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Research article

Synergy of machine learning and the Einstein Choquet integral with LOPCOW and fuzzy measures for sustainable solid waste management

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Abstract: Solid waste management (SWM) protects public health, the environment, and limited resources in densely populated and urbanized countries such as Singapore. This work presents an advanced framework for optimizing SWM using advanced mathematical models and decision-making techniques, including the circular q-rung orthopair fuzzy set (Cq-ROFS) for data, combined with the Choquet integral (CI) and logarithmic percentage change-driven objective weighting (LOPCOW) methods, enhanced by the aggregation operators (AOs) circular q-rung orthopair fuzzy Einstein Choquet integral weighted averaging (Cq-ROFECIWA) and circular q-rung orthopair fuzzy Einstein Choquet integral weighted geometric (Cq-ROFECIWG) aggregation operators. By conducting a systematic evaluation, these methods classified different alternatives to SWM, evaluating them according to criteria such as their environmental impact, cost-effectiveness, waste reduction efficiency, feasibility of implementation, health safety, and public acceptance. The operators Cq-ROFECIWA and Cq-ROFECIWG perform better than previous approaches in the effective management of multifaceted and dynamic SWM scenarios. The comparison study demonstrates that the integration of these operators with LOPCOW and the Choquet integral offers decision-making conclusions that are more reliable and sustainable. The study conducted in Singapore successfully finds the most feasible SWM alternatives and emphasizes the possibility of implementing more environmentally sustainable practices in the urban environment. This research offers practical insights for policymakers and emphasizes the need to improve and enhance these approaches to improve SWM in various urban environments.

Keywords: circular q-rung orthopair fuzzy set; Einstein operations; Choquet integral; fuzzy measure; machine learning; sensitivity analysis; solid waste management
Mathematics Subject Classification: 03E72, 90B50, 94D05

1. Introduction

Urban areas face a crucial challenge in SWM, which has substantial effects on public health, environmental sustainability, and economic progress. Singapore, with a population of 6.03 million and a highly urbanized city-state, encounters distinctive difficulties in handling its solid waste due to its dense population. To achieve efficient and effective SWM, it is necessary to employ innovative and strong approaches that can handle the complexity and interconnections of many criteria in SWM. Using the LOPCOW approach, we guarantee that the weights of the criteria precisely represent the relative significance of each criterion within the SWM framework. A fuzzy measure is used to quantify the level of reliance between different criteria. This measure is crucial for comprehending the interconnections among factors, which can greatly impact the ultimate determination of the optimal SWM strategy. The fuzzy measure enables a comprehensive and intricate examination of these interdependencies, offering profound insights into the interactions and impacts of many criteria on the overall SWM approach. To consolidate the information gathered from the experts, we employ the Cq-ROFECIWA and Cq-ROFECIWG operators. These operators excel at capturing the interaction and synergy between diverse criteria, ensuring that the aggregation process accurately represents the underlying complexity of the SWM decision-making environment.

1.1. An overview of Cq-ROFS

Lotfi Zadeh [1] introduced fuzzy set (FS) theory in 1965 as a mathematical approach to dealing with uncertainty and imprecision. Fuzzy sets provide a range of membership degrees from 0 to 1, which is different from classical sets. Fuzzy sets are well-suited for representing complex, real-world problems with ambiguous boundaries due to their adaptability and their ability to assist in the process of making decisions. Atanassov [2] proposed the notion of intuitionistic fuzzy sets (IFS), which integrate degrees of membership and non-membership, providing a more thorough framework for managing uncertainty. The idea was expanded to include interval-valued intuitionistic fuzzy sets (IVIFS), which make it easier to show uncertainty by giving us more options and accuracy [3]. Yager [4] proposed the concept of q-rung orthopair fuzzy sets (q-ROFS), which allow for more flexibility in representing uncertainty. Atanassov [5] proposed the concept of circular intuitionistic fuzzy sets (C-IFS), which are an expansion of IFS. The author also developed several relationships and operations for C-IFS. Yusoff et al. [6] introduced the circular q-rung orthopair fuzzy set (Cq-ROFS), a generalization of C-IFS. This set extends the space of imprecision and establishes a variety of algebraic operations.

1.2. An overview of aggregation operators, the Choquet integral and the LOPCOW method

Aggregation operators and multi-criteria decision-making (MCDM) techniques are crucial for decision-making in complex situations. It is easier to combine criteria or expert opinions with aggregation operators, and MCDM evaluates criteria that are at odds with each other, prioritizing and choosing good alternatives. This makes them useful in a wide range of situations. Wang and Zhang [7] created the T-spherical fuzzy interaction power heronian mean operator, which combines degrees of membership, non-membership, and abstention to help make strong decisions. Xu and Wang [8] introduced the induced generalized intuitionistic fuzzy ordered weighted averaging (I-GIFOWA) operator, which extends current aggregation operators to incorporate both intuitionistic and IVIFS in group decision-making. Garg and Rani [9] introduced innovative AOs for complicated IFS, which improve the representation and decision-making process by integrating phase terms with two-dimensional data. Mahmood et al. [10] created hybrid AOs for triangular IFS. They improved MCDM by adding new weighted, geometric, and hybrid operators and better operational rules. Yu et al. [11] looked at current MCDM methods and pointed out the problems with standard FS. They emphasized the need for more advanced methods like q-rung orthopair cubic fuzzy sets to deal with more complicated decision-making situations. Yasin et al. [12] suggested using cubic intuitionistic fuzzy sets and Schweizer-Sklar aggregation operators, like CIFSSSWA, CIFSSSOWA, CIFSSSWG, and CIFSSSOWG, to handle the complexity of assessment well. Garg et al. [13] introduced CIVq-ROFSs and their AOs, such as AAO and GAO, and discussed their applications in the AHP and TOPSIS methods for improved MCDM. Pinar and Boran [14] discussed higher-order FS and distance metrics in data mining and decision-making. They used a unique distance metric for q-RPFS in the q-RPF ELECTRE combined with TOPSIS to improve group decision-making and categorization. Farid and Riaz [15] focused on q-ROFSs in decision-making because they effectively convey preferences. They defined new q-ROFS aggregation operators based on Aczel-Alsina procedures, such as the q-ROFAAWA operator, and apply them to MCDM situations. Hamid et al. [16] provided an overview of the algebraic structures and operations of q-ROFSSs, as well as their applications in decision-making. They presented the q-ROFS TOPSIS and q-ROFS VIKOR approaches for MCDM, illustrating their usefulness through real-world applications. Jameel et al. [17] used T-spherical fuzzy interactive Dubois-Prade operators, like T-SFDP, T-SFIDPWA, T-SFIDPOWA, T-SFIDPWG, and T-SFIDPOWG, to figure out what low-carbon technologies and environmental protection methods mean. Adding these operators to the CRITIC-EDAS framework shows a good way to carefully evaluate and rank changes that will make the power system more sustainable, which is a big step forward in the field of environmental optimization. Many researchers have thoroughly examined the LOPCOW technique and used the Choquet integral to improve decision-making processes in a variety of domains. Table 1 shows how this strategy has been effective in numerous applications.

Table 1. A comprehensive research work on the Choquet integral and LOPCOW.

Authors	Year	Method	Application
Khan [18]	2019	PFECIA	Supplier selection
Liang et al. [19]	2019	q-ROFCI	Differentiated two-sided matching decision-making based on multiple factors
Bektas [20]	2022	LOPCOW-EDAS	Analyze the efficiency and effectiveness of the Turkish insurance industry
Jia & Wang [21]	2022	CIIFAA and CIIFHAA	Multi-criteria decision-making
Ecer & Pamucar [22]	2022	LOPCOW-Dombi	The sustainability of banks in impoverished nations is evaluated
Karczmarek et al. [23]	2022	CI-based aggregation	Examining deviations in sustainable transportation systems
Mahmood et al. [24]	2022	A-IFHCIA	Select a multi-year investment business decision-making challenge
Ecer et al. [25]	2023	LOPCOW-CoCoSo	Evaluating the sustainability of micro-mobility systems in urban transportation
Garg et al. [26]	2023	AIVIFC-IAAA,	Recognition of human behavior using IVIFS information
Altıntaş [27]	2023	LOPCOW-CRADIS	Examining the economic performance of the G7 nations
Riaz et al. [28]	2023	LDFCIA and GLDFCIA	Project management and risk analysis
Sha & Shao [29]	2023	FHFCOA	Medical decision-making within the framework of FHFs
Putra et al. [30]	2024	LOPCOW-MARCOS	Choosing the most eminent educator.
Kakati et al. [31]	2024	rCTSFA$_\lambda$	Detection of diabetic retinopathy
Rong et al. [32]	2024	LOPCOW-ARAS	Evaluation of industrial robot offline programming system R&D project risks
Qin et al. [33]	2024	q-ROHFE VIKOR based on CI	Best investment in five ports
Wang et al. [34]	2024	IVSF-CRAIDS	Risk prioritization in Fine-Kinney

1.3. An overview of SWM and MCDM

The significance of SWM in advancing environmental sustainability and public health has made it a central area of study for numerous scientists. Researchers have investigated different facets of SWM, such as novel trash reduction strategies, effective recycling procedures, sophisticated composting techniques, and sustainable incineration practices. Research has also concentrated on enhancing landfill management to reduce environmental harm and investigating the socio-economic aspects of waste management policy. Researchers are working together to create comprehensive and long-lasting SWM systems that tackle the intricate problems associated with trash production and disposal in both urban and rural areas. Mallick [35] evaluated landfill site appropriateness in Saudi Arabia's Asir Region using GIS-based fuzzy-AHP-MCDA, including drainage density and land use. The analysis shows significant regional variety in prospective landfill sites, providing a solid framework for future site selection. Abdallah et al. [36] analysed 85 AI studies in SWM and found AI beneficial in waste forecasting, bin level detection, process parameter prediction, vehicle routing, and planning. The review highlights AI's ability to manage complicated, nonlinear SWM processes and explores problems and insights. Garg and Rani [37] presented a MULTIMOORA-based MCDM method for SWM evaluation under IFS theory. They used particle swarm optimization to determine attribute weight and offer new operational principles and AOs for IFS. Eghtesadifard et al. [38] used GIS, k-means clustering, and multi-criteria decision analysis to identify municipal solid waste landfills. In Shiraz, Iran, they used Delphi, DEMATEL, and ANP to define and weigh 13 criteria, and then used fuzzy logic using MOORA, WASPAS, and COPRAS to evaluate dump sites. Hoque and Rahman [39] used 2012–2016 data to create an ANN model to anticipate solid waste collection at the Matuail dump in Dhaka. Their 2-5-1-1 topology model achieved great accuracy with R^2 values of 0.85 and 0.86 for training and testing. The study shows that ANN-based forecasting can optimize landfill space needs, potentially reducing them by 28.6%. Shanta et al. [40] used fuzzy Delphi and fuzzy DEMATEL methodologies to develop and evaluate criteria to choose SWM technologies in Bangladesh. Their research identified 14 causal and 7 effect criteria, highlighting critical issues such as technology availability, feasibility, and infrastructure needs for effective SWM. Narayanamoorthy et al. [41] introduced FUCOM and MABAC to evaluate inorganic SWM approaches in India. Using IVq-ROFS, their work identifies effective ISW disposal strategies and illustrates the model's resilience through comparison analysis. Farid et al. [42] proposed a hybrid q-ROF method that combines CRITIC and EDAS to manage end-of-life automobile fuel cells in road freight trucks. This approach prioritizes sustainable strategies and shows their practicality through a case study, providing valuable suggestions to improve FCEV performance in transportation firms. The advantages over traditional methods are outlined below.

- Using Cq-ROFS, the Choquet integral, and LOPCOW together makes it easier to control uncertainty and ambiguity in SWM, although traditional methods might not work.
- The Choquet integral lets the model account for interdependence between criteria such as environmental impact and cost-effectiveness. Traditional methods may ignore complicated relationships by assuming that the criteria are independent.
- The LOPCOW method makes weighting objective and data-driven, which reduces subjective bias in judging the importance of criteria, which is a problem with many other weighting methods.
- This combined approach, by addressing a wider range of criteria and dynamically integrating them, aligns more effectively with sustainability goals than many traditional methods that often overlook long-term feasibility.
- The integration of these methods is especially effective in complex and variable urban environments, improving the reliability of SWM strategies over time, in contrast to traditional

methods that tend to be more rigid and less adaptable to dynamic conditions.

1.4. Motivation and objectives of the study

- SWM is a critical challenge in urban environments, especially in Singapore, due to limited land space and high waste-generation rates.

- Traditional SWM methods often fail to address the dynamic nature of waste management.

- Recent mathematical modeling and decision-making techniques offer promising avenues for optimizing SWM practices.

- The integration of Cq-ROFFs with the Einstein t-norm and Einstein t-conorm, the LOPCOW technique, and the Choquet integral provide robust AOs that are Cq-ROFECIWA and Cq-ROFECIWG for handling uncertainties and complexities in MCDM.

- The research aims to identify the most effective and efficient SWM method for Singapore, providing actionable recommendations for policymakers.

- The study also aims to identify the least effective SWM technique, highlighting its deficiencies and areas for improvement.

1.5. Organization of the study

The paper is structured as follows: The study is introduced in Section 1, and a concise summary of each method is provided. Additionally, relevant literature is reviewed. Detailed in Section 2 are the operational laws and preliminaries of Cq-ROFS. Section 3 investigates the Einstein t-norm, t-conorm, and their fundamental operational laws. Subsequently, it comprehensively analyses the Cq-ROFECIWA and Cq-ROFECIWG operators, including rigorous proofs. The MCDM framework that employs these operators is delineated in Section 4. A case study is presented in Section 5, which delineates the entire decision-making process and emphasizes the advantages of the proposed ranking. Section 6 concludes with managerial implications, directions for future research, and conclusions.

2. Preliminaries

Definition 2.1. [6] Let X be the universe. A Cq-ROFS $\mathscr{C}_r$ can be defined as follows:

$$\mathscr{C}_r = \left\{\left\langle x, \mathfrak{M}^q_{\mathscr{C}}(x), \mathfrak{Y}^q_{\mathscr{C}}(x); \mathscr{T}\right\rangle \mid x \in \mathrm{X}\right\},$$

where

$$0 \leq \mathfrak{M}^q_c(x) + \mathfrak{Y}^q_c(x) \leq 1,$$

and $\mathscr{T} \in [0, \sqrt{2}]$ is the radius of the circle around each element $x \in X$. The functions $\mathfrak{M}^q_{\mathscr{C}} : X \to [0, 1]$ and $\mathfrak{Y}^q_{\mathscr{C}} : X \to [0, 1]$ represent the degree of membership and degree of non-membership, respectively, of an element $x \in X$. The degree of indeterminacy is calculated as follows:

$$\pi_c(x) = \sqrt[q]{1 - \mathfrak{M}^q_c(x) - \mathfrak{Y}^q_c(x)}.$$

The radius of the $\langle \mu_c^q, \mathfrak{Y}_c^q \rangle$ can be calculated by Eq (1).

$$\mathscr{T}_i^q = \max_{1 \le j \le k_i} \sqrt[q]{\left(\left(\mathfrak{M}_{c_i} - \mathfrak{M}_{ij}\right)^2 + \left(\mathfrak{Y}_{c_i} - \mathfrak{Y}_{ij}\right)^2\right)}, \tag{1}$$

where $\langle \mathfrak{M}_{c_i}, \mathfrak{Y}_{c_i} \rangle = \left\langle \frac{1}{z} \sum_{m=1}^{z} \mathfrak{M}_{ij}^{q,z}, \frac{1}{z} \sum_{m=1}^{z} \mathfrak{Y}_{ij}^{q,z} \right\rangle$.

Definition 2.2. Let $\mathscr{C} = \langle (\mathfrak{M}_{\mathscr{C}}^q, \mathfrak{Y}_{\mathscr{C}}^q); \mathscr{T} \rangle$ be a Cq-ROFN, then the score function $S(\mathscr{C})$ is defined as follows:

$$S(\mathscr{C}) = \frac{\mathfrak{M}_{\mathscr{C}}^q - \mathfrak{Y}_{\mathscr{C}}^q + \sqrt{2}\mathscr{T}(2\wp - 1)}{3}, \tag{2}$$

where $S(\mathscr{C}) \in [-1, 1]$ and $\wp \in [0, 1]$ reflects the decision-maker's perspective of the model.

Definition 2.3. Let $\mathscr{C} = \langle (\mathfrak{M}_{\mathscr{C}}^q, \mathfrak{Y}_{\mathscr{C}}^q); \mathscr{T} \rangle$ be a Cq-ROFN, then the accuracy function $H(\mathscr{C})$ is defined as:

$$H(\mathscr{C}) = \mathfrak{M}_{\mathscr{C}}^q + \mathfrak{Y}_{\mathscr{C}}^q, \tag{3}$$

where $H(\mathscr{C}) \in [0, 1]$.

Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be two Cq-ROF numbers, and then the ranking rules are as follows:

- If $S(\mathscr{C}_1) > S(\mathscr{C}_2)$, then $\mathscr{C}_1 > \mathscr{C}_2$.
- If $S(\mathscr{C}_1) = S(\mathscr{C}_2)$, then $\mathscr{C}_1 = \mathscr{C}_2$.
- If $H(\mathscr{C}_1) > H(\mathscr{C}_2)$, then $\mathscr{C}_1 > \mathscr{C}_2$.
- If $H(\mathscr{C}_1) = H(\mathscr{C}_2)$, then $\mathscr{C}_1 = \mathscr{C}_2$.

2.1. Operational laws on Cq-ROFSs

Let $\mathscr{C}_1 = \langle (\mathfrak{M}_1^q, \mathfrak{Y}_1^q); \mathscr{T}_1^q \rangle$ and $\mathscr{C}_2 = \langle (\mathfrak{M}_2^q, \mathfrak{Y}_2^q); \mathscr{T}_2^q \rangle$ be two Cq-ROFSs. The minimum and maximum radii, which indicate the degree of uncertainty, with a smaller radius indicating less vagueness and larger radii indicating greater vagueness, determine the operations.

- $\mathscr{C}_1 \cap_{\min} \mathscr{C}_2 = \left\{\left\langle x, \min\left(\mathfrak{M}_{c_1}^q(x), \mathfrak{M}_{c_2}^q(x)\right), \max\left(\mathfrak{Y}_{c_1}^q(x), \mathfrak{Y}_{c_2}^q(x)\right); \min\left(\mathscr{T}_1^q, \mathscr{T}_2^q\right)\right\rangle \mid x \in X\right\}$.
- $\mathscr{C}_1 \cap_{\max} \mathscr{C}_2 = \left\{\left\langle x, \min\left(\mathfrak{M}_{c_1}^q(x), \mathfrak{M}_{c_2}^q(x)\right), \max\left(\mathfrak{Y}_{c_1}^q(x), \mathfrak{Y}_{c_2}^q(x)\right); \max\left(\mathscr{T}_1^q, \mathscr{T}_2^q\right)\right\rangle \mid x \in X\right\}$.
- $\mathscr{C}_1 \cup_{\min} \mathscr{C}_2 = \left\{\left\langle x, \max\left(\mathfrak{M}_{c_1}^q(x), \mathfrak{M}_{c_2}^q(x)\right), \min\left(\mathfrak{Y}_{c_1}^q(x), \mathfrak{Y}_{c_2}^q(x)\right); \min\left(\mathscr{T}_1^q, \mathscr{T}_2^q\right)\right\rangle \mid x \in X\right\}$.
- $\mathscr{C}_1 \cup_{\max} \mathscr{C}_2 = \left\{\left\langle x, \max\left(\mathfrak{M}_{c_1}^q(x), \mathfrak{M}_{c_2}^q(x)\right), \min\left(\mathfrak{Y}_{c_1}^q(x), \mathfrak{Y}_{c_2}^q(x)\right); \max\left(\mathscr{T}_1^q, \mathscr{T}_2^q\right)\right\rangle \mid x \in X\right\}$.
- $\mathscr{C}_1 \oplus_{\min} \mathscr{C}_2 = \left\{\left\langle x, \mathfrak{M}_{c_1}^q(x) + \mathfrak{M}_{c_2}^q(x) - \mathfrak{M}_{c_1}^q(x) * \mathfrak{M}_{c_2}^q(x), \mathfrak{Y}_{c_1}^q(x) * \mathfrak{Y}_{c_2}^q(x); \min\left(\mathscr{T}_1^q, \mathscr{T}_2^q\right)\right\rangle \mid x \in X\right\}$.
- $\mathscr{C}_1 \oplus_{\max} \mathscr{C}_2 = \left\{\left\langle x, \mathfrak{M}_{c_1}^q(x) + \mathfrak{M}_{c_2}^q(x) - \mathfrak{M}_{c_1}^q(x) * \mathfrak{M}_{c_2}^q(x), \mathfrak{Y}_{c_1}^q(x) * \mathfrak{Y}_{c_2}^q(x); \max\left(\mathscr{T}_1^q, \mathscr{T}_2^q\right)\right\rangle \mid x \in X\right\}$.
- $\mathscr{C}_1 \otimes_{\min} \mathscr{C}_2 = \left\{\left\langle x, \mathfrak{M}_{c_1}^q(x) * \mathfrak{M}_{c_2}^q(x), \mathfrak{Y}_{c_1}^q(x) + \mathfrak{Y}_{c_2}^q(x) - \mathfrak{Y}_{c_1}^q(x) * \mathfrak{Y}_{c_2}^q(x); \min\left(\mathscr{T}_1^q, \mathscr{T}_2^q\right)\right\rangle \mid x \in X\right\}$.
- $\mathscr{C}_1 \otimes_{\max} \mathscr{C}_2 = \left\{\left\langle x, \mathfrak{M}_{c_1}^q(x) * \mathfrak{M}_{c_2}^q(x), \mathfrak{Y}_{c_1}^q(x) + \mathfrak{Y}_{c_2}^q(x) - \mathfrak{Y}_{c_1}^q(x) * \mathfrak{Y}_{c_2}^q(x); \max\left(\mathscr{T}_1^q, \mathscr{T}_2^q\right)\right\rangle \mid x \in X\right\}$.

2.2. Cq-ROFS aggregation operators

Definition 2.4. Let $\mathscr{C}_i = \langle(\mathfrak{M}_{c_i}^q, \mathfrak{Y}_{c_i}^q); \mathscr{T}_i^q\rangle$ $(i = 1, \ldots, n)$ be a family of Cq-ROF numbers and Cq-ROFECWA: $\eth^n \to \eth$ if

$$Cq\text{-}ROFECIWA(\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n) = \bigoplus_{i=1}^{n} [(\mathscr{M}(\mathscr{A}_{\sigma(i)}) - \mathscr{M}(\mathscr{A}_{\sigma(i-1)}))\mathscr{C}_i]$$
$$= \left[\sqrt[q]{1 - \prod_{i=1}^{n}\left(1 - \mathfrak{M}_i^q\right)^{\mathscr{M}(\mathscr{A}_{\sigma(i)}) - \mathscr{M}(\mathscr{A}_{\sigma(i-1)})}}, \sqrt[q]{\prod_{i=1}^{n} \mathfrak{Y}_i^{q}.^{\left(\mathscr{M}(\mathscr{A}_{\sigma(i)}) - \mathscr{M}(\mathscr{A}_{\sigma(i-1)})\right)}}, \sqrt[q]{\prod_{i=1}^{n} \mathscr{T}_i^{q}.^{\left(\mathscr{M}(\mathscr{A}_{\sigma(i)}) - \mathscr{M}(\mathscr{A}_{\sigma(i-1)})\right)}}\right],$$

where $(\mathscr{M}(\mathscr{A}_{\sigma(i)}) - \mathscr{M}(\mathscr{A}_{\sigma(i-1)})) = ((\mathscr{M}(\mathscr{A}_{\sigma(1)}) - \mathscr{M}(\mathscr{A}_{\sigma(0)})), \ldots, (\mathscr{M}(\mathscr{A}_{\sigma(n)}) - \mathscr{M}(\mathscr{A}_{\sigma(n-1)})))$ is the weight vector of $(\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n)$ such that $0 \leq (\mathscr{M}(\mathscr{A}_{\sigma(i)}) - \mathscr{M}(\mathscr{A}_{\sigma(i-1)})) \leq 1$. Then C$q$-ROFECIWA is called a circular q-rung orthopair fuzzy Einstein Choquet integral weighted averaging operator.

Definition 2.5. Let $\mathscr{C}_i = \langle(\mathfrak{M}_{c_i}^q, \mathfrak{Y}_{c_i}^q); \mathscr{T}_i^q\rangle$ $(i = 1, \ldots, n)$ be a family of Cq-ROF numbers and Cq-ROFECIWG: $\eth^n \to \eth$ if

$$Cq\text{-}ROFECIWG(\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n) = \bigotimes_{i=1}^{n} (\mathscr{C}_i)^{\left(\mathscr{M}(\mathscr{A}_{\sigma(i)}) - \mathscr{M}(\mathscr{A}_{\sigma(i-1)})\right)}$$
$$= \left[\sqrt[q]{\prod_{i=1}^{n} \mathfrak{M}_i^{q\left(\mathscr{M}(\mathscr{A}_{\sigma(i)}) - \mathscr{M}(\mathscr{A}_{\sigma(i-1)})\right)}}, \sqrt[q]{1 - \prod_{i=1}^{n}\left(1 - \mathfrak{M}_i^q\right)^{\left(\mathscr{M}(\mathscr{A}_{\sigma(i)}) - \mathscr{M}(\mathscr{A}_{\sigma(i-1)})\right)}}, \sqrt[q]{\prod_{i=1}^{n} \mathscr{T}_i^{q\left(\mathscr{M}(\mathscr{A}_{\sigma(i)}) - \mathscr{M}(\mathscr{A}_{\sigma(i-1)})\right)}}\right],$$

where $(\mathscr{M}(\mathscr{A}_{\sigma(i)}) - \mathscr{M}(\mathscr{A}_{\sigma(i-1)})) = ((\mathscr{M}(\mathscr{A}_{\sigma(1)}) - \mathscr{M}(\mathscr{A}_{\sigma(0)})), \ldots, (\mathscr{M}(\mathscr{A}_{\sigma(n)}) - \mathscr{M}(\mathscr{A}_{\sigma(n-1)})))$ is the weight vector of $(\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n)$ such that $0 \leq (\mathscr{M}(\mathscr{A}_{\sigma(i)}) - \mathscr{M}(\mathscr{A}_{\sigma(i-1)})) \leq 1$. Then C$q$-ROFECIWG is called circular q-rung orthopair fuzzy Einstein Choquet integral weighted geometric operator.

3. Cq-ROF Einstein operational laws

Definition 3.1. [44] The t-norm T and t-conorm $\mathfrak{S}$ are Einstein products $\mathrm{T_E}$ and Einstein sums $\mathfrak{S}_{\mathfrak{E}}$, respectively, as defined in Eqs (4) and (5).

$$\mathrm{T_E}(x, y) = \frac{xy}{1 + (1 - x)(1 - y)}. \tag{4}$$

$$\mathfrak{S}_E(x, y) = \frac{x + y}{1 + xy}. \tag{5}$$

3.1. Circular q-ROF Einstein operations

$$\alpha \oplus_{\mathrm{T_E}} \beta = \left[\sqrt[q]{\frac{\mathfrak{M}_\alpha^q + \mathfrak{M}_\beta^q}{1 + \mathfrak{M}_\alpha^q \cdot \mathfrak{M}_\beta^q}}, \sqrt[q]{\frac{\mathfrak{Y}_\alpha^q \cdot \mathfrak{Y}_\beta^q}{1 + (1 - \mathfrak{Y}_\alpha^q) \cdot \left(1 - \mathfrak{Y}_\beta^q\right)}}, \sqrt[q]{\frac{\mathscr{T}_\alpha^q + \mathscr{T}_\beta^q}{1 + \mathscr{T}_\alpha^q \cdot \mathscr{T}_\beta^q}}\right]. \tag{6}$$

$$\alpha \oplus_{\mathfrak{S}_E} \beta = \left[\sqrt[q]{\frac{\mathfrak{M}_\alpha^q + \mathfrak{M}_\beta^q}{1 + \mathfrak{M}_\alpha^q \cdot \mathfrak{M}_\beta^q}}, \sqrt[q]{\frac{\mathfrak{Y}_\alpha^q \cdot \mathfrak{Y}_\beta^q}{1 + (1 - \mathfrak{Y}_\alpha^q) \cdot \left(1 - \mathfrak{Y}_\beta^q\right)}}, \sqrt[q]{\frac{\mathscr{T}_\alpha^q \cdot \mathscr{T}_\beta^q}{1 + (1 - \mathscr{T}_\alpha^q) \cdot \left(1 - \mathscr{T}_\beta^q\right)}}\right]. \tag{7}$$

$$\alpha \otimes_{\mathrm{T_E}} \beta = \left[\sqrt[q]{\frac{\mathfrak{M}_\alpha^q \cdot \mathfrak{M}_\beta^q}{1 + (1 - \mathfrak{M}_\alpha^q) \cdot \left(1 - \mathfrak{M}_\beta^q\right)}}, \sqrt[q]{\frac{\mathfrak{Y}_\alpha^q + \mathfrak{Y}_\beta^q}{1 + \mathfrak{Y}_\alpha^q \cdot \mathfrak{Y}_\beta^q}}, \sqrt[q]{\frac{\mathscr{T}_\alpha^q \cdot \mathscr{T}_\beta^q}{1 + (1 - \mathscr{T}_\alpha^q) \cdot \left(1 - \mathscr{T}_\beta^q\right)}} \right]. \quad (8)$$

$$\alpha \otimes_{\Xi_E} \beta = \left[\sqrt[q]{\frac{\mathfrak{M}_\alpha^q \cdot \mathfrak{M}_\beta^q}{1 + (1 - \mathfrak{M}_\alpha^q) \cdot \left(1 - \mathfrak{M}_\beta^q\right)}}, \sqrt[q]{\frac{\mathfrak{Y}_\alpha^q + \mathfrak{Y}_\beta^q}{1 + \mathfrak{Y}_\alpha^q \cdot \mathfrak{Y}_\beta^q}}, \sqrt[q]{\frac{\mathscr{T}_\alpha^q + \mathscr{T}_\beta^q}{1 + \mathscr{T}_\alpha^q \cdot \mathscr{T}_\beta^q}} \right]. \quad (9)$$

$$\lambda._{\mathrm{T_E}}\alpha = \left[\sqrt[q]{\frac{(1 + \mathfrak{M}_\alpha^q)^\lambda - (1 - \mathfrak{M}_\alpha^q)^\lambda}{(1 + \mathfrak{M}_\alpha^q)^\lambda + (1 - \mathfrak{M}_\alpha^q)^\lambda}}, \sqrt[q]{\frac{2\,(\mathfrak{Y}_\alpha^q)^\lambda}{(2 - \mathfrak{Y}_\alpha^q)^\lambda + (\mathfrak{Y}_\alpha^q)^\lambda}}, \sqrt[q]{\frac{(1 + \mathscr{T}_\alpha^q)^\lambda - (1 - \mathscr{T}_\alpha^q)^\lambda}{(1 + \mathscr{T}_\alpha^q)^\lambda + (1 - \mathscr{T}_\alpha^q)^\lambda}} \right]. \quad (10)$$

$$\lambda._{\Xi_E}\alpha = \left[\sqrt[q]{\frac{(1 + \mathfrak{M}_\alpha^q)^\lambda - (1 - \mathfrak{M}_\alpha^q)^\lambda}{(1 + \mathfrak{M}_\alpha^q)^\lambda + (1 - \mathfrak{M}_\alpha^q)^\lambda}}, \frac{\sqrt[q]{2}\,(\mathfrak{Y}_\alpha)^\lambda}{\sqrt[q]{(2 - \mathfrak{Y}_\alpha^q)^\lambda + (\mathfrak{Y}_\alpha^q)^\lambda}}, \sqrt[q]{\frac{2\,(\mathscr{T}_\alpha^q)^\lambda}{(2 - \mathscr{T}_\alpha^q)^\lambda + (\mathscr{T}_\alpha^q)^\lambda}} \right]. \quad (11)$$

$$(\alpha)^{\lambda_{\mathrm{T_E}}} = \left[\frac{\sqrt[q]{2}\,(\mathfrak{M}_\alpha)^\lambda}{\sqrt[q]{(2 - \mathfrak{M}_\alpha^q)^\lambda + (\mathfrak{M}_\alpha^q)^\lambda}}, \sqrt[q]{\frac{(1 + \mathfrak{Y}_\alpha^q)^\lambda - (1 - \mathfrak{Y}_\alpha^q)^\lambda}{(1 + \mathfrak{Y}_\alpha^q)^\lambda + (1 - \mathfrak{Y}_\alpha^q)^\lambda}}, \sqrt[q]{\frac{2\,(\mathscr{T}_\alpha^q)^\lambda}{(2 - \mathscr{T}_\alpha^q)^\lambda + (\mathscr{T}_\alpha^q)^\lambda}} \right]. \quad (12)$$

$$\alpha^{\lambda_{\Xi_E}} = \left[\frac{\sqrt[q]{2}\,(\mathfrak{M}_\alpha)^\lambda}{\sqrt[q]{(2 - \mathfrak{M}_\alpha^q)^\lambda + (\mathfrak{M}_\alpha^q)^\lambda}}, \sqrt[q]{\frac{(1 + \mathfrak{Y}_\alpha^q)^\lambda - (1 - \mathfrak{Y}_\alpha^q)^\lambda}{(1 + \mathfrak{Y}_\alpha^q)^\lambda + (1 - \mathfrak{Y}_\alpha^q)^\lambda}}, \sqrt[q]{\frac{(1 + \mathscr{T}_\alpha^q)^\lambda - (1 - \mathscr{T}_\alpha^q)^\lambda}{(1 + \mathscr{T}_\alpha^q)^\lambda + (1 - \mathscr{T}_\alpha^q)^\lambda}} \right]. \quad (13)$$

Theorem 3.2. *Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be Cq-ROFSs and $\lambda \geq 0$ be any real number. Then,*

- $\mathscr{C}_1 \otimes_{\mathrm{T_E}} \mathscr{C}_2 = \mathscr{C}_2 \otimes_{\mathrm{T_E}} \mathscr{C}_1$.
- $\mathscr{C}_1 \otimes_{\Xi_E} \mathscr{C}_2 = \mathscr{C}_2 \otimes_{\Xi_E} \mathscr{C}_1$.
- $(\mathscr{C}_1 \otimes_{\mathrm{T_E}} \mathscr{C}_2)^\lambda = (\mathscr{C}_1)^\lambda \otimes_{\mathrm{T_E}} (\mathscr{C}_2)^\lambda$.
- $(\mathscr{C}_1 \otimes_{\Xi_E} \mathscr{C}_2)^\lambda = (\mathscr{C}_1)^\lambda \otimes_{\Xi_E} (\mathscr{C}_2)^\lambda$.
- $\lambda._{\mathrm{T_E}}(\mathscr{C}_1 \oplus_{\mathrm{T_E}} \mathscr{C}_2) = \lambda._{\mathrm{T_E}}(\mathscr{C}_1) \oplus_{\mathrm{T_E}} \lambda._{\mathrm{T_E}}(\mathscr{C}_2)$.
- $\lambda._{\Xi_E}(\mathscr{C}_1 \oplus_{\Xi_E} \mathscr{C}_2) = \lambda._{\Xi_E}(\mathscr{C}_1) \oplus_{\Xi_E} \lambda._{\Xi_E}(\mathscr{C}_2)$.
- $\lambda_1._{\mathrm{T_E}}(\lambda_2._{\mathrm{T_E}}\mathscr{C}_1) = (\lambda_1._{\mathrm{T_E}}\lambda_2)._{\mathrm{T_E}}\mathscr{C}_1$.
- $\lambda_1._{\Xi_E}(\lambda_2._{\Xi_E}\mathscr{C}_1) = (\lambda_1._{\Xi_E}\lambda_2)._{\Xi_E}\mathscr{C}_1$.
- $({\mathscr{C}_1}^{\lambda_1})^{\lambda_2} = (\mathscr{C}_1)^{\lambda_1}._{\mathrm{T_E}}\lambda_2$.
- $({\mathscr{C}_1}^{\lambda_1})^{\lambda_2} = (\mathscr{C}_1)^{\lambda_1}._{\Xi_E}\lambda_2$.

3.2. Cq-ROFECI weighted averaging operator

Definition 3.3. [43] The set function $\mathscr{M} : P(x) \in [0, 1]$ that satisfies the following axioms is a fuzzy measure $\mathscr{M}$ on the set X:

- $\mathscr{M}(\phi) = 0, \mathscr{M}(X) = 1$.
- $\mathfrak{S}_1 \subseteq \mathfrak{S}_2$ implies $\mathscr{M}(\mathfrak{S}_1) \leq \mathscr{M}(\mathfrak{S}_2)$, for all $\mathfrak{S}_1, \mathfrak{S}_2 \subseteq X$.
- $\mathscr{M}(\mathfrak{S}_1 \cup \mathfrak{S}_2) = \mathscr{M}(\mathfrak{S}_1) + \mathscr{M}(\mathfrak{S}_2) + \rho\mathscr{M}(\mathfrak{S}_1)\mathscr{M}(\mathfrak{S}_2)$ for all $\mathfrak{S}_1, \mathfrak{S}_2 \in P(X)$, and $\mathfrak{S}_1 \cap \mathfrak{S}_2 = \varnothing, \rho > -1$.

Particularly, the above condition is reduced to the axiom of additive measure when $\rho = 0$.

$$\mathscr{M}(\mathfrak{S}_1 \cup \mathfrak{S}_2) = \mathscr{M}(\mathfrak{S}_1) + \mathscr{M}(\mathfrak{S}_2), \text{ for all } \mathfrak{S}_1, \mathfrak{S}_2 \subseteq X \text{ and } \mathfrak{S}_1 \cap \mathfrak{S}_2 = \varnothing. \tag{14}$$

In this instance, all elements of X are independent, and we have the following:

$$\mathscr{M}(\mathfrak{S}_1) = \sum_{x_i \in \mathfrak{S}_1} \mathscr{M}(\{x_i\}). \tag{15}$$

If $\rho > 0$, then $\mathscr{M}(\mathfrak{S}_1 \cup \mathfrak{S}_2) > \mathscr{M}(\mathfrak{S}_1) + \mathscr{M}(\mathfrak{S}_2)$, which implies that the set $\{\mathfrak{S}_1, \mathfrak{S}_2\}$ has a multiplicative effect. If $\rho < 0$, then $\mathscr{M}(\mathfrak{S}_1 \cup \mathfrak{S}_2) < \mathscr{M}(\mathfrak{S}_1) + \mathscr{M}(\mathfrak{S}_2)$ shows a substitutive effect. By parameter ρ, the interaction between sets or elements of a set can be represented.

$$\mathscr{M}(X) = \mathscr{M}\left(\cup_{i=1}^{n} x_i\right) = \begin{cases} \frac{1}{\rho}\left(\prod_{i=1}^{n} [1 + \rho\mathscr{M}(x_i)] - 1\right), & \text{if } \rho \neq 0, \\ \sum_{i=1}^{n} \mathscr{M}(x_i), & \text{if } \rho = 0. \end{cases} \tag{16}$$

Particularly for each subset $\mathfrak{S}_1 \subseteq X$, we have the following:

$$\mathscr{M}(\mathfrak{S}_1) = \begin{cases} \frac{1}{\rho}\left(\prod_{x_i \in \mathfrak{S}_1} [1 + \rho\mathscr{M}(x_i)] - 1\right), & \text{if } \rho \neq 0, \\ \sum_{x_i \in \mathfrak{S}_1} \mathscr{M}(x_i), & \text{if } \rho = 0. \end{cases} \tag{17}$$

Equation (16) uniquely determines ρ from $\mathscr{M}(X) = 1$, allowing for the solution:

$$\rho + 1 = \prod_{i=1}^{n} (1 + \rho\mathscr{M}(x_i)). \tag{18}$$

Definition 3.4. Let $\mathscr{C}_i = \langle(\mathfrak{M}_{c_i}^q, \mathfrak{Y}_{c_i}^q); \mathscr{T}_i^q\rangle$ $(i = 1, \ldots, n)$ be a family of Cq-ROF numbers and Cq-ROFECI : $\eth^n \to \eth$ if

$$\begin{aligned} \text{C}q\text{-ROFECIWA}(\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n) &= \bigoplus_{i=1\ \mathrm{T_E}}^{n} ((\mathscr{M}(\mathscr{A}_{\sigma(i)}) - \mathscr{M}(\mathscr{A}_{\sigma(i-1)}))\mathscr{C}_i) \\ &= ((\mathscr{M}(\mathscr{A}_{\sigma(1)}) - \mathscr{M}(\mathscr{A}_{\sigma(0)}))\mathscr{C}_{\sigma(1)} \oplus_{\mathrm{T_E}} (\mathscr{M}(\mathscr{A}_{\sigma(2)}) - \mathscr{M}(\mathscr{A}_{\sigma(1)}))\mathscr{C}_{\sigma(2)} \oplus_{\mathrm{T_E}} \ldots \\ &\quad \oplus_{\mathrm{T_E}} (\mathscr{M}(\mathscr{A}_{\sigma(n)}) - \mathscr{M}(\mathscr{A}_{\sigma(n-1)}))\mathscr{C}_{\sigma(n)}), \end{aligned} \tag{19}$$

where $(\sigma(1), \sigma(2), \ldots, \sigma(n)$ is a permutation such that $\mathscr{C}_{\sigma(j)} \geq \mathscr{C}_{\sigma(j+1)}$ for all $j = 1, 2, 3, \ldots, n$ and moreover $\mathscr{M}(\mathscr{A}_{\sigma(j)} = \{x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(j)}\}$ $0 \leq (\mathscr{M}(\mathscr{A}_{\sigma(i)}) - \mathscr{M}(\mathscr{A}_{\sigma(i-1)})) \leq 1$. Then a mapping

Cq-ROFECIWA is called a circular q-rung orthopair fuzzy Einstein Choquet integral weighted averaging operator.

Theorem 3.5. Let $\mathscr{C}_i = \langle(\mathfrak{M}_i^q, \mathfrak{Y}_i^q); \mathscr{T}_i^q\rangle$ $(i = 1, \ldots, n)$ be a family of Cq-ROF numbers, and then the aggregated value by using Cq-ROFECIWA operational laws is defined in (20).

$$\begin{aligned} &\mathrm{C}q\text{-ROFECIWA}(\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n) = \bigoplus_{i=1 \ \mathrm{T_E}}^{n} ((\mathscr{M}(\mathscr{A}_{\sigma(i)}) - \mathscr{M}(\mathscr{A}_{\sigma(i-1)}))\mathscr{C}_i) \\ &= \left[\begin{array}{c} \sqrt[q]{\dfrac{\prod_{i=1}^{n}\left(1+\mathfrak{M}_i^q\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))} - \prod_{i=1}^{n}\left(1-\mathfrak{M}_i^q\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}{\prod_{i=1}^{n}\left(1+\mathfrak{M}_i^q\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))} + \prod_{i=1}^{n}\left(1-\mathfrak{M}_i^q\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}}, \\ \dfrac{\sqrt[q]{2}\prod_{i=1}^{n}(\mathfrak{Y}_i)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}{\sqrt[q]{\prod_{i=1}^{n}\left(2-\mathfrak{Y}_i^q\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))} + \prod_{i=1}^{n}\left(\mathfrak{Y}_i^q\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}}, \\ \sqrt[q]{\dfrac{\prod_{i=1}^{n}\left(1+\mathscr{T}_i^q\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))} - \prod_{i=1}^{n}\left(1-\mathscr{T}_i^q\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}{\prod_{i=1}^{n}\left(1+\mathscr{T}_i^q\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))} + \prod_{i=1}^{n}\left(1-\mathscr{T}_i^q\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}} \end{array} \right], \end{aligned} \tag{20}$$

where $(\mathscr{M}(\mathscr{A}_{\sigma(i)}) - \mathscr{M}(\mathscr{A}_{\sigma(i-1)})) = ((\mathscr{M}(\mathscr{A}_{\sigma(1)}) - \mathscr{M}(\mathscr{A}_{\sigma(0)})), \ldots, (\mathscr{M}(\mathscr{A}_{\sigma(n)}) - \mathscr{M}(\mathscr{A}_{\sigma(n-1)})))$ is the weight vector of $(\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n)$ such that $(\mathscr{M}(\mathscr{A}_{\sigma(i)}) - \mathscr{M}(\mathscr{A}_{\sigma(i-1)})) \in [0, 1]$ and $\sum_{i=1}^{n} (\mathscr{M}(\mathscr{A}_{\sigma(i)}) - \mathscr{M}(\mathscr{A}_{\sigma(i-1)})) = 1$.

Proof. This theorem is proven via mathematical induction. Let $\varpi_j = (\mathscr{M}(\mathscr{A}_{\sigma(j)}) - \mathscr{M}(\mathscr{A}_{\sigma(j-1)}))$. For $n = 2$, Cq-ROFECIWA$(\mathscr{C}_1, \mathscr{C}_2) = (\varpi_1\mathscr{C}_1 \oplus_{\mathrm{T_E}} \varpi_2\mathscr{C}_2)$.

By using the Cq-RFE operation defined above, we know that

$$\varpi_1 \cdot_{\mathrm{T_E}} \mathscr{C}_1 = \left[\sqrt[q]{\frac{\left(1+\mathfrak{M}_1^q\right)^{\varpi_1} - \left(1-\mathfrak{M}_1^q\right)^{\varpi_1}}{\left(1+\mathfrak{M}_1^q\right)^{\varpi_1} + \left(1-\mathfrak{M}_1^q\right)^{\varpi_1}}}, \frac{\sqrt[q]{2}\,(\mathfrak{Y}_1)^{\varpi_1}}{\sqrt[q]{\left(2-\mathfrak{Y}_1^q\right)^{\varpi_1} + \left(\mathfrak{Y}_1^q\right)^{\varpi_1}}}, \sqrt[q]{\frac{\left(1+\mathscr{T}_1^q\right)^{\varpi_1} - \left(1-\mathscr{T}_1^q\right)^{\varpi_1}}{\left(1+\mathscr{T}_1^q\right)^{\varpi_1} + \left(1-\mathscr{T}_1^q\right)^{\varpi_1}}} \right].$$

$$\varpi_2 \cdot_{\mathrm{T_E}} \mathscr{C}_2 = \left[\sqrt[q]{\frac{\left(1+\mathfrak{M}_2^q\right)^{\varpi_2} - \left(1-\mathfrak{M}_2^q\right)^{\varpi_2}}{\left(1+\mathfrak{M}_2^q\right)^{\varpi_2} + \left(1-\mathfrak{M}_2^q\right)^{\varpi_2}}}, \frac{\sqrt[q]{2}\,(\mathfrak{Y}_2)^{\varpi_2}}{\sqrt[q]{\left(2-\mathfrak{Y}_2^q\right)^{\varpi_2} + \left(\mathfrak{Y}_2^q\right)^{\varpi_2}}}, \sqrt[q]{\frac{\left(1+\mathscr{T}_2^q\right)^{\varpi_2} - \left(1-\mathscr{T}_2^q\right)^{\varpi_2}}{\left(1+\mathscr{T}_2^q\right)^{\varpi_2} + \left(1-\mathscr{T}_2^q\right)^{\varpi_2}}} \right].$$

Then,

$$Cq\text{-ROFECIWA}(\mathscr{C}_1,\mathscr{C}_2) = \varpi_1.\mathscr{C}_1 \oplus_{T_E} \varpi_2.\mathscr{C}_2$$

$$= \left[\sqrt[q]{\frac{(1+\mathfrak{M}_1^q)^{\varpi_1} - (1-\mathfrak{M}_1^q)^{\varpi_1}}{(1+\mathfrak{M}_1^q)^{\varpi_1} + (1-\mathfrak{M}_1^q)^{\varpi_1}}}, \frac{\sqrt[q]{2}(\mathfrak{Y}_1)^{\varpi_1}}{\sqrt[q]{(2-(\mathfrak{Y}_1^q))^{\varpi_1} + (\mathfrak{Y}_1^q)^{\varpi_1}}}, \sqrt[q]{\frac{(1+\mathscr{T}_1^q)^{\varpi_1} - (1-\mathscr{T}_1^q)^{\varpi_1}}{(1+\mathscr{T}_1^q)^{\varpi_1} + (1-\mathscr{T}_1^q)^{\varpi_1}}} \right].$$

$$\bigoplus_{T_E} \left[\sqrt[q]{\frac{(1+\mathfrak{M}_2^q)^{\varpi_2} - (1-\mathfrak{M}_2^q)^{\varpi_2}}{(1+\mathfrak{M}_2^q)^q)^{\varpi_2} + (1-\mathfrak{M}_2^q)^{\varpi_2}}}, \frac{\sqrt[q]{2}(\mathfrak{Y}_2)^{\varpi_2}}{\sqrt[q]{(2-\mathfrak{Y}_2^q)^{\varpi_2} + (\mathfrak{Y}_2^q)^{\varpi_2}}}, \sqrt[q]{\frac{(1+\mathscr{T}_2^q)^{\varpi_2} - (1-\mathscr{T}_2^q)^{\varpi_2}}{(1+\mathscr{T}_2^q)^{\varpi_2} + (1-\mathscr{T}_2^q)^{\varpi_2}}} \right]$$

$$= \left[\begin{array}{c} \sqrt[q]{\dfrac{\dfrac{(1+\mathfrak{M}_1^q)^{\varpi_1} - (1-\mathfrak{M}_1^q)^{\varpi_1}}{(1+\mathfrak{M}_1^q)^{\varpi_1} + (1-\mathfrak{M}_1^q)^{\varpi_1}} + \dfrac{(1+\mathfrak{M}_2^q)^{\varpi_2} - (1-\mathfrak{M}_2^q)^{\varpi_2}}{(1+\mathfrak{M}_2^q)^{\varpi_2} + (1-\mathfrak{M}_2^q)^{\varpi_2}}}{1+\left(\dfrac{(1+\mathfrak{M}_1^q)^{\varpi_1} - (1-\mathfrak{M}_1^q)^{\varpi_1}}{(1+\mathfrak{M}_1^q)^{\varpi_1} + (1-\mathfrak{M}_1^q)^{\varpi_1}}\right).\left(\dfrac{(1+\mathfrak{M}_2^q)^{\varpi_2} - (1-\mathfrak{M}_2^q)^{\varpi_2}}{(1+\mathfrak{M}_2^q)^{\varpi_2} + (1-\mathfrak{M}_2^q)^{\varpi_2}}\right)}}, \\ \sqrt[q]{\dfrac{\left(\dfrac{2(\mathfrak{Y}_1^q)^{\varpi_1}}{(2-\mathfrak{Y}_1^q)^{\varpi_1} + (\mathfrak{Y}_1^q)^{\varpi_1}}\right).\dfrac{2(\mathfrak{Y}_2^q)^{\varpi_2}}{(2-\mathfrak{Y}_2^q)^{\varpi_2} + (\mathfrak{Y}_2^q)^{\varpi_2}}}{1+\left(1-\dfrac{2((\mathfrak{Y}_1^q))^{\varpi_1}}{(2-\mathfrak{Y}_1^q)^{\varpi_1} + (\mathfrak{Y}_1^q)^{\varpi_1}}\right).\left(1-\dfrac{2((\mathfrak{Y}_2^q))^{\varpi_2}}{(2-\mathfrak{Y}_2^q)^{\varpi_2} + (\mathfrak{Y}_2^q)^{\varpi_2}}\right)}}, \\ \sqrt[q]{\dfrac{\dfrac{(1+\mathscr{T}_1^q)^{\varpi_1} - (1-\mathscr{T}_1^q)^{\varpi_1}}{(1+\mathscr{T}_1^q)^{\varpi_1} + (1-\mathscr{T}_1^q)^{\varpi_1}} + \dfrac{(1+\mathscr{T}_2^q)^{\varpi_2} - (1-\mathscr{T}_2^q)^{\varpi_2}}{(1+\mathscr{T}_2^q)^{\varpi_2} + (1-\mathscr{T}_2^q)^{\varpi_2}}}{1+\left(\dfrac{(1+\mathscr{T}_1^q)^{\varpi_1} - (1-\mathscr{T}_1^q)^{\varpi_1}}{(1+\mathscr{T}_1^q)^{\varpi_1} + (1-\mathscr{T}_1^q)^{\varpi_1}}\right).\left(\dfrac{(1+\mathscr{T}_2^q)^{\varpi_2} - (1-\mathscr{T}_2^q)^{\varpi_2}}{(1+\mathscr{T}_2^q)^{\varpi_2} + (1-\mathscr{T}_2^q)^{\varpi_2}}\right)}} \end{array} \right]$$

$$= \left[\begin{array}{c} \sqrt[q]{\dfrac{(1+\mathfrak{M}_1^q)^{\varpi_1}.(1+\mathfrak{M}_2^q)^{\varpi_2} - (1-\mathfrak{M}_1^q)^{\varpi_1}.(1-\mathfrak{M}_2^q)^{\varpi_2}}{(1+\mathfrak{M}_1^q)^{\varpi_1}.(1+\mathfrak{M}_2^q)^{\varpi_2} + (1-\mathfrak{M}_1^q)^{\varpi_1}.(1-\mathfrak{M}_2^q)^{\varpi_2}}}, \\ \dfrac{\sqrt[q]{2}(\mathfrak{Y}_1^{\varpi_1}\mathfrak{Y}_2^{\varpi_2})}{\sqrt[q]{(2-\mathfrak{Y}_1^q)^{\varpi_1}.(2-\mathfrak{Y}_2^q)^{\varpi_2} + (2-\mathfrak{Y}_1^q)^{\varpi_1}.(2-\mathfrak{Y}_2^q)^{\varpi_2}}}, \\ \sqrt[q]{\dfrac{(1+\mathscr{T}_1^q)^{\varpi_1}.(1+\mathscr{T}_2^q)^{\varpi_2} - (1-\mathscr{T}_1^q)^{\varpi_1}.(1-\mathscr{T}_2^q)^{\varpi_2}}{(1+\mathscr{T}_1^q)^{\varpi_1}.(1+\mathscr{T}_2^q)^{\varpi_2} + (1-\mathscr{T}_1^q)^{\varpi_1}.(1-\mathscr{T}_2^q)^{\varpi_2}}} \end{array} \right]$$

$$= \left[\begin{array}{c} \sqrt[q]{\dfrac{\prod_{i=1}^{2}\left(1+\mathfrak{M}_i^q\right)^{\varpi_i}-\prod_{i=1}^{2}\left(1-\mathfrak{M}_i^q\right)^{\varpi_i}}{\prod_{i=1}^{2}\left(1+\mathfrak{M}_i^q\right)^{\varpi_i}+\prod_{i=1}^{n}\left(1-\mathfrak{M}_i^q\right)^{\varpi_i}}}, \\ \dfrac{\sqrt[q]{2}\prod_{i=1}^{2}(\mathfrak{Y}_i)^{\varpi_i}}{\sqrt[q]{\prod_{i=1}^{2}\left(2-\mathfrak{Y}_i^q\right)^{\varpi_i}+\prod_{i=1}^{2}\left(\mathfrak{Y}_i^q\right)^{\varpi_i}}}, \\ \sqrt[q]{\dfrac{\prod_{i=1}^{2}\left(1+\mathscr{T}_i^q\right)^{\varpi_i}-\prod_{i=1}^{2}\left(1-\mathscr{T}_i^q\right)^{\varpi_i}}{\prod_{i=1}^{2}\left(1+\mathscr{T}_i^q\right)^{\varpi_i}+\prod_{i=1}^{n}\left(1-\mathscr{T}_i^q\right)^{\varpi_i}}} \end{array} \right].$$

That is, for $n = 2$, it holds.

Suppose that for $n = k$, the equation holds, that is:

$$\text{C}q\text{-ROFECIWA}(\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_k) = \left[\begin{array}{c} \sqrt[q]{\dfrac{\prod_{i=1}^{k}\left(1+\mathfrak{M}_i^q\right)^{\varpi_i}-\prod_{i=1}^{k}\left(1-\mathfrak{M}_i^q\right)^{\varpi_i}}{\prod_{i=1}^{k}\left(1+\mathfrak{M}_i^q\right)^{\varpi_i}+\prod_{i=1}^{k}\left(1-\mathfrak{M}_i^q\right)^{\varpi_i}}}, \\ \dfrac{\sqrt[q]{2}\prod_{i=1}^{k}(\mathfrak{Y}_i)^{\varpi_i}}{\sqrt[q]{\prod_{i=1}^{k}\left(2-\mathfrak{Y}_i^q\right)^{\varpi_i}+\prod_{i=1}^{k}\left(\mathfrak{Y}_i^q\right)^{\varpi_i}}}, \\ \sqrt[q]{\dfrac{\prod_{i=1}^{k}\left(1+\mathscr{T}_i^q\right)^{\varpi_i}-\prod_{i=1}^{k}\left(1-\mathscr{T}_i^q\right)^{\varpi_i}}{\prod_{i=1}^{k}\left(1+\mathscr{T}_i^q\right)^{\varpi_i}+\prod_{i=1}^{k}\left(1-\mathscr{T}_i^q\right)^{\varpi_i}}} \end{array} \right].$$

Now we will prove the same for $n = k + 1$.

$\text{C}q\text{-ROFECIWA}(\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_{k+1})$

$$= \text{C}q\text{-ROFECIWA}(\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_k) \oplus_{\text{T}_\text{E}} (\varpi_{k+1}\mathscr{C}_{k+1})$$

$$= \left[\begin{array}{l} \sqrt[q]{\frac{\prod_{i=1}^{k}\left(1+\mathfrak{M}_i^q\right)^{\varpi_i}-\prod_{i=1}^{k}\left(1-\mathfrak{M}_i^q\right)^{\varpi_i}}{\prod_{i=1}^{k}\left(1+\mathfrak{M}_i^q\right)^{\varpi_i}+\prod_{i=1}^{k}\left(1-\mathfrak{M}_i^q\right)^{\varpi_i}}} \oplus_{\mathrm{T_E}} \sqrt[q]{\frac{\left(1+\mathfrak{M}_{k+1}^q\right)^{\varpi_{k+1}}-\left(1-\mathfrak{M}_{k+1}^q\right)^{\varpi_{k+1}}}{\left(1+\mathfrak{M}_{k+1}^q\right)^{\varpi_{k+1}}+\left(1-\mathfrak{M}_{K+1}^q\right)^{\varpi_{K+1}}}}, \\ \frac{\sqrt[q]{2}\prod_{i=1}^{k}(\mathfrak{Y}_i)^{\varpi_i}}{\sqrt[q]{\prod_{i=1}^{k}\left(2-\mathfrak{Y}_i^q\right)^{\varpi_i}+\prod_{i=1}^{k}\left(\mathfrak{Y}_i^q\right)^{\varpi_i}}} \oplus_{\mathrm{T_E}} \frac{\sqrt[q]{2}\left(\mathfrak{Y}_{k+1}\right)^{\varpi_{K+1}}}{\sqrt[q]{\left(2-\mathfrak{Y}_{k+1}^q\right)^{\varpi_{k+1}}+\left(\mathfrak{Y}_{k+1}^q\right)^{\varpi_{k+1}}}}, \\ \sqrt[q]{\frac{\prod_{i=1}^{k}\left(1+\mathscr{T}_i^q\right)^{\varpi_i}-\prod_{i=1}^{k}\left(1-\mathscr{T}_i^q\right)^{\varpi_i}}{\prod_{i=1}^{k}\left(1+\mathscr{T}_i^q\right)^{\varpi_i}+\prod_{i=1}^{k}\left(1-\mathscr{T}_i^q\right)^{\varpi_i}}} \oplus_{\mathrm{T_E}} \sqrt[q]{\frac{\left(1+\mathscr{T}_{k+1}^q\right)^{\varpi_{k+1}}-\left(1-\mathscr{T}_{k+1}^q\right)^{\varpi_{k+1}}}{\left(1+\mathscr{T}_{k+1}^q\right)^{\varpi_{k+1}}+\left(1-\mathscr{T}_{K+1}^q\right)^{\varpi_{K+1}}}} \end{array} \right]$$

$$= \left[\begin{array}{l} \sqrt[q]{\frac{\prod_{i=1}^{k+1}(1+\mathfrak{M}_i)^{\varpi_i}-\prod_{i=1}^{k+1}(1-\mathfrak{M}_i)^{\varpi_i}}{\prod_{i=1}^{k+1}(1+\mathfrak{M}_i)^{\varpi_i}+\prod_{i=1}^{k+1}(1-\mathfrak{M}_i)^{\varpi_i}}}, \\ \sqrt[q]{\frac{2\prod_{i=1}^{k+1}(\mathfrak{Y}_i)^{\varpi_i}}{\prod_{i=1}^{k+1}(2-\mathfrak{Y}_i)^{\varpi_i}+\prod_{i=1}^{k+1}(\mathfrak{Y}_i)^{\varpi_i}}}, \\ \sqrt[q]{\frac{\prod_{i=1}^{k+1}(1+\Upsilon_i)^{\varpi_i}-\prod_{i=1}^{k+1}(1-\Upsilon_i)^{\varpi_i}}{\prod_{i=1}^{k+1}(1+\Upsilon_i)^{\varpi_i}+\prod_{i=1}^{k+1}(1-\Upsilon_i)^{\varpi_i}}} \end{array} \right],$$

where $\varpi_i = (\mathscr{M}(\mathscr{A}_{\sigma(i)}) - \mathscr{M}(\mathscr{A}_{\sigma(i-1)}))$ and it confirms that the above equation holds for $n = k + 1$, thus proving the required result. □

Theorem 3.6. Idempotency

Let $\mathscr{C}_i = \langle(\mathfrak{M}^q_{\mathscr{C}_i}, \mathfrak{Y}^q_{\mathscr{C}_i}); \mathscr{T}_i^q\rangle$ be a family of Cq-ROF numbers. Then, if all $\mathscr{C}_i$ are equal, i.e., $\mathscr{C}_i = C$ for all $i = 1, 2, \ldots, n$, then

$$\mathrm{C}q\text{-ROFECIWA}\,(\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n) = C.$$

Proof. Since $\mathscr{C}_i = \mathscr{C}$ for all $i = 1, 2, \ldots, n$, i.e., $\mathfrak{M}^q_{\mathscr{C}_i} = \mathfrak{M}^q_{\mathscr{C}}$, $\mathfrak{Y}^q_{\mathscr{C}_i} = \mathfrak{Y}^q_{\mathscr{C}}$, and $\mathscr{T}^q_{\mathscr{C}_i} = \mathscr{T}^q_{\mathscr{C}}$, $i = 1, 2, \ldots, n$, then

$$
\begin{aligned}
&\mathrm{C}q\text{-ROFECIWA}\left(\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n\right) \\
&= \left[\begin{array}{c}
\sqrt[q]{\dfrac{\prod_{i=1}^{n}\left(1+\mathfrak{M}^q_{\mathscr{C}_i}\right)^{\sum_{i=1}^{n}(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))} - \prod_{p=1}^{n}\left(1-\mathfrak{M}^q_{\mathscr{C}_i}\right)^{\sum_{i=1}^{n}(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}{\prod_{i=1}^{n}\left(1+\mathfrak{M}^q_{\mathscr{C}_i}\right)^{\sum_{i=1}^{n}(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))} + \prod_{i=1}^{n}\left(1-\mathfrak{M}^q_{\mathscr{C}_i}\right)^{\sum_{i=1}^{n}(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}}, \\
\sqrt[q]{\dfrac{2\prod_{i=1}^{n}\left(\mathfrak{Y}^q_{\mathscr{C}_i}\right)^{\sum_{i=1}^{n}(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}{\prod_{i=1}^{n}\left(2-\mathfrak{Y}^q_{\mathscr{C}_i}\right)^{\sum_{i=1}^{n}(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))} + \prod_{i=1}^{n}\left(\mathfrak{Y}^q_{\mathscr{C}_i}\right)^{\sum_{i=1}^{n}(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}}, \\
\sqrt[q]{\dfrac{\prod_{i=1}^{n}\left(1+\mathscr{T}^q_{\mathscr{C}_i}\right)^{\sum_{i=1}^{n}(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))} - \prod_{p=1}^{n}\left(1-\mathscr{T}^q_{\mathscr{C}_i}\right)^{\sum_{i=1}^{n}(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}{\prod_{i=1}^{n}\left(1+\mathscr{T}^q_{\mathscr{C}_i}\right)^{\sum_{i=1}^{n}(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))} + \prod_{i=1}^{n}\left(1-\mathscr{T}^q_{\mathscr{C}_i}\right)^{\sum_{i=1}^{n}(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}}
\end{array} \right]
\end{aligned}
$$

$$
= \left[\begin{array}{c}
\dfrac{\sum_{i=1}^{n}\left(1+\mathfrak{M}^q_{\mathscr{C}}\right)^{\sum_{i=1}^{n}\left(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)})\right)} - \sum_{i=1}^{n}\left(1-\mathfrak{M}^q_{\mathscr{C}}\right)^{\sum_{i=1}^{n}\left(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)})\right)}}{\sum_{i=1}^{n}\left(1+\mathfrak{M}^q_{\mathscr{C}}\right)^{\sum_{i=1}^{n}\left(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)})\right)} + \sum_{i=1}^{n}\left(1-\mathfrak{M}^q_{\mathscr{C}}\right)^{\sum_{i=1}^{n}\left(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)})\right)}}, \\
\dfrac{2\sum_{i=1}^{n}\left(\mathfrak{Y}^q_{\mathscr{C}}\right)^{\sum_{i=1}^{n}\left(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)})\right)}}{\sum_{i=1}^{n}\left(2-\mathfrak{Y}^q_{\mathscr{C}}\right)^{\sum_{i=1}^{n}\left(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)})\right)} + \sum_{i=1}^{n}\left(\mathfrak{Y}^q_{\mathscr{C}}\right)^{\sum_{i=1}^{n}\left(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)})\right)}}, \\
\dfrac{\sum_{i=1}^{n}\left(1+\mathscr{T}^q_{\mathscr{C}}\right)^{\sum_{i=1}^{n}\left(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)})\right)} - \sum_{i=1}^{n}\left(1-\mathscr{T}^q_{\mathscr{C}}\right)^{\sum_{i=1}^{n}\left(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)})\right)}}{\sum_{i=1}^{n}\left(1+\mathscr{T}^q_{\mathscr{C}}\right)^{\sum_{i=1}^{n}\left(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)})\right)} + \sum_{i=1}^{n}\left(1-\mathscr{T}^q_{\mathscr{C}}\right)^{\sum_{i=1}^{n}\left(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)})\right)}}
\end{array} \right]
$$

$$= \begin{bmatrix} \frac{\left(1+\mathfrak{M}_{\mathscr{C}}^{q}\right)-\left(1-\mathfrak{M}_{\mathscr{C}}^{q}\right)}{\left(1+\mathfrak{M}_{\mathscr{C}}^{q}\right)+\left(1-\mathfrak{M}_{\mathscr{C}}^{q}\right)}, \\ \frac{2\mathfrak{Y}_{\mathscr{C}}^{q}}{\left(2-\mathfrak{Y}_{\mathscr{C}}^{q}\right)+\mathfrak{Y}_{\mathscr{C}}^{q}}, \\ \frac{\left(1+\mathscr{T}_{\mathscr{C}}^{q}\right)-\left(1-\mathscr{T}_{\mathscr{C}}^{q}\right)}{\left(1+\mathscr{T}_{\mathscr{C}}^{q}\right)+\left(1-\mathscr{T}_{\mathscr{C}}^{q}\right)} \end{bmatrix} = \left(\mathfrak{M}_{\mathscr{C}}^{q}, \mathfrak{Y}_{\mathscr{C}}^{q}, \mathscr{T}_{\mathscr{C}}^{q}\right) = \mathscr{C}. \qquad \square$$

Theorem 3.7. Boundary

Let $\mathscr{C}_i = \langle(\mathfrak{M}_{\mathscr{C}_i}^{q}, \mathfrak{Y}_{\mathscr{C}_i}^{q}); \mathscr{T}_{\mathscr{C}_i}^{q}\rangle$ be a family of Cq-ROF numbers. Then,

$$\mathscr{C}_{\min} \leq \text{C}q\text{-ROFECIWA}\,(\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n) \leq \mathscr{C}_{\max}.$$

Where $\mathscr{C}_{\min} = \min\{\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n\}$ and $\mathscr{C}_{\max} = \max\{\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n\}$.

Proof. Let $f(r) = \frac{1-r}{1+r}$, $r \in [0,1]$, and then $f'(r) = \left[\frac{1-r}{1+r}\right]' = \frac{-2}{(1+r)^2} < 0$. Thus, $f(r)$ is a decreasing function. Since $\mathfrak{M}_{\mathscr{C}_{\min}}^{q} \leq \mathfrak{M}_{\mathscr{C}_i}^{q} \leq \mathfrak{M}_{\mathscr{C}_{\max}}^{q}$ for all i, then $f\left(\mathfrak{M}_{\mathscr{C}_{\max}}^{q}\right) \leq f\left(\mathfrak{M}_{\mathscr{C}_i}^{q}\right) \leq f\left(\mathfrak{M}_{\mathscr{C}_{\min}}^{q}\right)$ for all i, i.e., $\sqrt[q]{\frac{1-\mathfrak{M}_{\mathscr{C}_{\max}}^{q}}{1+\mathfrak{M}_{\mathscr{C}_{\max}}^{q}}} \leq \sqrt[q]{\frac{1-\mathfrak{M}_{\mathscr{C}_i}^{q}}{1+\mathfrak{M}_{\mathscr{C}_i}^{q}}} \leq \sqrt[q]{\frac{1-\mathfrak{M}_{\mathscr{C}_{\min}}^{q}}{1+\mathfrak{M}_{\mathscr{C}_{\min}}^{q}}}$ $(i = 1, 2, \ldots n)$. We have

$$\sqrt[q]{\left(\frac{1-\mathfrak{M}_{\mathscr{C}_{\max}}^{q}}{1+\mathfrak{M}_{\mathscr{C}_{\max}}^{q}}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}} \leq \sqrt[q]{\left(\frac{1-\mathfrak{M}_{\mathscr{C}_i}^{q}}{1+\mathfrak{M}_{\mathscr{C}_i}^{q}}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}} \leq \sqrt[q]{\left(\frac{1-\mathfrak{M}_{\mathscr{C}_{\min}}^{q}}{1+\mathfrak{M}_{\mathscr{C}_{\min}}^{q}}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}$$

$$\sqrt[q]{\prod_{i=1}^{n}\left(\frac{1-\mathfrak{M}_{\mathscr{C}_{\max}}^{q}}{1+\mathfrak{M}_{\mathscr{C}_{\max}}^{q}}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}} \leq \sqrt[q]{\prod_{i=1}^{n}\left(\frac{1-\mathfrak{M}_{\mathscr{C}_i}^{q}}{1+\mathfrak{M}_{\mathscr{C}_i}^{q}}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}} \leq \sqrt[q]{\prod_{i=1}^{n}\left(\frac{1-\mathfrak{M}_{\mathscr{C}_{\min}}^{q}}{1+\mathfrak{M}_{\mathscr{C}_{\min}}^{q}}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}$$

$$\sqrt[q]{\sum_{i=1}^{n}\left(\frac{1-\mathfrak{M}_{\mathscr{C}_{\max}}^{q}}{1+\mathfrak{M}_{\mathscr{C}_{\max}}^{q}}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}} \leq \sqrt[q]{\prod_{i=1}^{n}\left(\frac{1-\mathfrak{M}_{\mathscr{C}_p}^{q}}{1+\mathfrak{M}_{\mathscr{C}_i}^{q}}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}} \leq \sqrt[q]{\sum_{i=1}^{n}\left(\frac{1-\mathfrak{M}_{\mathscr{C}_{\max}}^{q}}{1+\mathfrak{M}_{\mathscr{C}_{\max}}^{q}}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}$$

$$\Leftrightarrow \sqrt[q]{\left(\frac{1-\mathfrak{M}_{\mathscr{C}_{\max}}^{q}}{1+\mathfrak{M}_{\mathscr{C}_{\max}}^{q}}\right)} \leq \sqrt[q]{\prod_{i=1}^{n}\left(\frac{1-\mathfrak{M}_{\mathscr{C}_p}^{q}}{1+\mathfrak{M}_{\mathscr{C}_i}^{q}}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}} \leq \sqrt[q]{\left(\frac{1-\mathfrak{M}_{\mathscr{C}_{\min}}^{q}}{1+\mathfrak{M}_{\mathscr{C}_{\min}}^{q}}\right)}$$

$$\Leftrightarrow \sqrt[q]{\frac{2}{1+\mathfrak{M}_{\mathscr{C}_{\max}}^{q}}} \leq \sqrt[q]{1+\prod_{i=1}^{n}\left(\frac{1-\mathfrak{M}_{\mathscr{C}_i}^{q}}{1+\mathfrak{M}_{\mathscr{C}_i}^{q}}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)})}}\,) \leq \sqrt[q]{\frac{2}{1+\mathfrak{M}_{\beta_{\min}}^{q}}}$$

$$\Leftrightarrow \sqrt[q]{\frac{1+\mathfrak{M}^q_{\beta_{\min}}}{2}} \leq \sqrt[q]{\frac{1}{1+\prod_{i=1}^{n}\left(\frac{1-\mathfrak{M}^q_{\mathscr{C}_i}}{1+\mathfrak{M}^q_{\mathscr{C}_i}}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}} \leq \sqrt[q]{\frac{1+\mathfrak{M}^q_{\mathscr{C}_{\max}}}{2}}$$

$$\Leftrightarrow 1+\mathfrak{M}^q_{\mathscr{C}_{\min}} \leq \frac{2}{1+\prod_{i=1}^{n}\left(\frac{1-\mathfrak{M}^q_{\mathscr{C}_i}}{1+\mathfrak{M}^q_{\mathscr{C}_i}}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}} \leq 1+\mathfrak{M}^q_{\mathscr{C}_{\max}}$$

$$\Leftrightarrow \mathfrak{M}^q_{\mathscr{C}_{\min}} \leq \frac{2}{1+\prod_{i=1}^{n}\left(\frac{1-\mathfrak{M}^q_{\mathscr{C}_i}}{1+\mathfrak{M}^q_{\mathscr{C}_i}}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}} - 1 \leq \mathfrak{M}^q_{\beta_{\max}},$$

i.e.,

$$\Leftrightarrow \mathfrak{M}^q_{\mathscr{C}_{\min}} \leq \frac{\prod_{i=1}^{n}\left(1+\mathfrak{M}^q_{\mathscr{C}_i}\right)^{\left(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)})\right)} - \prod_{i=1}^{n}\left(1-\mathfrak{M}^q_{\mathscr{C}_i}\right)^{\left(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)})\right)}}{\prod_{i=1}^{n}\left(1+\mathfrak{M}^q_{\mathscr{C}_i}\right)^{\left(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)})\right)} + \prod_{i=1}^{n}\left(1-\mathfrak{M}^q_{\mathscr{C}_i}\right)^{\left(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)})\right)}} \leq \mathfrak{M}^q_{\mathscr{C}_{\max}}.$$

Let $g(y) = \frac{2-y}{y}$, $y \in (0,1]$, then $g'(y) = \frac{-2}{y^2} < 0$, which is a decreasing function on $(0,1]$. Since $\mathfrak{Y}^q_{\mathscr{C}_{\max}} \leq \mathfrak{Y}^q_{\mathscr{C}_i} \leq \mathfrak{Y}^q_{\mathscr{C}_{\min}}$, for all i, where $0 < \mathfrak{Y}^q_{\mathscr{C}_{\max}}$, we have $g\left(\mathfrak{Y}^q_{\mathscr{C}_{\min}}\right) \leq g\left(\mathfrak{Y}^q_{\mathscr{C}_i}\right) \leq g\left(\mathfrak{Y}^q_{\mathscr{C}_{\max}}\right)$, for all i, i.e.,

$$\sqrt[q]{\frac{2-\mathfrak{Y}^q_{\mathscr{C}_{\min}}}{\mathfrak{Y}^q_{\mathscr{C}_{\min}}}} \leq \sqrt[q]{\frac{2-\mathfrak{Y}^q_{\mathscr{C}_i}}{\mathfrak{Y}^q_{\mathscr{C}_i}}} \leq \sqrt[q]{\frac{2-\mathfrak{Y}^q_{\mathscr{C}_{\max}}}{\mathfrak{Y}^q_{\mathscr{C}_{\max}}}}, (i = 1,2,\ldots,n).$$

Let

$$\varpi = ((\mathscr{M}(\mathscr{A}_{\sigma(1)}) - \mathscr{M}(\mathscr{A}_{\sigma(0)})), (\mathscr{M}(\mathscr{A}_{\sigma(2)}) - \mathscr{M}(\mathscr{A}_{\sigma(1)})), \ldots, (\mathscr{M}(\mathscr{A}_{\sigma(n)}) - \mathscr{M}(\mathscr{A}_{\sigma(n-1)})))^T$$

be the weight vector of $\mathscr{C}_i$, $(i = 1,2,\ldots,n)$ such that $\mathscr{M}(\mathscr{A}_{\sigma(i)}) - \mathscr{M}(\mathscr{A}_{\sigma(i-1)}) \in [0,1]$ and $\sum_{i=1}^{n}\left(\mathscr{M}(\mathscr{A}_{\sigma(i)}) - \mathscr{M}(\mathscr{A}_{\sigma(i-1)})\right) = 1$. Then for all i, we have

$$\sqrt[q]{\left(\frac{2-\mathfrak{Y}^q_{\mathscr{C}_{\min}}}{\mathfrak{Y}^q_{\mathscr{C}_{\min}}}\right)^{\left(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)})\right)}} \leq \sqrt[q]{\left(\frac{2-\mathfrak{Y}^q_{\mathscr{C}_i}}{\mathfrak{Y}^q_{\mathscr{C}_i}}\right)^{\left(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)})\right)}}$$
$$\leq \sqrt[q]{\left(\frac{2-\mathfrak{Y}^q_{\mathscr{C}_{\max}}}{\mathfrak{Y}^q_{\mathscr{C}_{\max}}}\right)^{\left(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)})\right)}}.$$

Thus,

$$\sqrt[q]{\prod_{i=1}^{n}\left(\frac{2-\mathfrak{Y}^q_{\mathscr{C}_{\min}}}{\mathfrak{Y}^q_{\mathscr{C}_{\min}}}\right)^{\left(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)})\right)}} \leq \sqrt[q]{\prod_{i=1}^{n}\left(\frac{2-\mathfrak{Y}^q_{\mathscr{C}_i}}{\mathfrak{Y}^q_{\mathscr{C}_i}}\right)^{\left(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)})\right)}}$$
$$\leq \sqrt[q]{\prod_{i=1}^{n}\left(\frac{2-\mathfrak{Y}^q_{\mathscr{C}_{\max}}}{\mathfrak{Y}^q_{\mathscr{C}_{\max}}}\right)^{\left(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)})\right)}}.$$

$$\Leftrightarrow \sqrt[q]{\left(\frac{2-\mathfrak{Y}^q_{\mathscr{C}_{\min}}}{\mathfrak{Y}^q_{\mathscr{C}_{\min}}}\right)} \leq \sqrt[q]{\prod_{i=1}^{n}\left(\frac{2-\mathfrak{Y}^q_{\mathscr{C}_i}}{\mathfrak{Y}^q_{\mathscr{C}_i}}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}} \leq \sqrt[q]{\left(\frac{2-\mathfrak{Y}^q_{\mathscr{C}_{\max}}}{\mathfrak{Y}^q_{\mathscr{C}_{\max}}}\right)}$$
$$\Leftrightarrow \frac{2}{\mathfrak{Y}^q_{\mathscr{C}_{\min}}} \leq \prod_{i=1}^{n}\left(\frac{2-\mathfrak{Y}^q_{\mathscr{C}_i}}{\mathfrak{Y}^q_{\mathscr{C}_i}}+1\right) \leq \frac{2}{\mathfrak{Y}^q_{\mathscr{C}_{\max}}}$$
$$\Leftrightarrow \frac{\mathfrak{Y}^q_{\mathscr{C}_{\min}}}{2} \leq \frac{1}{\prod_{i=1}^{n}\left(\frac{2-\mathfrak{Y}^q_{\mathscr{C}_i}}{\mathfrak{Y}^q_{\mathscr{C}_i}}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}+1} \leq \frac{\mathfrak{Y}^q_{\mathscr{C}_{\max}}}{2}$$
$$\Leftrightarrow \mathfrak{Y}^q_{\mathscr{C}_{\max}} \leq \frac{2}{\prod_{i=1}^{n}\left(\frac{2-\mathfrak{Y}^q_{\mathscr{C}_i}}{\mathfrak{Y}^q_{\mathscr{C}_i}}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}+1} \leq \mathfrak{Y}^q_{\mathscr{C}_{\min}}$$

i.e.,

$$\Leftrightarrow \mathfrak{Y}^q_{\mathscr{C}_{\max}} \leq \frac{2\prod_{i=1}^{n}\mathfrak{Y}^q_{\mathscr{C}_i}}{\prod_{i=1}^{n}\left(2-\mathfrak{Y}^q_{\mathscr{C}_i}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}+\prod_{i=1}^{n}\mathfrak{Y}^q_{\mathscr{C}_i}} \leq \mathfrak{Y}^q_{\mathscr{C}_{\min}}.$$

Similarly

$$\Leftrightarrow \mathscr{T}^q_{\mathscr{C}_{\min}} \leq \frac{\prod_{i=1}^{n}\left(1+\mathscr{T}^q_{\mathscr{C}_i}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}-\prod_{i=1}^{n}\left(1-\mathscr{T}^q_{\mathscr{C}_i}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}{\prod_{i=1}^{n}\left(1+\mathscr{T}^q_{\mathscr{C}_i}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}+\prod_{i=1}^{n}\left(1-\mathscr{T}^q_{\mathscr{C}_i}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}} \leq \mathscr{T}^q_{\mathscr{C}_{\max}}.$$

So, it concludes that $\mathfrak{M}^q_{\mathscr{C}_{\min}} \leq \mathfrak{M}^q_{\mathscr{C}} \leq \mathfrak{M}^q_{\mathscr{C}_{\max}}$, $\mathfrak{Y}^q_{\mathscr{C}_{\max}} \leq \mathfrak{Y}^q_{\mathscr{C}} \leq \mathfrak{Y}^q_{\mathscr{C}_{\min}}$, and $\mathscr{T}^q_{\mathscr{C}_{\min}} \leq \mathscr{T}^q_{\mathscr{C}} \leq \mathscr{T}^q_{\mathscr{C}_{\max}}$. □

Theorem 3.8. Monotonicity

Let $\mathscr{C}_i = \left\langle(\mathfrak{M}^q_{\mathscr{C}_i}, \mathfrak{Y}^q_{\mathscr{C}_i}); \mathscr{T}^q_{\mathscr{C}_i}\right\rangle$ and $\mathscr{C}^*_i = \left\langle(\mathfrak{M}^q_{\mathscr{C}^*_i}, \mathfrak{Y}^q_{\mathscr{C}^*_i}); \mathscr{T}^q_{\mathscr{C}^*_i}\right\rangle$ $(i = 1, 2, \ldots, n)$ be two families of Cq-ROF numbers, and $\mathscr{C}_i \leq \mathscr{C}_i^*$, i.e., $\mathfrak{M}^q_{\mathscr{C}_i} \leq \mathfrak{M}^q_{\mathscr{C}_i^*}, \mathfrak{Y}^q_{\mathscr{C}_i} \geq \mathfrak{Y}^q_{\mathscr{C}_i^*}$, and $\mathscr{T}^q_{\mathscr{C}_i} \leq \mathscr{T}^q_{\mathscr{C}_i^*}$, for all i. Then

$$\text{C}q\text{-ROFECIWA}\left(\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n\right) \leq \text{C}q\text{-ROFECIWA}\left(\mathscr{C}_1^*, \mathscr{C}_2^*, \ldots, \mathscr{C}_n^*\right).$$

Proof. Let $f(r) = \dfrac{1-r}{1+r}, r \in [0,1]$, be a decreasing function, $\mathfrak{M}^q_{\mathscr{C}_i} \leq \mathfrak{M}^q_{\mathscr{C}_i^*}$, and then $f\left(\mathfrak{M}^q_{\mathscr{C}_i^*}\right) \leq f\left(\mathfrak{M}^q_{\mathscr{C}_i}\right)$, i.e., $\sqrt[q]{\dfrac{1-\mathfrak{M}^q_{\mathscr{C}_i^*}}{1+\mathfrak{M}^q_{\mathscr{C}_i^*}}} \leq \sqrt[q]{\dfrac{1-\mathfrak{M}^q_{\mathscr{C}_i}}{1+\mathfrak{M}^q_{\mathscr{C}_i}}}, (i = 1, 2, \ldots n)$. Let $\varpi = ((\mathscr{M}(\mathscr{A}_{\sigma(1)}) - \mathscr{M}(\mathscr{A}_{\sigma(0)})), \ldots, (\mathscr{M}(\mathscr{A}_{\sigma(n)}) - \mathscr{M}(\mathscr{A}_{\sigma(n-1)})))$ be the weighting vector of $\mathscr{C}_i$, such that $(\mathscr{M}(\mathscr{A}_{\sigma(i)}) - \mathscr{M}(\mathscr{A}_{\sigma(i-1)})) \in [0,1]$ and $\sum_{i=1}^{n}(\mathscr{M}(\mathscr{A}_{\sigma(i)}) - \mathscr{M}(\mathscr{A}_{\sigma(i-1)})) = 1$. Then for all i, we have

$$\sqrt[q]{\left(\frac{1-\mathfrak{M}^q_{\mathscr{C}_i^*}}{1+\mathfrak{M}^q_{\mathscr{C}_i^*}}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}} \leq \sqrt[q]{\left(\frac{1-\mathfrak{M}^q_{\mathscr{C}_i}}{1+\mathfrak{M}^q_{\mathscr{C}_i}}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}.$$

Thus,

$$\begin{aligned}
&\sqrt[q]{\prod_{i=1}^{n}\left(\frac{1-\mathfrak{M}^q_{\mathscr{C}_i^*}}{1+\mathfrak{M}^q_{\mathscr{C}_i^*}}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}} \leq \sqrt[q]{\prod_{i=1}^{n}\left(\frac{1-\mathfrak{M}^q_{\mathscr{C}_i}}{1+\mathfrak{M}^q_{\mathscr{C}_i}}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}} \\
&\Leftrightarrow 1+\sqrt[q]{\prod_{i=1}^{n}\left(\frac{1-\mathfrak{M}^q_{\mathscr{C}_i^*}}{1+\mathfrak{M}^q_{\mathscr{C}_i^*}}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}} \leq 1+\sqrt[q]{\prod_{i=1}^{n}\left(\frac{1-\mathfrak{M}^q_{\mathscr{C}_i}}{1+\mathfrak{M}^q_{\mathscr{C}_i}}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}} \\
&\Leftrightarrow \frac{1}{1+\sqrt[q]{\prod_{i=1}^{n}\left(\frac{1-\mathfrak{M}^q_{\mathscr{C}_i}}{1+\mathfrak{M}^q_{\mathscr{C}_i}}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}} \leq \frac{1}{1+\sqrt[q]{\prod_{i=1}^{n}\left(\frac{1-\mathfrak{M}^q_{\mathscr{C}_i^*}}{1+\mathfrak{M}^q_{\mathscr{C}_i^*}}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}} \\
&\Leftrightarrow \frac{2}{1+\sqrt[q]{\prod_{i=1}^{n}\left(\frac{1-\mathfrak{M}^q_{\mathscr{C}_i}}{1+\mathfrak{M}^q_{\mathscr{C}_i}}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}} \leq \frac{2}{1+\sqrt[q]{\prod_{i=1}^{n}\left(\frac{1-\mathfrak{M}^q_{\mathscr{C}_i^*}}{1+\mathfrak{M}^q_{\mathscr{C}_i^*}}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}} \\
&\Leftrightarrow \frac{2}{1+\sqrt[q]{\prod_{i=1}^{n}\left(\frac{1-\mathfrak{M}^q_{\mathscr{C}_i}}{1+\mathfrak{M}^q_{\mathscr{C}_i}}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}} - 1 \leq \frac{2}{1+\sqrt[q]{\prod_{i=1}^{n}\left(\frac{1-\mathfrak{M}^q_{\mathscr{C}_i^*}}{1+\mathfrak{M}^q_{\mathscr{C}_i^*}}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}} - 1
\end{aligned}$$

$$\Leftrightarrow \sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\mathfrak{M}_{\mathscr{C}_i}^{q}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}-\prod_{i=1}^{n}\left(1-\mathfrak{M}_{\mathscr{C}_i}^{q}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}{\prod_{i=1}^{n}\left(1+\mathfrak{M}_{\mathscr{C}_i}^{q}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}+\prod_{i=1}^{n}\left(1-\mathfrak{M}_{\mathscr{C}_i}^{q}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}} \leq$$

$$\sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\mathfrak{M}_{\mathscr{C}_i^*}^{q}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}-\prod_{i=1}^{n}\left(1-\mathfrak{M}_{\mathscr{C}_i^*}^{q}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}{\prod_{i=1}^{n}\left(1+\mathfrak{M}_{\mathscr{C}_i^*}^{q}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}+\prod_{i=1}^{n}\left(1-\mathfrak{M}_{\mathscr{C}_i^*}^{q}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}}.$$

Similarly, it can be proved for $\mathfrak{Y}_{\mathscr{C}_i}^{q} \geq \mathfrak{Y}_{\mathscr{C}_i^*}^{q}$ such that:

$$\frac{\sqrt[q]{2}\prod_{i=1}^{n}\mathfrak{Y}_{\mathscr{C}_i}}{\sqrt[q]{\prod_{i=1}^{n}\left(2-\mathfrak{Y}_{\mathscr{C}_i}^{q}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}+\prod_{i=1}^{n}\mathfrak{Y}_{\mathscr{C}_i}^{q}}} \geq$$

$$\frac{\sqrt[q]{2}\prod_{i=1}^{n}\mathfrak{Y}_{\mathscr{C}_i^*}}{\sqrt[q]{\prod_{i=1}^{n}\left(2-\mathfrak{Y}_{\mathscr{C}_i^*}^{q}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}+\prod_{i=1}^{n}(\mathfrak{Y}_{\mathscr{C}_i^*}^{q})^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}}.$$

In the same way as proving for μ, it can be proved for $\mathscr{T}_{\mathscr{C}_i}^{q} \leq \mathscr{T}_{\mathscr{C}_i^*}^{q}$ that:

$$\Leftrightarrow \sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\mathscr{T}_{\mathscr{C}_i}^{q}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}-\prod_{i=1}^{n}\left(1-\mathscr{T}_{\mathscr{C}_i}^{q}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}{\prod_{i=1}^{n}\left(1+\mathscr{T}_{\mathscr{C}_i}^{q}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}+\prod_{i=1}^{n}\left(1-\mathscr{T}_{\mathscr{C}_i}^{q}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}} \leq$$

$$\sqrt[q]{\frac{\prod_{i=1}^{n}\left(1+\mathscr{T}_{\mathscr{C}_i^*}^{q}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}-\prod_{i=1}^{n}\left(1-\mathscr{T}_{\mathscr{C}_i^*}^{q}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}{\prod_{i=1}^{n}\left(1+\mathscr{T}_{\mathscr{C}_i^*}^{q}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}+\prod_{i=1}^{n}\left(1-\mathscr{T}_{\mathscr{C}_i^*}^{q}\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}}.$$

□

3.3. Cq-ROFECI weighted geometric operator

Theorem 3.9. Let $\mathscr{C}_i = \langle(\mathfrak{M}^q_{c_i}, \mathfrak{Y}^q_{c_i}); \mathscr{T}_i^q\rangle$ $(i = 1, \ldots, n)$ be a family of Cq-ROF numbers, and then the aggregated value by using Cq-ROF Einstein operational laws is defined as:

$$\text{C}q\text{-RFECIWG}(\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n) = \bigotimes_{i=1}^{n}{}_{\mathrm{T_E}} (\mathscr{C}_i)^{\varpi_i} \tag{21}$$

$$= \left[\begin{array}{c} \dfrac{\sqrt[q]{2}\prod_{i=1}^{n}(\mathfrak{M}_i)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}{\sqrt[q]{\prod_{i=1}^{n}\left(2-\mathfrak{M}_i^q\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}+\prod_{i=1}^{n}\left(\mathfrak{M}_i^q\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}}, \\ \sqrt[q]{\dfrac{\prod_{i=1}^{n}\left(1+\mathfrak{Y}_i^q\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}-\prod_{i=1}^{n}\left(1-\mathfrak{Y}_i^q\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}{\prod_{i=1}^{n}\left(1+\mathfrak{Y}_i^q\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}+\prod_{i=1}^{n}\left(1-\mathfrak{Y}_i^q\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}}, \\ \dfrac{\sqrt[q]{2}\prod_{i=1}^{n}(\Upsilon_i)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}{\sqrt[q]{\prod_{i=1}^{n}\left(2-\mathscr{T}_i^q\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}+\prod_{i=1}^{n}\left(\mathscr{T}_i^q\right)^{(\mathscr{M}(\mathscr{A}_{\sigma(i)})-\mathscr{M}(\mathscr{A}_{\sigma(i-1)}))}}} \end{array}\right], \tag{22}$$

where $(\mathscr{M}(\mathscr{A}_{\sigma(i)}) - \mathscr{M}(\mathscr{A}_{\sigma(i-1)})) = ((\mathscr{M}(\mathscr{A}_{\sigma(1)}) - \mathscr{M}(\mathscr{A}_{\sigma(0)})), \ldots, (\mathscr{M}(\mathscr{A}_{\sigma(n)}) - \mathscr{M}(\mathscr{A}_{\sigma(n-1)})))^T$ is the weight vector of $(\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n)$ and $\sum_{i=1}^{n} (\mathscr{M}(\mathscr{A}_{\sigma(i)}) - \mathscr{M}(\mathscr{A}_{\sigma(i-1)})) = 1$.

Theorem 3.10. Idempotency

Let $\mathscr{C}_i = \langle(\mathfrak{M}^q_i, \mathfrak{Y}^q_i); \mathscr{T}_i^q\rangle$ be a family of Cq-ROF numbers. Then if all $\mathscr{C}_i$ are equal, i.e., $\mathscr{C}_i = \mathscr{C}$ for all $i = 1, 2, \ldots, n$, then

$$\text{C}q\text{-RFECIWG}(\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n) = \mathscr{C}.$$

Theorem 3.11. Boundary

Let $\mathscr{C}_i = \langle(\mathfrak{M}^q_i, \mathfrak{Y}^q_i); \mathscr{T}_i^q\rangle$ be a family of Cq-ROF numbers. Then,

$$\mathscr{C}_{\min} \leq \text{C}q\text{-RFECIWG}(\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n) \leq \mathscr{C}_{\max},$$

where $\mathscr{C}_{\min} = \min\{\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n\}$ and $\mathscr{C}_{\max} = \max\{\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n\}$.

Theorem 3.12. Monotonicity

Let $\mathscr{C}_i = \left(\mathfrak{M}^q_{\mathscr{C}_i}, \mathfrak{Y}^q_{\mathscr{C}_i}, \mathscr{T}^q_{\mathscr{C}_i}\right)$ and $\mathscr{C}_i^* = \left(\mathfrak{M}^q_{\mathscr{C}_i^*}, \mathfrak{Y}^q_{\mathscr{C}_i^*}, \mathscr{T}^q_{\mathscr{C}_i^*}\right)(i = 1, 2, \ldots, n)$ be Cq-ROFECIWA of Cq-ROF numbers, and $\mathscr{C}_i \leq \mathscr{C}_i^*$, i.e., $\mathfrak{M}^q_{\mathscr{C}_i} \leq \mathfrak{M}^q_{\mathscr{C}_i^*}, \mathfrak{Y}^q_{\mathscr{C}_i} \geq \mathfrak{Y}^q_{\mathscr{C}_i^*}$ and $\mathscr{T}^q_{\mathscr{C}_i} \leq \mathscr{T}^q_{\mathscr{C}_i^*}$, for all i; then Cq-RFECIWG$(\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n) \leq$ Cq-RFECIWG$\left(\mathscr{C}_1^*, \mathscr{C}_2^*, \ldots, \mathscr{C}_n^*\right)$.

4. MCDM framework utilizing the Cq-ROFECI operator

Step 1. The proposed case study requires decision-makers with a thorough background in this sector. Decision-makers' weight vectors will be assessed using linguistic terms from Table 2.

Table 2. Linguistic terms for DMs.

Qualification	Expertise	Experience (Years)	Cq-POFN
PhD	Public health expert	≥5	([0.95,0.10],Υ_1)
MS	Environmental scientist	(3,5)	([0.75,0.20],Υ_2)
MSc	Economist	$[0, 3)$	([0.55,0.30],Υ_3)

Step 2. Find the radius $\mathscr{T}$ by using Eq (1) and then apply Eq (2) to find the score value of each Cq-ROFN. Next, normalize the significance of the DMs by applying the Eq (23).

$$\xi_k = \frac{S(\tilde{\mathscr{C}})}{\sum\limits_{t=1}^{z} S(\tilde{\mathscr{C}}^t)} = \frac{\sum\limits_{k=1}^{n} \dfrac{\mathfrak{M}_{ij}^{q,k} - \mathfrak{Y}_{ij}^{q,k} + \sqrt{2}\mathscr{T}_{ij}^{q}(2\wp - 1)}{3}}{\sum\limits_{t=1}^{z} \dfrac{\mathfrak{M}_{ij}^{q,t} - \mathfrak{Y}_{ij}^{q,t} + \sqrt{2}\mathscr{T}_{ij}^{q,t}(2\wp - 1)}{3}}. \tag{23}$$

Here, $\xi = (\xi_1, \xi_2, ..., \xi_z)$ shows the important vector of the DMs, with the conditions $\xi \in [0, 1]$ and $\sum\limits_{k}^{z} \xi_k = 1$.

Step 3. Decision-makers input the Cq-ROF dataset against the suitable alternatives $\mathscr{L}_p; (p = 1, 2, ..., m)$ and under the effect of various criteria $\mathfrak{Cr}_p; (p = 1, 2, ..., n)$ with the help of linguistic terms defined in Table 3.

Table 3. Generalized linguistic terms and their corresponding Cq-ROFNs.

Linguistic term	Abbreviation	Cq-ROFNs
Extremely High	E.H	$([\mathfrak{M}_1^q, \mathfrak{Y}_1^q], \mathscr{T}_1^q)$
Highly Elevated	H.E	$([\mathfrak{M}_2^q, \mathfrak{Y}_2^q], \mathscr{T}_2^q)$
⋮	⋮	⋮
Moderate	M	$([\mathfrak{M}_g^q, \mathfrak{Y}_g^q], \mathscr{T}_g^q)$
⋮	⋮	⋮
Extremely Low	E.L	$([\mathfrak{M}_b^q, \mathfrak{Y}_b^q], \mathscr{T}_b^q)$

Step 4. Find the radius $\mathscr{T}$ by using Eq (1).

Step 5. Compute the aggregated values using the Cq-ROFECIWA operator described in Eq (24).

$$\mathscr{C}_{ij} = \text{C}q\text{-ROFECIWA}(\mathfrak{S}_{ij}^{1}, \mathfrak{S}_{ij}^{l}, ..., \mathfrak{S}_{ij}^{z})$$
$$= \left[\begin{array}{c} \sqrt[q]{\dfrac{\prod_{t=1}^{z}\left(1+\mathfrak{M}_{ij}^{q,t}\right)^{\xi_t} - \prod_{t=1}^{z}\left(1-\mathfrak{M}_{ij}^{q,t}\right)^{\xi_t}}{\prod_{t=1}^{z}\left(1+\mathfrak{M}_{ij}^{q,t}\right)^{\xi_t} + \prod_{t=1}^{z}\left(1-\mathfrak{M}_{ij}^{q,t}\right)^{\xi_t}}}, \\ \dfrac{\sqrt[q]{2}\prod_{t=1}^{z}\left(\mathfrak{Y}_{ij}^{t}\right)^{\xi_t}}{\sqrt[q]{\prod_{t=1}^{z}\left(2-\mathfrak{Y}_{ij}^{q,t}\right)^{\xi_t} + \prod_{t=1}^{z}\left(\mathfrak{Y}_{ij}^{q,t}\right)^{\xi_t}}}, \\ \sqrt[q]{\dfrac{\prod_{t=1}^{z}\left(1+\mathscr{T}_{i}^{q}\right)^{\xi_t} - \prod_{t=1}^{z}\left(1-\mathscr{T}_{i}^{q}\right)^{\xi_t}}{\prod_{t=1}^{z}\left(1+\mathscr{T}_{i}^{q}\right)^{\xi_t} + \prod_{t=1}^{z}\left(1-\mathscr{T}_{i}^{q}\right)^{\xi_t}}} \end{array} \right]. \tag{24}$$

Step 6. Find the importance of each criterion, which is fuzzy density $\mathscr{M}(Cr_j)$, by applying the LOPCOW method. The steps of the LOPCOW method are as follows:
Step 6.1. Find the score matrix Sc_{ij} of the aggregated decision matrix by applying Eq (2). Then normalize this matrix using (25).

$$\widetilde{Sc}_{ij} = \begin{cases} \frac{Sc_{ij}-Sc_j^-}{Sc_j^+-Sc_j^-}, & j \in \mathfrak{Cr}_b \\ \frac{Sc_j^+-Sc_{ij}}{Sc_j^+-Sc_j^-}, & j \in \mathfrak{Cr}_c \end{cases}, \tag{25}$$

where $Sc_j^+ = \max_i Sc_{ij}$, $Sc_j^- = \min_i Sc_{ij}$, and $\mathfrak{Cr}_b$ and $\mathfrak{Cr}_c$ represent the benefit-type and cost-type criteria, respectively.
Step 6.2. Obtain the percentage value (PV) for the criteria by using Eq (26).

$$P_j = \left| \ln \left(\frac{\sqrt{\frac{\sum_{i=1}^{m} \widetilde{Sc}_{ij}^2}{m}}}{\sigma} \right) \cdot 100 \right|, \tag{26}$$

where σ is the standard deviation of the performance values of the alternatives under a specific criterion.

Step 6.3. The fuzzy density $\mathscr{M}(Cr_j)$ weight for the j^{th} criterion is calculated by using Eq (27).

$$\mathscr{M}(Cr_j) = \frac{P_j}{\sum_{j=1}^{n} P_j}, \tag{27}$$

where $\sum_{j=1}^{n} \mathscr{M}(Cr_j) = 1$.

Step 7. Find the value of ρ by using Eq (18) and the normalized measure on X by using Eq (16).

Step 8. Apply Eq (28) to aggregate into one column.

$$\text{C}q\text{-ROFECIWA}(\mathscr{C}_{i1}, \mathscr{C}_{i2}, \ldots, \mathscr{C}_{in})$$

$$G_i = \left[\begin{array}{c} \sqrt[q]{\dfrac{\prod_{j=1}^{n}\left(1+\mathfrak{M}_{i\sigma(j)}^{q}\right)^{(\mathscr{M}(\mathscr{L}_{i\sigma(j)})-\mathscr{M}(\mathscr{L}_{i\sigma(j-1)}))} - \prod_{j=1}^{n}\left(1-\mathfrak{M}_{i\sigma(j)}^{q}\right)^{(\mathscr{M}(\mathscr{L}_{i\sigma(j)})-\mathscr{M}(\mathscr{L}_{i\sigma(j-1)}))}}{\prod_{j=1}^{n}\left(1+\mathfrak{M}_{i\sigma(j)}^{q}\right)^{(\mathscr{M}(\mathscr{L}_{i\sigma(j)})-\mathscr{M}(\mathscr{L}_{i\sigma(j-1)}))} + \prod_{j=1}^{n}\left(1-\mathfrak{M}_{i\sigma(j)}^{q}\right)^{(\mathscr{M}(\mathscr{L}_{i\sigma(j)})-\mathscr{M}(\mathscr{L}_{i\sigma(j-1)}))}}}, \\ \dfrac{\sqrt[q]{2}\prod_{j=1}^{n}\left(\mathfrak{Y}_{i\sigma(j)}^{q}\right)^{(\mathscr{M}(\mathscr{L}_{i\sigma(j)})-\mathscr{M}(\mathscr{L}_{i\sigma(j-1)}))}}{\sqrt[q]{\prod_{j=1}^{n}\left(2-\mathfrak{Y}_{i\sigma(j)}^{q}\right)^{(\mathscr{M}(\mathscr{L}_{i\sigma(j)})-\mathscr{M}(\mathscr{L}_{i\sigma(j-1)}))} + \prod_{j=1}^{n}\left(\mathfrak{Y}_{i\sigma(j)}^{q}\right)^{(\mathscr{M}(\mathscr{L}_{i\sigma(j)})-\mathscr{M}(\mathscr{L}_{i\sigma(j-1)}))}}}, \\ \sqrt[q]{\dfrac{\prod_{j=1}^{n}\left(1+\mathscr{T}_{i\sigma(j)}^{q}\right)^{(\mathscr{M}(\mathscr{L}_{i\sigma(j)})-\mathscr{M}(\mathscr{L}_{i\sigma(j-1)}))} - \prod_{j=1}^{n}\left(1-\mathscr{T}_{i\sigma(j)}^{q}\right)^{(\mathscr{M}(\mathscr{L}_{i\sigma(j)})-\mathscr{M}(\mathscr{L}_{i\sigma(j-1)}))}}{\prod_{j=1}^{n}\left(1+\mathscr{T}_{i\sigma(j)}^{q}\right)^{(\mathscr{M}(\mathscr{L}_{i\sigma(j)})-\mathscr{M}(\mathscr{L}_{i\sigma(j-1)}))} + \prod_{j=1}^{n}\left(1-\mathscr{T}_{i\sigma(j)}^{q}\right)^{(\mathscr{M}(\mathscr{L}_{i\sigma(j)})-\mathscr{M}(\mathscr{L}_{i\sigma(j-1)}))}}} \end{array} \right]. \tag{28}$$

Step 9. Apply Eq (2) and find the score values. Based on the score values, we can find the best alternative.

5. Case analysis

SWM encompasses the collection, transportation, processing, recycling, and disposal of waste materials generated by human activities. Effective waste management is crucial for maintaining public health, preserving the environment, and conserving resources. Singapore, a highly urbanized and densely populated city with 6.05 million people, faces significant challenges in managing its solid waste sustainably. The city produces approximately 7.39 million tons of waste annually. Singapore uses many different SWM methods and each needs its own facilities and costs. Because they are not as efficient as burning or recycling, Singapore does not do a lot of composting, vermiculture, bioremediation, or pyrolysis. See Table 4 for more information. Table 5 presents comprehensive explanations of several alternative approaches to SWM. Figure 1 emphasizes the essential elements of each criterion for sustainable and successful decision-making.

Table 4. Solid waste management techniques in Singapore.

Technique	Annual waste treated (tons)	Cost (USD/ton)	Location/Facilities
Incineration	2.9 million	77	Tuas Incineration Plant, Senoko Waste-to-Energy Plant, Keppel Seghers Tuas Waste-to-Energy Plant
Sanitary landfills	0.3 million (residue)	38	Semakau Landfill
Recycling	4.24 million	Varies	Public Waste Collection centers, various private facilities
Composting	Limited scale	Varies	Various small-scale facilities
Vermiculture	Limited scale	Low	Community-based projects, small-scale urban farms
Bioremediation	Limited (specific sites)	Variable	Specific contaminated sites
Pyrolysis	Limited scale (pilot)	High initial	Research facilities, pilot project sites
Open dumping	Practically zero	N/A	N/A

Table 5. Solid waste management alternatives with descriptions.

Alternative	Description
Landfills $\mathscr{L}_1$	To isolate garbage from the environment, landfills are used. Waste is compacted and covered with soil to reduce air exposure. Liners and garbage collection systems protect soil and groundwater in modern landfills. Due to organic waste decomposition, landfills emit methane, which contributes to climate change. Despite these concerns, landfills are nevertheless commonly utilized to manage massive amounts of trash.
Pyrolysis $\mathscr{L}_2$	Pyrolysis produces syngas, oil, and char by thermally decomposing waste at high temperatures without oxygen. High temperatures and oxygen-free conditions break down trash into simpler molecules in this process. Pyrolysis minimizes landfill waste and provides energy-producing byproducts. This process has lower emissions than incineration and is regulated for safety and emissions. Sustainable waste management may be possible with pyrolysis.
Vermiculture $\mathscr{L}_3$	Vermiculture or vermicomposting, refers to the technique of using worms to decompose organic waste and produce vermicast, also known as worm castings. Worms consume organic waste and generate nutrient-dense castings as a result. The procedure produces a valuable soil supplement that enhances soil quality and reduces the amount of organic waste disposed in landfills. Various organizations dedicated to enhancing the environment and the agricultural industry have promoted vermiculture as a method of waste management.
Bioremediation $\mathscr{L}_4$	Bioremediation employs living organisms such as plants, fungi, or bacteria to eradicate or counteract contaminants in water, soil, and waste. To mitigate the pollution in a certain area, the introduction of plants or microbes is employed to break down the harmful substances present. Bioremediation is an excellent choice for remedying contaminated areas and enhancing the potability of water and soil. Environmental agencies often allocate cash for this approach due to its sustainability and environmental friendliness.
Open dumping $\mathscr{L}_5$	Open dumping is the unrestricted disposal of garbage on unlicensed land. This technique degrades the environment without therapy or containment. Large amounts of waste exposed to nature in open dumps pollute soil and groundwater, release air pollutants from burning garbage, and pose health risks from bugs. Due to its environmental and health dangers, many countries ban or restrict this activity.
Incineration $\mathscr{L}_6$	Burning garbage at high temperatures reduces its volume and bulk and can generate energy. This process turns waste into ash, flue gas, and heat for electricity or heating. Incineration reduces trash volume but pollutes the air with dioxins and other pollutants. Modern incinerators have emission control systems that decrease these effects. Densely populated locations with little landfill space employ incineration.
Sanitary landfill $\mathscr{L}_7$	Sanitary landfills are designed to separate human waste from nature. Liners, leachate collection, and gas extraction systems reduce environmental effects. Daily, garbage is piled up, crushed, and covered with dirt or other material. Sanitary landfills clean soil and groundwater and reduce methane emissions more than regular landfills. This technology must be strictly regulated for environmental and public health reasons.
Composting $\mathscr{L}_8$	Composting is the biological decomposition of organic waste, such as food leftovers and garden detritus, which produces nutrient-rich compost. This process involves layering organic waste and promoting its decomposition through controlled conditions with the assistance of microorganisms. Composting reduces the amount of organic waste that is thrown away in landfills and produces valuable compost that may be used to improve soil fertility. This technique is an environmentally friendly method of managing garbage that reduces the emission of greenhouse gases and supports sustainable agriculture.

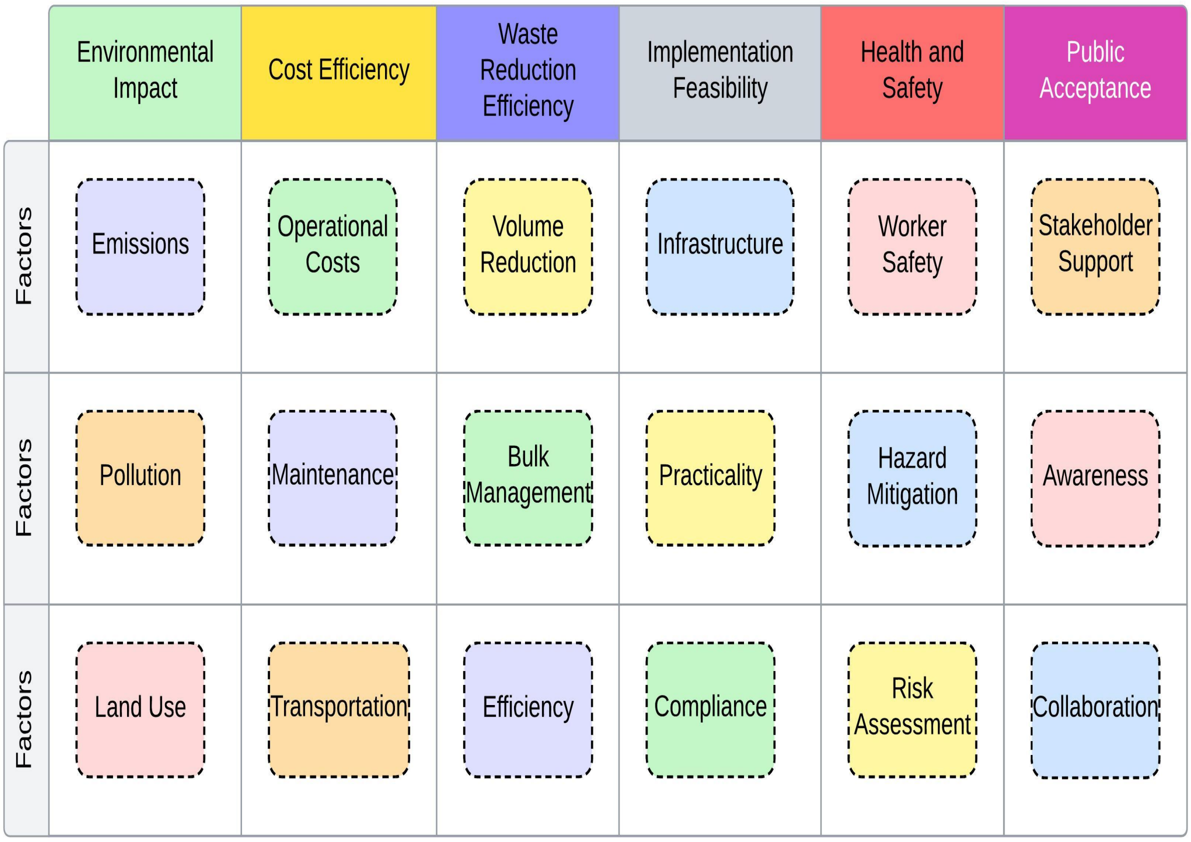

Figure 1. Overview of the six key criteria for evaluating SWM method.

(1) **Environmental impact** $\mathfrak{Cr}_1$**:** This criterion evaluates many environmental aspects, including emissions (such as greenhouse gases and pollutants), land usage, and the potential for soil and water pollution. We favor techniques that have a minimal impact on the environment.
(2) **Cost efficiency** $\mathfrak{Cr}_2$**:** This criterion evaluates the overall expenses that are connected with putting each waste management method into operation and keeping it running, including the costs of transportation, maintenance, and operations. In general, there is a preference for lower expenses.
(3) **Waste reduction efficiency** $\mathfrak{Cr}_3$**:** This criterion assesses the efficacy of each method in terms of its ability to reduce the volume and bulk of waste generated. It is recommended that waste elimination be conducted with greater efficiency.
(4) **Implementation feasibility** $\mathfrak{Cr}_4$**:** This criterion checks if each method is possible by looking at things like how hard it is to use, how easy it is to get technology, what kind of infrastructure is needed, and how well it follows the rules.
(5) **Health and safety** $\mathfrak{Cr}_5$**:** Using this standard, an evaluation is made of the possible health and safety risks that come with each method for both the public and the workers. It is better to use ways that are less likely to put health and safety at risk.
(6) **Public acceptance** $\mathfrak{Cr}_6$**:** Each waste management technique is evaluated according to this criterion, which measures the degree to which the community and other stakeholders accept and support the method. It would be best if there was more public acceptance.

5.1. Decision-making process

Step 1. Three individuals with decision-making authority participated in the present case study. The Cq-ROFN linguistic scale was utilized to evaluate different DM, as presented in Table 6, and corresponding values in Table 7.

Table 6. Information about the DMs.

DMs	Qualification	Background (Expertise)	Experience (Years)
D_1	PhD	Environmental scientist	2.5
D_2	MS	Public health expert	4
D_3	MSc	Economist	7

Table 7. Information about the DMs in term of Cq-ROFN.

DMs	Qualification	Background (Expertise)	Experience (Years)
D_1	$([0.95,0.10],\mathscr{T}_{11}^{q})$	$([0.75,0.20],\mathscr{T}_{12}^{q})$	$([0.55,0.30],\mathscr{T}_{13}^{q})$
D_2	$([0.75,0.20],\mathscr{T}_{21}^{q})$	$([0.95,0.10],\mathscr{T}_{22}^{q})$	$([0.75,0.20],\mathscr{T}_{23}^{q})$
D_3	$([0.55,0.30],\mathscr{T}_{31}^{q})$	$([0.55,0.30],\mathscr{T}_{32}^{q})$	$([0.95,0.10],\mathscr{T}_{33}^{q})$

Step 2. The radius $\mathscr{T}_{ij}^{q}$ is found by using Eqs (1) and (2) to find the score value of each Cq-ROFN. The normalized weight of each decision-maker by using Eq (23) are $\xi_1 = 0.3333, \xi_2 = 0.3943,$ and $\xi_3 = 0.2723$.

Step 3. The DMs utilized the linguistic words specified in Table 8 and allocated them based on their proficiency for each alternative about each criterion, as seen in Table 9 and the corresponding values in Table 10.

Table 8. Linguistic term and corresponding Cq-ROF numbers.

Linguistic term	Abbreviation	Cq-ROF numbers
Excellent	EX	$\langle(0.95, 0.05), \mathscr{T}\rangle$
Very satisfactory	VS	$\langle(0.85, 0.10), \mathscr{T}\rangle$
Satisfactory	S	$\langle(0.75, 0.20), \mathscr{T}\rangle$
Slightly satisfactory	SS	$\langle(0.65, 0.30), \mathscr{T}\rangle$
Neutral	N	$\langle(0.50, 0.45), \mathscr{T}\rangle$
Slightly unsatisfactory	SU	$\langle(0.40, 0.55), \mathscr{T}\rangle$
Unsatisfactory	U	$\langle(0.30, 0.65), \mathscr{T}\rangle$
Very unsatisfactory	VU	$\langle(0.20, 0.75), \mathscr{T}\rangle$
Terrible	T	$\langle(0.10, 0.85), \mathscr{T}\rangle$

Table 9. Linguistic decision matrix for assessments of criteria and alternatives by DMs.

DMs	Alternatives	$\mathfrak{Cr}_1$	$\mathfrak{Cr}_2$	$\mathfrak{Cr}_3$	$\mathfrak{Cr}_4$	$\mathfrak{Cr}_5$	$\mathfrak{Cr}_6$
DM_1	$\mathscr{L}_1$	EX	VS	S	N	S	SS
	$\mathscr{L}_2$	EX	S	VS	SS	EX	SS
	$\mathscr{L}_3$	SU	S	SU	N	SS	VU
	$\mathscr{L}_4$	T	SS	VS	S	SU	N
	$\mathscr{L}_5$	N	SU	VU	SS	S	EX
	$\mathscr{L}_6$	VS	S	SS	SS	EX	S
	$\mathscr{L}_7$	EX	N	SS	S	VS	SS
	$\mathscr{L}_8$	SU	EX	T	N	SS	VU
DM_2	$\mathscr{L}_1$	S	SS	U	N	VS	SU
	$\mathscr{L}_2$	EX	SU	N	S	VU	SS
	$\mathscr{L}_3$	VU	N	S	T	SS	SU
	$\mathscr{L}_4$	SS	U	VU	EX	N	S
	$\mathscr{L}_5$	S	SS	T	N	SU	VU
	$\mathscr{L}_6$	EX	S	SS	VS	S	VS
	$\mathscr{L}_7$	SU	VU	S	N	EX	T
	$\mathscr{L}_8$	VS	S	T	SU	S	N
DM_3	$\mathscr{L}_1$	SS	S	SU	T	N	VU
	$\mathscr{L}_2$	N	VU	SS	S	SU	SU
	$\mathscr{L}_3$	S	T	VU	N	SU	SS
	$\mathscr{L}_4$	EX	SU	S	SS	VU	T
	$\mathscr{L}_5$	SU	N	T	VU	EX	SS
	$\mathscr{L}_6$	VS	N	S	SS	EX	S
	$\mathscr{L}_7$	EX	SS	N	SU	S	VS
	$\mathscr{L}_8$	VU	EX	N	S	T	SS

Table 10. Linguistic decision matrix in terms of Cq-ROFN without the radius.

DMs	Alternatives	$\mathfrak{Cr}_1$	$\mathfrak{Cr}_2$	$\mathfrak{Cr}_3$	$\mathfrak{Cr}_4$	$\mathfrak{Cr}_5$	$\mathfrak{Cr}_6$
DM_1	$\mathscr{L}_1$	$\langle(0.95, 0.05), \mathscr{T}^q_{11}\rangle$	$\langle(0.85, 0.10), \mathscr{T}^q_{12}\rangle$	$\langle(0.75, 0.20), \mathscr{T}^q_{13}\rangle$	$\langle(0.50, 0.45), \mathscr{T}^q_{14}\rangle$	$\langle(0.75, 0.20), \mathscr{T}^q_{15}\rangle$	$\langle(0.65, 0.30), \mathscr{T}^q_{16}\rangle$
	$\mathscr{L}_2$	$\langle(0.95, 0.05), \mathscr{T}^q_{21}\rangle$	$\langle(0.75, 0.20), \mathscr{T}^q_{22}\rangle$	$\langle(0.85, 0.10), \mathscr{T}^q_{23}\rangle$	$\langle(0.65, 0.30), \mathscr{T}^q_{24}\rangle$	$\langle(0.95, 0.05), \mathscr{T}^q_{25}\rangle$	$\langle(0.65, 0.30), \mathscr{T}^q_{26}\rangle$
	$\mathscr{L}_3$	$\langle(0.40, 0.55), \mathscr{T}^q_{31}\rangle$	$\langle(0.75, 0.20), \mathscr{T}^q_{32}\rangle$	$\langle(0.40, 0.55), \mathscr{T}^q_{33}\rangle$	$\langle(0.50, 0.45), \mathscr{T}^q_{34}\rangle$	$\langle(0.65, 0.30), \mathscr{T}^q_{35}\rangle$	$\langle(0.20, 0.75), \mathscr{T}^q_{36}\rangle$
	$\mathscr{L}_4$	$\langle(0.10, 0.85), \mathscr{T}^q_{41}\rangle$	$\langle(0.65, 0.30), \mathscr{T}^q_{42}\rangle$	$\langle(0.85, 0.10), \mathscr{T}^q_{43}\rangle$	$\langle(0.75, 0.20), \mathscr{T}^q_{44}\rangle$	$\langle(0.40, 0.55), \mathscr{T}^q_{45}\rangle$	$\langle(0.50, 0.45), \mathscr{T}^q_{46}\rangle$
	$\mathscr{L}_5$	$\langle(0.50, 0.45), \mathscr{T}^q_{51}\rangle$	$\langle(0.40, 0.55), \mathscr{T}^q_{52}\rangle$	$\langle(0.20, 0.75), \mathscr{T}^q_{53}\rangle$	$\langle(0.65, 0.30), \mathscr{T}^q_{54}\rangle$	$\langle(0.75, 0.20), \mathscr{T}^q_{55}\rangle$	$\langle(0.95, 0.05), \mathscr{T}^q_{56}\rangle$
	$\mathscr{L}_6$	$\langle(0.85, 0.10), \mathscr{T}^q_{61}\rangle$	$\langle(0.75, 0.20), \mathscr{T}^q_{62}\rangle$	$\langle(0.65, 0.30), \mathscr{T}^q_{63}\rangle$	$\langle(0.65, 0.30), \mathscr{T}^q_{64}\rangle$	$\langle(0.95, 0.05), \mathscr{T}^q_{65}\rangle$	$\langle(0.75, 0.20), \mathscr{T}^q_{66}\rangle$
	$\mathscr{L}_7$	$\langle(0.95, 0.05), \mathscr{T}^q_{71}\rangle$	$\langle(0.50, 0.45), \mathscr{T}^q_{72}\rangle$	$\langle(0.65, 0.30), \mathscr{T}^q_{73}\rangle$	$\langle(0.75, 0.20), \mathscr{T}^q_{74}\rangle$	$\langle(0.85, 0.10), \mathscr{T}^q_{75}\rangle$	$\langle(0.65, 0.30), \mathscr{T}^q_{76}\rangle$
	$\mathscr{L}_8$	$\langle(0.40, 0.55), \mathscr{T}^q_{81}\rangle$	$\langle(0.95, 0.05), \mathscr{T}^q_{82}\rangle$	$\langle(0.10, 0.85), \mathscr{T}^q_{83}\rangle$	$\langle(0.50, 0.45), \mathscr{T}^q_{84}\rangle$	$\langle(0.65, 0.30), \mathscr{T}^q_{85}\rangle$	$\langle(0.20, 0.75), \mathscr{T}^q_{86}\rangle$
DM_2	$\mathscr{L}_1$	$\langle(0.75, 0.20), \mathscr{T}^q_{11}\rangle$	$\langle(0.65, 0.30), \mathscr{T}^q_{12}\rangle$	$\langle(0.30, 0.65), \mathscr{T}^q_{13}\rangle$	$\langle(0.50, 0.45), \mathscr{T}^q_{14}\rangle$	$\langle(0.85, 0.10), \mathscr{T}^q_{15}\rangle$	$\langle(0.40, 0.55), \mathscr{T}^q_{16}\rangle$
	$\mathscr{L}_2$	$\langle(0.95, 0.05), \mathscr{T}^q_{21}\rangle$	$\langle(0.40, 0.55), \mathscr{T}^q_{22}\rangle$	$\langle(0.50, 0.45), \mathscr{T}^q_{23}\rangle$	$\langle(0.75, 0.20), \mathscr{T}^q_{24}\rangle$	$\langle(0.20, 0.75), \mathscr{T}^q_{25}\rangle$	$\langle(0.65, 0.30), \mathscr{T}^q_{26}\rangle$
	$\mathscr{L}_3$	$\langle(0.20, 0.75), \mathscr{T}^q_{31}\rangle$	$\langle(0.50, 0.45), \mathscr{T}^q_{32}\rangle$	$\langle(0.75, 0.20), \mathscr{T}^q_{33}\rangle$	$\langle(0.10, 0.85), \mathscr{T}^q_{34}\rangle$	$\langle(0.65, 0.30), \mathscr{T}^q_{35}\rangle$	$\langle(0.40, 0.55), \mathscr{T}^q_{36}\rangle$
	$\mathscr{L}_4$	$\langle(0.65, 0.30), \mathscr{T}^q_{41}\rangle$	$\langle(0.30, 0.65), \mathscr{T}^q_{42}\rangle$	$\langle(0.20, 0.75), \mathscr{T}^q_{43}\rangle$	$\langle(0.95, 0.05), \mathscr{T}^q_{44}\rangle$	$\langle(0.50, 0.45), \mathscr{T}^q_{45}\rangle$	$\langle(0.75, 0.20), \mathscr{T}^q_{46}\rangle$
	$\mathscr{L}_5$	$\langle(0.75, 0.20), \mathscr{T}^q_{51}\rangle$	$\langle(0.65, 0.30), \mathscr{T}^q_{52}\rangle$	$\langle(0.10, 0.85), \mathscr{T}^q_{53}\rangle$	$\langle(0.50, 0.45), \mathscr{T}^q_{54}\rangle$	$\langle(0.40, 0.55), \mathscr{T}^q_{55}\rangle$	$\langle(0.20, 0.75), \mathscr{T}^q_{56}\rangle$
	$\mathscr{L}_6$	$\langle(0.95, 0.05), \mathscr{T}^q_{61}\rangle$	$\langle(0.75, 0.20), \mathscr{T}^q_{62}\rangle$	$\langle(0.65, 0.30), \mathscr{T}^q_{63}\rangle$	$\langle(0.85, 0.10), \mathscr{T}^q_{64}\rangle$	$\langle(0.75, 0.20), \mathscr{T}^q_{65}\rangle$	$\langle(0.85, 0.10), \mathscr{T}^q_{66}\rangle$
	$\mathscr{L}_7$	$\langle(0.40, 0.55), \mathscr{T}^q_{71}\rangle$	$\langle(0.20, 0.75), \mathscr{T}^q_{72}\rangle$	$\langle(0.75, 0.20), \mathscr{T}^q_{73}\rangle$	$\langle(0.50, 0.45), \mathscr{T}^q_{74}\rangle$	$\langle(0.95, 0.05), \mathscr{T}^q_{75}\rangle$	$\langle(0.10, 0.85), \mathscr{T}^q_{76}\rangle$
	$\mathscr{L}_8$	$\langle(0.85, 0.10), \mathscr{T}^q_{81}\rangle$	$\langle(0.75, 0.20), \mathscr{T}^q_{82}\rangle$	$\langle(0.10, 0.85), \mathscr{T}^q_{83}\rangle$	$\langle(0.40, 0.55), \mathscr{T}^q_{84}\rangle$	$\langle(0.75, 0.20), \mathscr{T}^q_{85}\rangle$	$\langle(0.50, 0.45), \mathscr{T}^q_{86}\rangle$
DM_3	$\mathscr{L}_1$	$\langle(0.65, 0.30), \mathscr{T}^q_{11}\rangle$	$\langle(0.75, 0.20), \mathscr{T}^q_{12}\rangle$	$\langle(0.40, 0.55), \mathscr{T}^q_{13}\rangle$	$\langle(0.10, 0.85), \mathscr{T}^q_{14}\rangle$	$\langle(0.50, 0.45), \mathscr{T}^q_{15}\rangle$	$\langle(0.20, 0.75), \mathscr{T}^q_{16}\rangle$
	$\mathscr{L}_2$	$\langle(0.50, 0.45), \mathscr{T}^q_{21}\rangle$	$\langle(0.20, 0.75), \mathscr{T}^q_{22}\rangle$	$\langle(0.65, 0.30), \mathscr{T}^q_{23}\rangle$	$\langle(0.75, 0.20), \mathscr{T}^q_{24}\rangle$	$\langle(0.40, 0.55), \mathscr{T}^q_{25}\rangle$	$\langle(0.40, 0.55), \mathscr{T}^q_{26}\rangle$
	$\mathscr{L}_3$	$\langle(0.75, 0.20), \mathscr{T}^q_{31}\rangle$	$\langle(0.10, 0.85), \mathscr{T}^q_{32}\rangle$	$\langle(0.20, 0.75), \mathscr{T}^q_{33}\rangle$	$\langle(0.50, 0.45), \mathscr{T}^q_{34}\rangle$	$\langle(0.40, 0.55), \mathscr{T}^q_{35}\rangle$	$\langle(0.65, 0.30), \mathscr{T}^q_{36}\rangle$
	$\mathscr{L}_4$	$\langle(0.95, 0.05), \mathscr{T}^q_{41}\rangle$	$\langle(0.40, 0.55), \mathscr{T}^q_{42}\rangle$	$\langle(0.75, 0.20), \mathscr{T}^q_{43}\rangle$	$\langle(0.65, 0.30), \mathscr{T}^q_{44}\rangle$	$\langle(0.20, 0.75), \mathscr{T}^q_{45}\rangle$	$\langle(0.10, 0.85), \mathscr{T}^q_{46}\rangle$
	$\mathscr{L}_5$	$\langle(0.40, 0.55), \mathscr{T}^q_{51}\rangle$	$\langle(0.50, 0.45), \mathscr{T}^q_{52}\rangle$	$\langle(0.10, 0.85), \mathscr{T}^q_{53}\rangle$	$\langle(0.20, 0.75), \mathscr{T}^q_{54}\rangle$	$\langle(0.95, 0.05), \mathscr{T}^q_{55}\rangle$	$\langle(0.65, 0.30), \mathscr{T}^q_{56}\rangle$
	$\mathscr{L}_6$	$\langle(0.85, 0.10), \mathscr{T}^q_{61}\rangle$	$\langle(0.50, 0.45), \mathscr{T}^q_{62}\rangle$	$\langle(0.75, 0.20), \mathscr{T}^q_{63}\rangle$	$\langle(0.65, 0.30), \mathscr{T}^q_{64}\rangle$	$\langle(0.95, 0.05), \mathscr{T}^q_{65}\rangle$	$\langle(0.75, 0.20), \mathscr{T}^q_{66}\rangle$
	$\mathscr{L}_7$	$\langle(0.95, 0.05), \mathscr{T}^q_{71}\rangle$	$\langle(0.65, 0.30), \mathscr{T}^q_{72}\rangle$	$\langle(0.50, 0.45), \mathscr{T}^q_{73}\rangle$	$\langle(0.40, 0.55), \mathscr{T}^q_{74}\rangle$	$\langle(0.75, 0.20), \mathscr{T}^q_{75}\rangle$	$\langle(0.85, 0.10), \mathscr{T}^q_{76}\rangle$
	$\mathscr{L}_8$	$\langle(0.20, 0.75), \mathscr{T}^q_{81}\rangle$	$\langle(0.95, 0.05), \mathscr{T}^q_{82}\rangle$	$\langle(0.50, 0.45), \mathscr{T}^q_{83}\rangle$	$\langle(0.75, 0.20), \mathscr{T}^q_{84}\rangle$	$\langle(0.10, 0.85), \mathscr{T}^q_{85}\rangle$	$\langle(0.65, 0.30), \mathscr{T}^q_{86}\rangle$

Step 4. Find the radius $\mathscr{T}$ by using Eq (1), and the corresponding values are represented in Table 11.

Table 11. Linguistic decision matrix in term of Cq-ROFN.

DMs	Alternatives	$\mathfrak{Cr}_1$	$\mathfrak{Cr}_2$	$\mathfrak{Cr}_3$	$\mathfrak{Cr}_4$	$\mathfrak{Cr}_5$	$\mathfrak{Cr}_6$
DM_1	$\mathscr{L}_1$	⟨(0.95, 0.05), 0.7782⟩	⟨(0.85, 0.10), 0.5088⟩	⟨(0.75, 0.20), 0.3285⟩	⟨(0.50, 0.45), 0.1721⟩	⟨(0.75, 0.20), 0.3085⟩	⟨(0.65, 0.30), 0.2211⟩
	$\mathscr{L}_2$	⟨(0.95, 0.05), 0.7136⟩	⟨(0.75, 0.20), 0.3438⟩	⟨(0.85, 0.10), 0.5181⟩	⟨(0.65, 0.30), 0.1732⟩	⟨(0.95, 0.05), 0.8197⟩	⟨(0.65, 0.30), 0.1804⟩
	$\mathscr{L}_3$	⟨(0.40, 0.55), 0.0517⟩	⟨(0.75, 0.20), 0.3671⟩	⟨(0.40, 0.55), 0.0517⟩	⟨(0.50, 0.45), 0.1721⟩	⟨(0.65, 0.30), 0.1804⟩	⟨(0.20, 0.75), 0.1777⟩
	$\mathscr{L}_4$	⟨(0.10, 0.85), 0.3455⟩	⟨(0.65, 0.30), 0.1975⟩	⟨(0.85, 0.10), 0.5266⟩	⟨(0.75, 0.20), 0.2801⟩	⟨(0.40, 0.55), 0.0635⟩	⟨(0.50, 0.45), 0.1598⟩
	$\mathscr{L}_5$	⟨(0.50, 0.45), 0.0623⟩	⟨(0.40, 0.55), 0.0514⟩	⟨(0.20, 0.75), 0.1371⟩	⟨(0.65, 0.30), 0.2116⟩	⟨(0.75, 0.20), 0.2958⟩	⟨(0.95, 0.05), 0.8097⟩
	$\mathscr{L}_6$	⟨(0.85, 0.10), 0.3747⟩	⟨(0.75, 0.20), 0.3138⟩	⟨(0.65, 0.30), 0.2038⟩	⟨(0.65, 0.30), 0.2403⟩	⟨(0.95, 0.05), 0.6377⟩	⟨(0.75, 0.20), 0.2945⟩
	$\mathscr{L}_7$	⟨(0.95, 0.05), 0.7226⟩	⟨(0.50, 0.45), 0.1939⟩	⟨(0.65, 0.30), 0.2450⟩	⟨(0.75, 0.20), 0.3172⟩	⟨(0.85, 0.10), 0.4299⟩	⟨(0.65, 0.30), 0.3850⟩
	$\mathscr{L}_8$	⟨(0.40, 0.55), 0.0391⟩	⟨(0.95, 0.05), 0.6378⟩	⟨(0.10, 0.85), 0.0001⟩	⟨(0.50, 0.45), 0.0623⟩	⟨(0.65, 0.30), 0.1778⟩	⟨(0.20, 0.75), 0.1946⟩
DM_2	$\mathscr{L}_1$	⟨(0.75, 0.20), 0.2801⟩	⟨(0.65, 0.30), 0.1654⟩	⟨(0.30, 0.65), 0.0883⟩	⟨(0.50, 0.45), 0.1721⟩	⟨(0.85, 0.10), 0.5141⟩	⟨(0.40, 0.55), 0.0537⟩
	$\mathscr{L}_2$	⟨(0.95, 0.05), 0.7136⟩	⟨(0.40, 0.55), 0.0517⟩	⟨(0.50, 0.45), 0.0633⟩	⟨(0.75, 0.20), 0.3111⟩	⟨(0.20, 0.75), 0.1805⟩	⟨(0.65, 0.30), 0.1804⟩
	$\mathscr{L}_3$	⟨(0.20, 0.75), 0.1799⟩	⟨(0.50, 0.45), 0.1598⟩	⟨(0.75, 0.20), 0.3438⟩	⟨(0.10, 0.85), 0.3207⟩	⟨(0.65, 0.30), 0.1804⟩	⟨(0.40, 0.55), 0.0537⟩
	$\mathscr{L}_4$	⟨(0.65, 0.30), 0.2370⟩	⟨(0.30, 0.65), 0.0862⟩	⟨(0.20, 0.75), 0.2104⟩	⟨(0.95, 0.05), 0.7782⟩	⟨(0.50, 0.45), 0.1253⟩	⟨(0.75, 0.20), 0.3671⟩
	$\mathscr{L}_5$	⟨(0.75, 0.20), 0.3190⟩	⟨(0.65, 0.30), 0.1826⟩	⟨(0.10, 0.85), 0.0685⟩	⟨(0.50, 0.45), 0.1021⟩	⟨(0.40, 0.55), 0.0606⟩	⟨(0.20, 0.75), 0.2085⟩
	$\mathscr{L}_6$	⟨(0.95, 0.05), 0.6672⟩	⟨(0.75, 0.20), 0.3138⟩	⟨(0.65, 0.30), 0.2038⟩	⟨(0.85, 0.10), 0.5441⟩	⟨(0.75, 0.20), 0.1396⟩	⟨(0.85, 0.10), 0.5001⟩
	$\mathscr{L}_7$	⟨(0.40, 0.55), 0.0902⟩	⟨(0.20, 0.75), 0.0918⟩	⟨(0.75, 0.20), 0.3608⟩	⟨(0.50, 0.45), 0.0635⟩	⟨(0.95, 0.05), 0.7224⟩	⟨(0.10, 0.85), 0.1713⟩
	$\mathscr{L}_8$	⟨(0.85, 0.10), 0.5244⟩	⟨(0.75, 0.20), 0.1401⟩	⟨(0.10, 0.85), 0.0001⟩	⟨(0.40, 0.55), 0.0532⟩	⟨(0.75, 0.20), 0.3157⟩	⟨(0.50, 0.45), 0.1021⟩
DM_3	$\mathscr{L}_1$	⟨(0.65, 0.30), 0.1423⟩	⟨(0.75, 0.20), 0.3032⟩	⟨(0.40, 0.55), 0.0254⟩	⟨(0.10, 0.85), 0.3207⟩	⟨(0.50, 0.45), 0.0606⟩	⟨(0.20, 0.75), 0.1777⟩
	$\mathscr{L}_2$	⟨(0.50, 0.45), 0.0473⟩	⟨(0.20, 0.75), 0.1799⟩	⟨(0.65, 0.30), 0.1745⟩	⟨(0.75, 0.20), 0.3111⟩	⟨(0.40, 0.55), 0.0485⟩	⟨(0.40, 0.55), 0.0611⟩
	$\mathscr{L}_3$	⟨(0.75, 0.20), 0.3438⟩	⟨(0.10, 0.85), 0.3338⟩	⟨(0.20, 0.75), 0.1799⟩	⟨(0.50, 0.45), 0.1721⟩	⟨(0.40, 0.55), 0.0611⟩	⟨(0.65, 0.30), 0.2211⟩
	$\mathscr{L}_4$	⟨(0.95, 0.05), 0.8217⟩	⟨(0.40, 0.55), 0.0256⟩	⟨(0.75, 0.20), 0.3274⟩	⟨(0.65, 0.30), 0.1423⟩	⟨(0.20, 0.75), 0.1668⟩	⟨(0.10, 0.85), 0.3338⟩
	$\mathscr{L}_5$	⟨(0.40, 0.55), 0.0532⟩	⟨(0.50, 0.45), 0.0627⟩	⟨(0.10, 0.85), 0.0685⟩	⟨(0.20, 0.75), 0.1946⟩	⟨(0.95, 0.05), 0.7930⟩	⟨(0.65, 0.30), 0.1942⟩
	$\mathscr{L}_6$	⟨(0.85, 0.10), 0.3748⟩	⟨(0.50, 0.45), 0.0651⟩	⟨(0.75, 0.20), 0.3752⟩	⟨(0.65, 0.30), 0.3854⟩	⟨(0.95, 0.05), 0.6377⟩	⟨(0.75, 0.20), 0.2945⟩
	$\mathscr{L}_7$	⟨(0.95, 0.05), 0.7226⟩	⟨(0.65, 0.30), 0.2007⟩	⟨(0.50, 0.45), 0.3502⟩	⟨(0.40, 0.55), 0.0285⟩	⟨(0.75, 0.20), 0.2243⟩	⟨(0.85, 0.10), 0.5466⟩
	$\mathscr{L}_8$	⟨(0.20, 0.75), 0.0305⟩	⟨(0.95, 0.05), 0.6382⟩	⟨(0.50, 0.45), 0.0625⟩	⟨(0.75, 0.20), 0.3190⟩	⟨(0.10, 0.85), 0.0033⟩	⟨(0.65, 0.30), 0.2116⟩

Step 5. Compute the aggregated values using the Cq-ROFECIWA operator described in Eq (24), is represented in Table 12.

Table 12. Aggregated decision matrix.

$\mathfrak{Cr}_1$	$\mathfrak{Cr}_2$	$\mathfrak{Cr}_3$	$\mathfrak{Cr}_4$	$\mathfrak{Cr}_5$	$\mathfrak{Cr}_6$
⟨(0.8437, 0.1407), 0.6005⟩	⟨(0.7633, 0.1863), 0.3975⟩	⟨(0.5871, 0.4229), 0.2476⟩	⟨(0.4619, 0.5423), 0.3762⟩	⟨(0.7639, 0.1900), 0.4181⟩	⟨(0.5154, 0.4935), 0.2357⟩
⟨(0.9112, 0.0911), 0.6607⟩	⟨(0.5876, 0.4323), 0.2793⟩	⟨(0.7145, 0.2445), 0.3947⟩	⟨(0.7219, 0.2290), 0.2844⟩	⟨(0.7796, 0.2849), 0.6272⟩	⟨(0.6088, 0.3545), 0.1660⟩
⟨(0.5596, 0.4787), 0.2859⟩	⟨(0.6040, 0.4162), 0.3911⟩	⟨(0.6087, 0.4066), 0.2860⟩	⟨(0.4413, 0.5885), 0.4126⟩	⟨(0.6088, 0.3545), 0.1660⟩	⟨(0.4943, 0.5222), 0.2455⟩
⟨(0.7768, 0.2672), 0.6161⟩	⟨(0.5147, 0.4828), 0.1662⟩	⟨(0.7245, 0.2717), 0.4165⟩	⟨(0.8583, 0.1293), 0.6251⟩	⟨(0.4283, 0.5560), 0.2290⟩	⟨(0.6213, 0.3962), 0.3936⟩
⟨(0.6282, 0.3463), 0.2510⟩	⟨(0.5573, 0.4109), 0.1446⟩	⟨(0.1565, 0.8168), 0.4392⟩	⟨(0.5400, 0.4557), 0.2355⟩	⟨(0.7978, 0.2052), 0.5830⟩	⟨(0.7978, 0.2410), 0.6224⟩
⟨(0.9028, 0.1026), 0.5491⟩	⟨(0.7071, 0.2497), 0.2896⟩	⟨(0.6825, 0.3887), 0.3270⟩	⟨(0.7553, 0.2636), 0.4790⟩	⟨(0.9064, 0.0864), 0.5636⟩	⟨(0.7966, 0.1700), 0.4135⟩
⟨(0.8819, 0.1886), 0.6395⟩	⟨(0.5153, 0.6382), 0.3033⟩	⟨(0.6717, 0.3461), 0.3972⟩	⟨(0.6116, 0.3440), 0.2404⟩	⟨(0.8889, 0.0630), 0.5905⟩	⟨(0.6806, 0.6200), 0.5672⟩
⟨(0.6947, 0.2826), 0.4131⟩	⟨(0.9064, 0.1573), 0.5641⟩	⟨(0.3617, 0.8500), 0.4489⟩	⟨(0.5881, 0.3919), 0.2292⟩	⟨(0.6577, 0.1570), 0.2553⟩	⟨(0.5219, 0.4825), 0.2453⟩

Steps 6 and 6.1. Find the importance of each criterion, which is fuzzy density $\mathscr{M}(Cr_j)$, by applying the LOPCOW method. The score matrix Sc_{ij} of the aggregated decision matrix by applying Eq (2) is shown in the matrix below. Then normalizing this matrix using (25) is represented in the matrix below.

$$Sc_{ij} = \begin{bmatrix} 0.1933 & 0.1174 & 0.0297 & -0.0099 & 0.1188 & 0.0043 \\ 0.2657 & 0.0292 & 0.0903 & 0.0908 & 0.1501 & 0.0407 \\ 0.0165 & 0.0388 & 0.0379 & -0.0219 & 0.0407 & -0.0042 \\ 0.1469 & 0.0054 & 0.0957 & 0.2096 & -0.0201 & 0.0460 \\ 0.0479 & 0.0227 & -0.1411 & 0.0146 & 0.1563 & 0.1622 \\ 0.2385 & 0.0834 & 0.0669 & 0.1168 & 0.2440 & 0.1394 \\ 0.2327 & -0.0302 & 0.0677 & 0.0426 & 0.2310 & 0.0418 \\ 0.0810 & 0.2439 & -0.1606 & 0.0325 & 0.0630 & 0.0074 \end{bmatrix} \quad \widetilde{Sc_{ij}} = \begin{bmatrix} 0.7096 & 0.4614 & 0.2577 & 0.0518 & 0.5260 & 0.0513 \\ 1.0000 & 0.7832 & 0.0212 & 0.4870 & 0.6445 & 0.2696 \\ 0.0000 & 0.7483 & 0.2255 & 0.0000 & 0.2302 & 0.0000 \\ 0.5233 & 0.8701 & 0.0000 & 1.0000 & 0.0000 & 0.3014 \\ 0.1261 & 0.8070 & 0.9239 & 0.1574 & 0.6678 & 1.0000 \\ 0.8909 & 0.5856 & 0.1125 & 0.5990 & 1.0000 & 0.8631 \\ 0.8677 & 1.0000 & 0.1091 & 0.2786 & 0.9508 & 0.2762 \\ 0.2590 & 0.0000 & 1.0000 & 0.2350 & 0.3146 & 0.0694 \end{bmatrix}.$$

Step 6.2. Obtain the PV for the criteria by using Eq (26). The values are represented in Table 13.

Table 13. Percentage value for each criterion.

$\mathfrak{Cr}_1$	$\mathfrak{Cr}_2$	$\mathfrak{Cr}_3$	$\mathfrak{Cr}_4$	$\mathfrak{Cr}_5$	$\mathfrak{Cr}_6$
53.9857	83.2572	22.2943	34.5448	59.7390	28.2700

Step 6.3. The fuzzy density $\mathscr{M}(Cr_j)$ weight for the j^{th} criterion is calculated by using Eq (27) and is represented in Table 14.

Table 14. Fuzzy density weight for each criterion.

$\mathfrak{Cr}_1$	$\mathfrak{Cr}_2$	$\mathfrak{Cr}_3$	$\mathfrak{Cr}_4$	$\mathfrak{Cr}_5$	$\mathfrak{Cr}_6$
0.1914	0.2951	0.0790	0.1225	0.2118	0.1002

Step 7. The equation for value ρ is represented in (29) and is formed by using Eq (18). The corresponding real values are −26.3241, −14.4080, and 0.

$$0.000012 \cdot p^6 + 0.00051 \cdot p^5 + 0.00903 \cdot p^4 + 0.08191 \cdot p^3 + 0.40007 \cdot p^2 = 0. \tag{29}$$

The normalized measure on X by using Eq (16) is mentioned here. Since $\rho > -1$, let us take the value of $\rho = 0$, so we have:

$\mathscr{M}(\{\}) = 0, \mathscr{M}(\{Cr_1\}) = 0.19138,\ \mathscr{M}(\{Cr_2\}) = 0.29514,\ \mathscr{M}(\{Cr_3\}) = 0.079032,\ \mathscr{M}(\{Cr_4\}) = 0.12246,\ \mathscr{M}(\{Cr_5\}) = 0.21177,\ \mathscr{M}(\{Cr_6\}) = 0.10022,$
$\mathscr{M}(\{Cr_1, Cr_2\}) = 0.48652,\ \mathscr{M}(\{Cr_1, Cr_3\}) = 0.27041,\ \mathscr{M}(\{Cr_1, Cr_4\}) = 0.31384,\ \mathscr{M}(\{Cr_1, Cr_5\}) = 0.40315,\ \mathscr{M}(\{Cr_1, Cr_6\}) = 0.29159,$
$\mathscr{M}(\{Cr_2, Cr_3\}) = 0.37418,\ \mathscr{M}(\{Cr_2, Cr_4\}) = 0.4176,\ \mathscr{M}(\{Cr_2, Cr_5\}) = 0.50692,\ \mathscr{M}(\{Cr_2, Cr_6\}) = 0.39536,\ \mathscr{M}(\{Cr_3, Cr_4\}) = 0.20149,$
$\mathscr{M}(\{Cr_3, Cr_5\}) = 0.2908,\ \mathscr{M}(\{Cr_3, Cr_6\}) = 0.17925,\ \mathscr{M}(\{Cr_4, Cr_5\}) = 0.33423,\ \mathscr{M}(\{Cr_4, Cr_6\}) = 0.22268,\ \mathscr{M}(\{Cr_5, Cr_6\}) = 0.31199,$
$\mathscr{M}(\{Cr_1, Cr_2, Cr_3\}) = 0.56555,\ \mathscr{M}(\{Cr_1, Cr_2, Cr_4\}) = 0.60898,\ \mathscr{M}(\{Cr_1, Cr_2, Cr_5\}) = 0.69829,\ \mathscr{M}(\{Cr_1, Cr_2, Cr_6\}) = 0.58674,$
$\mathscr{M}(\{Cr_1, Cr_3, Cr_4\}) = 0.39287,\ \mathscr{M}(\{Cr_1, Cr_3, Cr_5\}) = 0.48218,\ \mathscr{M}(\{Cr_1, Cr_3, Cr_6\}) = 0.37062,\ \mathscr{M}(\{Cr_1, Cr_4, Cr_5\}) = 0.52561,$
$\mathscr{M}(\{Cr_1, Cr_4, Cr_6\}) = 0.41405,\ \mathscr{M}(\{Cr_1, Cr_5, Cr_6\}) = 0.50336,\ \mathscr{M}(\{Cr_2, Cr_3, Cr_4\}) = 0.49664,\ \mathscr{M}(\{Cr_2, Cr_3, Cr_5\}) = 0.58595,$
$\mathscr{M}(\{Cr_2, Cr_3, Cr_6\}) = 0.47439,\ \mathscr{M}(\{Cr_2, Cr_4, Cr_5\}) = 0.62938,\ \mathscr{M}(\{Cr_2, Cr_4, Cr_6\}) = 0.51782,\ \mathscr{M}(\{Cr_2, Cr_5, Cr_6\}) = 0.60713,$
$\mathscr{M}(\{Cr_3, Cr_4, Cr_5\}) = 0.41326,\ \mathscr{M}(\{Cr_3, Cr_4, Cr_6\}) = 0.30171,\ \mathscr{M}(\{Cr_3, Cr_5, Cr_6\}) = 0.39102,\ \mathscr{M}(\{Cr_4, Cr_5, Cr_6\}) = 0.43445,$
$\mathscr{M}(\{Cr_1, Cr_2, Cr_3, Cr_4\}) = 0.68801,\ \mathscr{M}(\{Cr_1, Cr_2, Cr_3, Cr_5\}) = 0.77732,\ \mathscr{M}(\{Cr_1, Cr_2, Cr_3, Cr_6\}) = 0.66577,\ \mathscr{M}(\{Cr_1, Cr_2, Cr_4, Cr_5\}) = 0.82075,$
$\mathscr{M}(\{Cr_1, Cr_2, Cr_4, Cr_6\}) = 0.7092,\ \mathscr{M}(\{Cr_1, Cr_2, Cr_5, Cr_6\}) = 0.79851,\ \mathscr{M}(\{Cr_1, Cr_3, Cr_4, Cr_5\}) = 0.60464,\ \mathscr{M}(\{Cr_1, Cr_3, Cr_4, Cr_6\}) = 0.49308,$
$\mathscr{M}(\{Cr_1, Cr_3, Cr_5, Cr_6\}) = 0.5824,\ \mathscr{M}(\{Cr_1, Cr_4, Cr_5, Cr_6\}) = 0.62582,\ \mathscr{M}(\{Cr_2, Cr_3, Cr_4, Cr_5\}) = 0.70841,\ \mathscr{M}(\{Cr_2, Cr_3, Cr_4, Cr_6\}) = 0.59685,$
$\mathscr{M}(\{Cr_2, Cr_3, Cr_5, Cr_6\}) = 0.68616,\ \mathscr{M}(\{Cr_2, Cr_4, Cr_5, Cr_6\}) = 0.72959,\ \mathscr{M}(\{Cr_3, Cr_4, Cr_5, Cr_6\}) = 0.51348,$
$\mathscr{M}(\{Cr_1, Cr_2, Cr_3, Cr_4, Cr_5\}) = 0.89978,\ \mathscr{M}(\{Cr_1, Cr_2, Cr_3, Cr_4, Cr_6\}) = 0.78823,\ \mathscr{M}(\{Cr_1, Cr_2, Cr_3, Cr_5, Cr_6\}) = 0.87754,$
$\mathscr{M}(\{Cr_1, Cr_2, Cr_4, Cr_5, Cr_6\}) = 0.92097,\ \mathscr{M}(\{Cr_1, Cr_3, Cr_4, Cr_5, Cr_6\}) = 0.70486,\ \mathscr{M}(\{Cr_2, Cr_3, Cr_4, Cr_5, Cr_6\}) = 0.80862,$
$\mathscr{M}(\{Cr_1, Cr_2, Cr_3, Cr_4, Cr_5, Cr_6\}) = 1.$

Steps 8 and 9. Apply Eq (28) to aggregate into one column and then apply Eq (2) to find the score values. We can find the best alternative based on the score values, as represented in Table 15. Similarly, we can determine the best alternative by computing the score values for varying the parameter $\wp$ for a specific $q = 4$ by applying the Cq-ROFECIWA and Cq-ROFECIWG operators, as shown in Tables 16 and 17, respectively. The graphical representation is shown in Figure 2. More precisely, the bar graph representation is shown in Figure 3.

Table 15. Final aggregated decision matrix and ranking.

Alternative	CqROFNs	Score values	Ranking
$\mathscr{L}_1$	$\langle(0.7394, 0.2382), 0.4524\rangle$	0.1065	5
$\mathscr{L}_2$	$\langle(0.7608, 0.2550), 0.5223\rangle$	0.1243	3
$\mathscr{L}_3$	$\langle(0.5743, 0.4407), 0.3363\rangle$	0.0261	8
$\mathscr{L}_4$	$\langle(0.6767, 0.3553), 0.4779\rangle$	0.0744	6
$\mathscr{L}_5$	$\langle(0.6691, 0.3498), 0.4578\rangle$	0.0701	7
$\mathscr{L}_6$	$\langle(0.8258, 0.1688), 0.4757\rangle$	0.1644	1
$\mathscr{L}_7$	$\langle(0.7691, 0.2748), 0.5204\rangle$	0.1286	2
$\mathscr{L}_8$	$\langle(0.7621, 0.2532), 0.4482\rangle$	0.1187	4

Table 16. The impact of the parameter $\wp$ by applying the Cq-ROFECIWA operator.

q	$\wp$	$Sc(\mathscr{L}_1)$	$Sc(\mathscr{L}_2)$	$Sc(\mathscr{L}_3)$	$Sc(\mathscr{L}_4)$	$Sc(\mathscr{L}_5)$	$Sc(\mathscr{L}_6)$	$Sc(\mathscr{L}_7)$	$Sc(\mathscr{L}_8)$	Ranking
	$\wp = 0.0$	0.0715	0.0757	0.0130	0.0465	0.0426	0.1295	0.0770	0.0839	$\mathscr{L}_6 > \mathscr{L}_8 > \mathscr{L}_7 > \mathscr{L}_2 > \mathscr{L}_1 > \mathscr{L}_4 > \mathscr{L}_5 > \mathscr{L}_3$
	$\wp = 0.2$	0.0841	0.0911	0.0178	0.0559	0.0491	0.1407	0.0947	0.0960	$\mathscr{L}_6 > \mathscr{L}_8 > \mathscr{L}_7 > \mathscr{L}_2 > \mathscr{L}_1 > \mathscr{L}_4 > \mathscr{L}_5 > \mathscr{L}_3$
$q = 4$	$\wp = 0.5$	0.0977	0.1111	0.0229	0.0674	0.0618	0.1554	0.1150	0.1092	$\mathscr{L}_6 > \mathscr{L}_7 > \mathscr{L}_8 > \mathscr{L}_2 > \mathscr{L}_1 > \mathscr{L}_4 > \mathscr{L}_5 > \mathscr{L}_3$
	$\wp = 0.8$	0.1109	0.1309	0.0276	0.0778	0.0742	0.1688	0.1353	0.1236	$\mathscr{L}_6 > \mathscr{L}_7 > \mathscr{L}_8 > \mathscr{L}_2 > \mathscr{L}_1 > \mathscr{L}_4 > \mathscr{L}_5 > \mathscr{L}_3$
	$\wp = 1$	0.1195	0.1441	0.0307	0.0844	0.0821	0.1772	0.1488	0.1337	$\mathscr{L}_6 > \mathscr{L}_7 > \mathscr{L}_8 > \mathscr{L}_2 > \mathscr{L}_1 > \mathscr{L}_4 > \mathscr{L}_5 > \mathscr{L}_3$

Table 17. The impact of the parameter $\wp$ by applying the Cq-ROFECIWG operator.

q	$\wp$	$Sc(\mathscr{L}_1)$	$Sc(\mathscr{L}_2)$	$Sc(\mathscr{L}_3)$	$Sc(\mathscr{L}_4)$	$Sc(\mathscr{L}_5)$	$Sc(\mathscr{L}_6)$	$Sc(\mathscr{L}_7)$	$Sc(\mathscr{L}_8)$	Ranking
	$\wp = 0.0$	0.0895	0.0927	0.0234	0.0562	0.0523	0.1342	0.0958	0.1013	$\mathscr{L}_6 > \mathscr{L}_8 > \mathscr{L}_7 > \mathscr{L}_2 > \mathscr{L}_1 > \mathscr{L}_4 > \mathscr{L}_5 > \mathscr{L}_3$
	$\wp = 0.2$	0.1023	0.1079	0.0612	0.0667	0.0286	0.1497	0.1165	0.1129	$\mathscr{L}_6 > \mathscr{L}_8 > \mathscr{L}_7 > \mathscr{L}_2 > \mathscr{L}_1 > \mathscr{L}_4 > \mathscr{L}_3 > \mathscr{L}_5$
$q = 4$	$\wp = 0.5$	0.0794	0.1285	0.0342	0.1154	0.0743	0.1651	0.1378	0.1270	$\mathscr{L}_6 > \mathscr{L}_7 > \mathscr{L}_8 > \mathscr{L}_2 > \mathscr{L}_4 > \mathscr{L}_1 > \mathscr{L}_5 > \mathscr{L}_3$
	$\wp = 0.8$	0.1293	0.1489	0.0396	0.0865	0.0907	0.1798	0.1587	0.1424	$\mathscr{L}_6 > \mathscr{L}_7 > \mathscr{L}_8 > \mathscr{L}_2 > \mathscr{L}_1 > \mathscr{L}_5 > \mathscr{L}_4 > \mathscr{L}_3$
	$\wp = 1$	0.1389	0.1630	0.0435	0.0984	0.0941	0.1883	0.1728	0.1539	$\mathscr{L}_6 > \mathscr{L}_7 > \mathscr{L}_8 > \mathscr{L}_2 > \mathscr{L}_1 > \mathscr{L}_4 > \mathscr{L}_5 > \mathscr{L}_3$

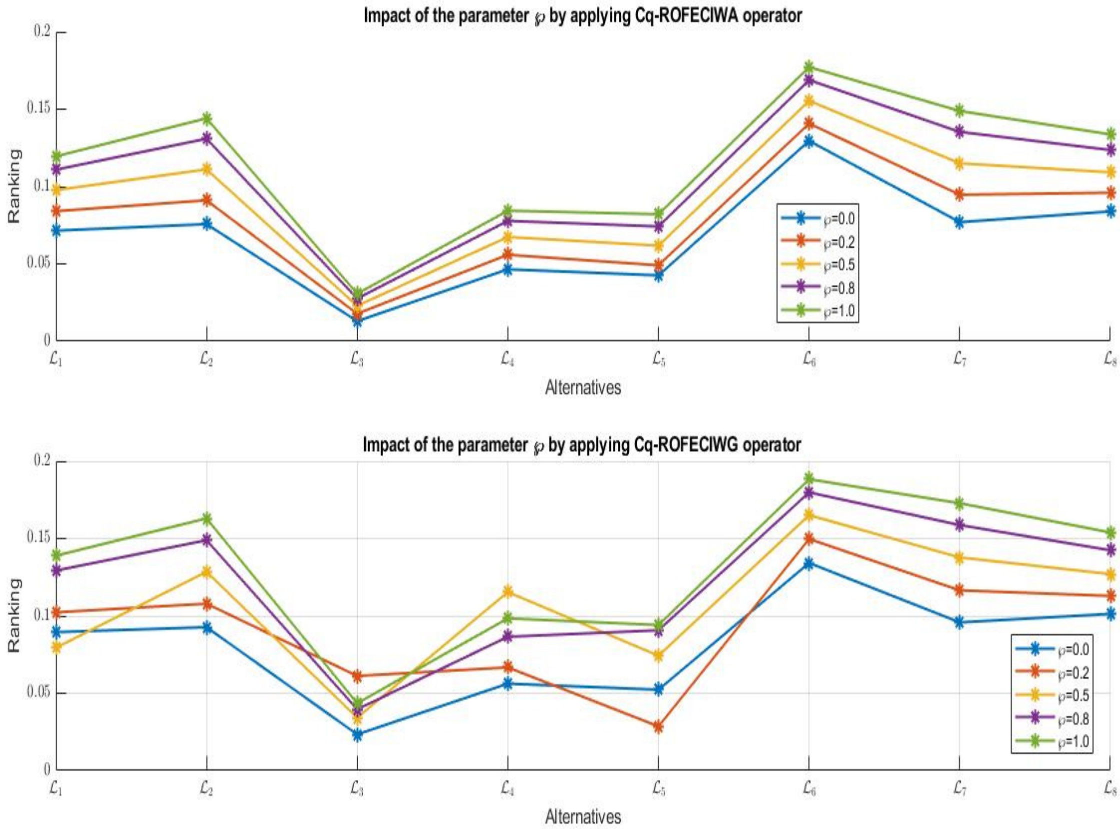

Figure 2. Impact of q and $\wp$ by using Cq-ROFECIWA and Cq-ROFECIWG.

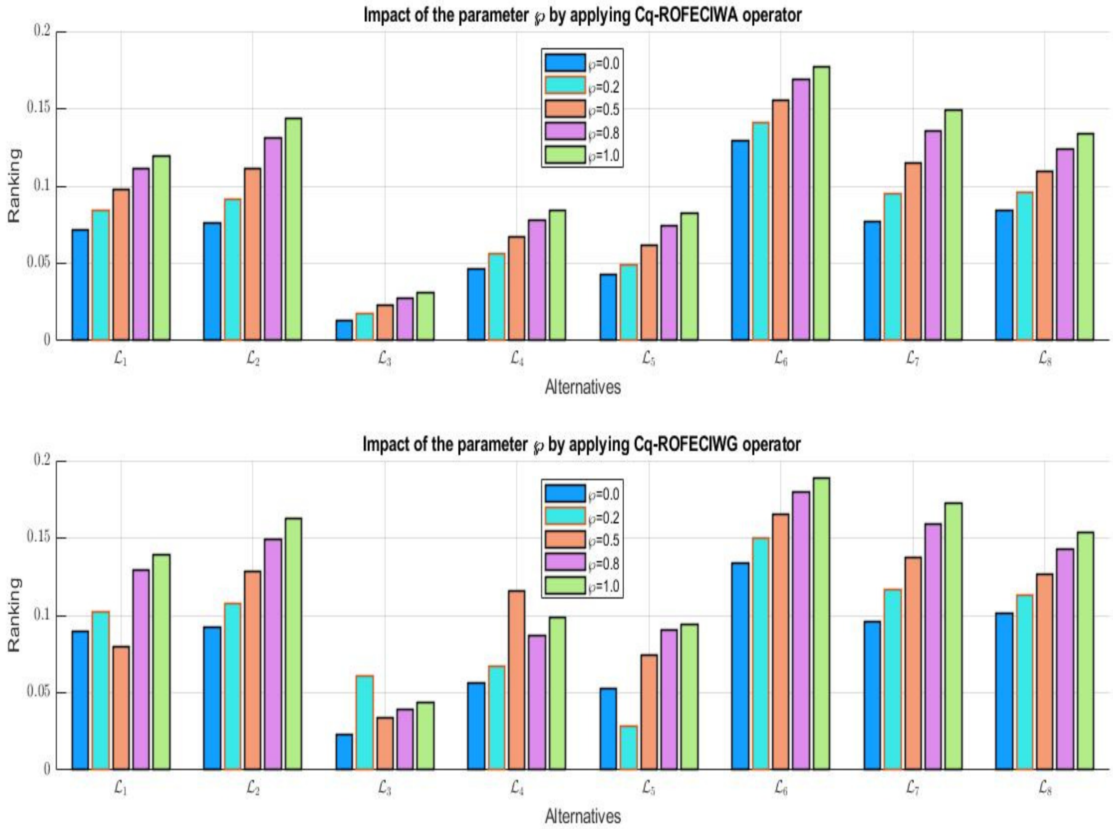

Figure 3. Bar graph representation showing the impact of the parameters on ranking.

5.2. Benefits

Incineration, composting, and sanitary landfills are the top three SWM strategies, each with its benefits. Incineration drastically reduces trash volume by 90%, saving landfill space. It generates renewable energy from waste products. Incineration kills bacteria and poisons, which makes it excellent for hazardous waste management. Organic waste becomes nutrient-rich compost, improving soil quality and fertility. This approach reduces the use of landfill methane and chemical fertilizers, reducing pollution. Composting reduces disposal costs and produces a valuable product for agriculture and landscaping. Engineered liners and leachate management systems protect groundwater and soil in sanitary landfills. Modern, sanitary landfills can trap methane emissions from decomposing waste to generate renewable energy. Their versatility and ability to handle vast amounts of varied waste make them essential for waste management systems. Open dumping, which harms the environment and health, is the worst method. Uncontrolled trash disposal pollutes groundwater and soil. This process releases greenhouse gases, polluting the air and causing climate change. It also invites pests and causes unclean conditions, endangering adjacent towns. Due to its negative effects, open dumping is banned in many countries and is not a future trash management solution. Open dumping is being replaced with incineration, composting, and sanitary landfills as a global effort to improve environmental health and waste management.

6. Comparative analysis

The study evaluated the performance of Cq-ROFECIWA and Cq-ROFECIWG AOs against existing operators in decision-making. When the proposed operators are combined with the LOPCOW technique and Choquet integral, they offer better stability and sustainability. They can handle fuzzy information, capture how criteria interact, and give more accurate results, which makes them good for situations that are complex and change over time. Table 18 displays the findings of the comparison analysis that was performed with several different operators on Cq-ROFSs. Graphical representation is shown in Figure 4.

Table 18. Comparison with newly proposed methods.

Authors	Methods	Ranking of alternatives
Ali and Mahmood [45]	Cq-ROFDWA and Cq-ROFDWG	$\mathscr{L}_6 > \mathscr{L}_7 > \mathscr{L}_8 > \mathscr{L}_2 > \mathscr{L}_1 > \mathscr{L}_5 > \mathscr{L}_4 > \mathscr{L}_3$
Liu et al. [46]	Cq-ROFPWA and Cq-ROFPWG	$\mathscr{L}_6 > \mathscr{L}_8 > \mathscr{L}_7 > \mathscr{L}_1 > \mathscr{L}_2 > \mathscr{L}_4 > \mathscr{L}_5 > \mathscr{L}_3$
Otay and Kahraman [47]	C-IF AHP and C-IF VIKOR	$\mathscr{L}_8 > \mathscr{L}_6 > \mathscr{L}_7 > \mathscr{L}_2 > \mathscr{L}_1 > \mathscr{L}_5 > \mathscr{L}_3 > \mathscr{L}_4$
Chen [48]	C-IF VIKOR	$\mathscr{L}_7 > \mathscr{L}_6 > \mathscr{L}_8 > \mathscr{L}_1 > \mathscr{L}_2 > \mathscr{L}_4 > \mathscr{L}_5 > \mathscr{L}_3$
Alinejad et al. [49]	CIF-SWARA and CIF-DEMATEL	$\mathscr{L}_6 > \mathscr{L}_8 > \mathscr{L}_7 > \mathscr{L}_1 > \mathscr{L}_2 > \mathscr{L}_5 > \mathscr{L}_4 > \mathscr{L}_3$
Hussain et al. [50]	IFAADHM and IFAAWDHM	$\mathscr{L}_8 > \mathscr{L}_6 > \mathscr{L}_7 > \mathscr{L}_2 > \mathscr{L}_1 > \mathscr{L}_5 > \mathscr{L}_4 > \mathscr{L}_3$
Proposed	Cq-ROFECIWA	$\mathscr{L}_6 > \mathscr{L}_8 > \mathscr{L}_7 > \mathscr{L}_2 > \mathscr{L}_1 > \mathscr{L}_4 > \mathscr{L}_5 > \mathscr{L}_3$
Proposed	Cq-ROFECIWG	$\mathscr{L}_6 > \mathscr{L}_8 > \mathscr{L}_7 > \mathscr{L}_1 > \mathscr{L}_2 > \mathscr{L}_5 > \mathscr{L}_4 > \mathscr{L}_3$

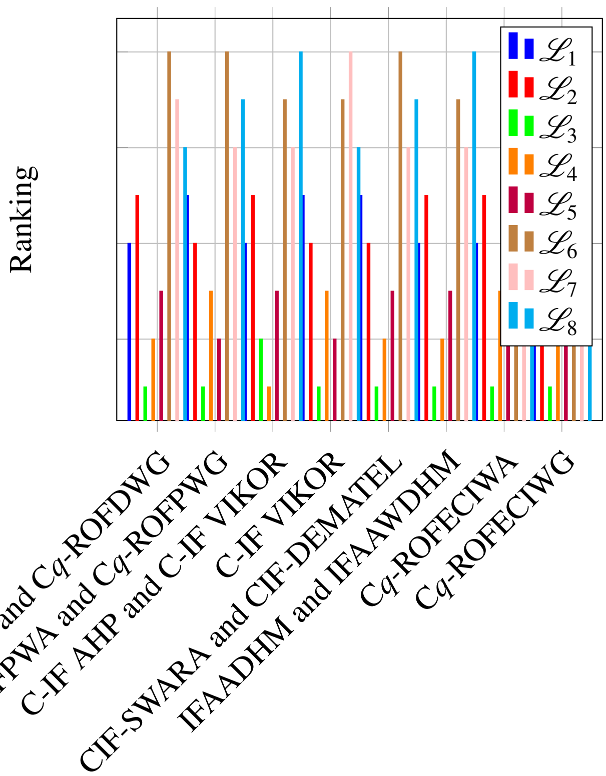

Figure 4. Ranking of alternatives for different methods.

6.1. *Managerial limitations*

Despite the robust methodology and thorough research offered in this paper, several managerial limitations need to be highlighted. The implementation of Cq-ROFS information requires a deep understanding and proficiency in advanced mathematical models like the Choquet integral and LOPCOW techniques, as well as the specific aggregation operators Cq-ROFECIWA and Cq-ROFECIWG. However, it should be noted that such expertise may not be easily accessible in all contexts. Consequently, there could be challenges regarding the training and allocation of resources. Furthermore, the data utilized in the case study was specific to Singapore, and it is plausible that the findings may not be transferable to other urban areas with various characteristics in terms of waste management. Hence, managers should use moderation when making sweeping assertions regarding the outcomes. Furthermore, the ever-changing nature of SWM requires ongoing modifications and enhancements to models and procedures to effectively adapt to shifts in trash generation patterns, regulatory mandates, and technological advancements. Therefore, it is imperative to dedicate yourself to continuous learning and adjustment, which can require a substantial allocation of resources. In the end, implementing decision-making frameworks that take into account more than one factor, like using Cq-ROFECIWA and Cq-ROFECIWG operators, might be met with resistance because the process is complicated and could make current practices more difficult. Considering these limitations is crucial to ensure the effective execution of the suggested approaches in practical SWM situations.

7. Conclusions

This research introduces a novel approach to optimizing SWM in Singapore by incorporating sophisticated mathematical models and decision-making methods. It employs Cq-ROFS for initial information gathering, the Choquet integral, LOPCOW techniques, and the AOs Cq-ROFECIWA and Cq-ROFECIWG. The study conducts a systematic analysis of numerous SWM methods, identifying the most effective and efficient strategies and the least effective and inefficient ones. The findings indicate the subsequent insights.

- Most effective alternative: Recycling and incineration are prioritized for their effectiveness in waste reduction and their practical applicability in the context of Singapore. Recycling promotes environmental sustainability and is consistent with public acceptance.
- Least effective alternative: Open dumping is considered the most ineffective method due to its considerable environmental and health hazards, as well as its lack of public approval.
- Decision-making advantages: The suggested technique offers a resilient and adaptable decision-making framework capable of addressing evolving urban challenges and changing policy agendas.

The proposed framework offers stakeholders and policy-makers an effective tool for making informed decisions, as it can navigate the complexities and unknowns of SWM. The Singapore case study not only demonstrates the efficacy of the proposed strategies in a real-world context, but also provides practical insights. We will develop MATLAB code to implement the entire decision-making process for practical use. Even if initial conditions or situations change, this tool will let stakeholders evaluate and choose the best SWM strategies. To enhance the practical utility of the methodologies, future research should further refine them, investigate the framework's applicability in a variety of urban areas, and resolve these constraints. The distinct insights that this study has provided to the field of SWM can be used to implement more sustainable and effective waste management methods in urban settings.

Author contributions

Yasir Yasin: Writing, Data curation, Investigation, Conceptualization; Muhammad Riaz: Supervision, Editing, Formal analysis, Conceptualization; Kholood Alsager: Review, Software, Resources. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The author declares that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors have no conflict of interest.

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