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Research article

Existence and asymptotic behavior of normalized solutions for the mass supercritical fractional Kirchhoff equations with general nonlinearities

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Abstract: In this paper, we studied a fractional Kirchhoff equation with mass supercritical general nonlinearities. Under some suitable conditions, we obtained the existence of ground state normalized solutions for this equation. Moreover, we presented the asymptotic behavior of normalized solutions to the above equation as $c \to 0^+$ and $c \to +\infty$.

Keywords: Kirchhoff equation; fractional; normalized solution; asymptotic behavior
Mathematics Subject Classification: 35A15, 35J20

1. Introduction and main results

The purpose of this paper is to study the existence and blow-up behavior of positive solutions for the following fractional Kirchhoff equation:

$$\begin{cases} \left(a + b\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx\right)(-\Delta)^s u + V(x)u + \omega u = f(u), \quad x \in \mathbb{R}^N, \\ u \in H^s(\mathbb{R}^N), \end{cases} \tag{$\mathcal{A}$}$$

having prescribed mass

$$\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = c, \tag{1.1}$$

where a, b, c are positive constants, $1 \leqslant N < 4s$, $s \in (0,1)$, $V \in C(\mathbb{R}^N, \mathbb{R})$, $f \in C(\mathbb{R}, \mathbb{R})$, $\omega \in \mathbb{R}$ is a Lagrange multiplier, and $(-\Delta)^s$ denotes the fractional Laplacian operator defined as

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s}\mathcal{F}(u)), \quad \forall \xi \in \mathbb{R}^N,$$

where $\mathcal{F}$ denotes the Fourier transform on $\mathbb{R}^N$. It is well-known that it can also be computed by

$$(-\Delta)^s v(x) = C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{v(x) - v(y)}{|x-y|^{N+2s}} dy,$$

if v is smooth enough, where $C_{N,s}$ is the normalization constant, and $P.V.$ denotes a Cauchy principle value, see [13, 15, 18].

When $s = 1$, $f(u) = |u|^{p-2}u$, $V(x) = 0$, $(\mathcal{A})$ reduces to the following case, i.e.,

$$\begin{cases} -\left(a + b\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x\right)\Delta u + \omega u = |u|^{p-2}u, & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{1.2}$$

The sharp existence and the concentration behavior of the normalized solution of (1.2) in the mass subcritical, supercritical and critical cases were established in [7, 14, 22]. In fact, the author obtained the solutions by looking for critical points of the following functional:

$$I(u) = \frac{a}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x\right)^2 - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p \mathrm{d}x$$

constrained on the L^1 sphere in $H^1(\mathbb{R}^N)$,

$$S_c = \left\{u \in H^1(\mathbb{R}^N) : \|u\|_{L^2(\mathbb{R}^N)} = c > 0\right\}.$$

In [22], Ye showed that the constrained minimization problem

$$I_{c^2} := \inf_{S_c} I(u) \tag{1.3}$$

admits a minimizer if, and only if, $c > c_p^*$ with $p \in (0, 2 + \frac{4}{N}]$ or $c \geqslant c_p^*$ with $p \in (2 + \frac{4}{N}, 2 + \frac{8}{N})$, and there is no minimizer of (1.3) if $p \in [2 + \frac{8}{N}, +\infty)$. In [23], Ye studied (1.3) with $p = 2 + \frac{8}{N}$ and obtained that there exists a mountain pass critical point of $I(u)$ on S_c if $c > c^*$. In the mass subcritical case, a complete classification with respect to the exponent p for its L^2 normalized critical points can be deduced from some simple energy estimates in [24]. To be precise, they gained existence and the uniqueness of the mountain pass type minimum for (1.3) with $p \in (2 + \frac{8}{N}, 2^*)$ or $p = 2 + \frac{8}{N}$ and $c > c^*$. Moreover, if $f(u)$ satisfies some suitable conditions, the authors of [8] studied the blow-up behavior of minimizers (1.3).

To the best of our knowledge, if $b \equiv 0$, then Eq $(\mathcal{A})$ with a prescribed mass has been studied in [16, 17, 21]. Frank-Lenzmann-Silvestre [4] established the uniqueness result of the positive ground state solution of the equation

$$(-\Delta)^s u + u = |u|^{\frac{4s}{N}} u,$$

which is an important foundation for blow-up analysis. In [2], Du et al. explored the existence, nonexistence and mass concentration of L^2-normalized solutions for nonlinear fractional Schrödinger equations with nonnegative potentials

$$(-\Delta)^s u + V(x)u = \mu u + \beta f(u),$$

under the following assumptions:
(f_1): $f \in C(\mathbb{R}, \mathbb{R})$, $|f(t)| \leqslant c_1(|t| + |t|^{p-1})$ for some $c_1 > 0$ and $2 < p < 2 + \frac{4s}{N}$.

(f_2): $f \in C(\mathbb{R}, \mathbb{R}), |f(t)| \leqslant c_1(|t| + |t|^{p-1})$ for some $c_1 > 0$ and $2 + \frac{4s}{N} < p < 2_s^* = \frac{2N}{N-2s}$.
(f_3): there exist $\nu > 2 + \frac{4s}{N}$ and $r_0 > 0$ such that

$$0 < \nu F(t) \leqslant t f(t), \quad \text{for all } |t| \geqslant r_0.$$

For the case of power function $f(t) = t^{p-2}t$ with $2 < p < 2_s^*$, Du et al. conducted a complete classification of the existence and nonexistence of minimizers, except that $p = 2+4s/N$. Very recently, Bao-Lv-Ou [1] investigated the following fractional Schrödinger equation with prescribed mass:

$$(-\Delta)^s u = \mu u + a(x)|u|^{p-2}u,$$

where $s \in (0, 1), 2 + \frac{4s}{N} < p < 2_s^*$. The existence of the bounded state normalized solution under various conditions on $a(x)$ was demonstrated in [1]. For more recent works about the fractional Schrödinger or Kirchhoff equation, see [10, 12, 20] and the references therein.

It is worth pointing out that $p = 2 + 4s/N$ is the mass critical exponent related to $(\mathscr{A})$. However, Eq $(\mathscr{A})$ involving the general potential and nonlinearities has not yet been resolved. An interesting question now is whether the same existence or nonexistence results occurs for the nonhomogeneous nonlinearities and mass supercritical case of $(\mathscr{A})$. On the other hand, there have been no previous articles studying the asymptotic behavior of solutions to $(\mathscr{A})$. Therefore, our goal is to fill the gaps in these areas. More precisely, in the first part of the paper, we prove the existence of solutions for $(\mathscr{A})$ with $V(x) \neq 0$.

Before describing more details, let's introduce the following fractional Gagliardo-Nirenberg-Sobolev inequality in [2] and Hardy inequality in [5].

Lemma 1.1. *[2] For $u \in H^s(\mathbb{R}^N)$ and $q \in (0, 2_s^* - 2)$, the fractional Gagliardo-Nirenberg-Sobolev inequality*

$$\int_{\mathbb{R}^N} |u|^{q+2} \mathrm{d}x \leqslant C_{opt} \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x \right)^{\frac{Nq}{4s}} \left(\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x \right)^{\frac{q+2}{2} - \frac{Nq}{4s}} \tag{1.4}$$

is attained at a function $\phi_q(x)$ with the following properties:
(i) $\phi_q(x)$ is radial, positive, and strictly decreasing in $|x|$.
(ii) $\phi_q(x)$ belongs to $H^{2s+1}(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$ and satisfies

$$\frac{C_1}{1 + |x|^{2s+N}} \leq \phi_q(x) \leq \frac{C_2}{1 + |x|^{2s+N}}, \quad x \in \mathbb{R}^N,$$

where $C_i (i = 1, 2)$ are positive constants.
(iii) $\phi_q(x)$ is the unique solution of the fractional Schrödinger equation

$$\begin{cases} \frac{Nq}{4s}(-\Delta)^s u + \left[1 + \frac{q}{4}\left(2 - \frac{N}{s}\right)\right] u - u^{q+1} = 0, \\ u \in H^s(\mathbb{R}^N), \ q \in (0, 2_s^* - 2). \end{cases}$$

(iv) $C_{opt} = \frac{q+2}{2\|\phi_q\|_2^q}$.

Lemma 1.2. *[5] Let $s \in (0, 1)$ and $N > 2s$. Then, for all $u \in D^{s,2}(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}} \mathrm{d}x \leqslant H_{N,s} \|u\|^2_{D^{s,2}(\mathbb{R}^N)}, \tag{1.5}$$

where

$$H_{N,s} = 2\pi^{\frac{N}{2}} \frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})} \frac{\Gamma^2(\frac{N+2s}{2})}{|\Gamma(-s)|}.$$

Lemma 1.3. *[13] Let $s \in (0,1)$ and $N > 2s$. Then, there exists a constant $S > 0$ such that for any $u \in D^{s,2}(\mathbb{R}^N)$,*

$$\|u\|^2_{L^{2_s^*}(\mathbb{R}^N)} \leqslant S^{-1}\|u\|^2_{D^{s,2}(\mathbb{R}^N)}.$$

In this paper, we will require $f(x)$ to satisfy the following conditions:
(H_1) $f \in C(\mathbb{R},\mathbb{R})$ and odd.
(H_2) There exist some $\lambda, \gamma \in \mathbb{R}^+ \times \mathbb{R}^+$ with

$$\begin{cases} 2 + \frac{8s}{N} < \lambda \leqslant \gamma < 2_s^* := \frac{2N}{N-2s}, & \text{if } N \neq 2s, \\ 2 + \frac{8s}{N} < \lambda \leqslant \gamma < 2_s^* := +\infty, & \text{if } N = 2s, \end{cases}$$

such that

$$0 < \lambda F(t) \leqslant f(t)t \leqslant \gamma F(t), \ \text{ for } \ t \neq 0, \ \text{ where } \ F(t) = \int_0^t f(\tau)d\tau.$$

(H_3) The function $\widetilde{F}(t) := \frac{1}{2}f(t)t - F(t)$ is of class C^1 and

$$\widetilde{F}'(t)t \geqslant \lambda\widetilde{F}(t), \ \ \forall t \in \mathbb{R},$$

where λ is given in (H_2).

We assume that $V(x)$ is a radial function and satisfies the following assumptions:
(V_1) $V(x) \in C^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N), p \in (\frac{N}{2},\infty)$, $\lim_{|x|\to\infty} V(x) = 0$, $\inf_{x\in\mathbb{R}^N} V(x) = 0$.
(V_2) There exists $\kappa_1 \in [0, s)$ such that one of the following two conditions holds
(i)$\nabla V(x) \cdot x \leqslant \frac{2a\kappa_1}{H_{N,s}|x|^{2s}}$, for any $x \in \mathbb{R}^N \setminus \{0\}$, where $H_{N,s}$ is given in Lemma 1.2;
(ii)$\|\max\{\nabla V(x) \cdot x, 0\}\|_{L^{\frac{N}{2s}}(\mathbb{R}^N)} \leqslant 2a\kappa_1 S$, where S is given in Lemma 1.3.
(V_3) There exists $\kappa_2 \in \left(0, \frac{N(\lambda-2)-4s}{4}\right)$ such that

$$\nabla V(x) \cdot x \geqslant -\frac{a\kappa_2}{2H_{N,s}|x|^{2s}},$$

for any $x \in \mathbb{R}^N \setminus \{0\}$.

Let us introduce the space of radial functions in $H^s(\mathbb{R}^N)$ defined by

$$H^s_{rad}(\mathbb{R}^N) = \left\{u \in H^s(\mathbb{R}^N) : u(x) = u(|x|)\right\}.$$

It is standard to see that critical points of the energy functional

$$J(u) := \frac{a}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 dx\right)^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 dx - \int_{\mathbb{R}^N} F(u)dx$$

restricted to the (mass) constraint

$$\mathcal{S}_c := \left\{u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 dx = c\right\}$$

are normalized solutions of ($\mathcal{A}$).

From (H_1) and (H_2), there exist $C_1, C_2 > 0$ such that for each $t \in \mathbb{R}$,

$$C_1 \min\{|t|^\lambda, |t|^\gamma\} \leqslant F(t) \leqslant C_2 \max\{|t|^\lambda, |t|^\gamma\} \leqslant C_2(|t|^\lambda + |t|^\gamma), \tag{1.6}$$

$$\left(\frac{\lambda}{2} - 1\right) F(t) \leqslant \widetilde{F}(t) \leqslant \left(\frac{\gamma}{2} - 1\right) F(t), \tag{1.7}$$

where $C_2 = F(1)$. It follows from (1.4) that there exists $C_3 > 0$, for any $u \in H^s(\mathbb{R}^N)$,

$$\|u\|^\lambda_{L^\lambda(\mathbb{R}^N)} \leqslant C_3 \|u\|^{\lambda'}_{D^{s,2}(\mathbb{R}^N)} \|u\|^{\lambda-\lambda'}_{L^2(\mathbb{R}^N)}, \quad \|u\|^\gamma_{L^\gamma(\mathbb{R}^N)} \leqslant C_3 \|u\|^{\gamma'}_{D^{s,2}(\mathbb{R}^N)} \|u\|^{\gamma-\gamma'}_{L^2(\mathbb{R}^N)}, \tag{1.8}$$

where $\lambda' = \frac{(\lambda-2)N}{2s} \in (2, \lambda), \gamma' = \frac{(\gamma-2)N}{2s} \in (2, \gamma)$. Let

$$L := \frac{sa - \kappa_1 a}{N(\frac{\gamma}{2} - 1)C_2 C_3}, \tag{1.9}$$

$$\varrho := \min\left\{\left(\frac{L^2/2}{c^{\lambda-\lambda'} + c^{\gamma-\gamma'}}\right)^{\frac{1}{\lambda'-2}}, 1\right\}, \tag{1.10}$$

$$K := \frac{2N - (N-2s)\gamma}{(\gamma-2)c}\left[\left(\frac{1}{2} - \frac{2s}{(\lambda-2)N} - \frac{2\kappa_2}{(\lambda-2)N}\right) a\varrho + \left(\frac{1}{4} - \frac{2s}{(\lambda-2)N}\right) b\varrho^2\right]. \tag{1.11}$$

(V_4) $\sup_{x\in\mathbb{R}^N}\left(V(x) + \frac{1}{2s}\nabla V(x) \cdot x\right) < K$.

Our first result is as follows.

Theorem 1. *Let $s \in (0, 1)$ and $N > 2s$. Assume that (V_1)–(V_4) and (H_1)–(H_3) hold. Then, Eq ($\mathcal{A}$) admits at least a radial solution.*

The second purpose of this article is to establish the existence results of ground state solutions for ($\mathcal{A}$) wih $V(x) = 0$.

Theorem 2. *Let $1 \leqslant N < 4s$, $s \in (0, 1)$, $V(x) = 0$, and suppose that f satisfies (H_1)–(H_3). Then, for all $c > 0$ fixed, Eq ($\mathcal{A}$) admits a ground state normalized solution (ω_c, u_c) with $\omega_c > 0$ and $u_c \in H^s_{rad}(\mathbb{R}^N)$.*

Remark 1. *Theorems 1 and 2 extend and complement the previous results on the fractional Kirchhoff equation with a mass super critical general nonlinearities. Particularly, unlike [8], in which $V(x) = 0$, we apply a new deformation argument for the constrained functional on $\mathcal{S}_c$.*

Remark 2. *We now point out some difficulties faced in Theorems 1 and 2.*

(i) When $V(x) > 0$ and the nonlinearity f is general mass supercritical, it prevents us from obtaining the compactness. We will apply a new deformation argument for the constraint functional with a new type of Palais-Smale condition denoted by $(PSP)_m$.

(ii) The simplest case of the function f satisfying the assumptions (H_1)–(H_3) is $f(t) = |t|^{p-2}t$ with $2+8s/N < p < 2^*_s$. Naturally, the class of general nonlinearities satisfying these assumptions is much more difficult than this homogeneous case.

(iii) Due to the appearance of the Kirchhoff nonlocal term

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x (-\Delta)^s u,$$

Eq ($\mathcal{A}$) is no longer point by point identity. Compared with [8], it is worth noting that ($\mathcal{A}$) is a double nonlocal equation, and the decay estimates of test function near infinity are different from those in the case of the classical local problem; we thus borrow ideas of [3] for nonlocal operators to establish the decay estimates. This phenomenon has caused some mathematical difficulties, making research on such problems particularly interesting.

Our other aim is to study the behavior of the normalized solution u_c given in Theorem 2 as $c \to 0$ and $c \to +\infty$. In this direction, we need to assume that
(H_4) $\lim_{t\to 0^+} \frac{f(t)}{t^{\lambda-1}} = \mu_1 > 0$.
(H_5) $\lim_{t\to +\infty} \frac{f(t)}{t^{\gamma-1}} = \mu_2 > 0$.

The following results demonstrate the asymptotic behavior of u_c in the sense of $C^{2,\alpha}_{loc}(\mathbb{R}^N)$ as well as $H^s(\mathbb{R}^N)$.

Theorem 3. *Let $1 \leqslant N < 4s$, $s \in (\frac{1}{2}, 1)$, $V(x) = 0$, and suppose that f satisfies (H_1)–(H_5). For $c > 0$, let (ω_c, u_c) be given by Theorem 2, then*

$$v_c(x) := \omega_c^{\frac{1}{2-\lambda}} u_c\left(\frac{x}{\omega_c^{\frac{1}{2s}}}\right) \to Q(x) \ \ in \ \ C^{2,\alpha}_{loc}(\mathbb{R}^N), \ \ as\ c \to +\infty,$$

where Q is the unique radial positive solution of

$$\begin{cases} a(-\Delta)^s Q + Q = \mu_1 Q^{\lambda-1}, \ \ in\ \mathbb{R}^N, \\ \lim_{|x|\to+\infty} Q(x) = 0. \end{cases}$$

Theorem 4. *Let $1 \leqslant N < 4s$, $s \in (\frac{1}{2}, 1)$, $V(x) = 0$, and suppose that f satisfies (H_1)–(H_5). For $c > 0$, let v_c and Q be given by Theorem 3, then*

$$v_c(x) \to Q(x) \ \ in \ \ H^s(\mathbb{R}^N), \ \ as\ c \to +\infty.$$

Theorem 5. *Let $1 \leqslant N < 4s$, $s \in (\frac{1}{2}, 1)$, $V(x) = 0$, and suppose that f satisfies (H_1)–(H_5). For $c > 0$, let (ω_c, u_c) be given by Theorem 2, then*

$$\bar{v}_c(x) := \omega_c^{\frac{1}{2-\lambda}} u_c\left(\frac{\|u_c\|^{\frac{1}{s}}_{D^{s,2}(\mathbb{R}^N)}}{\omega_c^{\frac{1}{2s}}} x\right) \to U(x) \ \ in \ \ C^{2,\alpha}_{loc}(\mathbb{R}^N), \ \ as\ c \to 0^+,$$

where U is the unique radial positive solution of

$$\begin{cases} b(-\Delta)^s U + U = \mu_2 U^{\gamma-1}, \ \ in\ \mathbb{R}^N, \\ \lim_{|x|\to+\infty} U(x) = 0. \end{cases}$$

Theorem 6. *Let $1 \leqslant N < 4s$, $s \in (\frac{1}{2}, 1)$, $V(x) = 0$, and suppose that f satisfies (H_1)–(H_5). For $c > 0$, let $\tilde{v}_c$ and U be given by Theorem 5, then*

$$\bar{v}_c(x) \to U(x) \ \ in \ \ H^s(\mathbb{R}^N), \ \ as\ c \to 0^+.$$

Throughout the paper, we use the following notations:

- $L^q(\mathbb{R}^N)$ denotes the Lebesgue space with the norm

$$\|u\|_{L^q(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |u|^q \mathrm{d}x\right)^{1/q}.$$

- For any $x \in \mathbb{R}^N$ and $R > 0$, $B_R(x) := \{y \in \mathbb{R}^N : |y - x| < R\}$.
- C indicates positive numbers that may be different in different lines.

The rest of this paper is organized as follows. Section 2 is dedicated to some preliminary notations and lemmas. In Section 3, we obtain the radial solutions for Eq ($\mathscr{A}$) with $V(x) \neq 0$ and Theorem 1 will be proved there. In Section 4, we derive the existence of ground state normalized solution for problem ($\mathscr{A}$) and give the proof of Theorems 2. In Section 5, we deal with asymptotic property of minimizers to problem ($\mathscr{A}$) by proving Theorems 3–6.

2. Preliminaries

In this section, we provide some lemmas that will be frequently used in the rest of this article.

We claim that the condition (V_2) yields that for any $u \in H^s(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \nabla V(x) \cdot x u^2 \mathrm{d}x \leqslant 2a\kappa_1 \|u\|^2_{D^{s,2}(\mathbb{R}^N)}. \tag{2.1}$$

Indeed, if (i) of (V_2) holds, from Lemma 1.2, we deduce that

$$\int_{\mathbb{R}^N} \nabla V(x) \cdot x u^2 \mathrm{d}x \leqslant \frac{2a\kappa_1}{H_{N,s}} \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} \mathrm{d}x \leqslant 2a\kappa_1 \|u\|^2_{D^{s,2}(\mathbb{R}^N)}.$$

If (ii) of (V_2) holds, by the Sobolev embedding inequality, we observe that

$$\begin{aligned}\int_{\mathbb{R}^N} \nabla V(x) \cdot x u^2 \mathrm{d}x \leqslant& \left(\int_{\mathbb{R}^N} |\max\{\nabla V(x) \cdot x, 0\}|^{\frac{N}{2s}} \mathrm{d}x\right)^{\frac{2s}{N}} \left(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2s}} \mathrm{d}x\right)^{\frac{N-2s}{N}} \\ \leqslant& S^{-1} \|\max\{\nabla V(x) \cdot x, 0\}\|_{L^{\frac{N}{2s}}(\mathbb{R}^N)} \|u\|^2_{D^{s,2}(\mathbb{R}^N)} \leqslant 2a\kappa_1 \|u\|^2_{D^{s,2}(\mathbb{R}^N)}.\end{aligned}$$

Define

$$\mathcal{P}(u) := sa\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + sb\|u\|^4_{D^{s,2}(\mathbb{R}^N)} - N\int_{\mathbb{R}^N} \widetilde{F}(u) \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} (\nabla V(x) \cdot x) u^2 \mathrm{d}x, \tag{2.2}$$

and

$$\mathcal{M}_c := \{u \in \mathcal{S}_c : \mathcal{P}(u) < 0\}.$$

Then

$$\partial\mathcal{M}_c := \{u \in \mathcal{S}_c : \mathcal{P}(u) = 0\}.$$

Set

$$m_c := \inf_{\partial\mathcal{M}_c} J(u).$$

Lemma 2.1. *Let $s \in (0,1)$ and $N > 2s$. Assume that (H_1)–(H_3) and (V_1)–(V_4) hold, then there exists $\bar{m} > 0$ such that $m_c \geqslant \bar{m} > 0$. Moreover,*

$$\bar{m} \geqslant \left(\frac{1}{2} - \frac{2s}{(\lambda-2)N} - \frac{2\kappa_2}{(\lambda-2)N}\right) a\varrho + \left(\frac{1}{4} - \frac{2s}{(\lambda-2)N}\right) b\varrho^2, \tag{2.3}$$

where ϱ is given in (1.10).

Proof. For any $u \in \partial\mathcal{M}_c$, applying (1.7), (1.8), (2.1), and (2.2), we obtain that

$$\begin{aligned} 0 =& sa\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + sb\|u\|^4_{D^{s,2}(\mathbb{R}^N)} - N\int_{\mathbb{R}^N} \widetilde{F}(u)\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} (\nabla V(x)\cdot x)u^2\mathrm{d}x \\ \geqslant& sa\|u\|^2_{D^{s,2}(\mathbb{R}^N)} - N\left(\frac{\gamma}{2}-1\right)\int_{\mathbb{R}^N} F(u)\mathrm{d}x - a\kappa_1\|u\|^2_{D^{s,2}(\mathbb{R}^N)} \\ \geqslant& (s-\kappa_1)a\|u\|^2_{D^{s,2}(\mathbb{R}^N)} - N\left(\frac{\gamma}{2}-1\right)C_2\left(\|u\|^\lambda_{L^\lambda(\mathbb{R}^N)} + \|u\|^\gamma_{L^\gamma(\mathbb{R}^N)}\right) \\ \geqslant& (s-\kappa_1)a\|u\|^2_{D^{s,2}(\mathbb{R}^N)} - N\left(\frac{\gamma}{2}-1\right)C_2C_3\left(\|u\|^{\lambda'}_{D^{s,2}(\mathbb{R}^N)}c^{\frac{\lambda-\lambda'}{2}} + \|u\|^{\gamma'}_{D^{s,2}(\mathbb{R}^N)}c^{\frac{\gamma-\gamma'}{2}}\right). \end{aligned} \tag{2.4}$$

Set

$$M_1 = N\left(\frac{\gamma}{2}-1\right)C_2C_3c^{\frac{\lambda-\lambda'}{2}}, \quad M_2 = N\left(\frac{\gamma}{2}-1\right)C_2C_3c^{\frac{\gamma-\gamma'}{2}}, \tag{2.5}$$
$$g(t) = (s-\kappa_1)a - M_1t^{\lambda'-2} - M_2t^{\gamma'-2}.$$

Since $g(t)$ is decreasing on $[0,+\infty)$, there exists a unique $t_0 > 0$ such that

$$(s-\kappa_1)a - M_1t_0^{\lambda'-2} - M_2t_0^{\gamma'-2} = 0, \tag{2.6}$$

and t_0 is dependent on $\kappa_1, \lambda, \gamma, C_2, C_3, c$. It follows from (2.4) that $g(\|u\|_{D^{s,2}(\mathbb{R}^N)}) \leqslant 0$. Therefore, for any $u \in \partial\mathcal{M}_c$, we observe that $\|u\|_{D^{s,2}(\mathbb{R}^N)} \geqslant t_0$ and

$$\begin{aligned} J(u) =& \frac{a}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\right)^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2\mathrm{d}x - \int_{\mathbb{R}^N} F(u)\mathrm{d}x \\ \geqslant& \frac{a}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\right)^2 - \frac{2}{\lambda-2}\int_{\mathbb{R}^N} \widetilde{F}(u)\mathrm{d}x \\ =& \left(\frac{1}{2} - \frac{2s}{(\lambda-2)N}\right)a\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + \left(\frac{1}{4} - \frac{2s}{(\lambda-2)N}\right)b\|u\|^4_{D^{s,2}(\mathbb{R}^N)} + \frac{1}{\lambda-2}\int_{\mathbb{R}^N}(\nabla V(x)\cdot x)u^2\mathrm{d}x \\ \geqslant& \left(\frac{1}{2} - \frac{2s}{(\lambda-2)N} - \frac{2\kappa_2}{\lambda-2}\right)a\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + \left(\frac{1}{4} - \frac{2s}{(\lambda-2)N}\right)b\|u\|^4_{D^{s,2}(\mathbb{R}^N)}, \end{aligned}$$

due to (1.7), (2.2), (V_3), and Lemma 1.2. Let

$$\bar{m} := \left(\frac{1}{2} - \frac{2s}{(\lambda-2)N} - \frac{2\kappa_2}{\lambda-2}\right)at_0^2 + \left(\frac{1}{4} - \frac{2s}{(\lambda-2)N}\right)bt_0^4,$$

then, $m_c \geqslant \bar{m} > 0$. From (1.9), (2.5), and (2.6), we conclude that

$$L = c^{\frac{\lambda-\lambda'}{2}}t_0^{\lambda'-2} + c^{\frac{\gamma-\gamma'}{2}}t_0^{\gamma'-2} \leqslant \left(c^{\lambda-\lambda'} + c^{\gamma-\gamma'}\right)^{\frac{1}{2}}\left(t_0^{2(\lambda'-2)} + t_0^{2(\gamma'-2)}\right)^{\frac{1}{2}},$$

which yields that

$$t_0^{2(\lambda'-2)} + t_0^{2(\gamma'-2)} \geqslant \frac{L^2}{c^{\lambda-\lambda'} + c^{\gamma-\gamma'}}.$$

If $L < c^{\frac{\gamma-\gamma'}{2}} + c^{\frac{\lambda-\lambda'}{2}}$, thus $g(1) < 0$, and we conclude that $0 < t_0 < 1$ and

$$t_0^{2(\lambda'-2)} > \frac{1}{2}\left(t_0^{2(\lambda'-2)} + t_0^{2(\gamma'-2)}\right) \geqslant \frac{L^2/2}{c^{\lambda-\lambda'} + c^{\gamma-\gamma'}},$$

that is, $t_0 > \left(\frac{L^2/2}{c^{\lambda-\lambda'}+c^{\gamma-\gamma'}}\right)^{\frac{1}{2(\lambda'-2)}}$. If $L \geqslant c^{\frac{\gamma-\gamma'}{2}} + c^{\frac{\lambda-\lambda'}{2}}$, thus $g(1) \geqslant 0$, and $t_0 \geqslant 1$. Therefore, $t_0^2 \geqslant \varrho$ and (2.3) holds. □

3. Nontrivial solutions for ($\mathcal{A}$) with $V(x) \neq 0$

As in [9], we define

$$m_\Gamma = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)),$$

where

$$\Gamma = \left\{\gamma \in C([0,1], \mathcal{S}_c)) : \gamma(0) \in \mathcal{M}_c,\ \gamma(1) \in \mathcal{S}_c \setminus \overline{\mathcal{M}}_c,\ J(\gamma(0)) < \frac{1}{2}m_c,\ J(\gamma(1)) < \frac{1}{2}m_c\right\}.$$

We will show that m_Γ is well-defined.

Lemma 3.1. *Let $s \in (0,1)$ and $N > 2s$. Assume that (H_1)–(H_3) and (V_1)–(V_4) hold, then $\Gamma \neq \emptyset$.*

Proof. For any $u \in \mathcal{S}_c$, define

$$u_\tau(x) := \tau^{\frac{N}{2}} u(\tau x), \quad \tau > 0.$$

It's not difficult to see $u_\tau(x) \in \mathcal{S}_c$. From (1.6), for any $\tau \geqslant 1$,

$$\int_{\mathbb{R}^N} F(u_\tau)\mathrm{d}x \geqslant C_1 \min\{\|u\|^\lambda_{L^\lambda(\mathbb{R}^N)}, \|u\|^\gamma_{L^\gamma(\mathbb{R}^N)}\}\tau^{\frac{\lambda-2}{2}N}, \tag{3.1}$$

and for any $0 < \tau < 1$,

$$C_1 \min\{\|u\|^\lambda_{L^\lambda(\mathbb{R}^N)}, \|u\|^\gamma_{L^\gamma(\mathbb{R}^N)}\}\tau^{\frac{\gamma-2}{2}N} \leqslant \int_{\mathbb{R}^N} F(u_\tau)\mathrm{d}x \leqslant C_2\left(\|u\|^\lambda_{L^\lambda(\mathbb{R}^N)}\tau^{\frac{\lambda-2}{2}N} + \|u\|^\gamma_{L^\gamma(\mathbb{R}^N)}\tau^{\frac{\gamma-2}{2}N}\right). \tag{3.2}$$

According to (3.1) and

$$0 \leqslant \int_{\mathbb{R}^N} V\left(\frac{x}{\tau}\right)u^2\mathrm{d}x \leqslant \|V\|_{L^\infty(\mathbb{R}^N)}c,$$

we observe that

$$\begin{aligned} J(u_\tau) =& \frac{a\tau^{2s}}{2}\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + \frac{b\tau^{4s}}{4}\|u\|^4_{D^{s,2}(\mathbb{R}^N)} + \frac{1}{2}\int_{\mathbb{R}^N} V\left(\frac{x}{\tau}\right)u^2\mathrm{d}x - \int_{\mathbb{R}^N} F(u_\tau)\mathrm{d}x \\ \leqslant& \frac{a\tau^{2s}}{2}\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + \frac{b\tau^{4s}}{4}\|u\|^4_{D^{s,2}(\mathbb{R}^N)} + \frac{1}{2}\|V\|_{L^\infty(\mathbb{R}^N)}c - C_1 \min\{\|u\|^\lambda_{L^\lambda(\mathbb{R}^N)}, \|u\|^\gamma_{L^\gamma(\mathbb{R}^N)}\}\tau^{\frac{\lambda-2}{2}N}, \end{aligned}$$

which yields that $J(u_\tau) \to -\infty$, as $\tau \to +\infty$. On the other side, from (3.2), we conclude that

$$\begin{aligned} J(u_\tau) =& \frac{a\tau^{2s}}{2}\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + \frac{b\tau^{4s}}{4}\|u\|^4_{D^{s,2}(\mathbb{R}^N)} + \frac{1}{2}\int_{\mathbb{R}^N} V\left(\frac{x}{\tau}\right)u^2 \mathrm{d}x - \int_{\mathbb{R}^N} F(u_\tau)\mathrm{d}x \\ \geqslant& \frac{a\tau^{2s}}{2}\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + \frac{b\tau^{4s}}{4}\|u\|^4_{D^{s,2}(\mathbb{R}^N)} - C_2\left(\|u\|^\lambda_{L^\lambda(\mathbb{R}^N)}\tau^{\frac{\lambda-2}{2}N} + \|u\|^\gamma_{L^\gamma(\mathbb{R}^N)}\tau^{\frac{\gamma-2}{2}N}\right), \end{aligned}$$

and

$$J(u_\tau) \leqslant \frac{a\tau^{2s}}{2}\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + \frac{b\tau^{4s}}{4}\|u\|^4_{D^{s,2}(\mathbb{R}^N)} + \frac{1}{2}\int_{\mathbb{R}^N} V\left(\frac{x}{\tau}\right)u^2\mathrm{d}x - C_1 \min\{\|u\|^\lambda_{L^\lambda(\mathbb{R}^N)}, \|u\|^\gamma_{L^\gamma(\mathbb{R}^N)}\}\tau^{\frac{\gamma-2}{2}N},$$

which leads to $J(u_\tau) \to 0^+$, as $\tau \to 0^+$, recalling $\lim_{|x|\to\infty} V(x) = 0$. From (V_2), (V_3), and (1.5), we see that

$$-2\kappa_2\tau^{2s}\|u\|^2_{D^{s,2}(\mathbb{R}^N)} \leqslant \int_{\mathbb{R}^N}(\nabla V(x)\cdot x)u_\tau^2(x)\mathrm{d}x \leqslant 2\kappa_1\tau^{2s}\|u\|^2_{D^{s,2}(\mathbb{R}^N)}.$$

Combining with (1.7), (2.2), and (3.1), we deduce that

$$\begin{aligned} \mathcal{P}(u_\tau) =& as\tau^{2s}\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + bs\tau^{4s}\|u\|^4_{D^{s,2}(\mathbb{R}^N)} - N\int_{\mathbb{R}^N}\widetilde{F}(u_\tau)\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N}(\nabla V(x)\cdot x)u_\tau^2\mathrm{d}x \\ \leqslant& (s+\kappa_2)a\tau^{2s}\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + bs\tau^{4s}\|u\|^4_{D^{s,2}(\mathbb{R}^N)} - N\left(\frac{\lambda}{2}-1\right)C_1\min\{\|u\|^\lambda_{L^\lambda(\mathbb{R}^N)}, \|u\|^\gamma_{L^\gamma(\mathbb{R}^N)}\}\tau^{\frac{\lambda-2}{2}N}, \end{aligned}$$

and

$$\mathcal{P}(u_\tau) \geqslant (s-\kappa_1)a\tau^{2s}\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + bs\tau^{4s}\|u\|^4_{D^{s,2}(\mathbb{R}^N)} - N\left(\frac{\gamma}{2}-1\right)C_2\left(\|u\|^\lambda_{L^\lambda(\mathbb{R}^N)}\tau^{\frac{\lambda-2}{2}N} + \|u\|^\gamma_{L^\gamma(\mathbb{R}^N)}\tau^{\frac{\gamma-2}{2}N}\right),$$

which yields that $\mathcal{P}(u_\tau) \to -\infty$ as $\tau \to +\infty$, and $\mathcal{P}(u_\tau) \to 0^+$ as $\tau \to 0^+$. Therefore, for any given $u \in \mathcal{S}_c$, we can take $\tau_0 > 0$ large enough, and $\tau_1 \in (0,1)$ small enough such that

$$J(u_{\tau_1}) < \frac{m_c}{2}, \;\; \mathcal{P}(u_{\tau_1}) > 0, \;\; J(u_{\tau_0}) < 0, \;\; \mathcal{P}(u_{\tau_0}) < 0.$$

Thus, taking $\gamma_0(t) := u_{\tau_0(1-t)+\tau_1 t}$, we see that $\gamma_0(t) \in \Gamma$. □

Lemma 3.2. *Let $s \in (0,1)$ and $N > 2s$. Assume that (H_1)–(H_3) and (V_1)–(V_4) hold, then $m_\Gamma \geqslant m_c \geqslant \bar{m}$.*

Proof. For any $\gamma \in \Gamma$, there exists $t_\gamma \in [0,1]$ such that $\gamma(t_\gamma) \in \partial\mathcal{M}_c$. Therefore,

$$\max_{t\in[0,1]} J(\gamma(t)) \geqslant J(\gamma(t_\gamma)) \geqslant \inf_{\partial\mathcal{M}_c} J(u) = m_c.$$

Together with Lemma 2.1, the conclusion holds. □

Now in view of Lemma 3.1, we can apply a new deformation argument for the constraint functional on $\mathcal{S}_c$ with a new type of Palais-Smale condition denoted by $(PSP)_m$. The functional J satisfies the $(PSP)_m$ condition on $\mathcal{S}_c$, if, and only if, any $(PSP)_m$ sequence $\{u_n\} \subset \mathcal{S}_c$ satisfying

$$J(u_n) \to m, \;\; \|J'(u_n)\|_{T^*_{u_n}\mathcal{S}_c} \to 0, \;\; \mathcal{P}(u_n) \to 0, \tag{3.3}$$

has a strongly convergent subsequence.

Lemma 3.3. *Let $s \in (0,1)$ and $N > 2s$. Assume that (H_1)–(H_3) and (V_1)–(V_4) hold. If $\{u_n\} \subset \mathcal{S}_c$ is a $(PSP)_m$ sequence satisfying* (3.3) *with $m \geqslant \bar{m}$, then*

(i) $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$.

(ii) there exists $\omega > 0$ such that the sequence of Lagrange multipliers ω_n satisfying

$$\left(a + b\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx\right)(-\Delta)^s u_n + V(x)u_n + \omega_n u_n = f(u_n) + o_n(1)$$

converges to ω in the sense of subsequence.

Proof. (i) For every $m \in \mathbb{R}$, let $\{u_n\} \subset \mathcal{S}_c$ be a $(PSP)_m$ sequence satisfying (3.3). From (3.3), (V_1), (V_3) and (1.7), we conclude that

$$\begin{aligned}
m + o_n(1) &= J(u_n) - \frac{2}{N(\lambda-2)}\mathcal{P}(u_n)\\
&= \left(\frac{1}{2} - \frac{2s}{N(\lambda-2)}\right)a\|u_n\|^2_{D^{s,2}(\mathbb{R}^N)} + \left(\frac{1}{4} - \frac{2s}{N(\lambda-2)}\right)b\|u_n\|^4_{D^{s,2}(\mathbb{R}^N)} + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u_n^2 \mathrm{d}x\\
&\quad + \frac{2}{\lambda-2}\int_{\mathbb{R}^N} \widetilde{F}(u_n)\mathrm{d}x - \int_{\mathbb{R}^N} F(u_n)\mathrm{d}x + \frac{1}{N(\lambda-2)}\int_{\mathbb{R}^N} (\nabla V(x)\cdot x)u_n^2 \mathrm{d}x\\
&\geqslant \left(\frac{1}{2} - \frac{2s}{N(\lambda-2)} - \frac{2\kappa_2}{N(\lambda-2)}\right)a\|u_n\|^2_{D^{s,2}(\mathbb{R}^N)} + \left(\frac{1}{4} - \frac{2s}{N(\lambda-2)}\right)b\|u_n\|^4_{D^{s,2}(\mathbb{R}^N)},
\end{aligned}$$

which implies that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$, recalling that $\kappa_2 \in \left(0, \frac{N(\lambda-2)-4s}{4}\right)$ and $\lambda > 2 + \frac{8s}{N}$. Then, up to a subsequence, $u_n \rightharpoonup u$ weakly in $H^s(\mathbb{R}^N)$. From (3.3), there exists $\{\omega_n\} \subset \mathbb{R}$ such that

$$\left(a + b\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx\right)(-\Delta)^s u_n + V(x)u_n + \omega_n u_n = f(u_n) + o_n(1) \text{ in } H^{-s}(\mathbb{R}^N).$$

(ii) By (3.3), we see that

$$\begin{aligned}
m + o_n(1) &= J(u_n) - \frac{1}{2s}\mathcal{P}(u_n)\\
&= -\frac{b}{4}\|u_n\|^4_{D^{s,2}(\mathbb{R}^N)} + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u_n^2\mathrm{d}x + \frac{N}{2s}\int_{\mathbb{R}^N}\widetilde{F}(u_n)\mathrm{d}x - \int_{\mathbb{R}^N} F(u_n)\mathrm{d}x + \frac{1}{4s}\int_{\mathbb{R}^N}(\nabla V(x)\cdot x)u_n^2\mathrm{d}x.
\end{aligned}$$

Combining with (1.7), we observe that

$$2\left[\frac{N}{2s}\left(\frac{\gamma}{2}-1\right)-1\right]\int_{\mathbb{R}^N} F(u_n)\mathrm{d}x + \int_{\mathbb{R}^N}\left[V + \frac{1}{2s}(\nabla V(x)\cdot x)\right]u_n^2\mathrm{d}x \geqslant 2m + o_n(1).$$

Set

$$\begin{aligned}
&y_n := \int_{\mathbb{R}^N} F(u_n)\mathrm{d}x, \quad z_n := \int_{\mathbb{R}^N}\left[V + \frac{1}{2s}(\nabla V(x)\cdot x)\right]u_n^2\mathrm{d}x,\\
&\bar{\lambda} := 2\left[\frac{N}{2s}\left(\frac{\lambda}{2}-1\right)-1\right], \quad \bar{\gamma} := 2\left[\frac{N}{2s}\left(\frac{\gamma}{2}-1\right)-1\right], \quad \bar{\alpha} := \frac{2N-(N-2s)\gamma}{2s}.
\end{aligned}$$

Obviously, $y_n \geqslant 0$, $\bar{\gamma}, \bar{\alpha} > 0$, and

$$\bar{\gamma}y_n + z_n \geqslant 2m + o_n(1). \tag{3.4}$$

Applying (V_4), (1.11), and Lemma 2.1, there exists $\delta > 0$ such that

$$\sup_{x\in\mathbb{R}^N}\left(V + \frac{1}{2s}(\nabla V(x)\cdot x)\right) < \frac{2s\bar{\alpha}\bar{m}}{(\bar{\gamma}+\bar{\alpha})c} - \delta.$$

Noting that $m \geqslant \bar{m}$, we obtain

$$V + \frac{1}{2s}(\nabla V(x)\cdot x) \leqslant \sup_{x\in\mathbb{R}^N}\left(V + \frac{1}{2s}(\nabla V(x)\cdot x)\right) < \frac{2s\bar{\alpha}m}{(\bar{\gamma}+\bar{\alpha})c} - \delta. \tag{3.5}$$

Let

$$z_{c,m} := \frac{2s\bar{\alpha}m}{(\bar{\gamma}+\bar{\alpha})} - \frac{\delta c}{2} < 2m,$$

and from (3.5), we deduce that

$$z_n \leqslant \frac{2s\bar{\alpha}m}{(\bar{\gamma}+\bar{\alpha})} - \delta c < z_{c,m}. \tag{3.6}$$

Using (3.4)–(3.6), we deduce that

$$z_n < \bar{\alpha}y_n - \hat{m} + o(1), \tag{3.7}$$

where $\hat{m} = \bar{\alpha}\bar{y} - \bar{z}$, and $(\bar{y}, \bar{z})$ satisfies

$$\begin{cases} \bar{\gamma}\bar{y} + \bar{z} = 2m, \\ \bar{z} = z_{c,m}. \end{cases}$$

Therefore,

$$\hat{m} = \frac{2\bar{\alpha}}{\bar{\gamma}}m - \frac{\bar{\alpha}+\bar{\gamma}}{\bar{\gamma}}z_{c,m} = \frac{\bar{\alpha}+\bar{\gamma}}{2\bar{\gamma}}\delta c > 0. \tag{3.8}$$

Gathering (1.7), (2.2), (3.7), and (3.8), we conclude that

$$\begin{aligned}
\omega_n c =& \omega_c\|u_n\|^2_{L^2(\mathbb{R}^N)} \\
=& -a\|u_n\|^2_{D^{s,2}(\mathbb{R}^N)} - b\|u_n\|^4_{D^{s,2}(\mathbb{R}^N)} - \int_{\mathbb{R}^N} V(x)u_n^2\mathrm{d}x + \int_{\mathbb{R}^N} f(u_n)u_n\mathrm{d}x + o_n(1) \\
=& -\frac{N}{s}\int_{\mathbb{R}^N}\widetilde{F}(u_n)\mathrm{d}x - \frac{1}{2s}\int_{\mathbb{R}^N}(\nabla V(x)\cdot x)u_n^2\mathrm{d}x - \int_{\mathbb{R}^N} V(x)u_n^2\mathrm{d}x + \int_{\mathbb{R}^N} f(u_n)u_n\mathrm{d}x + o_n(1) \\
=& \int_{\mathbb{R}^N}\left(-\frac{N-2s}{2s}f(u_n)u_n + \frac{N}{s}F(u_n)\right)\mathrm{d}x - \int_{\mathbb{R}^N}\left(V + \frac{1}{2s}\nabla V(x)\cdot x\right)u_n^2\mathrm{d}x + o_n(1) \\
\geqslant& \bar{\alpha}y_n - z_n + o_n(1) \\
>& \hat{m} + o_n(1).
\end{aligned} \tag{3.9}$$

Meanwhile, from (3.9), (H_2), (V_1), (V_3), and Lemmas 1.1 and 1.2, we infer that

$$\begin{aligned}
\omega_n c =& \int_{\mathbb{R}^N}\left(-\frac{N-2s}{2s}f(u_n)u_n + \frac{N}{s}F(u_n)\right)\mathrm{d}x - \int_{\mathbb{R}^N}\left(V + \frac{1}{2s}\nabla V(x)\cdot x\right)u_n^2\mathrm{d}x + o_n(1) \\
\leqslant& \left(-\frac{(N-2s)\lambda}{2s} + \frac{N}{s}\right)\int_{\mathbb{R}^N} F(u_n)\mathrm{d}x + \frac{a\kappa_2}{4s}\|u_n\|^2_{D^{s,2}(\mathbb{R}^N)} + o_n(1) \\
\leqslant& C + o_n(1).
\end{aligned} \tag{3.10}$$

(3.9) and (3.10) imply that $\omega_n \to \omega > 0$. □

In what follows, we consider that $V(x)$ is radial, and then we can choose $H^s_{rad}(\mathbb{R}^N)$ as the workspace. We take $\mathcal{S}^{rad}_c := \mathcal{S}_c \cap H^s_{rad}(\mathbb{R}^N)$.

Lemma 3.4. *Assume that* (V_1)–(V_4) *and* (H_1)–(H_3) *hold. Then,* $J(u)$ *satisfies the* $(PSP)_m$ *condition on* $\mathcal{S}^{rad}_c$ *for* $m \geqslant \bar{m}$.

Proof. Let $\{u_n\} \subset \mathcal{S}^{rad}_c$ be a $(PSP)_m$ sequence satisfying (3.3) with $m \geqslant \bar{m}$. Due to Lemma 3.3, there exist $u \in H^s_{rad}(\mathbb{R}^N)$ and $\omega > 0$ such that, up to a subsequence, $u_n \rightharpoonup u$ weakly in $H^s_{rad}(\mathbb{R}^N)$ and

$$\left(a + b\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x\right)(-\Delta)^s u + V(x)u + \omega u = f(u). \tag{3.11}$$

Then, using the compact embedding $H^s_{rad}(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^q(\mathbb{R}^N)$ for all $q \in (2, 2^*_s)$, we deduce that $u_n \to u$ strongly in $L^q(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} f(u_n)u_n \mathrm{d}x \to \int_{\mathbb{R}^N} f(u)u \mathrm{d}x. \tag{3.12}$$

Due to (V_1), for $p \in (\frac{N}{2s}, +\infty)$, $\|u_n - u\|_{L^{2p'}(\mathbb{R}^N)} \to 0$ as $n \to \infty$, where $p^{'} = \frac{p}{p-1}$. Hence, by $u_n \rightharpoonup u$ weakly in $H^s_{rad}(\mathbb{R}^N)$, we observe that

$$\int_{\mathbb{R}^N} V(x)u_n^2 \mathrm{d}x - \int_{\mathbb{R}^N} V(x)u^2 \mathrm{d}x = \int_{\mathbb{R}^N} V(x)(u-u_n)^2 \mathrm{d}x + o_n(1) \leqslant \|V\|_{L^p(\mathbb{R}^N)}\|u_n - u\|_{L^{2p'}(\mathbb{R}^N)} = o_n(1). \tag{3.13}$$

Therefroe, by (3.11)–(3.13), we see that

$$a(\|u_n\|^2_{D^{s,2}(\mathbb{R}^N)} - \|u\|^2_{D^{s,2}(\mathbb{R}^N)}) + b(\|u_n\|^4_{D^{s,2}(\mathbb{R}^N)} - \|u\|^4_{D^{s,2}(\mathbb{R}^N)}) + \omega(\|u_n\|^2_{L^2(\mathbb{R}^N)} - \|u\|^2_{L^2(\mathbb{R}^N)}) = o_n(1),$$

which implies that $u_n \to u$ strongly in $H^s_{rad}(\mathbb{R}^N)$. □

To prove the existence of nontrivial solutions, we use the following deformation result given by [9]. Define

$$\mathcal{K}_m := \{u \in \mathcal{S}_c : J(u) = m, dJ|_{\mathcal{S}_c} = 0, \mathcal{P}(u) = 0\}.$$

Lemma 3.5. *([9, Proposition 4.5]) Assume that* J *satisfies the* $(PSP)_m$ *condition on* $\mathcal{S}_c$. *For any neighborhood* D *of* $\mathcal{K}_m$ *(if* $\mathcal{K}_m = \emptyset$, $D = \emptyset$ *) and any* $\tilde{\varepsilon} > 0$, *there exists* $\varepsilon \in (0, \tilde{\varepsilon})$ *and* $\eta \in C([0,1] \times \mathcal{S}_c, \mathcal{S}_c)$ *such that*

(i) $\eta(0, u) = u$, *for* $u \in \mathcal{S}_c$;

(ii) $\eta(t, u) = u$, *for* $t \in [0, 1]$ *if* $u \in [J \leqslant m - \tilde{\varepsilon}]_{\mathcal{S}_c}$;

(iii) $t \longmapsto J(\eta(t, u))$ *is nonincreasing for* $u \in \mathcal{S}_c$;

(iv) $\eta(1, [J \leqslant m - \varepsilon]_{\mathcal{S}_c} \setminus D) \subset [J \leqslant m - \varepsilon]_{\mathcal{S}_c}, \eta(1, [J \leqslant m + \varepsilon]_{\mathcal{S}_c}) \subset [J \leqslant m - \varepsilon]_{\mathcal{S}_c} \cup D$.

Proof of Theorem 1. Let

$$\mathcal{K}_{m_\Gamma} := \{u \in \mathcal{S}_c : J(u) = m_\Gamma, dJ|_{\mathcal{S}^{rad}_c} = 0, \mathcal{P}(u) = 0\} = \emptyset,$$

then $U = \emptyset$. It follows from Lemmas 3.2 and 3.4 that J satisfies the $(PSP)_{m_\Gamma}$ condition. Due to $m_\Gamma \geqslant m_c > 0$, taking $\tilde{\varepsilon} = m_\Gamma - \frac{m_c}{2}$, and using Lemma 3.5, we obtain that there exist $\varepsilon \in (0, \tilde{\varepsilon}), \eta \in C([0,1] \times \mathcal{S}^{rad}_c, \mathcal{S}^{rad}_c)$ satisfying

$$\eta(t, u) = u, \quad \forall t \in [0, 1], \quad u \in [J \leqslant m_\Gamma - \tilde{\varepsilon}]_{\mathcal{S}^{rad}_c}, \tag{3.14}$$

$$\eta(1, [J \leqslant m_\Gamma + \varepsilon]_{\mathcal{S}_c^{rad}} \setminus U) \subset [J \leqslant m_\Gamma - \varepsilon]_{\mathcal{S}_c^{rad}}, \tag{3.15}$$

According to the definition of m_Γ and Lemma 3.1, there exists $\gamma \in \Gamma$ such that

$$\max_{t\in[0,1]} J(\gamma(t)) < m_\Gamma + \varepsilon.$$

Let us define $\tilde{\gamma} := \eta(1, \gamma(t))$, and claim that $\tilde{\gamma} \in \Gamma$. Indeed, $J(\gamma(0)) < \frac{1}{2}m_c = m_\Gamma - \tilde{\varepsilon}$, which implies that $\gamma(0) \in [J \leqslant m_\Gamma - \tilde{\varepsilon}]_{\mathcal{S}_c^{rad}}$. By (3.14), we observe that $\tilde{\gamma}(0) = \eta(1, \gamma(0)) = \gamma(0)$ and $\tilde{\gamma}(1) = \gamma(1)$ analogously. By (3.15), we conclude that

$$J(\eta(1, \gamma(t))) \leqslant m_\Gamma - \varepsilon.$$

Therefore, $J(\tilde{\gamma}(t)) \leqslant m_\Gamma - \varepsilon$ for any $t \in [0, 1]$. We have

$$m_\Gamma = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)) \leqslant \max_{t\in[0,1]} J(\tilde{\gamma}(t)) \leqslant m_\Gamma - \varepsilon,$$

which yields a contradiction. Hence, $\mathcal{K}_{m_\Gamma} \neq \emptyset$. Thie implies that $(\mathcal{A})$ admits a radial solution. □

4. Ground satate solution for $(\mathcal{A})$ with $V(x) = 0$

In this section, we consider the existence of ground state solutions for $(\mathcal{A})$ with $V(x) = 0$. From now on, in this article, we always assume that (H_1)–(H_3) hold and will not further mention it.

Lemma 4.1. *Let $1 \leqslant N < 4s$, $s \in (0, 1)$, and suppose that f satisfies (H_1)–(H_3). If $u \in H^s(\mathbb{R}^N)$ is a nontrivial solution of Eq $(\mathcal{A})$, then $u \in \mathcal{M}$, where*

$$\mathcal{M} := \{u \in H^s(\mathbb{R}^N) : \mathcal{P}(u) = 0\},$$

and

$$\mathcal{P}(u) := sa\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + sb\|u\|^4_{D^{s,2}(\mathbb{R}^N)} - N\int_{\mathbb{R}^N} \widetilde{F}(u)\mathrm{d}x.$$

Proof. Let u be a solution to Eq $(\mathcal{A})$, and we derive that

$$a\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + b\|u\|^4_{D^{s,2}(\mathbb{R}^N)} + \omega\|u\|^2_{L^2(\mathbb{R}^N)} = \int_{\mathbb{R}^N} f(u)u\mathrm{d}x. \tag{4.1}$$

Meanwhile, u satisfies the following Pohožaev identity:

$$(N - 2s)\left(a\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + b\|u\|^4_{D^{s,2}(\mathbb{R}^N)}\right) + N\omega\|u\|^2_{L^2(\mathbb{R}^N)} = 2N\int_{\mathbb{R}^N} F(u)\mathrm{d}x. \tag{4.2}$$

Combining (4.1) and (4.2), we conclude that

$$sa\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + sb\|u\|^4_{D^{s,2}(\mathbb{R}^N)} = N\int_{\mathbb{R}^N} \widetilde{F}(u)\mathrm{d}x. \tag{4.3}$$

□

Define

$$J_u(\tau) := J(u_\tau), \quad \tau > 0,$$

where $u_\tau(x) := \tau^{\frac{N}{2}} u(\tau x), \tau > 0$. We observe that

$$\mathcal{P}(u_\tau) = \tau (J_u)^{'}(\tau), \quad \tau > 0. \tag{4.4}$$

Particularly, it holds that $\mathcal{P}(u) = (J_u)^{'}(1)$. By exploiting it, we can deduce the following result.

Lemma 4.2. *Let $u \in \mathcal{S}_c$. Then, $\tau > 0$ is a critical point of $J_u(\tau)$ if, and only if, $u_\tau \in \mathcal{M}_c$, where*

$$\mathcal{M}_c := \mathcal{M} \cap \mathcal{S}_c.$$

Lemma 4.3. *For each critical point of $J|_{\mathcal{M}_c}$, if $(J_u)^{''}(1) \neq 0$, then there exists a $\omega \in \mathbb{R}$ such that*

$$J'(u) + \omega u = 0.$$

Proof. Let u be a critical point of $J(u)$ constrained on $\mathcal{M}_c$, then there exist $\omega, \nu \in \mathbb{R}$ such that

$$J'(u) + \omega u + \nu \mathcal{P}'(u) = 0. \tag{4.5}$$

Therefore, we need to show that $\nu = 0$. Let

$$\Phi(u) := J(u) + \frac{\omega}{2} \|u\|^2_{L^2(\mathbb{R}^N)} + \nu \mathcal{P}(u),$$

and it is the corresponding energy functional of (4.5). Thanks to (4.5), one sees that u satisfies the corresponding Pohozaev identity

$$(\Phi_u)'(1) := \frac{d}{d\tau} \Phi(u_\tau)|_{\tau=1} = 0. \tag{4.6}$$

By (4.4), we observe that

$$\Phi_u(\tau) = \Phi(u_\tau) = J(u_\tau) + \frac{\omega}{2} \|u\|^2_{L^2(\mathbb{R}^N)} + \nu \mathcal{P}(u_\tau) = J(u_\tau) + \frac{\omega}{2} \|u\|^2_{L^2(\mathbb{R}^N)} + \nu \tau (J_u)^{'}(\tau),$$

which yields

$$(\Phi_u)^{'}(\tau) = (1 + \nu)(J_u)^{'}(\tau) + \nu \tau (J_u)^{''}(\tau).$$

Together with (4.6), one gets

$$0 = (\Phi_u)'(1) = (1 + \nu)(J_u)^{'}(1) + \nu (J_u)^{''}(1) = (1 + \nu)\mathcal{P}(u) + \nu (J_u)^{''}(1) = \nu (J_u)^{''}(1).$$

According to $(J_u)^{''}(1) \neq 0$, we derive that $\nu = 0$. □

Lemma 4.4. *Let $1 \leqslant N < 4s$, $s \in (0, 1)$, and suppose that f satisfies (H_1)–(H_3). Then if $\omega \leqslant 0$, Eq $(\mathcal{A})$ has no nontrivial solution.*

Proof. Assume by contradiction that $u \in H^s(\mathbb{R}^N)$ is a nontrivial solution of Eq $(\mathscr{A})$ with $\omega \leqslant 0$. It follows from (4.1) and (4.2), $\omega \leqslant 0$, and (H_2) that

$$0 \geqslant s\omega\|u\|^2_{L^2(\mathbb{R}^N)} = \int_{\mathbb{R}^N}\left[NF(u) - \frac{N-2s}{2}f(u)u\right]\mathrm{d}x \geqslant 0.$$

Therefore, we derive that $\omega = 0$ and

$$\int_{\mathbb{R}^N} F(u)\mathrm{d}x = \int_{\mathbb{R}^N} f(u)u\mathrm{d}x = \int_{\mathbb{R}^N} \widetilde{F}(u)\mathrm{d}x = 0.$$

For $\omega = 0$, by Lemma 4.1, we deduce that $u \in \mathcal{M}$, that is, (4.3) holds. This yields

$$\|u\|_{D^{s,2}(\mathbb{R}^N)} = 0,$$

which contradicts $u \not\equiv 0$ in $H^s(\mathbb{R}^N)$. □

Lemma 4.5. *Let* $1 \leqslant N < 4s$, $s \in (0,1)$*, and suppose that* f *satisfies* (H_1)–(H_3)*. Then, for any* $c > 0$ *fixed, there exists a* $\sigma_c > 0$ *such that*

$$\inf\{\tau > 0 : \exists u \in \mathcal{S}_c \text{ with } \|u\|_{D^{s,2}(\mathbb{R}^N)} = 1 \text{ such that } u_\tau \in \mathcal{M}_c\} \geqslant \sigma_c,$$

i.e.,

$$\inf\{\|u\|_{D^{s,2}(\mathbb{R}^N)} : u \in \mathcal{M}_c\} \geqslant \sigma_c.$$

Proof. Since $u_\tau \in \mathcal{M}_c$, we obtain that $\mathcal{P}(u_\tau) = 0$. According to (4.3), we derive that

$$sa\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + sb\tau^{2s}\|u\|^4_{D^{s,2}(\mathbb{R}^N)} = N\tau^{-N-2s}\int_{\mathbb{R}^N}\widetilde{F}(\tau^{\frac{N}{2}}u(x))\mathrm{d}x.$$

Together with (H_2), for $\|u\|_{D^{s,2}(\mathbb{R}^N)} = 1$, we get that

$$sa < sa + sb\tau^{2s} \leqslant N\left(\frac{1}{2} - \frac{1}{\gamma}\right)\tau^{-N-2s}\int_{\mathbb{R}^N} f(\tau^{\frac{N}{2}}u(x))\tau^{\frac{N}{2}}u(x)\mathrm{d}x. \tag{4.7}$$

From (H_2), there exists some $C > 0$ such that

$$f(t)t \leqslant C(|t|^\lambda + |t|^\gamma), \quad \forall t \in \mathbb{R}.$$

Note that for $u \in \mathcal{M}_c$ with $\|u\|_{D^{s,2}(\mathbb{R}^N)} = 1$, by Lemma 1.1, there exists $C > 0$ such that

$$\|u\|^\lambda_{L^\lambda(\mathbb{R}^N)} \leq C, \quad \|u\|^\gamma_{L^\gamma(\mathbb{R}^N)} \leq C. \tag{4.8}$$

Combining (4.7) and (4.8), we deduce that

$$sa < CN\left(\frac{1}{2} - \frac{1}{\gamma}\right)\left(\tau^{\frac{N\lambda}{2}-N-2s} + \tau^{\frac{N\gamma}{2}-N-2s}\right),$$

which implies that $\sigma_c > 0$. □

Lemma 4.6. *Let* $1 \leqslant N < 4s$, $s \in (0,1)$, *and suppose that* f *satisfies* (H_1)–(H_3). *Then,* $(J_u)''(1) < 0$, *for each* $u \in \mathcal{M}_c$, *and* $\mathcal{M}_c$ *is a natural constraint of* $J|_{\mathcal{S}_c}$.

Proof. From (4.4), by a direct caculation, we derive that

$$\begin{aligned}(J_u)''(\tau) =& as(2s-1)\tau^{2s-2}\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + bs(4s-1)\tau^{4s-2}\|u\|^4_{D^{s,2}(\mathbb{R}^N)} \\ &+ N(N+1)\tau^{-N-2}\int_{\mathbb{R}^N}\widetilde{F}(\tau^{\frac{N}{2}}u(x))\mathrm{d}x - \frac{N^2}{2}\tau^{-\frac{N}{2}-2}\int_{\mathbb{R}^N}\widetilde{F}'(\tau^{\frac{N}{2}}u(x))u(x)\mathrm{d}x.\end{aligned}$$

Then,

$$(J_u)''(1) = as(2s-1)\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + bs(4s-1)\|u\|^4_{D^{s,2}(\mathbb{R}^N)} + N(N+1)\int_{\mathbb{R}^N}\widetilde{F}(u)\mathrm{d}x - \frac{N^2}{2}\int_{\mathbb{R}^N}\widetilde{F}'(u)u\mathrm{d}x.$$

Together with (H_3) and $\mathcal{P}(u) = 0$, we conclude that

$$\begin{aligned}(J_u)''(1) \leqslant& as(2s-1)\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + bs(4s-1)\|u\|^4_{D^{s,2}(\mathbb{R}^N)} + N(N+1)\int_{\mathbb{R}^N}\widetilde{F}(u)\mathrm{d}x - \frac{N^2\lambda}{2}\int_{\mathbb{R}^N}\widetilde{F}(u)\mathrm{d}x \\ =& as(2s-1)\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + bs(4s-1)\|u\|^4_{D^{s,2}(\mathbb{R}^N)} + \left(N+1-\frac{N\lambda}{2}\right)\left(sa\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + sb\|u\|^4_{D^{s,2}(\mathbb{R}^N)}\right) \\ =& \left(N+2s-\frac{N\lambda}{2}\right)sa\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + \left(N+4s-\frac{N\lambda}{2}\right)sb\|u\|^4_{D^{s,2}(\mathbb{R}^N)}.\end{aligned}$$

According to $\lambda > 2 + \frac{8s}{N}$ and Lemma 4.5, one can see that

$$(J_u)''(1) \leqslant \left(N+2s-\frac{N\lambda}{2}\right)sa\sigma_c^2 + \left(N+4s-\frac{N\lambda}{2}\right)sb\sigma_c^4 < 0.$$

Recalling Lemma 4.3, we conclude that $\mathcal{M}_c$ is a natrual contraint of $J|_{\mathcal{S}_c}$. □

Corollary 4.1. *Let* $1 \leqslant N < 4s$, $s \in (0,1)$, *and suppose that* f *satisfies* (H_1)–(H_3). *Then, for each* $u \in H^s(\mathbb{R}^N) \setminus \{0\}$, *there exists a unique* $\tau_u > 0$ *such that* $u_{\tau_u} \in \mathcal{M}$. *Moreover,*

$$J(u_{\tau_u}) = \max_{\tau>0} J(u_\tau).$$

Proof. Set $c := \|u\|^2_{L^2(\mathbb{R}^N)}$. From (H_1) and (H_2), we observe that for each $\tau \geqslant 0$ and $t \in \mathbb{R}$,

$$\begin{cases}\tau^\gamma F(t) \leqslant F(\tau t) \leqslant \tau^\lambda F(t), & \text{if } \tau \leqslant 1,\\ \tau^\lambda F(t) \leqslant F(\tau t) \leqslant \tau^\gamma F(t), & \text{if } \tau \geqslant 1.\end{cases} \tag{4.9}$$

Hence, we obtain

$$\frac{\lambda-2}{\gamma-2}\min\{\tau^\lambda,\tau^\gamma\}\widetilde{F}(t) \leqslant \widetilde{F}(\tau t) \leqslant \frac{\gamma-2}{\lambda-2}\max\{\tau^\lambda,\tau^\gamma\}\widetilde{F}(t), \tag{4.10}$$

which implies that

$$\tau^{-N}\int_{\mathbb{R}^N}\widetilde{F}(\tau^{\frac{N}{2}}u)\mathrm{d}x = o(\tau^{4s}) \text{ as } \tau \to 0^+,$$

noting that $\lambda > 2 + \frac{8s}{N}$. Recalling that

$$\mathcal{P}(u_\tau) = sa\tau^{2s}\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + sb\tau^{4s}\|u\|^4_{D^{s,2}(\mathbb{R}^N)} - N\tau^{-N}\int_{\mathbb{R}^N}\widetilde{F}(\tau^{\frac{N}{2}}u(x))\mathrm{d}x,$$

we can deduce that $\mathcal{P}(u_\tau) > 0$ for $\tau > 0$ small enough. Meanwhile, according to (4.4), one can see that $(J_u)'(\tau) > 0$ for $\tau > 0$ small enough. Then, there exists a $\tau_0 > 0$ such that $J_u(\tau)$ is increasing in $\tau \in (0, \tau_0)$.

On the other hand, from $\int_{\mathbb{R}^N} F(u)\mathrm{d}x > 0$, (4.9), and $\lambda > 2 + \frac{8s}{N}$, we derive that

$$\tau^{-N-4s}\int_{\mathbb{R}^N} F(\tau^{\frac{N}{2}}u)\mathrm{d}x \geqslant \tau^{\frac{\lambda N}{2}-N-4s}\int_{\mathbb{R}^N} F(u)\mathrm{d}x \to +\infty, \quad \text{as } \tau \to +\infty,$$

which yields that

$$\lim_{\tau\to+\infty} J_u(\tau) = \lim_{\tau\to+\infty} \tau^{4s}\left(\frac{a\tau^{-2s}}{2}\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + \frac{b}{4}\|u\|^4_{D^{s,2}(\mathbb{R}^N)} - \tau^{-N-4s}\int_{\mathbb{R}^N} F(\tau^{\frac{N}{2}}u)\mathrm{d}x\right) = -\infty.$$

Hence, there exist some $\tau_1 > \tau_0$ such that

$$J_u(\tau_1) = \max_{\tau>0} J(u_\tau),$$

and $(J_u)'(\tau_1) = 0$. Then, by Lemma 4.2, we conclude that $u_{\tau_1} \in \mathcal{M}$. Next, we will prove that τ_1 is unique. Assume by contradiction that there exists $\tau_2 > 0$ such that $u_{\tau_2} \in \mathcal{M}$. From Lemma 4.6, we know that τ_1 and τ_2 are strict local maximum points of $J_u(\tau)$. Without loss of generality, we suppose that $\tau_1 < \tau_2$. Thus, there exist some $\tau_3 \in (\tau_1, \tau_2)$ such that

$$J_u(\tau_3) = \min_{\tau\in[\tau_1,\tau_2]} J(u_\tau),$$

which indicates that τ_3 is a local minimum point of $J_u(\tau)$. Then, $(J_u)'(\tau_1) = 0$ and $u_{\tau_3} \in \mathcal{M}$ with $(J_{u_{\tau_3}})''(1) = (J_u)''(\tau_3) \geqslant 0$, which contradicts Lemma 4.6. □

Corollary 4.2. *Let $1 \leqslant N < 4s$, $s \in (0,1)$, and suppose that f satisfies (H_1)–(H_3). For each $u \in H^s(\mathbb{R}^N) \setminus \{0\}$, let τ_u be given in Corollary 4.1. Then,*

$$\begin{cases} \tau_u = 1 \Leftrightarrow (J_u)'(1) = 0 \Leftrightarrow \mathcal{P}(u) = 0, \\ \tau_u > 1 \Leftrightarrow (J_u)'(1) > 0 \Leftrightarrow \mathcal{P}(u) > 0, \\ \tau_u < 1 \Leftrightarrow (J_u)'(1) < 0 \Leftrightarrow \mathcal{P}(u) < 0. \end{cases}$$

Proof. By Corollary 4.1, we obtain that

$$J_u(\tau_u) = \max_{\tau>0} J_u(\tau).$$

Moreover,

$$(J_u)'(\tau) > 0, \quad \text{for } 0 < \tau < \tau_u, \quad \text{and} \quad (J_u)'(\tau) < 0, \quad \text{for } \tau > \tau_u.$$

The conclusion follows from (4.4). □

Lemma 4.7. *Let $1 \leqslant N < 4s$, $s \in (0,1)$, and suppose that f satisfies (H_1)–(H_3). Then $J|_{\mathcal{M}_c}$ is coercive.*

Proof. For $u \in \mathcal{M}_c$, from (H_2), we observe that

$$sa\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + sb\|u\|^4_{D^{s,2}(\mathbb{R}^N)} = N\int_{\mathbb{R}^N} \widetilde{F}(u)\mathrm{d}x \geqslant \frac{N(\lambda-2)}{2}\int_{\mathbb{R}^N} F(u)\mathrm{d}x. \tag{4.11}$$

Therefore,

$$\begin{aligned} J(u) =& \frac{a}{2}\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + \frac{b}{4}\|u\|^4_{D^{s,2}(\mathbb{R}^N)} - \int_{\mathbb{R}^N} F(u)\mathrm{d}x \\ \geqslant& \frac{a}{2}\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + \frac{b}{4}\|u\|^4_{D^{s,2}(\mathbb{R}^N)} - \frac{2s}{N(\lambda-2)}\left(a\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + b\|u\|^4_{D^{s,2}(\mathbb{R}^N)}\right) \\ =& \frac{N(\lambda-2)-4s}{2N(\lambda-2)}a\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + \frac{N(\lambda-2)-8s}{4N(\lambda-2)}b\|u\|^4_{D^{s,2}(\mathbb{R}^N)}. \end{aligned} \tag{4.12}$$

Thanks to $\lambda > 2 + \frac{8s}{N}$, we complete the proof. □

For given $c > 0$, let us define

$$m_c := \inf_{u \in \mathcal{M}_c} J(u) = \inf_{u \in \mathcal{S}_c} \max_{\tau > 0} J(u_\tau).$$

Since u is a solution to $(\mathcal{A})$ satisfying (1.1), u must belong to $\mathcal{M}_c$. If u attains m_c, we can assert that u is the least energy solution, i.e., ground state solution. It follows from (4.12) and Lemma 4.5 that $m_c > 0$ for each $c > 0$.

For each $u \in H^s(\mathbb{R}^N)$, let u^* be the symmetric radial decreasing rearrangement of u. By (H_1), without loss of generality, we assume that u is nonnegative. Then, we can obtain that

$$\begin{aligned} \int_{\mathbb{R}^N} F(u)\mathrm{d}x &= \int_{\mathbb{R}^N}\left(\int_0^{u(x)} f(t)\mathrm{d}t\right)\mathrm{d}x = \int_0^\infty f(t)|\{x : u(x) > t\}|\mathrm{d}t \\ &= \int_0^\infty f(t)|\{x : u^*(x) > t\}|\mathrm{d}t = \int_{\mathbb{R}^N} F(u^*)\mathrm{d}x. \end{aligned}$$

From [11, Lemma 2.3], one can see that

$$\iint_{\mathbb{R}^{2N}} \frac{|u^*(x)-u^*(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \leqslant \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y.$$

Thus, we derive that

$$J(u^*) \leqslant J(u). \tag{4.13}$$

Set

$$\mathcal{S}_c^{rad} := \mathcal{S}_c \cap H^s_{rad}(\mathbb{R}^N), \ \ \mathcal{M}^{rad} := \mathcal{M} \cap H^s_{rad}(\mathbb{R}^N), \ \ \mathcal{P}_c^{rad} := \mathcal{P}_c \cap H^s_{rad}(\mathbb{R}^N).$$

Define

$$m_c^{rad} := \inf_{u \in \mathcal{S}_c^{rad}} \max_{\tau > 0} J(u_\tau),$$

and we can obtain that

$$m_c^{rad} = \inf_{u \in \mathcal{M}_c^{rad}} J(u).$$

Moreover, we find the following.

Lemma 4.8. *Let $1 \leqslant N < 4s$, $s \in (0,1)$, and suppose that f satisfies (H_1)–(H_3). Then,*

$$m_c^{rad} = m_c.$$

Proof. Since $\mathcal{S}_c^{rad} \subset \mathcal{S}_c$, it is clear that $m_c^{rad} \geqslant m_c$. On the other side, for each $t > 0$,

$$\begin{aligned}
&|\{x : u_\tau^*(x) > t\}| = |\{x : \tau^{\frac{N}{2}} u^*(\tau x) > t\}| = |\{y : \tau^{\frac{N}{2}} u^*(y) > t\}|\tau^{-N} \\
=&\tau^{-N}|\{y : u^*(y) > \tau^{-\frac{N}{2}} t\}| = \tau^{-N}|\{y : u(y) > \tau^{-\frac{N}{2}} t\}| = \tau^{-N}|\{y : \tau^{\frac{N}{2}} u(y) > t\}| \\
=&|\{x : \tau^{\frac{N}{2}} u(\tau x) > t\}| = |\{x : u_\tau(x) > t\}| = |\{x : (u_\tau(x))^* > t\}|.
\end{aligned}$$

Hence, it holds true that

$$u_\tau^* = (u_\tau)^*, \quad \forall \tau \in \mathbb{R}^+.$$

As a consequence of (4.13), for each $u \in \mathcal{M}_c$, we obtain that

$$J(u_\tau^*) = J((u_\tau)^*) \leqslant J(u_\tau) \leqslant \max_{t>0} J(u_t) = J(u), \quad \forall \tau \in \mathbb{R}^+,$$

which yields that $m_c^{rad} \leqslant m_c$, by the arbitrary of $u \in \mathcal{M}_c$, and this ends the proof. □

Proof of Theorem 2. Thanks to Lemma 4.8, let $\{u_n\} \subset \mathcal{M}_c^{rad}$ be such that $J(u_n) \to m_c > 0$. From Lemma 4.7, we obtain that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$. We may suppose that up to a subsequence, $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^N)$. Then, for $N = 2, 3$, using the compact embedding $H^s_{rad}(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^q(\mathbb{R}^N)$ for all $q \in (2, 2_s^*)$, we deduce that

$$\int_{\mathbb{R}^N} F(u_n)\mathrm{d}x \to \int_{\mathbb{R}^N} F(u)\mathrm{d}x. \tag{4.14}$$

For $N = 1$, we may suppose that $u_n = u_n^*, \forall n \in \mathbb{N}$. Then, (4.14) also holds true.

Now, we claim that $u \neq 0$. Suppose by contradiction that $u = 0$. Then, $\int_{\mathbb{R}^N} \widetilde{F}(u_n)\mathrm{d}x = o_n(1)$, and taking into account of $\{u_n\} \subset \mathcal{M}_c^{rad}$, we obtain that

$$sa\|u_n\|^2_{D^{s,2}(\mathbb{R}^N)} + sb\|u_n\|^4_{D^{s,2}(\mathbb{R}^N)} = o_n(1),$$

which contradicts Lemma 4.5.

Since $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$, it is obvious that

$$\omega_n := -\frac{1}{c}\langle J'(u_n), u_n\rangle$$

is a bounded sequence. Particularly, applying (4.3), the definition of $\widetilde{F}$, and (H_2), we derive that

$$\begin{aligned}
\omega_n c =&\omega_n\|u_n\|^2_{L^2(\mathbb{R}^N)} = -\langle J'(u_n), u_n\rangle \\
=&\int_{\mathbb{R}^N} f(u_n)u_n\mathrm{d}x - a\|u_n\|^2_{D^{s,2}(\mathbb{R}^N)} - b\|u_n\|^4_{D^{s,2}(\mathbb{R}^N)} \\
=&\int_{\mathbb{R}^N} f(u_n)u_n\mathrm{d}x - \frac{N}{s}\int_{\mathbb{R}^N} \widetilde{F}(u_n)\mathrm{d}x \\
=&\frac{N}{s}\int_{\mathbb{R}^N} F(u_n)\mathrm{d}x - \frac{N-2s}{2s}\int_{\mathbb{R}^N} f(u_n)u_n\mathrm{d}x \\
\geqslant&\left[\frac{N}{s} - \frac{(N-2s)\gamma}{2s}\right]\int_{\mathbb{R}^N} F(u_n)\mathrm{d}x.
\end{aligned} \tag{4.15}$$

On the other hand, from (H_2), it follows that

$$sa\|u_n\|^2_{D^{s,2}(\mathbb{R}^N)} + sb\|u_n\|^4_{D^{s,2}(\mathbb{R}^N)} = N\int_{\mathbb{R}^N} \widetilde{F}(u_n)\mathrm{d}x \leqslant \frac{(\gamma-2)N}{2}\int_{\mathbb{R}^N} F(u_n)\mathrm{d}x. \tag{4.16}$$

Due to $\gamma > 2 + \frac{8s}{N}$, by Lemma 4.5, there exist some $\delta_c > 0$ such that for any $n \in \mathbb{N}$,

$$\int_{\mathbb{R}^N} F(u_n)\mathrm{d}x \geqslant \delta_c.$$

Therefore, together with (4.15), there exist some $\eta_c > 0$ such that for any $n \in \mathbb{N}$,

$$\omega_n \geqslant \eta_c.$$

Hence, we can assume that $\omega_n \to \omega_c > 0$. Up to a subsequence, if necessary, we can assume that $\|u_n\|^2_{D^{s,2}(\mathbb{R}^N)} \to A \geqslant 0$. It is easily seen that $u_c \in H^s_{rad}(\mathbb{R}^N)$ is a solution of

$$(a + bA)\,(-\Delta)^s u + \omega_c u = f(u), \quad x \in \mathbb{R}^N. \tag{4.17}$$

Then, $u_c \in \mathcal{M}^{rad}$ and

$$\begin{aligned} &sa\|u_c\|^2_{D^{s,2}(\mathbb{R}^N)} + sbA\|u_c\|^2_{D^{s,2}(\mathbb{R}^N)} = N\int_{\mathbb{R}^N} \widetilde{F}(u_c)\mathrm{d}x \\ =&N\lim_{n\to\infty}\int_{\mathbb{R}^N} \widetilde{F}(u_n)\mathrm{d}x = \lim_{n\to\infty}\left(sa\|u_n\|^2_{D^{s,2}(\mathbb{R}^N)} + sb\|u_n\|^4_{D^{s,2}(\mathbb{R}^N)}\right) \\ =&saA + sbA^2. \end{aligned}$$

Hence, $s(a + bA)(A - \|u_c\|^2_{D^{s,2}(\mathbb{R}^N)}) = 0$. By $s > 0, a, b > 0$, we get $\|u_c\|^2_{D^{s,2}(\mathbb{R}^N)} = A$, which yields that $u_n \to u_c$ in $D^{s,2}_0(\mathbb{R}^N)$. So, (4.17) gives that u_c is a solution of

$$\left(a + b\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2\mathrm{d}x\right)(-\Delta)^s u + \omega_c u = f(u), \quad x \in \mathbb{R}^N.$$

As a consequence, we get that

$$\begin{aligned} &a\|u_c\|^2_{D^{s,2}(\mathbb{R}^N)} + b\|u_c\|^4_{D^{s,2}(\mathbb{R}^N)} + \omega_c\|u_c\|^2_{L^2(\mathbb{R}^N)} = \int_{\mathbb{R}^N} f(u_c)u_c\mathrm{d}x = \int_{\mathbb{R}^N} f(u_n)u_n\mathrm{d}x + o_n(1) \\ =&a\|u_n\|^2_{D^{s,2}(\mathbb{R}^N)} + b\|u_n\|^4_{D^{s,2}(\mathbb{R}^N)} + \omega_n\|u_n\|^2_{L^2(\mathbb{R}^N)} + o_n(1), \end{aligned}$$

which yields that

$$\omega_c(c - \|u_c\|^2_{L^2(\mathbb{R}^N)}) = 0.$$

So, $u_c \in \mathcal{S}_c$, and then $u_c \in \mathcal{M}_c$. Therefore,

$$m_c \leqslant J(u_c) = \lim_{n\to\infty} J(u_n) = m_c,$$

that is, (ω_c, u_c) is a ground state normalized solution to Eq $(\mathscr{A})$. Recalling Lemma 4.4, we complete the proof. □

5. Proofs of Theorems 3–6

The aim of this section is to consider the continuity and the limit behavior of m_c and ω_c as $c \to 0^+$ as well as $c \to +\infty$.

For $c > 0$, let (ω_c, u_c) be the solution to $(\mathscr{A})$, which is given by Theorem 2. We remark that $\omega_c > 0$ and $u_c \in \mathcal{S}_c^{rad}$ satisfy

$$\left(a + b \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_c|^2 \mathrm{d}x\right) (-\Delta)^s u_c + \omega_c u_c = f(u_c), \quad x \in \mathbb{R}^N,$$

and

$$J(u_c) = \frac{a}{2}\|u_c\|^2_{D^{s,2}(\mathbb{R}^N)} + \frac{b}{4}\|u_c\|^4_{D^{s,2}(\mathbb{R}^N)} - \int_{\mathbb{R}^N} F(u_c)\mathrm{d}x = m_c. \tag{5.1}$$

5.1. Preliminary results on m_c and ω_c

Lemma 5.1. *Let* $1 \leqslant N < 4s$, $s \in (0,1)$*, and suppose that* f *satisfies* (H_1)–(H_3)*. Then,* m_c *is continuous with respect to* $c \in (0, +\infty)$*.*

Proof. Thanks to Theorem 2, we note that m_c is attained by a symmetric decreasing function $u_c \in H^s_{rad}(\mathbb{R}^N)$. Let $c > 0$ be fixed, and for each $\{c_n\} \subset \mathbb{R}^+$ with $c_n \to c$ as $n \to +\infty$, for the sake of simplicity, we denote (ω_{c_n}, u_{c_n}) by (ω_n, u_n). Without loss of generality, we may suppose that for any $n \in \mathbb{N}$, $\frac{c}{2} < c_n < 2c$. Taking into account of (4.11), (4.16), and (5.1), we deduce that

$$\begin{aligned} &\left[\frac{1}{2} - \frac{2s}{(\gamma-2)N}\right] a\|u_n\|^2_{D^{s,2}(\mathbb{R}^N)} + \left[\frac{1}{4} - \frac{2s}{(\gamma-2)N}\right] b\|u_n\|^4_{D^{s,2}(\mathbb{R}^N)} \\ \leqslant & m_{c_n} = J(u_n) \leqslant \left[\frac{1}{2} - \frac{2s}{(\lambda-2)N}\right] a\|u_n\|^2_{D^{s,2}(\mathbb{R}^N)} + \left[\frac{1}{4} - \frac{2s}{(\lambda-2)N}\right] b\|u_n\|^4_{D^{s,2}(\mathbb{R}^N)}. \end{aligned} \tag{5.2}$$

Let $u_{c/2}$ attain $m_{c/2}$. For $\theta \in (1,4)$, $\theta u_{c/2} \in \mathcal{S}_{\theta^2 c/2}$. From Corollary 4.1, there exists a unique $\tau_\theta > 0$ such that $(\theta u_{c/2})_{\tau_\theta} \in \mathcal{M}_{\theta^2 c/2}$ and

$$sa\tau_\theta^{2s}\|\theta u_{c/2}\|^2_{D^{s,2}(\mathbb{R}^N)} + sb\tau_\theta^{4s}\|\theta u_{c/2}\|^4_{D^{s,2}(\mathbb{R}^N)} = N\int_{\mathbb{R}^N} \widetilde{F}((\theta u_{c/2})_{\tau_\theta})\mathrm{d}x.$$

From (4.10), we conclude that

$$\begin{aligned} & sa\tau_\theta^{-2s}\|\theta u_{c/2}\|^2_{D^{s,2}(\mathbb{R}^N)} + sb\|\theta u_{c/2}\|^4_{D^{s,2}(\mathbb{R}^N)} = N\tau_\theta^{-4s}\int_{\mathbb{R}^N} \widetilde{F}((\theta u_{c/2})_{\tau_\theta})\mathrm{d}x \\ =& N\tau_\theta^{-4s}\int_{\mathbb{R}^N} \widetilde{F}\left(\tau_\theta^{\frac{N}{2}}\theta u_{c/2}(\tau_\theta x)\right)\mathrm{d}x \\ \geqslant & C\tau_\theta^{-4s}\min\left\{\left(\tau_\theta^{\frac{N}{2}}\theta\right)^\lambda, \left(\tau_\theta^{\frac{N}{2}}\theta\right)^\gamma\right\}\int_{\mathbb{R}^N} \widetilde{F}(u_{c/2}(\tau_\theta x))\mathrm{d}x \\ =& C\tau_\theta^{-4s-N}\min\left\{\left(\tau_\theta^{\frac{N}{2}}\theta\right)^\lambda, \left(\tau_\theta^{\frac{N}{2}}\theta\right)^\gamma\right\}\int_{\mathbb{R}^N} \widetilde{F}(u_{c/2}(y))\mathrm{d}y, \end{aligned}$$

which leads to that τ_θ is bounded due to $\frac{N\lambda}{2} - 4s - N \geqslant \frac{N\gamma}{2} - 4s - N > 0$. It easily follows from $m_{\theta^2 c/2} \leqslant J((\theta u_{c/2})_{\tau_\theta})$ that $\{m_{\theta^2 c/2} : \theta \in (1,4)\}$ is bounded. Together with (5.2), we conclude that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$.

Similar as the proof of Theorem 2, we can suppose that $\omega_n \to \omega_c > 0$ and $u_n \rightharpoonup u_c$ in $H^s(\mathbb{R}^N)$, where $u_c \in \mathcal{S}_c$ is a solution of

$$\left(a + b\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 \mathrm{d}x\right)(-\Delta)^s u + \omega_c u = f(u), \quad x \in \mathbb{R}^N.$$

Therefore,

$$\lim_{n\to\infty} m_{c_n} = \lim_{n\to\infty} J(u_n) = J(u_c) \geqslant m_c.$$

If $J(u_c) \neq m_c$, there exist some $\delta > 0$ such that for n large enough,

$$m_{c_n} \geqslant m_c + \delta. \tag{5.3}$$

Recall that $\sqrt{\rho}u_c \in \mathcal{S}_{\rho c}$, and let $\tau_\rho > 0$ be the unique number such that $(\sqrt{\rho}u_c)_{\tau_\rho} = \sqrt{\rho}(u_c)_{\tau_\rho} \in \mathcal{M}_{\rho c}$, that is,

$$sa\rho\tau_\rho^{2s}\|u_c\|^2_{D^{s,2}(\mathbb{R}^N)} + sb\rho^2\tau_\rho^{4s}\|u_c\|^4_{D^{s,2}(\mathbb{R}^N)} - N\int_{\mathbb{R}^N} \widetilde{F}(\sqrt{\rho}(u_c)_{\tau_\rho})\mathrm{d}x = 0.$$

By Lemma 4.5, we can derive that τ_ρ is bounded away from 0 as ρ approaches to 1. Thus, the uniqueness indicates that $\tau_\rho \to 1$ as $\rho \to 1$. Therefore, for ρ close to 1 enough, we conclude that

$$m_{\rho c} \leqslant J(\sqrt{\rho}(u_c)_{\tau_\rho}) \to J(u_c) = m_c.$$

Hence, there exists $N_0 \in \mathbb{N}$ such that for $n \geqslant N_0$,

$$m_{c_n} \leqslant m_c + \frac{\delta}{2},$$

which contradicts (5.3). Consequently, we obtain that $J(u_c) = m_c$ and

$$\lim_{n\to\infty} m_{c_n} = m_c.$$

□

Lemma 5.2. *Let* $1 \leqslant N < 4s$, $s \in (0,1)$*, and suppose that* f *satisfies* (H_1)–(H_3)*. Then,*

$$\lim_{c\to 0^+} m_c = +\infty \text{ and } \lim_{c\to+\infty} m_c = 0.$$

Proof. Using (H_2) and Lemma 1.1, there exists $C > 0$ such that

$$\begin{aligned} sa\|u_c\|^2_{D^{s,2}(\mathbb{R}^N)} + sb\|u_c\|^4_{D^{s,2}(\mathbb{R}^N)} &= N\int_{\mathbb{R}^N} \widetilde{F}(u_c)\mathrm{d}x \leqslant \frac{(\gamma-2)N}{2}\int_{\mathbb{R}^N} F(u_c)\mathrm{d}x \\ \leqslant C(\|u\|^\lambda_{L^\lambda(\mathbb{R}^N)} + \|u\|^\gamma_{L^\gamma(\mathbb{R}^N)}) &\leqslant C\left(\|u\|^{\frac{N(\lambda-2)}{2s}}_{D^{s,2}(\mathbb{R}^N)} c^{\frac{\lambda}{2}-\frac{N(\lambda-2)}{4s}} + \|u\|^{\frac{N(\gamma-2)}{2s}}_{D^{s,2}(\mathbb{R}^N)} c^{\frac{\gamma}{2}-\frac{N(\gamma-2)}{4s}}\right). \end{aligned}$$

Therefore,

$$sa < sa + sb\|u_c\|^2_{D^{s,2}(\mathbb{R}^N)} \leqslant C\left(\|u\|^{\frac{N(\lambda-2)-4s}{2s}}_{D^{s,2}(\mathbb{R}^N)} c^{\frac{\lambda}{2}-\frac{N(\lambda-2)}{4s}} + \|u\|^{\frac{N(\gamma-2)-4s}{2s}}_{D^{s,2}(\mathbb{R}^N)} c^{\frac{\gamma}{2}-\frac{N(\gamma-2)}{4s}}\right),$$

which yields that

$$\|u\|_{D^{s,2}(\mathbb{R}^N)} \to +\infty,$$

as $c \to 0^+$, since $\frac{N(\lambda-2)-4s}{2s}, \frac{\lambda}{2} - \frac{N(\lambda-2)}{4s}, \frac{N(\gamma-2)-4s}{2s}, \frac{\gamma}{2} - \frac{N(\gamma-2)}{4s} > 0$. Together with (4.12), we observe that

$$\lim_{c\to 0^+} m_c = +\infty.$$

For the case $c \to +\infty$, we choose a positive $v \in H^s(\mathbb{R}^N)$ with $\|v\|_{L^2(\mathbb{R}^N)} = 1$. For $c > 0$, it is easy to check that $u = \sqrt{c}v \in \mathcal{S}_c$. From Corollary 4.1, there exists a unique $\tau_c > 0$ such that $u_{\tau_c} \in \mathcal{M}_c$. We observe that $c\tau_c^{2s} \to 0$ as $c \to +\infty$. If not, there exists a sequence $\{c_n\}$ with $c_n \to +\infty$ as $n \to +\infty$ and $c_n\tau_{c_n}^{2s} \geqslant \delta > 0$, for all $n \in \mathbb{N}$. Applying (4.9), we obtain that

$$\begin{aligned} c_n^{-2}\tau_{c_n}^{-4s}\int_{\mathbb{R}^N}\widetilde{F}(\sqrt{c_n}v_{\tau_{c_n}})\mathrm{d}x &= c_n^{-2}\tau_{c_n}^{-4s-N}\int_{\mathbb{R}^N}\widetilde{F}\left(\sqrt{c_n}\tau_{c_n}^{\frac{N}{2}}v(y)\right)\mathrm{d}y \\ \geqslant & c_n^{-2}\tau_{c_n}^{-4s-N}\frac{\lambda-2}{\gamma-2}\min\left\{\left(\sqrt{c_n}\tau_{c_n}^{\frac{N}{2}}\right)^{\lambda}, \left(\sqrt{c_n}\tau_{c_n}^{\frac{N}{2}}\right)^{\gamma}\right\}\int_{\mathbb{R}^N}\widetilde{F}(v(y))\mathrm{d}y. \end{aligned}$$

According to

$$c_n^{-2}\tau_{c_n}^{-4s-N}\left(\sqrt{c_n}\tau_{c_n}^{\frac{N}{2}}\right)^{\lambda} = \left(c_n\tau_{c_n}^{2s}\right)^{-2-\frac{N}{2s}+\frac{N\lambda}{4s}} c_n^{\frac{2N-(N-2s)\lambda}{4s}} \geqslant \delta^{-2-\frac{N}{2s}+\frac{N\lambda}{4s}} c_n^{\frac{2N-(N-2s)\lambda}{4s}} \to +\infty, \quad \text{as } n \to +\infty,$$

and

$$c_n^{-2}\tau_{c_n}^{-4s-N}\left(\sqrt{c_n}\tau_{c_n}^{\frac{N}{2}}\right)^{\gamma} \to +\infty \ \text{ as } n \to +\infty,$$

we conclude that

$$c_n^{-2}\tau_{c_n}^{-4s}\int_{\mathbb{R}^N}\widetilde{F}(\sqrt{c_n}v_{\tau_{c_n}})\mathrm{d}x \to +\infty, \quad \text{as } n \to +\infty.$$

Therefore,

$$\begin{aligned} 0 =&\mathcal{P}(u_{\tau_{c_n}}) = sa\tau_{c_n}^{2s}\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + sb\tau_{c_n}^{4s}\|u\|^4_{D^{s,2}(\mathbb{R}^N)} - N\int_{\mathbb{R}^N}\widetilde{F}(u_{\tau_{c_n}})\mathrm{d}x \\ =&sac_n\tau_{c_n}^{2s}\|v\|^2_{D^{s,2}(\mathbb{R}^N)} + sbc_n^2\tau_{c_n}^{4s}\|v\|^4_{D^{s,2}(\mathbb{R}^N)} - N\int_{\mathbb{R}^N}\widetilde{F}(\sqrt{c_n}v_{\tau_{c_n}})\mathrm{d}x \\ =&c_n^2\tau_{c_n}^{4s}\left(sac_n^{-1}\tau_{c_n}^{-2s}\|v\|^2_{D^{s,2}(\mathbb{R}^N)} + sb\|v\|^4_{D^{s,2}(\mathbb{R}^N)} - Nc_n^{-2}\tau_{c_n}^{-4s}\int_{\mathbb{R}^N}\widetilde{F}(\sqrt{c_n}v_{\tau_{c_n}})\mathrm{d}x\right) \\ <&0, \end{aligned}$$

for n large enough, which is a contradiction. Hence, our claim $c\tau_c^{2s} \to 0$ as $c \to +\infty$ holds true. Consequently, combining with $w := u_{\tau_c} = \sqrt{c}v_{\tau_c} \in \mathcal{M}_c$, we deduce that

$$\begin{aligned} m_c \leqslant J(w) =&\frac{a}{2}\|w\|^2_{D^{s,2}(\mathbb{R}^N)} + \frac{b}{4}\|w\|^4_{D^{s,2}(\mathbb{R}^N)} - \int_{\mathbb{R}^N}F(w)\mathrm{d}x \\ \leqslant&\frac{a}{2}\|w\|^2_{D^{s,2}(\mathbb{R}^N)} + \frac{b}{4}\|w\|^4_{D^{s,2}(\mathbb{R}^N)} - \frac{2}{\gamma-2}\int_{\mathbb{R}^N}\widetilde{F}(w)\mathrm{d}x \\ =&\frac{a}{2}\|w\|^2_{D^{s,2}(\mathbb{R}^N)} + \frac{b}{4}\|w\|^4_{D^{s,2}(\mathbb{R}^N)} - \frac{2}{(\gamma-2)N}\left(sa\|w\|^2_{D^{s,2}(\mathbb{R}^N)} + sb\|w\|^4_{D^{s,2}(\mathbb{R}^N)}\right) \\ =&\left(\frac{1}{2} - \frac{2s}{(\gamma-2)N}\right)a\|w\|^2_{D^{s,2}(\mathbb{R}^N)} + \left(\frac{1}{4} - \frac{2s}{(\gamma-2)N}\right)b\|w\|^4_{D^{s,2}(\mathbb{R}^N)} \\ =&\left(\frac{1}{2} - \frac{2s}{(\gamma-2)N}\right)ac\tau_c^{2s}\|v\|^2_{D^{s,2}(\mathbb{R}^N)} + \left(\frac{1}{4} - \frac{2s}{(\gamma-2)N}\right)b(c\tau_c^{2s})^2\|v\|^4_{D^{s,2}(\mathbb{R}^N)} \\ \to&0, \quad \text{as } c \to +\infty. \end{aligned}$$

On the other hand, it follows from (4.12) and Lemma 4.5 that $m_c > 0$ for each $c > 0$. This completes the proof. □

Lemma 5.3. *Let* $1 \leqslant N < 4s$*,* $s \in (0,1)$*, and suppose that* f *satisfies* (H_1)*–*(H_3)*. Then,*

$$\lim_{c\to 0^+} \omega_c c = +\infty \ \text{ and } \ \lim_{c\to +\infty} \omega_c c = 0.$$

Proof. Combining (5.2) and Lemma 5.2, we derive that

$$\lim_{c\to 0^+} \|u_c\|_{D^{s,2}(\mathbb{R}^N)} = +\infty, \ \text{ and } \ \lim_{c\to +\infty} \|u_c\|_{D^{s,2}(\mathbb{R}^N)} = 0. \tag{5.4}$$

Moreover, from $\mathcal{P}(u_c) = 0$, we conclude that

$$\int_{\mathbb{R}^N} \widetilde{F}(u_c)\mathrm{d}x = \frac{1}{N}\left(as\|u_c\|^2_{D^{s,2}(\mathbb{R}^N)} + bs\|u_c\|^4_{D^{s,2}(\mathbb{R}^N)}\right).$$

Together with (5.4), we see that

$$\lim_{c\to 0^+} \int_{\mathbb{R}^N} \widetilde{F}(u_c)\mathrm{d}x = +\infty, \ \text{ and } \ \lim_{c\to +\infty} \int_{\mathbb{R}^N} \widetilde{F}(u_c)\mathrm{d}x = 0. \tag{5.5}$$

Similar as (4.15), we derive that

$$\omega_c c = \omega_c\|u_c\|^2_{L^2(\mathbb{R}^N)} = \frac{N}{s}\int_{\mathbb{R}^N} F(u_c)\mathrm{d}x - \frac{N-2s}{2s}\int_{\mathbb{R}^N} f(u_c)u_c\mathrm{d}x.$$

By (H_2), there exist $C_1, C_2 > 0$ such that

$$C_1\int_{\mathbb{R}^N} \widetilde{F}(u_c)\mathrm{d}x \leqslant \omega_c c \leqslant C_2\int_{\mathbb{R}^N} \widetilde{F}(u_c)\mathrm{d}x,$$

which clearly means

$$\lim_{c\to 0^+} \omega_c c = +\infty \ \text{ and } \ \lim_{c\to +\infty} \omega_c c = 0,$$

recalling (5.5). □

Remark 3. *For the two quantities* $\zeta(c), \varphi(c)$*, if there exist* $C_1, C_2 > 0$ *independent of c such that*

$$C_1\zeta(c) \leqslant \varphi(c) \leqslant C_2\zeta(c),$$

we say $\zeta(c)$ *and* $\varphi(c)$ *are comparable. Thus, by the proofs above in Section 5, it is easy to see that any two elements in the set*

$$\left\{m_c, \omega_c c, \|u_c\|^2_{D^{s,2}(\mathbb{R}^N)} + \|u_c\|^4_{D^{s,2}(\mathbb{R}^N)} \int_{\mathbb{R}^N} F(u_c)\mathrm{d}x, \int_{\mathbb{R}^N} \widetilde{F}(u_c)\mathrm{d}x, \int_{\mathbb{R}^N} f(u_c)u_c\mathrm{d}x\right\},$$

are comparable.

5.2. The case of $c \to +\infty$

Let $c_n \to +\infty$. By Lemma 5.3, we know that $\omega_{c_n} \to 0$. For convenience, we denote (ω_{c_n}, u_{c_n}) as (ω_n, c_n). Set

$$e_n := \|u_n\|^2_{D^{s,2}(\mathbb{R}^N)}.$$

From (5.4), one sees that $e_n \to 0$ as $n \to +\infty$.

Lemma 5.4. *Let $1 \leqslant N < 4s$, $s \in (\frac{1}{2}, 1)$, and suppose that f satisfies (H_1)–(H_5). Let (ω_c, u_c) be the solution given by Theorem 2. Then,*

$$\limsup_{c\to+\infty} u_c(0) < +\infty.$$

Proof. Let us argue by contradiction. Assume that there exists a sequence $\{c_n\} \to +\infty$ such that

$$m_n := u_n(0) = \max_{x\in\mathbb{R}^N} u_n(x) \to +\infty.$$

Take $x = y/m_n^{\frac{\gamma-2}{2s}}$ and define

$$Q_n(y) = \frac{u_n\left(y/m_n^{\frac{\gamma-2}{2s}}\right)}{m_n}, \ \ y \in \mathbb{R}^N.$$

Thus, $\max_{y\in\mathbb{R}^N} Q_n(y) = 1$ and

$$(-\Delta)^s Q_n(y) = \frac{1}{a + be_n}\left[\frac{f(m_n Q_n(y))}{m_n^{\gamma-1}} - \frac{\omega_n}{m_n^{\gamma-2}} Q_n(y)\right]. \tag{5.6}$$

It follows from (H_1) and (H_2) that there exists a $C > 0$ such that

$$f(t) \leqslant C(t^{\gamma-1} + t^{\lambda-1}).$$

Therefore, we note that

$$\frac{1}{a + be_n}\left[\frac{f(m_n Q_n(y))}{m_n^{\gamma-1}} - \frac{\omega_n}{m_n^{\gamma-2}} Q_n(y)\right] \in L^\infty(\mathbb{R}^N).$$

Then, applying a similar argument to the proof of [19, Proposition 4.4], and passing to a subsequence if necessary, $Q_n \to Q$ in $C^{2,\alpha}_{loc}(\mathbb{R}^N)$, for some $\alpha \in (0, 1)$. It is easy to see that Q satisfies, in weak sense,

$$(-\Delta)^s Q = \frac{\mu_2}{a} Q^{\gamma-1}, \ \text{ in } \ \mathbb{R}^N.$$

According to [6, Theorem 1.5], we derive that $Q = 0$, which contradicts $Q(0) = 1$. □

Define

$$\tilde{u}_n(x) := \frac{1}{u_n(0)} u_n\left(\frac{x}{\omega_n^{\frac{1}{2s}}}\right).$$

By direct calculation, we see that $\tilde{u}_n(0) = \|\tilde{u}_n\|_{L^\infty(\mathbb{R}^N)} = 1$ and

$$(a + be_n)(-\Delta)^s \tilde{u}_n(x) + \tilde{u}_n(x) = \frac{1}{\omega_n u_n(0)} f(u_n(0)\tilde{u}_n(x)). \tag{5.7}$$

Lemma 5.5. *Let $1 \leqslant N < 4s$, $s \in (\frac{1}{2}, 1)$, and suppose that f satisfies (H_1)–(H_5). Let (ω_c, u_c) be the solution given by Theorem 2. Then,*

$$\liminf_{c\to+\infty} \frac{[u_c(0)]^{\lambda-2}}{\omega_c} > 0.$$

Proof. We assume by contradiction that there exists a sequence $c_n \to +\infty$ such that

$$[u_n(0)]^{\lambda-2} = o_n(\omega_n).$$

Letting $x = 0$ in (5.7), one sees that

$$\begin{aligned} 1 =&\tilde{u}_n(0) \leqslant (a + be_n)(-\Delta)^s\tilde{u}_n(0) + \tilde{u}_n(0) = \frac{1}{\omega_n u_n(0)} f(u_n(0)\tilde{u}(0)) \\ \leqslant&\frac{C}{\omega_n u_n(0)}\left([u_n(0)]^{\lambda-1} + [u_n(0)]^{\gamma-1}\right) \leqslant \frac{C[u_n(0)]^{\lambda-2}}{\omega_n} = o_n(1), \end{aligned}$$

which is a contradiction. □

Lemma 5.6. *Let $1 \leqslant N < 4s$, $s \in (\frac{1}{2}, 1)$, and suppose that f satisfies (H_1)–(H_5). Let (ω_c, u_c) be the solution given by Theorem 2. Then, $u_c(0) = \|u_c\|_{L^\infty(\mathbb{R}^N)} \to 0$, as $c \to +\infty$.*

Proof. Recalling that $\omega_c \to 0^+$ and $\|u_n\|_{D^{s,2}(\mathbb{R}^N)} \to 0$ as $c \to +\infty$, assume by contradiction that $\liminf_{n\to+\infty} u_n(0) > 0$. According to $f(u_n) - \omega_n u_n \in L^\infty(\mathbb{R}^N)$, applying a similar argument to the proof of [19, Proposition 4.4], and passing to a subsequence if necessary, we assume that $u_n \to u_c$ in $C^{2,\alpha}_{loc}(\mathbb{R}^N)$, for some $\alpha \in (0, 1)$ with $u_c(0) = \max_{x\in\mathbb{R}^N} u_c(x) > 0$, and u_c is a nonnegative bounded radial solution of

$$(-\Delta)^s u = \frac{1}{a} f(u) \geqslant 0 \ \text{ in } \ \mathbb{R}^N.$$

Then, by [6, Theorem 1.1], we derive that $u_c \equiv 0$, which contradicts $u_c(0) > 0$. □

Lemma 5.7. *Let $1 \leqslant N < 4s$, $s \in (\frac{1}{2}, 1)$, and suppose that f satisfies (H_1)–(H_5). Let (ω_c, u_c) be the solution given by Theorem 2. Then,*

$$\limsup_{c\to+\infty} \frac{[u_c(0)]^{\lambda-2}}{\omega_c} < +\infty.$$

Proof. Assume by contradiction that there exists a sequence $c_n \to +\infty$ such that

$$\frac{[u_n(0)]^{\lambda-2}}{\omega_n} \to +\infty.$$

Thus,

$$\lim_{n\to+\infty} \frac{\omega_n}{[u_n(0)]^{\lambda-2}} = 0. \tag{5.8}$$

Set

$$\hat{u}_n(x) := \frac{1}{u_n(0)} u_n\left(\frac{x}{[u_n(0)]^{\frac{\lambda-2}{2s}}}\right).$$

A direct calculation shows that $\hat{u}_n(0) = \|\hat{u}_n\|_{L^\infty(\mathbb{R}^N)} = 1$ and

$$(a + be_n)(-\Delta)^s \hat{u}_n = \frac{f(u_n(0)\hat{u})}{[u_n(0)]^{\lambda-1}} - \frac{\omega_n}{[u_n(0)]^{\lambda-2}} \hat{u}_n. \tag{5.9}$$

From (H_2), Lemma 5.4 and (5.8), we derive that the right side of (5.9) is of $L^\infty(\mathbb{R}^N)$. Therefore, applying a similar argument to the proof of [19, Proposition 4.4] and passing to a subsequence if necessary, we assume that $\hat{u}_n \to \hat{u}_c$ in $C^{2,\alpha}_{loc}(\mathbb{R}^N)$, for some $\alpha \in (0, 1)$. Combining Lemma 5.6 and (H_4), noting that $e_n \to 0$, we deduce that $\hat{u}_c$ is a nonnegative bounded radial solution of

$$(-\Delta)^s \hat{u}_c = \frac{\mu_1}{a} \hat{u}_c^{\lambda-1}, \ \text{ in } \ \mathbb{R}^N.$$

Thanks to [6, Theorem 1.5], we derive that $\hat{u}_c = 0$, which contradicts $\hat{u}_c(0) = 1$. □

Lemma 5.8. *Let $1 \leqslant N < 4s$, $s \in (\frac{1}{2}, 1)$, and suppose that f satisfies (H_1)–(H_5). Let (ω_c, u_c) be the solution given by Theorem 2. Then, $\tilde{u}_n \to 0$ as $|x| \to +\infty$ uniformly for large $n \in \mathbb{N}$.*

Proof. (5.7) can be rewritten as

$$(-\Delta)^s \tilde{u}_n + \tilde{u}_n = g_n(x) \quad \in \mathbb{R}^N,$$

where $g_n(x) = -\frac{\tilde{u}_n(x)}{(a+be_n)} + \frac{f(u_n(0)\tilde{u}_n(x))}{\omega_n u_n(0)(a+be_n)}$. By Lemma 5.4, it is clear that $g_n \in L^\infty(\mathbb{R}^N)$. Moreover, it is uniformly bounded for sufficiently large n. Due to the fact $\{u_n\}$ converges strongly in $H^s(\mathbb{R}^N)$, the interpolation inequality yields that there exists $g \in L^r(\mathbb{R}^N)$ such that $g_n \to g$ in $L^r(\mathbb{R}^N)$ for $r \in [2, +\infty)$. Thus, by [3], we observe that

$$\tilde{u}_n(x) = \int_{\mathbb{R}^N} K(x-y) g_n(y) \mathrm{d}y,$$

where K is a Bessel potential and it satisfies the following properties:
(D_1) K is positive, radially symmetric, and smooth in $\mathbb{R}^N \setminus \{0\}$.
(D_2) There exists $C > 0$ such that $K(x) \leq \frac{C}{|x|^{N+2s}}$ for $x \in \mathbb{R}^N \setminus \{0\}$.
(D_3) $K \in L^r(\mathbb{R}^N)$ for $r \in [1, \frac{1}{1-s})$. For any $\sigma > 0$, we see that

$$0 \leqslant \tilde{u}_n(x) \leqslant \int_{\mathbb{R}^N} K(x-y)|g_n(y)|\mathrm{d}y = \int_{\{|x-y| \geqslant \frac{1}{\sigma}\}} K(x-y)|g_n(y)|\mathrm{d}y + \int_{\{|x-y| < \frac{1}{\sigma}\}} K(x-y)|g_n(y)|\mathrm{d}y.$$

It follows from (D_2) that

$$\int_{\{|x-y| \geqslant \frac{1}{\sigma}\}} K(x-y)|g_n(y)|\mathrm{d}y \leqslant C\|g_n\|_\infty \int_{\{|x-y| \geqslant \frac{1}{\sigma}\}} \frac{1}{|x-y|^{N+2s}} \mathrm{d}y \leqslant C\sigma^{2s}. \tag{5.10}$$

By Hölder's inequality and (D_3), we obtain

$$\begin{aligned}
&\int_{\{|x-y| < \frac{1}{\sigma}\}} K(x-y)|g_n(y)|\mathrm{d}y \\
&\leqslant \int_{\{|x-y| < \frac{1}{\sigma}\}} K(x-y)|g_n(y) - g|\mathrm{d}y + \int_{\{|x-y| < \frac{1}{\sigma}\}} K(x-y)|g(y)|\mathrm{d}y \\
&\leqslant \left(\int_{\mathbb{R}^N} |K|^2 \mathrm{d}y\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} |g_n - g|^2 \mathrm{d}y\right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}^N} |K|^2 \mathrm{d}y\right)^{\frac{1}{2}} \left(\int_{\{|x-y| < \frac{1}{\sigma}\}} |g|^2 \mathrm{d}y\right)^{\frac{1}{2}}.
\end{aligned}$$

This yields that there exists $R_1 > 0$ independent of $\sigma > 0$ such that

$$\int_{|x-y|<\frac{1}{\sigma}} K(x-y)|g_n(y)|\mathrm{d}y \leqslant \sigma \quad \text{uniformly for large } n \text{ and } |x| \geq R_1. \tag{5.11}$$

Here, we use the fact $2 < \frac{1}{1-s}$ and $\left(\int_{\{|x-y|<\frac{1}{\sigma}\}} |g|^2\mathrm{d}y\right)^{\frac{1}{2}} \to 0$ as $|x| \to +\infty$. Combining (5.10) and (5.11), we obtain

$$0 \leq \tilde{u}_n(x) \leqslant C(\sigma^{2s} + \sigma^s), \quad \text{as } |x| \geq R_1 \text{ uniformly for large n},$$

and, thus, $\tilde{u}_n(x) \to 0$ as $|x| \to +\infty$ uniformly for large n. □

Proof of Theorem 3. For each sequence $\{c_n\} \to +\infty$, we define

$$v_n(x) := \omega_n^{\frac{1}{2-\lambda}} u_n\left(\frac{x}{\omega_n^{\frac{1}{2s}}}\right) = \frac{u_n(0)}{\omega_n^{1/(\lambda-2)}}\tilde{u}_n.$$

According to Lemma 5.7, we observe that

$$\limsup_{n\to+\infty} \sup_{x\in\mathbb{R}^N} v_n(x) = \limsup_{n\to+\infty} \omega_n^{\frac{1}{2-\lambda}} u_n(0) < +\infty. \tag{5.12}$$

Together with Lemmas 5.5, 5.7, and 5.8, we deserve that $\lim_{|x|\to+\infty} v_n(x) = 0$ uniformly for large n. Moreover, v_n is the solution of

$$(a + be_n)(-\Delta)^s v_n(x) + v_n(x) = \frac{f(\omega_n^{\frac{1}{\lambda-2}} v_n)}{\omega_n^{\frac{\lambda-1}{\lambda-2}}}. \tag{5.13}$$

Hence, a similar argument to the proof of [19, Proposition 4.4] implies that $v_n \to Q$ in $C^{2,\alpha}_{loc}(\mathbb{R}^N)$ for some $\alpha \in (0, 1)$. Recalling that $\omega_n \to 0^+$ and $\|u_n\|_{D^{s,2}(\mathbb{R}^N)} \to 0$ as $n \to +\infty$, from (H_4), we obtain that Q is nontrivial nonnegative solution of

$$\begin{cases} a(-\Delta)^s Q + Q = \mu_1 Q^{\lambda-1}, & \mathit{in}\ \mathbb{R}^N, \\ \lim_{|x|\to+\infty} Q(x) = 0. \end{cases}$$

From [4], we know that Q is a radial, positive, and strictly decreasing in x. □

Proof of Theorem 4. From (5.13), we have

$$(-\Delta)^s v_n(x) + \frac{1}{a + b\|u_n\|^2_{D^{s,2}(\mathbb{R}^N)}}\left[1 - \frac{f(\omega_n^{\frac{1}{\lambda-2}} v_n)}{\omega_n^{\frac{\lambda-1}{\lambda-2}} v_n}\right] v_n = 0. \tag{5.14}$$

Since (5.12) and $\omega_n \to 0^+$, by (H_4), we observe that

$$f(\omega_n^{\frac{1}{\lambda-2}} v_n) = (\mu_1 + o_n(1))\omega_n^{\frac{\lambda-1}{\lambda-2}}[v_n(x)]^{\lambda-1},$$

as $n \to +\infty$. Together with $\|u_n\|_{D^{s,2}(\mathbb{R}^N)} \to 0$ as $n \to +\infty$, it follows from (5.14) that

$$(-\Delta)^s v_n(x) + \left(\frac{1}{a} + o_n(1)\right)\left[1 - (\mu_1 + o_n(1))[v_n(x)]^{\lambda-2}\right] v_n = 0. \tag{5.15}$$

Noting that $\lim_{|x|\to+\infty} v_n(x) = 0$ uniformly in $n \in \mathbb{N}$, there exist $R > 0$ large enough and $N_0 \in \mathbb{N}$ such that

$$\left(\frac{1}{a} + o_n(1)\right)\left[1 - (\mu_1 + o_n(1))[v_n(x)]^{\lambda-2}\right] > \frac{1}{2a},$$

for $|x| > R$ and $n \geqslant N_0$. Therefore,

$$(-\Delta)^s v_n(x) + \frac{1}{2a} v_n(x) \leqslant 0, \ \text{ for } \ |x| > R, \ n \geqslant N_0.$$

Arguing as in the proof of [19, Lemma 5.6], we see that

$$v_n(x) \leqslant \frac{C}{1 + |x|^{N+2s}}, \ \text{ for } \ |x| > R, \ n \geqslant N_0.$$

Thus, $v_n \to Q$ in $L^2(\mathbb{R}^N)$. By direct calculation, we see that

$$\|v_n\|^2_{L^2(\mathbb{R}^N)} = \omega_n^{\frac{N(\lambda-2)-4s}{2s(\lambda-2)}} c_n, \ \text{ and } \ \|v_n\|^2_{D^{s,2}(\mathbb{R}^N)} = \omega_n^{\frac{N(\lambda-2)-4s}{2s(\lambda-2)}} e_n/\omega_n.$$

Combining with Remark 3 and recalling that $\omega_n \to 0$, we deduce that $\|v_n\|^2_{L^2(\mathbb{R}^N)}$ and $\|v_n\|^2_{D^{s,2}(\mathbb{R}^N)}$ are comparable. According to the fact $v_n \to Q$ in $L^2(\mathbb{R}^N)$, there exist $C_3, C_4 > 0$ such that

$$C_3 \leqslant \|v_n\|^2_{L^2(\mathbb{R}^N)} \leqslant C_4,$$

for all $n \in \mathbb{N}$. Therefore, there exist $C_5, C_6 > 0$ such that

$$C_5 \leqslant \|v_n\|^2_{D^{s,2}(\mathbb{R}^N)} \leqslant C_6.$$

Hence, $\{v_n\}$ is a bounded sequence in $H^s(\mathbb{R}^N)$. Up to a subsequence, $v_n \rightharpoonup v$ in $H^s(\mathbb{R}^N)$. Moreover, from Lemma 1.1 and the fact $v_n \to Q$ in $L^2(\mathbb{R}^N)$, one gets that $v_n \to Q$ in $L^q(\mathbb{R}^N), q \in [2, 2_s^*)$. Applying (5.15), we observe that $\|v_n\|^2_{D^{s,2}(\mathbb{R}^N)} \to \|Q\|^2_{D^{s,2}(\mathbb{R}^N)}$, which yields $v_n \to Q$ in $H^s(\mathbb{R}^N)$. □

5.3. The case of $c \to 0^+$

Let $c_n \to 0^+$. From Lemma 5.3 and (5.4), we obtain that $\omega_n \to +\infty$ and $e_n \to +\infty$ as $n \to +\infty$.

Lemma 5.9. *Let $1 \leqslant N < 4s$, $s \in (\frac{1}{2}, 1)$, and suppose that f satisfies (H_1)–(H_5). Let (ω_c, u_c) be the solution given by Theorem 2. Then,*

$$\liminf_{c\to 0^+} u_c(0) = +\infty,$$

and

$$\liminf_{c\to 0^+} \frac{[u_c(0)]^{\gamma-2}}{\omega_c} > 0.$$

Proof. For each sequence $\{c_n\} \to 0^+$, set

$$\tilde{v}_n(x) := \frac{1}{u_n(0)} u_n\left(\frac{e_n^{\frac{1}{2s}}}{\omega_n^{\frac{1}{2s}}} x\right).$$

A direct calculation shows that $\tilde{v}_n(0) = \|\tilde{v}_n\|_{L^\infty(\mathbb{R}^N)} = 1$ and

$$\left(\frac{a}{e_n} + b\right)(-\Delta)^s \tilde{v}_n + \tilde{v}_n = \frac{1}{\omega_n u_n(0)} f(u_n(0)\tilde{v}_n). \tag{5.16}$$

Letting $x = 0$ in (5.16), and applying (H_2), there exists $C > 0$ such that

$$\begin{aligned} 1 \leqslant &\left(\frac{a}{e_n} + b\right)(-\Delta)^s \tilde{v}_n(0) + \tilde{v}_n(0) = \frac{1}{\omega_n u_n(0)} f(u_n(0)\tilde{v}_n(0)) \\ \leqslant &\frac{C}{\omega_n u_n(0)} \left(u_n^{\lambda-1}(0) + u_n^{\gamma-1}(0)\right) = \frac{C}{\omega_n} \left(u_n^{\lambda-2}(0) + u_n^{\gamma-2}(0)\right), \end{aligned}$$

which yields that $u_n(0) \to +\infty$, since $\omega_n \to +\infty, \gamma > \lambda > 2$. Moreover, by $\lambda \leqslant \gamma$, we deserve that

$$\liminf_{n\to+\infty} \frac{[u_n(0)]^{\gamma-2}}{\omega_n} \geqslant \frac{1}{2C} > 0.$$

By the arbitrary of c_n, we complete the proof. □

Lemma 5.10. *Let $1 \leqslant N < 4s$, $s \in (\frac{1}{2}, 1)$, and suppose that f satisfies (H_1)–(H_5). Let (ω_c, u_c) be the solution given by Theorem 2. Then,*

$$\limsup_{c\to0^+} \frac{[u_c(0)]^{\gamma-2}}{\omega_c} < +\infty.$$

Proof. We assume by contradiction that there exists a sequence $c_n \to 0^+$ such that

$$\frac{[u_n(0)]^{\gamma-2}}{\omega_n} \to +\infty,$$

which implies that $\omega_n = o([u_n(0)]^{\gamma-2})$. Set

$$\hat{v}_n(x) := \frac{1}{u_n(0)} u_n\left(\frac{e_n^{\frac{1}{2s}}}{u_n^{\frac{\gamma-2}{2s}}(0)} x\right).$$

Then, $\hat{v}_n(0) = \|\hat{v}_n\|_{L^\infty(\mathbb{R}^N)} = 1$ and

$$\left(\frac{a}{e_n} + b\right)(-\Delta)^s \hat{v}_n + \frac{\omega_n}{u_n^{\gamma-2}(0)} \hat{v}_n = \frac{1}{u_n^{\gamma-1}(0)} f(u_n(0)\hat{v}_n). \tag{5.17}$$

Note that

$$\frac{1}{u_n^{\gamma-1}(0)} f(u_n(0)\hat{v}_n) \leqslant \frac{C}{u_n^{\gamma-1}(0)} \left(u_n^{\gamma-1}(0)\hat{v}_n^{\gamma-1} + u_n^{\lambda-1}(0)\hat{v}_n^{\lambda-1}\right) = C\left(\hat{v}_n^{\gamma-1} + u_n^{\lambda-\gamma}(0)\hat{v}_n^{\lambda-1}\right).$$

Together with $u_n(0) \to +\infty$ and $\lambda \leqslant \gamma$, we conclude that

$$\frac{1}{u_n^{\gamma-1}(0)} f(u_n(0)\hat{v}_n) \leqslant C\left(\hat{v}_n^{\gamma-1} + \hat{v}_n^{\lambda-1}\right).$$

This indicates that $\frac{1}{u_n^{\gamma-1}(0)} f(u_n(0)\hat{v}_n)$ is of $L^\infty(\mathbb{R}^N)$. Therefore, applying a similar argument to the proof of [19, Proposition 4.4], and passing to a subsequence if necessary, we assume that $\hat{v}_n \to \hat{v}$ in $C^{2,\alpha}_{loc}(\mathbb{R}^N)$, for some $\alpha \in (0,1)$. Combining Lemma 5.6 and (H_4), noting that $e_n \to +\infty$, we deduce that $\hat{v}$ is a nonnegative bounded radial solution of

$$(-\Delta)^s \hat{v} = \frac{\mu_2}{a} \hat{v}^{\gamma-1}, \text{ in } \mathbb{R}^N.$$

Thanks to [6, Theorem 1.5], we derive that $\hat{v} = 0$, which contradicts $\hat{v}(0) = 1$. □

Proof of Theorem 5. For each sequence $\{c_n\} \to 0^+$, set

$$\bar{v}_n(x) := \omega_n^{\frac{1}{2-\lambda}} u_n \left(\frac{\|u_n\|^{\frac{1}{s}}_{D^{s,2}(\mathbb{R}^N)}}{\omega_n^{\frac{1}{2s}}} x \right).$$

Furthermore, $\bar{v}_n$ is the solution of

$$(ae_n^{-1} + b)(-\Delta)^s \bar{v}_n(x) + \bar{v}_n(x) = \frac{f(\omega_n^{\frac{1}{\gamma-2}} \bar{v}_n)}{\omega_n^{\frac{\gamma-1}{\gamma-2}}}. \tag{5.18}$$

From (H_2), we obtain that

$$\frac{f(\omega_n^{\frac{1}{\gamma-2}} \bar{v}_n)}{\omega_n^{\frac{\gamma-1}{\gamma-2}}} \leqslant \frac{C}{\omega_n^{\frac{\gamma-1}{\gamma-2}}} \left(\omega_n^{\frac{\gamma-1}{\gamma-2}} \bar{v}_n^{\gamma-1} + \omega_n^{\frac{\lambda-1}{\gamma-2}} \bar{v}_n^{\lambda-1} \right) = C \left(\bar{v}_n^{\gamma-1} + \omega_n^{\frac{\lambda-\gamma}{\gamma-2}} \bar{v}_n^{\lambda-1} \right).$$

Combining $\omega_n \to +\infty$ and $\lambda \leqslant \gamma$, we deduce that $\frac{f(\omega_n^{\frac{1}{\gamma-2}} \bar{v}_n)}{\omega_n^{\frac{\gamma-1}{\gamma-2}}} \in L^\infty(\mathbb{R}^N)$. Hence, a similar discussion to the proof of Lemma 5.8 and Theorem 3 implies that $\bar{v}_n \to U$ in $C^{2,\alpha}_{loc}(\mathbb{R}^N)$, for some $\alpha \in (0,1)$. Recalling that $\omega_n \to +\infty$ and $e_n \to +\infty$ as $n \to +\infty$, from (H_4), we obtain that U is nontrivial nonnegative solution of

$$\begin{cases} b(-\Delta)^s U + U = \mu_2 U^{\gamma-1}, \ \ in\ \mathbb{R}^N, \\ \lim_{|x| \to +\infty} U(x) = 0. \end{cases}$$

From [4], we know that U is radial, positive, and strictly decreasing in x. □

Proof of Theorem 6. Letting $c_n \to 0^+$ and recalling (5.18), we derive that

$$(-\Delta)^s \bar{v}_n(x) + \frac{1}{(ae_n^{-1} + b)} \left[1 - \frac{f(\omega_n^{\frac{1}{\gamma-2}} \bar{v}_n)}{\omega_n^{\frac{\gamma-1}{\gamma-2}} \bar{v}_n(x)} \right] \bar{v}_n(x) = 0. \tag{5.19}$$

By the proof of Theorem 5, we also obtain that $\lim_{|x| \to +\infty} \bar{v}_n(x) = 0$ uniformly in $n \in \mathbb{N}$. It follows from (H_2) and $\omega_n \to +\infty$ as $n \to +\infty$ that

$$\frac{f(\omega_n^{\frac{1}{\gamma-2}} \bar{v}_n)}{\omega_n^{\frac{\gamma-1}{\gamma-2}} \bar{v}_n(x)} \leqslant C \left(\bar{v}_n^{\gamma-2} + \bar{v}_n^{\lambda-2} \right) \to 0,$$

as $|x| \to +\infty$ uniformly in $n \in \mathbb{N}$. Hence, there exist $R > 0$ large enough and $N_1 \in \mathbb{N}$ such that

$$\frac{1}{(ae_n^{-1} + b)} \left[1 - \frac{f(\omega_n^{\frac{1}{\gamma-2}} \bar{v}_n)}{\omega_n^{\frac{\gamma-1}{\gamma-2}} \bar{v}_n(x)} \right] > \frac{1}{2b},$$

for $|x| > R$ and $n \geqslant N_1$. Arguing similarly as in the proof of Theorem 4, we can show that $\bar{v}_n \to U$ in $L^2(\mathbb{R}^N)$. By direct calculation, we see that

$$\|\bar{v}_n\|^2_{L^2(\mathbb{R}^N)} = \omega_n^{\frac{N}{2s} - \frac{2}{\gamma-2}} e_n^{-\frac{N}{2s}} c_n, \text{ and } \|\bar{v}_n\|^2_{D^{s,2}(\mathbb{R}^N)} = \omega_n^{\frac{N}{2s} - \frac{2}{\gamma-2}} e_n^{-\frac{N}{2s}} \frac{e_n}{\omega_n}.$$

Combining with Remark 3 and recalling that $e_n \to +\infty$, we deduce that $\|\bar{v}_n\|^2_{L^2(\mathbb{R}^N)}$ and $\|\bar{v}_n\|^2_{D^{s,2}(\mathbb{R}^N)}$ are comparable. Similar as the proof of Theorem 4, we observe that $\bar{v}_n \to U$ in $H^s(\mathbb{R}^N)$. □

Author contributions

Min Shu, Haibo Chen and Jie Yang: Conceptualization, Methodology, Validation, Writing-original draft, Writing-review & editing. All authors contributed equally to this work.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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