcal{F}(q))\}^\beta$ iff $\Omega(q) \in (c, \ell_\infty) = (c_0, \ell_\infty)$ and the condition* (3.6) *is satisfied.*

Proof. This result is drawn in a manner analogous to the proof of Theorem 3.3, but using Lemma 3.1(iv) instead of Lemma 3.1(iii). We omit the detailed proof here to avoid unnecessary repetition. □

4. Matrix mappings

Herein, we aim to characterize the matrix classes $(c(\mathcal{F}(q)), \mathfrak{U})$ and $(\mathfrak{U}, c(\mathcal{F}(q)))$, where $\mathfrak{U}$ represents any chosen sequence space. To accomplish this, we employ the dual summability method of the new type as discussed by Şengönül and Başar in [25] (also see [5, Section 4.2.3]).

Consider two infinite matrices over the complex fields, $\Theta = (\theta_{km})$ and $\Phi = (\phi_{km})$, which are related in the following manner:

$$\theta_{km} = \sum_{l=m}^{\infty} q^{m+1} \frac{f_{m+1}(q)}{f_{l+3}(q) - 1} \phi_{kl} \left(\text{or } \phi_{km} = (f_{m+3}(q) - 1) \left\{ \frac{\theta_{km}}{q^{m+1} f_{m+1}(q)} - \frac{\theta_{k,m+1}}{q^{m+2} f_{m+2}(q)} \right\} \right), \tag{4.1}$$

for all $k, m \in \mathbb{Z}_0^+$. It is evident that the matrices Θ and Φ are dual matrices of a new type (cf. [25]).

Theorem 4.1. *Assume that $\Theta = (\theta_{km})$ and $\Phi = (\phi_{km})$ are dual matrices of a new type related by* (4.1)*, and $\mathfrak{U}$ is any given space. Then, $\Theta \in (c(\mathcal{F}(q)), \mathfrak{U})$ iff $\Phi \in (c, \mathfrak{U})$, and*

$$\left\{ \frac{f_{r+3}(q) - 1}{q^{r+1} f_{r+1}(q)} \theta_{kr} \right\}_{k \in \mathbb{Z}_0^+} \in c_0, \tag{4.2}$$

for each fixed $r \in \mathbb{Z}_0^+$.

Proof. Let $\mathfrak{U}$ be any arbitrary space. Suppose that $\Theta \in (c(\mathcal{F}(q)), \mathfrak{U})$ and select $v \in c$. Then, $\Phi\mathcal{F}(q)$ exists, and $\Theta_j \in \{c(\mathcal{F}(q))\}^\beta$, which implies that $\Phi_k \in \ell_1$ for each $k \in \mathbb{Z}_0^+$. Consequently, Φv exists for all $v \in c$. Now, consider the r^{th} partial sum of the series $\sum_{m=0}^{\infty} \phi_{km} v_m$, given by:

$$\sum_{m=0}^{r} \phi_{km} v_m = \sum_{m=0}^{r} \left(\sum_{l=m}^{r} q^{m+1} \frac{f_{m+1}(q)}{f_{l+3}(q) - 1} \phi_{kl} \right) u_m \tag{4.3}$$

for $r, k \in \mathbb{Z}_0^+$. Taking the limit as $r \to \infty$ in (4.3), we obtain that $\Phi v = \Theta u$. Therefore, $\Phi \in (c, \mathfrak{U})$.

Conversely, assume that $\Phi \in (c, \mathfrak{U})$ and the condition in (4.2) is satisfied. Let $u \in c(\mathcal{F}(q))$. It follows that $\Phi_k \in \ell_1$ for each $k \in \mathbb{Z}_0^+$. Combining this with (4.2), we deduce that $\Theta_k \in \{c(\mathcal{F}(q))\}^\beta$ for each $k \in \mathbb{Z}_0^+$. Consequently, Θu exists. This led to the derivation of the much needed equality:

$$\begin{aligned} \sum_{m=0}^{r} \theta_{km} u_m &= \sum_{m=0}^{r} \left\{ \sum_{l=m-1}^{m} (-1)^{m-l} \frac{f_{l+3}(q) - 1}{q^{m+1} f_{m+1}(q)} v_l \right\} \theta_{km} \\ &= \sum_{m=0}^{r-1} (f_{m+3}(q) - 1) \left\{ \frac{\theta_{km}}{q^{m+1} f_{m+1}(q)} - \frac{\theta_{k,m+1}}{q^{m+2} f_{m+2}(q)} \right\} v_m + \frac{f_{r+3}(q) - 1}{q^{r+1} f_{r+1}(q)} v_r \theta_{kr} \end{aligned} \tag{4.4}$$

which, when taking the limit as $r \to \infty$, yields that $\Theta u = \Phi v$. This confirms that $\Theta \in (c(\mathcal{F}(q)), \mathfrak{U})$. □

It is clear that Theorem 4.1 holds various implications depending on the selection of the space $\mathfrak{U}$. By substituting ℓ_∞, c, and c_0 for $\mathfrak{U}$, we derive the following corollary:

Corollary 4.2. *The following assertions hold true:*

(i) An infinite matrix $\Theta \in (c(\mathcal{F}(q)), \ell_\infty)$ iff (3.3) *holds with ϕ_{km} instead of θ_{km}, and* (4.2) *holds.*

(ii) *An infinite matrix* $\Theta \in (c(\mathcal{F}(q)), c$ *iff* (3.2)–(3.4) *hold with* ϕ_{km} *instead of* θ_{km}*, and* (4.2) *holds.*

(iii) *An infinite matrix* $\Theta \in (c(\mathcal{F}(q)), c_0$ *iff* (3.2) *with* $\xi_m = 0$ *for all* $m \in \mathbb{Z}_0^+$, (3.3)*, and* (3.4) *with* $\zeta = 0$ *hold, with* ϕ_{km} *instead of* θ_{km}*, and* (4.2) *holds.*

Lemma 4.3. [5, Lemma 4.3.24] *Let* $\mathfrak{U}, \mathfrak{B} \subset \omega$, Θ *be an infinite matrix, and* $\mathcal{T}$ *be a triangle. Then,* $\Theta \in (\mathfrak{U}, \mathfrak{B}_{\mathcal{T}})$ *iff* $\mathcal{T}\Theta \in (\mathfrak{U}, \mathfrak{B})$.

The aforementioned lemma has played a crucial role in characterizing matrix transformations between domains of triangles. An immediate application of this lemma is presented below without proof, as it is straightforward.

Lemma 4.4. *Let* $\mathfrak{B} \in \{\ell_\infty, c, c_0\}$*. Define the matrix* $\Psi = (\psi_{km})$ *in terms of the matrix* $\Theta = (\theta_{km})$ *by*

$$\psi_{km} = \sum_{l=0}^{k} q^{l+1} \frac{f_{l+1}(q)}{f_{k+3}(q) - 1} \theta_{lm} \tag{4.5}$$

for all $k, m \in \mathbb{Z}_0^+$*. Then,* $\Theta \in (\mathfrak{U}, \mathfrak{B}_{\mathcal{F}(q)})$ *iff* $\Psi \in (\mathfrak{U}, \mathfrak{B})$.

Next, we outline several significant corollaries as immediate implications of Lemma 4.3 or Lemma 4.4:

Corollary 4.5. *The following assertions hold true:*

(i) *An infinite matrix* $\Theta \in (\ell_\infty, c(\mathcal{F}(q)))$ *iff* (3.2) *and* (3.5) *are satisfied with* ψ_{km} *in place of* θ_{km}.

(ii) *An infinite matrix* $\Theta \in (c, c(\mathcal{F}(q)))$ *iff* (3.2), (3.3)*, and* (3.4) *are satisfied with* ψ_{km} *in place of* θ_{km}.

(iii) *An infinite matrix* $\Theta \in (c_0, c(\mathcal{F}(q)))$ *iff* (3.2) *with* $\xi_m = 0$ *for all* $m \in \mathbb{Z}_0^+$, (3.3)*, and* (3.4) *with* $\zeta = 0$ *are satisfied with* ψ_{km} *in place of* θ_{km}.

Corollary 4.6. *Suppose that entries of the matrices* $\Sigma = (\sigma_{km})$ *and* $\Theta = (\theta_{km})$ *are connected by the relation:*

$$\sigma_{km} = \sum_{l=0}^{k} \theta_{lm}$$

for all $k, m \in \mathbb{Z}_0^+$*. Then, we have the following assertions:*

(i) $\Theta \in (c(\mathcal{F}(q)), bs)$ *iff* $\Sigma \in (c(\mathcal{F}(q)), \ell_\infty)$*, and the required conditions follow immediately from* Corollary 4.2 (i).

(ii) $\Theta \in (c(\mathcal{F}(q)), cs)$ *iff* $\Sigma \in (c(\mathcal{F}(q)), c)$*, and the required conditions follow immediately from* Corollary 4.2 (ii).

(iii) $\Theta \in (c(\mathcal{F}(q)), cs_0)$ *iff* $\Sigma \in (c(\mathcal{F}(q)), c_0)$*, and the required conditions follow immediately from* Corollary 4.2 (iii).

Corollary 4.7. *Suppose that entries of the matrices* $\Psi = (\psi_{km})$ *and* $\Theta = (\theta_{km})$ *are related by* (4.5)*. Then, we have the following assertions:*

(i) $\Theta \in (c(\mathcal{F}(q)), \ell_\infty(\mathcal{F}(q)))$ *iff* $\Psi \in (c(\mathcal{F}(q)), \ell_\infty)$*, and the required conditions follow immediately from* Corollary 4.2 (i).

(ii) $\Theta \in (c(\mathcal{F}(q)), c(\mathcal{F}(q)))$ *iff* $\Psi \in (c(\mathcal{F}(q)), c)$*, and the required conditions follow immediately from* Corollary 4.2 (ii).

(iii) $\Theta \in (c(\mathcal{F}(q)), c_0(\mathcal{F}(q)))$ *iff* $\Psi \in (c(\mathcal{F}(q)), c_0)$*, and the required conditions follow immediately from* Corollary 4.2 (iii).

5. Compactness via Hmnc on the space $c_0(\mathcal{F}(q))$

Consider the unit sphere $\mathfrak{B}_{\mathfrak{U}}$ in a *BK*-space $\mathfrak{U} \supset \sigma$, and let $r = (r_k) \in \omega$. In this section, we employ the following notation:

$$\|r\|^*_{\mathfrak{U}} = \sup_{u \in \mathfrak{B}_{\mathfrak{U}}} \left| \sum_{m=0}^{\infty} r_m u_m \right|.$$

It should be noted that $r \in \mathfrak{U}^\beta$.

Lemma 5.1. [17, Lemma 6] *$\ell_\infty^\beta = c^\beta = c_0^\beta = \ell_1$ and $\|r\|^*_{\mathfrak{U}} = \|r\|_{\ell_1}$ for $\mathfrak{U} \in \{\ell_\infty, c, c_0\}$.*

The notation $\mathfrak{B}(\mathfrak{U}, \mathfrak{V})$ is used to denote the set of all bounded (continuous) linear operators from $\mathfrak{U}$ to $\mathfrak{V}$.

Lemma 5.2. [18, Theorem 1.23(a)] *Suppose $\mathfrak{U}$ and $\mathfrak{V}$ are arbitrary BK-spaces. For each $\Theta \in (\mathfrak{U}, \mathfrak{V})$, there exists a bounded linear operator $\mathcal{M}_\Theta \in \mathfrak{B}(\mathfrak{U}, \mathfrak{V})$ such that $\mathcal{M}_\Theta(u) = \Theta u$ for all $u \in \mathfrak{U}$.*

Lemma 5.3. [18] *Consider a BK-space $\mathfrak{U} \supset \sigma$ and $\mathfrak{V} \in \{c_0, c, \ell_\infty\}$. If $\Theta \in (\mathfrak{U}, \mathfrak{V})$, then the following holds:*

$$\|\mathcal{M}_\Theta\| = \|\Theta\|_{(\mathfrak{U},\mathfrak{V})} = \sup_{m \in \mathbb{Z}_0^+} \|\Theta_m\|^*_{\mathfrak{U}} < \infty.$$

In a metric space $\mathfrak{U}$, the Hausdorff measure of noncompactness (Hmnc) of a bounded set $\mathfrak{S}$ is denoted by $\chi(\mathfrak{S})$. It is given by:

$$\chi(\mathfrak{S}) = \inf \left\{ \epsilon > 0 : \mathfrak{S} \subset \bigcup_{m=0}^{k} \mathfrak{B}(c_m, a_m), c_m \in \mathfrak{U}, a_m < \epsilon,\ k \in \mathbb{Z}_0^+ \right\},$$

where $\mathfrak{B}(c_m, a_m)$ denotes the open ball centered at c_m with radius a_m. For further details on the Hmnc, consult [18] and the references therein.

Theorem 5.4. *For each $u = (u_m) \in c_0$ and $m \in \mathbb{Z}_0^+$, define the operator $\mathcal{T}_m : c_0 \to c_0$ by $\mathcal{T}_m(u) = (u_0, u_1, u_2, \ldots, u_m, 0, 0, \ldots)$. The Hmnc of any bounded set $\mathcal{S} \subset c_0$ is given by:*

$$\chi(\mathcal{S}) = \lim_{m \to \infty} \left(\sup_{u \in \mathcal{S}} \|(\mathcal{I} - \mathcal{T}_k)(u)\|_{c_0} \right),$$

where $\mathcal{I}$ represents the identity operator on c_0.

Consider two arbitrary Banach spaces, $\mathfrak{U}$ and $\mathfrak{V}$. A linear operator $\mathcal{M} : \mathfrak{U} \to \mathfrak{V}$ is compact provided its domain is all of $\mathfrak{U}$ and, for any bounded sequence $u = (u_k) \in \mathfrak{U}$, the sequence $(\mathcal{M}(u_k))$ contains a convergent subsequence in $\mathfrak{V}$.

The condition (necessary and sufficient) for $\mathcal{M}$ to be compact is that its Hmnc is zero, denoted by $\|\mathcal{M}\|_\chi = \chi(\mathcal{M}(\mathfrak{B}_{\mathfrak{U}})) = 0$.

In the field of sequence spaces, the Hmnc of an operator (linear) assumes a pivotal role in determining the compactness of the operators among Banach spaces. For further exploration, one may consult [19, 20, 23, 34].

Let $r=(r_m)\in\omega$, and define the sequence $s=(s_m)$ as follows:

$$s_m=(f_{m+3}(q)-1)\left\{\frac{r_m}{q^{m+1}f_{m+1}(q)}-\frac{r_{m+1}}{q^{m+2}f_{m+2}(q)}\right\}$$

for all $m\in\mathbb{Z}_0^+$.

Lemma 5.5. *Let $r=(r_m)\in[c_0(\mathcal{F}(q))]^\beta$. Then $s=(s_m)\in\ell_1$, and the equality*

$$\sum_{m=0}^{\infty}r_mu_m=\sum_{m=0}^{\infty}s_mv_m \tag{5.1}$$

is satisfied for all $u=(u_m)\in c_0(\mathcal{F}(q))$.

Lemma 5.6. *For all $r=(r_m)\in[c_0(\mathcal{F}(q))]^\beta$,*

$$\|r\|^*_{c_0(\mathcal{F}(q))}=\sum_{m=0}^{\infty}|s_m|<\infty.$$

Proof. Let $r=(r_m)\in[c_0(\mathcal{F}(q))]^\beta$. According to Lemma 5.5, $s=(s_m)\in\ell_1$ and (5.1) holds. Since $\|u\|_{c_0(\mathcal{F}(q))}=\|v\|_{c_0}$, it follows that $u\in\mathfrak{B}_{c_0(\mathcal{F}(q))}$ iff $v\in\mathfrak{B}_{c_0}$. Thus,

$$\|r\|^*_{c_0(\mathcal{F}(q))}=\sup_{u\in\mathfrak{B}_{c_0(\mathcal{F}(q))}}\left|\sum_{m=0}^{\infty}r_mu_m\right|=\sup_{v\in\mathfrak{B}_{c_0}}\left|\sum_{m=0}^{\infty}s_mv_m\right|=\|s\|^*_{c_0}.$$

It follows with Lemma 5.1 that

$$\|r\|^*_{c_0(\mathcal{F}(q))}=\|s\|^*_{c_0}=\|s\|_{\ell_1}=\sum_{m=0}^{\infty}|s_m|.$$

□

Let $\Omega=(\omega_{km})$ and $\Theta=(\theta_{km})$ be matrices related as follows:

$$\omega_{km}=(f_{m+3}(q)-1)\left\{\frac{\theta_{km}}{q^{m+1}f_{m+1}(q)}-\frac{\theta_{k,m+1}}{q^{m+2}f_{m+2}(q)}\right\}$$

for all $k,m\in\mathbb{Z}_0^+$.

Lemma 5.7. *Suppose $\mathfrak{U}\subset\omega$ and $\Theta=(\theta_{km})$ is an infinite matrix. If $\Theta\in(c_0(\mathcal{F}(q)),\mathfrak{U})$, then $\Omega\in(c_0,\mathfrak{U})$ and $\Theta u=\Omega v$ for all $u\in c_0(\mathcal{F}(q))$.*

Proof. This conclusion is drawn from Lemma 5.5. □

Lemma 5.8. *For $\Theta\in(c_0(\mathcal{F}(q)),\mathfrak{V})$, it holds that*

$$\|\mathcal{M}_\Theta\|=\|\Theta\|_{(c_0(\mathcal{F}(q)),\mathfrak{V})}=\sup_{k\in\mathbb{Z}_0^+}\left(\sum_{m=0}^{\infty}|\omega_{km}|\right)<\infty,$$

where $\mathfrak{V}\in\{c_0,c,\ell_\infty\}$.

Lemma 5.9. [19, Theorem 3.7] *Suppose $\mathfrak{U} \supset \sigma$ is a BK-space. Then, each of the following statements is true:*

(a) If $\Theta \in (\mathfrak{U}, \ell_\infty)$, then

$$0 \leq \|\mathcal{M}_\Theta\|_\chi \leq \limsup_{k\to\infty} \|\Theta_k\|^*_{\mathfrak{U}}.$$

(b) If $\Theta \in (\mathfrak{U}, c_0)$, then

$$\|\mathcal{M}_\Theta\|_\chi = \limsup_{k\to\infty} \|\Theta_k\|^*_{\mathfrak{U}}.$$

(c) If $\mathfrak{U}$ has AK or $\mathfrak{U} = \ell_\infty$ and $\Theta \in (\mathfrak{U}, c)$, then

$$\frac{1}{2}\limsup_{k\to\infty} \|\Theta_k - \theta\|^*_{\mathfrak{U}} \leq \|\mathcal{M}_\Theta\|_\chi \leq \limsup_{k\to\infty} \|\Theta_k - \theta\|^*_{\mathfrak{U}},$$

where $\theta = (\theta_m)$ and $\theta_m = \lim_{k\to\infty} \theta_{km}$ for each $m \in \mathbb{Z}_0^+$.

Lemma 5.10. [19, Theorem 3.11] *Suppose $\mathfrak{U} \supset \omega$ is any BK-space. If $\Theta \in (\mathfrak{U}, \ell_1)$, then*

$$\lim_{p\to\infty}\left(\sup_{\mathcal{K}\in\mathfrak{Z}_p}\left\|\sum_{k\in\mathcal{K}}\Theta_k\right\|^*_{\mathfrak{U}}\right) \leq \|\mathcal{M}_\Theta\|_\chi \leq 4\lim_{p\to\infty}\left(\sup_{\mathcal{J}\in\mathfrak{Z}_p}\left\|\sum_{k\in\mathcal{K}}\Theta_k\right\|^*_{\mathfrak{U}}\right)$$

and $\mathcal{M}_\Theta$ is compact iff $\lim_{p\to\infty}\left(\sup_{\mathcal{K}\in\mathfrak{Z}_p} \|\sum_{k\in\mathcal{K}}\Theta_k\|^*_{\mathfrak{U}}\right) = 0$, where $\mathfrak{Z}_p$ is the sub-family of $\mathfrak{Z}$ comprising subsets of $\mathbb{Z}_0^+$ with elements exceeding p.

Theorem 5.11.

(a) If $\Theta \in (c_0(\mathcal{F}(q)), \ell_\infty)$, then

$$0 \leq \|\mathcal{M}_\Theta\|_\chi \leq \limsup_{k\to\infty}\sum_{m=0}^{\infty}|\omega_{km}|$$

holds.

(b) If $\Theta \in (c_0(\mathcal{F}(q)), c)$, then

$$\frac{1}{2}\limsup_{k\to\infty}\sum_{m=0}^{\infty}|\omega_{km} - \omega_m| \leq \|\mathcal{M}_\Theta\|_\chi \leq \limsup_{k\to\infty}\sum_{m=0}^{\infty}|\omega_{km} - \omega_m|$$

holds.

(c) If $\Theta \in (c_0(\mathcal{F}(q)), c_0)$, then

$$\|\mathcal{M}_\Theta\|_\chi = \limsup_{k\to\infty}\sum_{m=0}^{\infty}|\omega_{km}|$$

holds.

(d) If $\Theta \in (c_0(\mathcal{F}(q)), \ell_1)$, then

$$\lim_{p\to\infty}\|\Theta\|^{(p)}_{(c_0(\mathcal{F}(q)),\ell_1)} \leq \|\mathcal{M}_\Theta\|_\chi \leq 4\lim_{p\to\infty}\|\Theta\|^{(p)}_{(c_0(\mathcal{F}(q)),\ell_1)}$$

holds, where $\|\Theta\|^{(p)}_{(c_0(\mathcal{F}(q)),\ell_1)} = \sup_{\mathcal{K}\in\mathfrak{Z}_p}\left(\sum_{m=0}^{\infty}|\sum_{k\in\mathcal{K}}\omega_{km}|\right)$ $(p \in \mathbb{Z}_0^+)$.

Proof. (a) Let $\Theta \in (c_0(\mathcal{F}(q)), \ell_\infty)$. As the series $\sum_{m=0}^{\infty} \theta_{km} u_m$ converges for each $k \in \mathbb{Z}_0^+$, it follows that $\Theta_k \in [c_0(\mathcal{F}(q))]^\beta$. Referring to Lemma 5.6, we express

$$\|\Theta_k\|^*_{c_0(\mathcal{F}(q))} = \|\Omega_k\|^*_{c_0} = \|\Omega_k\|_{\ell_1} = \sum_{m=0}^{\infty} |\omega_{km}|$$

for each $k \in \mathbb{Z}_0^+$. Utilizing Lemma 5.9(a), it leads us to the conclusion that

$$0 \leq \|\mathcal{M}_\Theta\|_\chi \leq \limsup_k \left(\sum_{m=0}^{\infty} |\omega_{km}| \right).$$

(b) Let $\Theta \in (c_0(\mathcal{F}(q)), c)$. According to Lemma 5.7, one gets $\Omega \in (c_0, c)$. Thus, by use of Lemma 5.9(c), it follows that

$$\frac{1}{2} \limsup_{k\to\infty} \|\Omega_k - \omega\|^*_{c_0} \leq \|\mathcal{M}_\Theta\|_\chi \leq \limsup_{k\to\infty} \|\Omega_k - \omega\|^*_{c_0},$$

where $\omega = (\omega_m)$ and $\omega_m = \lim_{k\to\infty} \omega_{km}$ for each $m \in \mathbb{Z}_0^+$. Moreover, Lemma 5.1 implies that

$$\|\Omega_k - \omega\|^*_{c_0} = \|\Omega_k - \omega\|_{\ell_1} = \sum_{m=0}^{\infty} |\omega_{km} - \omega_m|$$

for each $k \in \mathbb{Z}_0^+$. This completes the proof.

(c) Let $\Theta \in (c_0(\mathcal{F}(q)), c_0)$. Since

$$\|\Theta_k\|^*_{c_0(\mathcal{F}(q))} = \|\Omega_k\|^*_{c_0} = \|\Omega_k\|_{\ell_1} = \left(\sum_{m=0}^{\infty} |\omega_{km}| \right)$$

for each $k \in \mathbb{Z}_0^+$, it follows by use of Lemma 5.9(b) that

$$\|\mathcal{M}_\Theta\|_\chi = \limsup_{k\to\infty} \left(\sum_{m=0}^{\infty} |\omega_{km}| \right).$$

(d) Let $\Theta \in (c_0(\mathcal{F}(q)), \ell_1)$. According to Lemma 5.7, one gets $\Omega \in (c_0, \ell_1)$. It follows by use of Lemma 5.10 that

$$\lim_{p\to\infty} \left(\sup_{\mathcal{K}\in\mathfrak{Z}_p} \left\| \sum_{k\in\mathcal{K}} \Omega_k \right\|^*_{c_0} \right) \leq \|\mathcal{M}_\Theta\|_\chi \leq 4 \lim_{p\to\infty} \left(\sup_{\mathcal{K}\in\mathfrak{Z}_p} \left\| \sum_{k\in\mathcal{K}} \Omega_k \right\|^*_{c_0} \right).$$

Moreover, Lemma 5.1 implies that

$$\left\| \sum_{k\in\mathcal{K}} \Omega_k \right\|^*_{c_0} = \left\| \sum_{k\in\mathcal{K}} \Omega_k \right\|_{\ell_1} = \left(\sum_{m=0}^{\infty} \left| \sum_{k\in\mathcal{K}} \omega_{km} \right| \right),$$

which ends the proof.

□

This theorem entails the corollary below.

Corollary 5.12.

(a) $\mathcal{M}_\Theta$ *is compact for* $\Theta \in (c_0(\mathcal{F}(q)), \ell_\infty)$ *if*

$$\lim_{k\to\infty} \sum_{m=0}^{\infty} |\omega_{km}| = 0.$$

(b) $\mathcal{M}_\Theta$ *is compact for* $\Theta \in (c_0(\mathcal{F}(q)), c)$ *iff*

$$\lim_{k\to\infty} \sum_{m=0}^{\infty} |\omega_{km} - \omega_m| = 0.$$

(c) $\mathcal{M}_\Theta$ *is compact for* $\Theta \in (c_0(\mathcal{F}(q)), c_0)$ *iff*

$$\lim_{k\to\infty} \sum_{m=0}^{\infty} |\omega_{km}| = 0.$$

(d) $\mathcal{M}_\Theta$ *is compact for* $\Theta \in (c_0(\mathcal{F}(q)), \ell_1)$ *iff*

$$\lim_{p\to\infty} \|\Theta\|^{(p)}_{(c_0(\mathcal{F}(q)),\ell_1)} = 0,$$

where $\|\Theta\|^{(p)}_{(c_0(\mathcal{F}(q)),\ell_1)} = \sup_{\mathcal{K}\in\mathfrak{Z}_p} \left(\sum_{m=0}^{\infty} |\sum_{k\in\mathcal{K}} \omega_{km}|\right)$.

6. Conclusions

In recent times, there has been significant research interest in q-sequence spaces as the domains of q-analogues of well-known matrices. Some examples include q-Cesàro spaces [31], q-Euler spaces [29, 32], q-Pascal sequence spaces [28], q-Fibonacci difference spaces [4], q-difference spaces [2, 30, 33], and q-Catalan spaces [34]. Furthermore, Yaying et al. [35], quite recently, introduced q-Fibonacci matrix $\mathcal{F}(q)$, and examined its domain in spaces ℓ_p and ℓ_∞.

Our present investigation builds upon the Yaying et al. work [35], expanding the domain of the q-Fibonacci matrix $\mathcal{F}(q)$ in c and c_0. This extension led to the introduction of the spaces $c_0(\mathcal{F}(q))$ and $c(\mathcal{F}(q))$, and explored various properties such as the Schauder basis, α-, β-, and γ-duals, as well as matrix transformation related results. Additionally, a section is devoted to the investigation of compactness of linear operators on the space $c_0(\mathcal{F}(q))$. We established that as q tends to 1^-, the spaces $c_0(\mathcal{F}(q))$ and $c(\mathcal{F}(q))$ reduce to the classical Fibonacci spaces $c_0(\mathcal{F})$ and $c(\mathcal{F})$, as discussed by Debnath and Saha in [7]. Hence, our findings represent a generalization of the results presented in [7].

Consider the q-Fibonacci space $\mathfrak{U}_{\mathcal{F}(q)}$, where $\mathfrak{U}$ is any of the classical paranormed spaces $\ell(p), c_0(p), c(p)$, or $\ell_\infty(p)$. It is observed that the spaces $\mathfrak{U}_{\mathcal{F}(q)}$ and $\mathfrak{U}$ are paranorm isometric. Consequently, it is worthwhile to explore and investigate the following:

- Inclusion relations between the spaces $\mathfrak{U}_{\mathcal{F}(q)}$ and $\mathfrak{U}$.
- Evaluation of a Schauder basis and computation of the continuous dual of the space $\mathfrak{U}_{\mathcal{F}(q)}$.
- Calculation of duals such as α-, β-, and γ-duals of $\mathfrak{U}_{\mathcal{F}(q)}$ space.
- Characterization of the matrix classes $(\mathfrak{U}_{\mathcal{F}(q)}, \mathfrak{V})$ and $(\mathfrak{V}, \mathfrak{U}_{\mathcal{F}(q)})$, where $\mathfrak{V}$ is any paranormed space.

Author contributions

Taja Yaying, S. A. Mohiuddine and Jabr Aljedani: Conceptualization, Methodology, Validation, Writing – original draft, Writing – review & editing. The authors contributed equally to this work. All authors have read and approved the final version of this manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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