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Research article

Existence of solution for a Langevin equation involving the ψ-Hilfer fractional derivative: A variational approach

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Abstract: This study examines the existence of a solution for a nonvariational Langevin equation that involves the ψ-Hilfer fractional derivative. More specifically, we apply the mountain pass theorem, and then an iterative approach to establish the existence of a solution for the problem.

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1. Introduction

Fractional calculus is a field of mathematical analysis that generalizes the concepts of derivatives and integrals to non-integer orders. It has become increasingly important in numerous scientific and engineering disciplines. As a result, many researchers have investigated its applications across various domains, such as virotherapy (see [1]), quantum mechanics (see [2]), image processing (see [3]), other applications in physics and engineering can be found in the papers [4, 5]. Given their significance, numerous researchers have focused on addressing problems involving various fractional derivatives. For example, the works [6] (for the ψ-Caputo derivative), the papers [7–10] (Riemann-Liouville derivative combined with fixed points theorems), the papers [11–13] (Riemann-Liouville derivative combined with the variational method), and the work [14] (Hadamard derivative).

Recently, numerous papers have focused on studying problems involving the ψ-Hilfer fractional derivative, which was introduced in the work of Hilfer [15]. The ψ-Hilfer fractional derivative is based on the generalized Hilfer fractional derivative, which itself combines elements of the Riemann-Liouville and Caputo derivatives. The Hilfer fractional derivative is a more flexible formulation than other fractional derivatives because it incorporates both the memory of a function and its local behavior, thus providing more control over the fractional-order derivative. This fractional derivative type generalizes and extends several classical definitions in fractional calculus and has been applied to model complex processes in physics, biology, and engineering; see, for instance, [16–20].

In this work, we continue to address a problem involving the ψ-Hilfer fractional derivatives ${}_yD_T^{\nu,\beta,\psi}$ and ${}_0D_y^{\nu,\beta,\psi}$, which were introduced in Section 2. More precisely, we prove the existence of solutions for the following problem:

$$\begin{cases} {}_yD_T^{\nu,\beta,\psi}\left({}_0D_y^{\nu,\beta,\psi}u(y)\right) = g(y, u(y), {}_0D_y^{\nu,\beta,\psi}u(y)), \ y \in (0, T), \\ I_{0^+}^{\nu(\nu-1);\psi}u(0) = I_T^{\nu(\nu-1);\psi}u(T) = 0, \end{cases} \tag{1.1}$$

where $\nu \in \left(\frac{1}{2}, 1\right)$, ψ is a positive increasing function on $[0, T]$ with $T > 0$ and $\psi'(y) \neq 0$ for all $y \in (0, T)$. The function g is a measurable function on $[0, T] \times \mathbb{R} \times \mathbb{R}$ and satisfies additional conditions that will be stated later. We note that a function $u \in E$ is a solution for problem (1.1), if for every $v \in E$ we have

$$\int_0^T {}_0D_y^{\nu,\beta,\psi}u(y)\, {}_0D_y^{\nu,\beta,\psi}v(y)\, dy = \int_0^T g(y, u(y), {}_0D_y^{\nu,\beta,\psi}u(y))v(y)\, dy,$$

where E is a functional space introduced in Section 2. We also note that, in the special case where

$$g(y, u(y), {}_0D_y^{\nu,\beta,\psi}u(y)) = f(y, u(y)) - \lambda\, {}_0D_y^{\nu,\beta,\psi}u(y),$$

the problem (1.1) simplifies to a Langevin equation.

It is worth mentioning that the Langevin equation is the fundamental equation in the fields of statistical mechanics and stochastic processes. It describes the time evolution of a system's state, incorporating both deterministic and random forces. Albert Einstein provided a theoretical framework for Brownian motion, which refers to the random motion of particles suspended in a fluid. His work established the foundation for understanding the stochastic behavior of particles. Paul [21] further extended the concept of Brownian motion by introducing both a deterministic force and a random force (the thermal or stochastic force). He formulated what is now known as the Langevin equation, which describes the dynamics of a particle in a fluid, accounting for both frictional drag and random thermal fluctuations.

The Langevin equation is given by:

$$\mathfrak{m}\frac{d^2\varphi(y)}{dx^2} = -\gamma\frac{d\varphi(y)}{dx} + \eta(y),$$

where $\mathfrak{m}$ is the mass of the particle, γ is the damping coefficient, and $\eta(y)$ represents the random force following a Gaussian distribution. Recently, significant attention has been given to the study of Langevin equations with various fractional derivatives. Ahmad et al. [22, 23] applied a combination of the contraction mapping principle and the Krasnoselskii fixed point theorem to establish the existence of solutions for certain Langevin equations involving Caputo fractional derivatives. Almaghamsi [24] investigated weak solutions for boundary value problems with fractional Langevin inclusions, employing the Katugampola-Caputo derivative and Pettis integrability, and utilizing the Mönch fixed point theorem along with weak noncompactness. Lim et al. [25] examined a Langevin problem involving the Weyl and Riemann-Liouville fractional derivatives, focusing on

Gaussian processes and their relationship with fractional Brownian motion, thereby proving several existing results. Most recently, Torres [26] studied a Langevin equation incorporating the Riemann-Liouville fractional derivative. He combined a variational method with an iterative technique to demonstrate the existence of solutions for the problem under study.

In this paper, we further explore a Langevin problem involving the ψ-Hilfer fractional derivative by using the study of Xie et al. [27]. It is important to point out that the presence of the third variable in the function g prevents the application of variational methods to solve this problem. So, we begin by replacing the third variable of a function g with a fixed function in an appropriate function space, and in Section 3, we employ the mountain pass theorem to establish the existence of solutions for this auxiliary problem. Following this, in Section 4 we construct a sequence of solutions for related auxiliary problems and demonstrate that this sequence converges to a solution for the original problem (1.1). It is noted that our study generalizes other works in the literature; for example, if $\psi(x) = x$ and $\beta = 1$, then we obtain the result of Toress [26]. To our knowledge, no known work in the literature utilizes the theory of [27] to problems involving fractional derivatives with respect to a function.

2. Preliminaries and variational setting

In this section, we present some foundational background and theoretical concepts related to the ψ-Hilfer fractional derivative, which will be applied throughout this paper. We begin by recalling the definition of the fractional integral, as introduced by Kilbas et al. [4] and Samko et al. [28]. In this section, η and β are positive real numbers, Γ represents the Euler gamma function, and for $-\infty \leq a < b \leq \infty$, $[a, b]$ denotes a finite or infinite interval on the real line. Moreover, ψ is assumed to be an increasing positive function on $[a, b]$, with a continuously differentiable derivative $\psi'(y) \neq 0$ on (a, b).

Definition 2.1. [4,28] Let χ be an integrable function on the interval $[a, b]$, and let $\psi \in \mathfrak{C}^1([a, b], \mathbb{R})$ be a strictly increasing function with $\psi'(y) \neq 0$ for all $y \in [a, b]$. The fractional integrals of χ with respect to ψ, on the left and right sides, are given by:

$$I_{a^+}^{\beta,\psi}\chi(y) := \frac{1}{\Gamma(\beta)} \int_a^y \psi'(\sigma)(\psi(y) - \psi(\sigma))^{\beta-1}\chi(\sigma)\, d\sigma, \tag{2.1}$$

and

$$I_{b^-}^{\beta,\psi}\chi(y) := \frac{1}{\Gamma(\beta)} \int_y^b \psi'(\sigma)(\psi(\sigma) - \psi(y))^{\beta-1}\chi(\sigma)\, d\sigma. \tag{2.2}$$

Definition 2.2. [19, 29] Let χ be a function that is integrable over the interval $[a, b]$, and let $\psi \in \mathfrak{C}^1([a, b], \mathbb{R})$ be a strictly increasing function such that $\psi'(y) \neq 0$ for all $y \in [a, b]$. Below is the definition of the left and right ψ-Hilfer fractional derivatives of type $0 \leq \beta \leq 1$:

$$^H\mathcal{D}_{a^+}^{\eta,\beta,\psi}\chi(y) \;:=\; I_{a^+}^{\beta(m-\eta),\psi}\left(\frac{1}{\psi'(y)}\frac{d}{dy}\right)^m I_{a^+}^{(1-\beta)(m-\eta),\psi}\chi(y),$$

and

$$^H\mathcal{D}_{b^-}^{\eta,\beta,\psi}\chi(y) \;:=\; I_{b^-}^{\beta(m-\eta),\psi}\left(\frac{1}{\psi'(y)}\frac{d}{dy}\right)^m I_{b^-}^{(1-\beta)(m-\eta),\psi}\chi(y),$$

where m is an integer satisfying $m - 1 < \eta \leq m$.

It is important to highlight that the ψ-Hilfer fractional derivatives generalize previous concepts, including the ψ-Riemann-Liouville and ψ-Caputo fractional derivatives. Specifically, the following remark holds:

Remark 2.3. *The following statements are true:*

(i) From the ψ-Hilfer fractional derivatives, as β tends to zero,we obtain the ψ-Riemann-Liouville fractional derivatives:

$$\mathcal{D}_{a^+}^{\eta,\psi}\chi(y) = \left(\frac{1}{\psi'(y)}\frac{d}{dy}\right)^m I_{a^+}^{m-\eta,\psi}\chi(y),$$

and

$$\mathcal{D}_{b^-}^{\eta,\psi}\chi(y) = \left(-\frac{1}{\psi'(y)}\frac{d}{dy}\right)^m I_{b^-}^{m-\eta,\psi}\chi(y).$$

(ii) As β tends to 1, the ψ-Hilfer fractional derivatives become equivalent to the ψ-Caputo fractional derivatives, defined as:

$${}^C\mathcal{D}_{a^+}^{\eta,\psi}\chi(y) = I_{a^+}^{m-\eta,\psi}\left(\frac{1}{\psi'(y)}\frac{d}{dy}\right)^m \chi(y),$$

and

$${}^C\mathcal{D}_{b^-}^{\eta,\psi}\chi(y) = I_{b^-}^{m-\eta,\psi}\left(-\frac{1}{\psi'(y)}\frac{d}{dy}\right)^m \chi(y).$$

(iii) ψ-Hilfer fractional derivatives are directly related to ψ-Riemann-Liouville fractional derivatives. Below are the specific results:

$${}^H\mathcal{D}_{a^+}^{\eta,\beta,\psi}\chi(y) = I_{a^+}^{\xi-\eta,\psi}\mathcal{D}_{a^+}^{\xi,\psi}\chi(y),$$

and

$${}^H\mathcal{D}_{b^-}^{\eta,\beta,\psi}\chi(y) = I_{b^-}^{\xi-\eta,\psi}\mathcal{D}_{b^-}^{\xi,\psi}\chi(y),$$

where $\xi = \eta + \beta(m - \eta)$.

We note that we have an analogue integration by parts formula; for more details, the interested readers can consult [30, Theorem 12].

For $1 \leq s \leq \infty$, $L^s(a,b)$ denotes the set of all measurable functions χ on $[a,b]$, such that $\int_a^b |\chi(\sigma)|^s\, d\sigma < \infty$. Define the norm as

$$\|\chi\|_{L^s(a,b)} = \left(\int_a^b |\chi(\sigma)|^s\, d\sigma\right)^{\frac{1}{s}}, \text{ and } \|\chi\|_\infty = ess \sup_{a\leq\sigma\leq b} |\chi(\sigma)|.$$

Remark 2.4. *[13, 19] Let $0 < \eta \leq 1$, $s \geq 1$, and $q = \frac{s}{s-1}$. For each $\chi \in L^s(a,b)$, the following properties hold:*

(i) The operator $I_{a^+}^{\eta,\psi}\chi$ *is bounded in* $L^s(a,b)$*, and the following inequality holds:*

$$\|I_{a^+}^{\eta,\psi}\chi\|_{L^s(a,b)} \leq \frac{(\psi(b)-\psi(a))^\eta}{\Gamma(\eta+1)}\|\chi\|_{L^s(a,b)}.$$

(ii) If $\frac{1}{s} < \eta < 1$*, then* $I_{a^+}^{\eta,\psi}\chi$ *is Hölder continuous on* $[a,b]$ *with an exponent of* $\eta - \frac{1}{s}$*.*

(iii) If $\frac{1}{s} < \eta < 1$*, then* $\lim_{y\to a} I_{a^+}^{\eta,\psi}\chi(y) = 0$*, meaning that* $I_{a^+}^{\eta,\psi}\chi$ *can be continuously extended to zero at* $y = a$*. Therefore,* $I_{a^+}^{\eta,\psi}\chi$ *is continuous on* $[a,b]$*, and the following bound holds:*

$$\|I_{a^+}^{\eta,\psi}\chi\|_\infty \leq \frac{(\psi(b)-\psi(a))^{\eta-\frac{1}{s}}}{\Gamma(\eta)\,((\eta-1)q+1)^{\frac{1}{q}}}\|\chi\|_{L^s(a,b)}.$$

To establish the variational structure for problem (1.1), we introduce the fractional derivative space $E_{\eta,\beta,\psi}$, defined as the closure of $C_c^\infty([0,T],\mathbb{R})$ under the norm:

$$\|w\|_{E_p^{\eta,\beta,\psi}} = \left(\|w\|^p_{L^p(0,T)} + \|{}_0\mathcal{D}_t^{\eta,\beta,\psi}w\|^p_{L^p(0,T)}\right)^{\frac{1}{p}}.$$

The space $E_{\eta,\psi}$ can be characterized as follows:

$$E_p^{\eta,\beta,\psi} = \left\{v \in L^p([0,T],\mathbb{R}) : \mathcal{D}_{0^+}^{\eta,\beta,\psi}v \in L^p([0,T],\mathbb{R}), I_{0^+}^{\beta(\beta-1);\psi}(0) = I_T^{\beta(\beta-1);\psi}(T) = 0\right\}.$$

Remark 2.5. *[13,19] If* $0 < \eta \leq 1$ *and* $0 \leq \beta \leq 1$*, then for all* $\chi \in E_s^{\eta,\beta,\psi}$*, the following properties hold:*

(i) The space $E_s^{\eta,\beta,\psi}$ *is a Banach space that is both reflexive and separable.*

(ii) If $1-\eta > \frac{1}{s}$ *or* $\eta > \frac{1}{s}$*, we have the inequality:*

$$\|\chi\|_{L^s(0,T)} \leq \frac{(\psi(T)-\psi(0))^\eta}{\Gamma(\eta+1)}\|\mathcal{D}_{0^+}^{\eta,\beta,\psi}\chi\|_{L^s(0,T)}.$$

(iii) If $\frac{1}{s} < \eta$ *and* $q = \frac{s}{s-1}$*, then the following holds:*

$$\|\chi\|_\infty \leq \frac{(\psi(T)-\psi(0))^{\eta-\frac{1}{s}}}{\Gamma(\eta)\,((\eta-1)q+1)^{\frac{1}{q}}}\|\mathcal{D}_{0^+}^{\eta,\beta,\psi}\chi\|_{L^s(0,T)}.$$

Let

$$E := E_2^{\eta,\beta,\psi},$$

which defines a Hilbert space, and from the last remark, E can be equipped with the following equivalent norm:

$$\|\chi\| = \|{}_0\mathcal{D}_t^{\eta,\beta,\psi}\chi\|_{L^2(0,T)}.$$

Furthermore, based on Remark 2.5, there exists a constant $c_1 > 0$, such that for each $\chi \in E$, we have

$$\|\chi\|_\infty \leq c_1\|\chi\|. \tag{2.3}$$

3. Auxiliary problem

In this section, we fix a function φ in the space E and consider the following auxiliary problem:

$$(P_\varphi)\begin{cases} {}_yD_T^{\nu,\beta,\psi}\left({}_0D_y^{\nu,\beta,\psi}u(y)\right) = g(y, u(y), {}_0D_y^{\nu,\beta,\psi}\varphi(y)),\ y \in (0,T),\\ I_{0^+}^{\nu(\nu-1);\psi}u(0) = I_T^{\nu(\nu-1);\psi}u(T) = 0.\end{cases}$$

We assume the following conditions:
(H_1) We assume that there exist $q > 1$, $0 < \delta < \frac{\Gamma(\nu+1)}{(\psi(T)-\psi(0))^\nu}$, and $C_\delta > 0$ such that for all $(y,\xi,\eta) \in [0,T]\times\mathbb{R}\times\mathbb{R}$, the following inequalities hold:

$$|g(y,\xi,\eta)| \le \delta|\xi| + C_\delta|\xi|^q,$$

and

$$|G(y,\xi,\eta)| \le \frac{\delta}{2}|\xi|^2 + \frac{C_\delta}{q+1}|\xi|^{q+1}.$$

(H_2) There exist $\sigma > 2$, and positive constants c_1 and c_2, such that for all real values ξ and η, and for each $y \in [0,T]$, we have

$$G(y,\xi,\eta) \ge c_1|\xi|^\sigma - c_2,$$

where G is the antiderivative of the function g with respect to its second variable and equals zero when evaluated at zero.
(H_3) The function g satisfies the Ambrosetti-Rabinowitz condition, which means that there exists a constant $C > 0$ such that for all $(y,\eta) \in [0,T]\times\mathbb{R}$, we have:

$$0 < \sigma G(y,\xi,\eta) \le \xi g(y,\xi,\eta),\ \text{for all}\ |\xi| \ge C.$$

The first main result of this work is stated in the following theorem.

Theorem 3.1. *Under the conditions (H_1)–(H_3), problem (P_φ) has a nontrivial solution.*

To prove the existence of solutions for problem (P_φ), we will make use of the following theorem.

Theorem 3.2. *[31] Let X be a real Banach space, and let $J \in \mathfrak{C}^1(E,\mathbb{R})$ be a functional that satisfies the Palais-Smale condition. Assume the following:*

(i) $J(0) = 0$.
(ii) There exist constants $\rho > 0$ and $\chi > 0$ such that $J(z) \ge \chi$ for all $z \in E$ with $\|z\| = \rho$.
(iii) There exists $z_1 \in E$ with $\|z_1\| \ge \rho$ such that $J(z_1) < 0$.

Then, J has a critical value $c \ge \sigma$. Furthermore, c can be expressed as

$$c = \inf_{\gamma\in\Gamma}\max_{z\in[0,1]} J(\gamma(z)),$$

where $\Gamma = \{\gamma \in C([0,1],X) : \gamma(0) = 0, \gamma(1) = z_1\}$.

We recall that a functional J satisfies the Palais-Smale condition if any sequence $\{u_m\} \subset X$ such that $J(u_m)$ is bounded and $J'(u_m) \to 0$ in X', possesses a convergent subsequence.

The energy functional associated with problem (P_φ) is defined as:

$$L_\varphi(u) = \frac{1}{2}\int_0^T |{}_0D_s^{\nu,\beta,\psi}u(s)|^2\,ds - \int_0^T G(s,u(s),{}_0D_s^{\nu,\beta,\psi}\varphi(s))\,ds,$$

where $G(s,\xi,\eta) = \int_0^\xi g(s,\sigma,\eta)\,d\sigma$.
Since the function G is continuous, then we get $L_\varphi \in C^1(E,\mathbb{R})$. Moreover, for all pairs $(u,v) \in E^2$, we have:

$$\langle L'_\varphi(u), v\rangle = \int_0^T {}_0D_s^{\nu,\beta,\psi}u(s)\cdot {}_0D_s^{\nu,\beta,\psi}v(s)\,ds - \int_0^T g(s,u(s),{}_0D_s^{\nu,\beta,\psi}\varphi(s))v(s)\,ds. \tag{3.1}$$

Lemma 3.3. *Under assumptions* (H_1)–(H_3)*, the functional* L_φ *satisfies the Palais-Smale condition.*

Proof. Let $\{u_k\}$ be a sequence in E such that

$$L_\varphi(u_k) \text{ is bounded, and } L'_\varphi(u_k) \to 0 \text{ as } k \to \infty.$$

We first observe that, based on this information, there exists a constant $C_0 > 0$ such that

$$|L_\varphi(u_k)| \le C_0 \text{ and } |\langle L'_\varphi(u_k), u_k\rangle| \le C_0 \text{ for all } k \in \mathbb{N}. \tag{3.2}$$

Next, we show that the sequence $\{u_k\}$ is bounded. To do so, let $r \in \left(\frac{1}{\sigma}, \frac{1}{2}\right)$. Using assumption (H_3), we can derive the necessary bounds for any $k \in \mathbb{N}$.

$$\begin{aligned}
C_0 + rC_0 &\ge L_\varphi(u_k) - < L'_\varphi(u_k), u_k > \\
&= (\frac{1}{2} - r)\|u_k\|^2 \\
&\quad + \int_0^T \Big(ru_k(s)g(s,u_k(s),{}_0D_s^{\nu,\beta,\psi}\varphi(s)) - G(s,u_k(s),{}_0D_s^{\nu,\beta,\psi}\varphi(s))\Big)\,ds \\
&\ge (\frac{1}{2} - r)\|u_k\|^2 \\
&\quad + \int_{|u_k|\le C} \Big(ru_k(s)g(s,u_k(s),{}_0D_s^{\nu,\beta,\psi}\varphi(s)) - G(s,u_k(s),{}_0D_s^{\nu,\beta,\psi}\varphi(s))\Big)\,ds \\
&\quad + \int_{|u_k|\ge C} \Big(ru_k(s)g(s,u_k(s),{}_0D_s^{\nu,\beta,\psi}\varphi(s)) - G(s,u_k(s),{}_0D_s^{\nu,\beta,\psi}\varphi(s))\Big)\,ds \\
&\ge (\frac{1}{2} - r)\|u_k\|^2 + C_1 \\
&\quad +(r\sigma - 1)\int_{|u_k|\ge C} G(s,u_k(s),{}_0D_s^{\nu,\beta,\psi}\varphi(s))\,ds \\
&\ge (\frac{1}{2} - r)\|u_k\|^2 + C_1.
\end{aligned} \tag{3.3}$$

Since $r < \frac{1}{2}$, it is clear that the sequence $\{u_k\}$ is bounded. Moreover, since E is a reflexive space, this implies that, after possibly passing to a subsequence, there exists $u_* \in E$ such that u_k converges weakly

to u_* in E. Additionally, due to the compact embedding, u_k converges strongly to u_* in $C([0,T])$. Now, we will show that $\{u_k\}$ converges strongly to u_* in E.
From Eq (3.1), we obtain

$$\begin{aligned} \prec L'_\varphi(u_k), u_k - u_* \succ &= \int_0^T {}_0D_s^{\nu,\beta,\psi} u_k(s)\, {}_0D_s^{\nu,\beta,\psi}(u_k - u_*)(s)\, ds \\ &\quad - \int_0^T g(s, u_k(s), {}_0D_s^{\nu,\beta,\psi}\varphi(s))(u_k - u_*)(s)\, ds, \end{aligned}$$

and

$$\begin{aligned} \prec L'_\varphi(u_*), u_k - u_* \succ &= \int_0^T {}_0D_s^{\nu,\beta,\psi} u_*(s)\, {}_0D_s^{\nu,\beta,\psi}(u_k - u_*)(s)\, ds \\ &\quad - \int_0^T g(s, u_*(s), {}_0D_s^{\nu,\beta,\psi}\varphi(s))(u_k - u_*)(s)\, ds. \end{aligned}$$

So, we obtain

$$\prec L'_\varphi(u_k) - L'_\varphi(u_*), u_k - u_* \succ = \|u_k - u_*\|^2 - I_k, \tag{3.4}$$

where

$$I_k = \int_0^T \left(g(s, u_k(s), {}_0D_s^{\nu,\beta,\psi}\varphi(s)) - g(s, u_*(s), {}_0D_s^{\nu,\beta,\psi}\varphi(s)) \right)(u_k - u_*)(s)\, ds.$$

From hypothesis (H_2) and the fact that u_k converges strongly to u_* in $C([0,T])$, we obtain

$$\lim_{k\to\infty} I_k = 0. \tag{3.5}$$

Next, since $L'_\varphi(u_k) \to 0$ and $u_k - u_*$ is bounded, then we obtain

$$\lim_{k\to\infty} \prec L'_\varphi(u_k), u_k - u_* \succ = 0.$$

Moreover, from the fact that u_k converges strongly to u_* in $C([0,T])$

$$\lim_{k\to\infty} \prec L'_\varphi(u_*), u_k - u_* \succ = 0.$$

Hence, one has

$$\lim_{k\to\infty} \prec L'_\varphi(u_k) - L'_\varphi(u_*), u_k - u_* \succ = 0. \tag{3.6}$$

In conclusion, by utilizing Eqs (3.4)–(3.6), we can confirm that the sequence $\{u_k\}$ converges strongly to u_* in E.

□

Lemma 3.4. *Under assumption (H_1)–(H_3), if $q > 1$, then, there exist $\rho > 0$ and $\varpi > 0$ such that $L_\varphi(z) \geq \varpi$ for every $z \in E$ with $\|z\| = \rho$.*

Proof. Let $z \in E$, and $\delta > 0$. Then using Remark 2.5, hypothesis (H_1), and the decreasing embedding of the Lebegue spaces, we have

$$L_\varphi(z) = \frac{1}{2}\|z\|^2 - \int_0^T G(s, u(s), {}_0D_s^{\nu,\beta,\psi} z(s))ds$$

$$\begin{aligned}
&\geq \frac{1}{2}\|z\|^2 - \frac{\delta}{2}\|z\|_{L^2(0,T)} - \frac{C_\delta}{q+1}\|z\|^{q+1}_{L^{q+1}(0,T)} \\
&\geq \frac{1}{2}\|z\|^2 - \frac{\delta(\psi(T)-\psi(0))^\nu}{2\Gamma(\nu+1)}\|z\|^2 - \frac{TC_\delta\,(\psi(T)-\psi(0))^{\nu-\frac{1}{2}}}{(q+1)\Gamma(\nu)(\nu-1)^{\frac{1}{2}}}\|z\|^{q+1} \\
&\geq \|z\|^2\left(\frac{1}{2} - \frac{\delta(\psi(T)-\psi(0))^\nu}{2\Gamma(\nu+1)} - \frac{TC_\delta\,(\psi(T)-\psi(0))^{\nu-\frac{1}{2}}}{(q+1)\Gamma(\nu)(\nu-1)^{\frac{1}{2}}}\|z\|^{q-1}\right). \qquad (3.7)
\end{aligned}$$

Now, if we fix $\delta < \frac{\Gamma(\nu+1)}{(\psi(T)-\psi(0))^\nu}$ and

$$\rho < \left[\left(\frac{1}{2} - \frac{\delta(\psi(T)-\psi(0))^\nu}{2\Gamma(\nu+1)}\right)\frac{(q+1)\Gamma(\nu)(\nu-1)^{\frac{1}{2}}}{TC_\delta\,(\psi(T)-\psi(0))^{\nu-\frac{1}{2}}}\right]^{\frac{1}{q-1}}.$$

Then, we obtain

$$\varpi = \rho^2\left(\frac{1}{2} - \frac{\delta(\psi(T)-\psi(0))^\nu}{2\Gamma(\nu+1)} - \frac{TC_\delta\,(\psi(T)-\psi(0))^{\nu-\frac{1}{2}}}{(q+1)\Gamma(\nu)(\nu-1)^{\frac{1}{2}}}\rho^{q-1}\right) > 0.$$

So, we can see from (3.7) that if $\|z\| = \rho$, then $L_\varphi(z) > 0$. Thus Lemma 3.4 has been proven. □

Lemma 3.5. *Under assumptions* (H_1)–(H_3)*, there exists* $z_1 \in E$ *with* $\|z_1\| \geq \rho$ *such that* $L_\varphi(z_1) < 0$.

Proof. Let $\xi > 0$ and $z \in E$, then from hypothesis (H_2), we obtain

$$\begin{aligned}
L_\varphi(\xi z) &= \frac{\xi^2}{2}\|z\|^2 - \int_0^T G(s, \xi z(s), {}_0D_s^{\nu,\beta,\psi}\varphi)ds \\
&\leq \frac{\xi^2}{2}\|z\| - c_1\xi^\sigma\|z\|^\sigma_{L^\sigma(0,T)} + c_3T. \qquad (3.8)
\end{aligned}$$

Since we have $\sigma > 2$, then

$$\lim_{\xi\to\infty} L_\varphi(\xi u) = -\infty.$$

So, we can choose ξ large enough and $z_1 = \xi u$ so that $\|z_1\| \geq \rho$ and $L_\varphi(z_1) < 0$. □

Proof of Theorem 3.1. The proof of Theorem 3.1 naturally follows from the preceding lemmas in this section. Specifically, Lemma 3.3 confirms that L_φ satisfies the Palais-Smale condition, while Lemma 3.4 ensures that condition (*ii*) of Theorem 3.2 is met. Furthermore, Lemma 3.5 guarantees that condition (*iii*) of Theorem 3.2 is also fulfilled. Finally, noting that $L_\varphi(0) = 0$, we infer from Theorem 3.2 that L_φ possesses a critical point, which corresponds to a weak solution of the problem (P_φ). Additionally, Lemma 3.4 confirms that this solution is nontrivial.

For the remainder of this paper, a solution to problem (P_φ) will be referred to as $S(P_\varphi)$.

4. Main result and its proof

In this part, we introduce the second primary finding of this paper, which deals with the existence of solutions for the problem denoted as (1.1). The demonstration relies on the application of the outcome

established in Section 3 alongside some iterative techniques. Before revealing the main conclusion of this paper, we assume that the function g adheres to the condition outlined below:
(H_4) There exists $R > 0$ small enough, such that for every pair (η, η_1) belonging to the real numbers $\mathbb{R}$, the following holds true:

$$|g(s,\xi,\eta) - g(s,\xi_1,\eta)| \leq \alpha_1|\xi - \xi_1|,\ \forall\,(s,\xi,\xi_1) \in [0,T] \times [-R,R] \times [-R,R],$$

and

$$|g(s,\xi,\eta) - g(s,\xi,\eta_1)| \leq \alpha_2|\eta - \eta_1|, \quad \forall(s,\xi) \in [0,T] \times [-R,R],$$

where α_1 and α_2 are positive constants satisfying the condition:

$$0 < \theta := \alpha_2 \frac{\Gamma(\nu+1)(\psi(T) - \psi(0))^{\nu}}{(\Gamma(\nu+1))^2 - \alpha_1(\psi(T) - \psi(0))^{2\nu}} < 1. \tag{4.1}$$

The second result of this work is the following theorem.

Theorem 4.1. *Under the Assumptions* (H_1)–(H_4)*, problem* (1.1) *admits a nontrivial solution.*

Proof. To prove Theorem 4.1, we begin by remarking that, since the hypotheses (H_1)–(H_3) are satisfied, then, the sequence $\{u_k\} \subset E$ defined by

$$\begin{cases} u_0 \ \text{ is a fixed function in } \ E, \\ u_{k+1} = S(P_{u_k}),\ \forall\, k \in \mathbb{N}, \end{cases}$$

is well defined, moreover, for each $k \in \mathbb{N}$, we have

$$\int_0^T |{}_sD_T^{\nu,\beta,\psi}(u_k(s))|^2\,ds = \int_0^T g(s, u_k(s), {}_0D_s^{\nu,\beta,\psi}u_{k-1}(s))u_k(s)\,ds. \tag{4.2}$$

Let $0 < \delta < \frac{\Gamma(\nu+1)}{(\psi(T)-\psi(0))^{\nu}}$, then from Eq (4.2) and hypothesis (H_1), we obtain

$$\begin{aligned} \|u_k\|^2 &\leq \delta\|u\|^2_{L^2([0,T])} + C_\delta\|u\|^{q+1}_{L^{q+1}([0,T])} \\ &\leq \frac{\delta(\psi(T) - \psi(0))^{\nu}}{\Gamma(\nu+1)}\|u_k\|^2 + \frac{TC_\delta\,(\psi(T) - \psi(0))^{\nu-\frac{1}{2}}}{\Gamma(\nu)(\nu-1)^{\frac{1}{2}}}\|u_k\|^{q+1}. \end{aligned}$$

So, we obtain

$$\left(1 - \frac{\delta(\psi(T) - \psi(0))^{\nu}}{\Gamma(\nu+1)}\right)\|u_k\|^2 \leq \frac{TC_\delta\,(\psi(T) - \psi(0))^{\nu-\frac{1}{2}}}{\Gamma(\nu)(\nu-1)^{\frac{1}{2}}}\|u_k\|^{q+1}.$$

Since $q > 1$, then it follows that

$$\|u_k\| \geq \left(\frac{\Gamma(\nu)(\nu-1)^{\frac{1}{2}}}{TC_\delta\,(\psi(T) - \psi(0))^{\nu-\frac{1}{2}}}\left(1 - \frac{\delta(\psi(T) - \psi(0))^{\nu}}{\Gamma(\nu+1)}\right)\right)^{\frac{1}{q-1}} := M_0 > 0. \tag{4.3}$$

On the other hand, from the characterization of the mountain pass level (see [32]), we have

$$|L_{u_{k-1}}(u_k)| \leq \max_{s\geq 0} L_{u_{k-1}}(sz),$$

for some $z \in E$ with $\|z\| = 1$.
Now, from the last inequality, and as in Eq (3.8), we obtain

$$|L_{u_{k-1}}(u_k)| \leq \frac{s^2}{2}\|z\| - c_1 s^{\sigma} \|z\|^{\sigma}_{L^{\sigma}(0,T)} + c_2 T := f(s).$$

Since f is continuous on $[0, \infty)$, $f(0) = 0$, and $\lim_{s\to\infty} f(s) = -\infty$. Then we can deduce that f is upper bounded, so there exists $M > 0$ such that $|L_{u_{k-1}}(u_k)| \leq M$. Therefore, as in Eq (3.3), we obtain

$$\begin{aligned} (\frac{1}{2} - r)\|u_k\|^2 + C_1 &\leq L_{u_{k-1}}(u_k) - \prec L'_{u_{k-1}}(u_k), u_k \succ \\ &\leq M - \prec L'_{u_{k-1}}(u_k), u_k \succ . \end{aligned} \tag{4.4}$$

Now, from the fact that u_{k+1} is a critical point of the functional L_{u_k}, and u_k is a critical point of the functional $L_{u_{k-1}}$, one has

$$\prec L'_{u_k}(u_{k+1}), u_{k+1} - u_k \succ = 0, \text{ and } \prec L'_{u_{k-1}}(u_k), u_{k+1} - u_k \succ = 0. \tag{4.5}$$

So, by combining the last information with Eq (4.4), we can deduce the existence of $M_1 > 0$, such that

$$\|u_k\| \leq M_1. \tag{4.6}$$

Now, by combining Eq (4.6) with Eq (4.3), we get

$$0 < M_0 \leq \|u_k\| \leq M_1. \tag{4.7}$$

So, if we choose $R = c_1 M_1$, where R is given by hypothesis (H_4), and c_1 is given in Eq (2.3), then using Eq (4.7), we obtain

$$\|u_k\|_\infty \leq c_1 \|u_k\| \leq c_1 M_1 = R.$$

Next, from Eq (4.5), we deduce that

$$\prec L'_{u_k}(u_{k+1}) - L'_{u_{k-1}}(u_k), u_{k+1} - u_k \succ = 0,$$

which yields to

$$\begin{aligned} \|u_{k+1} - u_k\|^2 &= \int_0^T g\left((s, u_{k+1}(s), {}_0D_s^{\nu,\beta;\psi} u_k(s)\right)(u_{k+1} - u_k)\, ds \\ &\quad - \int_0^T g\left(s, u_k(s), {}_0D_s^{\nu,\beta;\psi} u_{k-1}(s))\right)(u_{k+1} - u_k)\, ds. \end{aligned}$$

So, from Remark 2.5, assumption (H_4), and the Hölder inequality, we obtain

$$\begin{aligned}
\|u_{k+1}-u_k\|^2 &= \int_0^T g\left((s,u_{k+1}(s),\, {}_0D_s^{\nu,\beta;\psi}u_k(s)) - g(s,u_k(s),\, {}_0D_s^{\nu,\beta;\psi}u_k(s))\right)(u_{k+1}-u_k)ds \\
&+ \int_0^T g\left((s,u_k(s),\, {}_0D_s^{\nu,\beta;\psi}u_k(s)) - g(s,u_k(s),\, {}_0D_s^{\nu,\beta;\psi}u_{k-1}(s))\right)(u_{k+1}-u_k)ds \\
&\le \alpha_1\|u_{k+1}-u_n\|^2_{L^2(0,T)} + \alpha_2\int_0^T |\,{}_0D_s^{\nu,\beta;\psi}(u_k(s)-u_{k-1}(s))|(u_{k+1}-u_k)ds \\
&\le \alpha_1\frac{(\psi(T)-\psi(0))^{2\nu}}{(\Gamma(\nu+1))^2}\|u_{k+1}-u_k\|^2 + \alpha_2\|u_k-u_{k-1}\|\,\|u_{k+1}-u_k\|_{L^2(0,T)} \\
&\le \alpha_1\frac{(\psi(T)-\psi(0))^{2\nu}}{(\Gamma(\nu+1))^2}\|u_{k+1}-u_k\|^2 \\
&+ \alpha_2\frac{(\psi(T)-\psi(0))^{\nu}}{\Gamma(\nu+1)}\|u_k-u_{k-1}\|\,\|\|u_{k+1}-u_k\|.
\end{aligned}$$

Hence, we deduce that

$$\|u_{k+1}-u_k\| \le \theta\|u_k-u_{k-1}\|,$$

where θ is given by Eq (4.1).

Based on the previous inequality, we can infer that the sequence $\{u_k\}$ is a Cauchy sequence in the Banach space E. Consequently, there is an element $u_* \in E$ to which the sequence $\{u_n\}$ converges in E. To conclude the proof, it is enough to demonstrate that u_* solves problem (1.1). To achieve this, it is necessary to show that as k approaches infinity, we get

$$\int_0^T g(s,u_{k+1}(s),\, {}_0D_s^{\nu,\beta;\psi}u_k(s))v(s)\,ds \to \int_0^T g(s,u_*(s),\, {}_0D_s^{\nu,\beta;\psi}u_*(s))v(s)\,ds. \tag{4.8}$$

Again, from Remark 2.5, assumption (H_4), and the Hölder inequality, we conclude that

$$\begin{aligned}
&\int_0^T \left(g(s,u_{k+1}(s),\, {}_0D_s^{\nu,\beta;\psi}u_k(s)) - g(s,u_*(s),\, {}_0D_s^{\nu,\beta;\psi}u_*(s))\right)v(s)\,ds \\
&= \int_0^T \left(g(s,u_{k+1}(s),\, {}_0D_s^{\nu,\beta;\psi}u_k(s)) - g(s,u_*(s),\, {}_0D_s^{\nu,\beta;\psi}u_k(s))\right)v(s)ds \\
&+ \int_0^T \left(g(s,u_*(s),\, {}_0D_s^{\nu,\beta;\psi}u_k(s)) - g(s,u_*(s),\, {}_0D_s^{\nu,\beta;\psi}u_*(s))\right)v(s)ds \\
&\le \alpha_1\|u_{k+1}-u_*\|_{L^2(0,T)}\|v\|_{L^2(0,T)} + \alpha_2\|u_k-u_*\|\|v\|_{L^2(0,T)} \\
&\le \|v\|_{L^2(0,T)}\left(\alpha_1\frac{(\psi(T)-\psi(0))^{\nu}}{\Gamma(\nu+1)}\|u_{k+1}-u_*\| + \alpha_2\|u_k-u_*\|\right).
\end{aligned}$$

Hence, we deduce that Eq (4.8) holds, and consequently the proof of Theorem 4.1 is completed. □

5. Example

In this section, we present the following illustrative example:

$$(P)\begin{cases} {}^{H}D_{1^-}^{\frac{6}{10},\frac{1}{2}}\left({}^{H}D_{0^+}^{\frac{6}{10},\frac{1}{2}}u(y)\right) = g\left(y, u(y), {}^{H}D_{0^+}^{\frac{6}{10},\frac{1}{2}}u(y)\right), \quad y \in (0,1), \\ I_{0+}^{-0.24,y}u(0) = I_{1}^{-0.24,y}u(\frac{1}{2}) = 0, \end{cases}$$

where ${}^{H}D_{1^-}^{\frac{6}{10}}$ and ${}^{H}D_{0^+}^{\frac{6}{10},\frac{1}{2}}$ are the left and right Hilfer derivatives of order $\frac{6}{10}$ and type $\frac{1}{2}$, that is, $\psi(x) = x$, $\nu = \frac{6}{10}, T = 1, \beta = \frac{1}{2}$, and

$$g(y,\xi,\eta) = (c_1 - c_2 sin\mu)|\xi|\xi,$$

with $c_1 > c_2$ are positive constants that satisfy an appropriate condition fixed later.
A simple calculation shows that, for any $\delta > 0$ small enough, we have

$$\begin{aligned} |g(y,\xi,\eta)| &\leq (c_1 + c_2)|\xi|^2 \\ &\leq \delta|\xi| + (c_1 + c_2)|\xi|^2. \end{aligned}$$

This means that the first inequality in hypothesis (H_1) is satisfied with $q = 2 > 1$. On the other hand, we have

$$\begin{aligned} |G(y,\xi,\eta)| &\leq \frac{c_1 + c_2}{3}|\xi|^3 \\ &\leq \frac{\delta}{2}|\xi|^2 + \frac{c_1 + c_2}{3}|\xi|^3. \end{aligned}$$

So, hypothesis (H_1) holds.

Now, since $-1 \leq sin\mu \leq 1$, then for any $c > 0$, we obtain

$$G(y,\xi,\eta) \geq \frac{c_1 - c_2}{3}|\xi|^3 - c,$$

which means that hypothesis (H_1) is also satisfied with $\sigma = 3 > 2$.
It is not difficult to see that

$$\xi g(y,\xi,\eta) = 3G(y,\xi,\eta) \geq 3G(y,\xi,\eta),$$

so hypothesis (H_3) is satisfied.
Next, we prove that hypothesis (H_4) holds. For this, we recall from [33, Lemme 2.1] the following elementary inequality:

$$\left||a|^{p-2}a - |b|^{p-2}b\right| \leq C(|a| + |b|)^{p-2}|a - b|, \forall\, p > 1, \tag{5.1}$$

provided that $|a| + |b| \neq 0$.
Now, we applied equation for $p = 2$, $a = \xi$ and $b = \xi_1$, we get

$$||\xi|\xi - |\xi_1|\xi_1| \leq C|\xi - \xi_1|.$$

This implies that

$$|g(y,\xi,\eta) - g(y,\xi_1,\eta)| \leq C(c_1 + c_2)|\xi - \xi_1|.$$

on the other hand, we have

$$\begin{aligned}|g(y,\xi,\eta)-g(y,\xi,\eta_1)| &\leq c_2\xi^2|sin\eta - sin\eta_1|\\ &\leq c_2R^2|\eta-\eta_1|.\end{aligned}$$

To finish, it suffices to prove that Eq (4.1) holds. From the above inequalities, we can see that $\alpha_1 = C(c_1+c_2)$ and $\alpha_2 = c_2R^2$. Since $\psi(1) = 1$ and $\psi(0) = o$, then we obtain

$$\theta = \frac{c_2R^2\Gamma(\frac{8}{5})}{\left(\Gamma(\frac{8}{5})\right)^2 - C(c_1+c_2)}.$$

If we choose $c_1 + c_2 < \frac{(\Gamma(\frac{8}{5}))^2}{C}$ and

$$0 < R < \left(\frac{\left(\Gamma(\frac{8}{5})\right)^2 - C(c_1+c_2)}{c_2\Gamma(\frac{8}{5})}\right)^{\frac{1}{2}},$$

then, all hypotheses of Theorem 4.1 hold. Hence, problem (P) admits a nontrivial solution.

6. Conclusions

In this work, we investigated the existence of solutions for a nonvariational problem involving the ψ-Hilfer fractional derivative. Precisely, we fixed a function in the functional space to transform a nonvariational problem into a variational auxiliary problem; after that, we used a mountain pass theorem to prove that the auxiliary problem admits a nontrivial solution. Later, we fix a function $u_0 \in E$, and we find a function u_1 as a solution for the auxiliary problem associated with u_0. by the same way, we determine a function u_2 as a solution for the auxiliary problem associated with u_1, so a sequence $\{u_k\}$ is constructed. Finally, we proved that this sequence is convergent and its limit is a nontrivial solution for our studied problem. This idea was first introduced by Xie et al. [27] and used later by Torres [26] for the particular case when $\psi(x) = x$ and $\beta = 1$. We hope to use this idea for a capillarity problem involving the ψ-Hilfer derivative and variable exponents, which will also be submitted to Aims Mathematics.

Author contributions

Lamya Almaghamsi: Conceptualization, writing-review & editing, funding acquisition; Aeshah Alghamdi: Conceptualization, writing-review & editing; Abdeljabbar Ghanmi: Conceptualization; Abdeljabbar Ghanmi: Resources. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no competing interests.

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