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Research article

Adaptive estimation: Fuzzy data-driven gamma distribution via Bayesian and maximum likelihood approaches

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Abstract: Integrating fuzzy concepts into statistical estimation offers considerable advantages by enhancing both the accuracy and reliability of parameter estimations, irrespective of the sample size and technique used. This study specifically examined the improvement of parameter estimation accuracy when dealing with fuzzy data, with a focus on the gamma distribution. We explored and evaluated a variety of estimation techniques for determining the scale parameter η and shape parameter ρ of the gamma distribution, employing both maximum likelihood (ML) and Bayesian methods. In the case of ML estimates, the expectation-maximization (EM) algorithm and the Newton-Raphson (NR) method were applied, with confidence intervals constructed using the Fisher information matrix. Additionally, the highest posterior density (HPD) intervals were derived through Gibbs sampling. For Bayesian estimates, the Tierney and Kadane (TK) approximation and Gibbs sampling were used to enhance the estimation process. A thorough performance comparison was undertaken using a simulated fuzzy dataset of the lifetimes of rechargeable batteries to assess the effectiveness of these methods. The methods were evaluated by comparing the estimated parameters to their true values using mean squared error (MSE) as a metric. Our findings demonstrate that the Bayesian approach, particularly when combined with the TK method, consistently produces more accurate and reliable parameter estimates compared to traditional methods. These results underscore the potential of Bayesian techniques in addressing fuzzy data and enhancing precision in statistical analyses.

Keywords: EM algorithm; fuzzy data; Gibbs sampling; gamma distribution; TK approximation
Mathematics Subject Classification: 62F86, 62F15

1. Introduction

An uncertain type of data that contains imprecision in the form of lack of clarity or incomplete information is referred to as fuzzy data. Irrespective of its limitations, fuzzy sets hold an important place in the field of research. This is mainly because most data in life cannot be expressed with a specific number, and therefore lessens the possibility of applying the vague theory. Various methods have been proposed regarding soft estimation. However, the effects of related estimates on time is an area that is still in nascent stages in the literature. Referred to as fuzzy data, the reliability of data sources is closely knitted with the fact that fuzzy logic effectively cuts across the uncertainties that may arise in the reliability analysis. It bases parameters such as failure coefficients on fuzzy numbers which help in modeling imprecision and incorporating expert opinion. The manner in which fuzzy reliability assists decision-making, especially in complicated systems, is by vague data and aiding in the diagnosis of faults. Such an approach tends to be more robust in the sense that it provides an opportunity to have better and clearer reliability assessments in cases of uncertainty or incomplete information.

The gamma distribution can provide a suitable model for reliability testing as it broadly covers data that possesses a positive skew, such as life spans or the length of time spent waiting for an event to take place. It provides a shape that incorporates fuzzy data and therefore addresses uncertainties faced in system parameter values, thus providing more realism and applicability in practice. This approach encompasses a wide range of activities including maintenance windows, risk analysis, quality control, and predicting conditions, which ultimately allow competing forces in an uncertain environment to be well-handled. The use of fuzzy data theoretically together with the gamma distribution presents an excellent model of practical variability as it encompasses multiple reliability situations and enhances the models used for decision-making. It renders probabilistic interpretations with limited precision points in fuzzy sets, which makes it useful for complicated systems when there are no exact inputs available.

The two-parameter gamma distribution is a widely used and adaptable probability distribution. The gamma distribution finds applications across various fields, including engineering, finance, hydrology, Bayesian statistics, queuing theory, survival analysis, and reliability testing. Research by Engelhardt and Bain [1], as well as Glaser [2], highlights its extensive use in life testing, reliability, and climate analysis. For further exploration of its applications in areas like hydrology, lifetime studies, meteorology, and medicine, consider reviewing the works of Aksoy [3], Mosino and Garcia [4], Gupta and Kundu [5], and Nadarajah and Gupta [6]. This distribution is particularly valuable for data analysis when working with known populations and has been extensively studied by numerous experts in recent years. The authors primarily developed moment estimators (MMEs) and maximum likelihood estimators (MLEs) for the parameters. Some researchers have also considered using Bayesian methods for parameter estimation. However, deriving closed-form solutions for these estimators is notably challenging. After calculating the ML estimates for the shape and scale parameters, Choi and Wette [7] quantitatively analyzed the bias in these estimates. Gilchrist [8] used the maximal invariant approach to evaluate the MLE of both parameters, while Son and Oh [9] applied the Gibbs sampling technique to derive the Bayes estimates for the gamma distribution parameters. Their numerical analysis, which incorporated adaptive rejection sampling with Gibbs sampling, demonstrated that Bayesian estimation outperformed both moment- and MLE-based estimators, as well as other Bayesian methods. Pradhan and Kundu [10] also explored Bayesian estimation of shape and scale parameters using the Gibbs sampling approach. A new sampling method based on ranked set sampling (RSS) was proposed in [11], called moving extremes RSS (MERSS), which is appropriate for estimating the location parameter of location families. The research derives the greatest likelihood estimator (MLE) associated with MERSS, establishes its equivariance property under location invariance, and evaluates its asymptotic efficiency with simple random sampling (SRS) for several distributions. Results show that, for the case of MERSS, the MLE estimator is a good alternative to the SRS-based MLE. As a way to overcome issues brought on by extreme weather conditions, [12] explored an improved methodology for reliability assessment of the offshore support structures of wind turbines. It enhances the accuracy and conservatism by integrating hybrid uncertainty analysis and an intelligent optimization-oriented support vector regression (SVR) modeling approach. Using a case study, the study demonstrated the usefulness of the framework and emphasized that the predicted fatigue reliability in such complex engineering systems could be improved.

Historically, discussions on the gamma distribution have not addressed fuzzy data, where uncertainty plays a role. Instead, research has focused on well-defined and precise data. In this study, we consider data that is not only random but also ambiguous in many real-world scenarios. While randomness pertains to uncertainty about experimental outcomes, vagueness involves uncertainty in interpreting the data itself. The probability density function (p.d.f.) of the gamma distribution is given by

$$f(x;\rho,\eta) = \frac{1}{\Gamma(\rho)\eta^{\rho}} x^{\rho-1} e^{-\frac{x}{\eta}}, \quad x > 0, \rho > 0, \eta > 0 \tag{1.1}$$

where its shape is governed by the parameter ρ. Distributions with lower ρ values are more skewed, while those with higher ρ values tend to be more symmetric and bell-shaped. The scale parameter η influences the spread of the distribution, with larger η values leading to greater variance and a wider spread. In practical settings, the lifespan of a unit may not be precisely measured due to human error, mechanical issues, or unforeseen circumstances. For instance, lifetime observations may be recorded as fuzzy in such cases, with the imprecise nature of lifespan data represented using fuzzy sets. In recent years, applying fuzzy sets to estimation theory has attracted the interest of numerous scholars. A novel technique for figuring out the reliability function and membership function of multiparameter lifespan distributions was put forward by Huang [13]. In 1991, Coppi et al. [14] introduced a few uses of fuzzy approaches in statistical analysis. Denoeux [15] examined the use of the EM method for ML estimates based on fuzzy data. A number of experiments were carried out by Pak et al. [16] in 2014 to create inferential methods for lifespan distributions based on fuzzy data, and a series of investigations were developed by him [16, 17] to draw the inferential algorithms for the lifespan distributions based on fuzzy data. When the lifetime observations are imprecise, Khoolenjani et al. [18] computed the mean parameter of the exponential distribution under the Type-II censoring strategy. In 2017, Kula and Dalkilic [19] introduced Type-II fuzzy logic and parameter estimation for the Pareto distribution. Fuzzy logic was first used to estimate the parameters of a combination of normal distributions using fuzzy clustering techniques; these methods were covered by Yang et al. in 1994 [20] and Gath and Geva [21]. Eventually, in 2019, Basharat et al. [22] demonstrated how to estimate the parameters of a linear combination of two exponentially distributed random variables using fuzzy data. This paper's major goal is to verify various inferential algorithms for the two-parameter gamma distribution where fuzzy numbers represent the available data. We shall talk about the core definitions and notations of fuzzy set theory.

Fuzzy set theory offers a robust framework for dealing with uncertainty and vagueness across different disciplines, making it useful for modeling complex systems and phenomena with more accuracy. Consider an experiment that can be described by a probability space $S = (X, \mathcal{B}_X, P_\theta)$, where

$(X, \mathcal{B}_X)$ is a measurable space and P_θ is a probability measure corresponding to a specific family defined on $(X, \mathcal{B}_X)$.

An indicator function $I_A : X \to [0, 1]$ is defined as:

$$I_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases}$$

which describes a standard (crisp) subset A within X. By contrast, a fuzzy subset $\tilde{A}$ of X is defined by a membership function $\mu_{\tilde{A}}(x)$, which assigns a degree of membership $\mu_{\tilde{A}}(x) \in [0, 1]$ to each point x in X. As per Zadeh's 1968 framework, the probability of a fuzzy event $\tilde{A}$ can be written as:

$$P(\tilde{A}) = \int \mu_{\tilde{A}}(x)\, dP.$$

The conditional probability density function for X, given that fuzzy observations $\tilde{X}$ occur, is expressed as:

$$f(x \mid X \in \tilde{A}) = \frac{\mu_{\tilde{A}}(x) f(x)}{\int \mu_{\tilde{A}}(x) f(x)\, dx}$$

In Section 2, we explain how to compute the maximum likelihood estimates (MLEs) of the parameters using both the expectation-maximization (EM) algorithm and Newton-Raphson (NR) methods. Section 3 focuses on deriving the Bayes estimates of the unknown parameters using TK and Gibbs sampling methods, with the assumption of gamma priors. Finally, Section 4 offers a numerical study of Monte Carlo simulation analysis, where all the estimation methods are compared and assessed.

Figure 1 under consideration describes parameter estimation methods for a fuzzy gamma distribution with data emphasis. It differentiates the two types of estimates: point estimation which may include maximum likelihood estimation or Bayesian approaches, Tierney Kadane's or Gibbs methods, and interval estimation utilizing HPD intervals and confidence intervals. It highlights simulation studies as well as real data analysis for operational use.

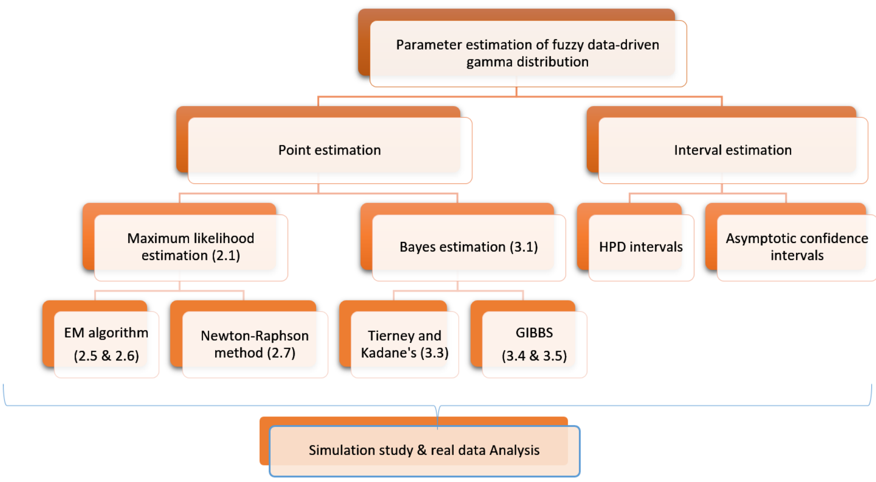

Figure 1. Framework for fuzzy gamma-based parameter estimation and analysis.

2. Maximum likelihood estimation

Consider the scenario where $X = (X_1, X_2, \ldots, X_n)$ represents an independent random sample of size n drawn from the distribution $G(\rho, \eta)$, with the probability density function given in equation (1.1). The complete data likelihood function for this sample is:

$$L(x; \rho, \eta) = \prod_{i=1}^{n} f(x_i; \rho, \eta).$$

Taking into account the scenario where the information that is currently accessible regarding x is fuzzy data rather than clear data, that is, $\tilde{X} = \tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n$ denotes its observations, where $\tilde{x}_i$ is the fuzzy observed number for the random variable x_i, the likelihood function based on fuzzy observations $\tilde{X}$ is defined as the following, with the membership function $\mu_{\tilde{x}_i}(x)$ as a function of θ:

$$L(\theta \mid \tilde{X}) = L(\rho, \eta; \tilde{x}) = \prod_{i=1}^{n} \int_0^{\infty} f_\theta(x) \mu_{\tilde{x}_i}(x)\, dx,$$

$$L(\rho, \eta; \tilde{x}) = \prod_{i=1}^{n} \int_0^{\infty} \frac{1}{\Gamma(\rho)\eta^\rho} x^{\rho-1} e^{-\frac{x}{\eta}} \mu_{\tilde{x}_i}(x)\, dx,$$

$$L^*(\rho, \eta; \tilde{x}) = \log L(\rho, \eta; \tilde{x}) = -n\rho \log \eta - n \log \Gamma(\rho) + \sum_{i=1}^{n} \log \int_0^{\infty} x^{\rho-1} e^{-\frac{x}{\eta}} \mu_{\tilde{x}_i}(x)\, dx. \tag{2.1}$$

The concept of maximum likelihood parameter estimation is to identify the parameters that maximize the probability (or likelihood) of the observed sample data. Maximum likelihood methods are highly adaptable and can be applied to a wide range of models and various types of data. The maximum likelihood estimate of the parameters is derived from the observed maximizing of the log-likelihood function $L^*(\theta)$ that is (2.1). By setting the partial derivatives of the log-likelihood with respect to the parameters ρ and η to zero, two resulting equations can be obtained for solving the parameters.

$$\frac{\partial}{\partial \rho} L^*(\rho, \eta; \tilde{x}) = -n \log \eta - n\psi(\rho) + \sum_{i=1}^{n} \frac{\int_0^{\infty} x^{\rho-1} \log x\, e^{-\frac{x}{\eta}} \mu_{\tilde{x}_i}(x)\, dx}{\int_0^{\infty} x^{\rho-1} e^{-\frac{x}{\eta}} \mu_{\tilde{x}_i}(x)\, dx} \tag{2.2}$$

$$\frac{\partial}{\partial \eta} L^*(\rho, \eta; \tilde{x}) = -\frac{n\rho}{\eta} - \sum_{i=1}^{n} \frac{\int_0^{\infty} x^{\rho-1} e^{-\frac{x}{\eta}} \left(\frac{x}{\eta^2}\right) \mu_{\tilde{x}_i}(x)\, dx}{\int_0^{\infty} x^{\rho-1} e^{-\frac{x}{\eta}} \mu_{\tilde{x}_i}(x)\, dx} \tag{2.3}$$

where $\psi(\rho) = \frac{\Gamma'(\rho)}{\Gamma(\rho)}$.

To obtain the estimation using the ML method, we find that it is tough to figure out the above nonlinear Eqs (2.2) and (2.3). In this case, iterative numerical methods called the expectation-maximization (EM) algorithm and Newton-Raphson method are used to obtain the MLEs.

2.1. Expectation-maximization algorithm

The EM algorithm repeatedly addresses the issue of maximizing the observed data log-likelihood function by utilizing the complete data log-likelihood function. Each iteration of the method consists of two stages: the expectation (E-step) and the maximization (M-step). The E-step includes iteratively calculating the expectation of the observed data log-likelihood function with the complete data log-likelihood function. The complete data log-likelihood function is expressed as follows:

$$\log L(x;\rho,\eta) = -n\rho\log\eta - n\log\Gamma(\rho) + (\rho-1)\sum_{i=1}^{n}\log x_i - \frac{1}{\eta}\sum_{i=1}^{n} x_i. \tag{2.4}$$

By differentiating the above Eq (2.4) with respect to ρ and simplifying, we get

$$\psi(\rho) = \frac{1}{n}\sum_{i=1}^{n}\log x_i + \log\eta.$$

By differentiating the Eq (2.4) with respect to η and simplifying, we get

$$\eta = \frac{\sum_{i=1}^{n} x_i}{n\rho}.$$

The iterative process of the EM algorithm is as follows:

(1) Give initial values (calculated by the sample mean and variance) of $\rho = \rho^{(0)} = \frac{\bar{x}^2}{s^2}$ and $\eta = \eta^{(0)} = \frac{\bar{x}}{\rho^{(0)}}$, and set $h = 0$.

(2) In the $(h+1)^{th}$ iterative process, the EM approach aims to maximize the log-likelihood function $L^*(\theta)$ of the observed data by repeatedly processing the log-likelihood function of the entire data.

(3) **E-step:** The E-step requires the computation of conditional expectations utilizing the expression:

$$E_{1i} = E_{\rho^{(h)},\eta^{(h)}}[\log(x_i) \mid \tilde{X}_i] = \frac{\int x^{\rho^{(h)}-1} e^{-\frac{x}{\eta^{(h)}}} \log x \mu_{\tilde{X}_i}(x)dx}{\int x^{\rho^{(h)}-1} e^{-\frac{x}{\eta^{(h)}}} \mu_{\tilde{X}_i}(x)dx}$$

$$E_{2i} = E_{\rho^{(h)},\eta^{(h)}}[x_i \mid \tilde{X}_i] = \frac{\int x^{\rho^{(h)}} e^{-\frac{x}{\eta^{(h)}}} \mu_{\tilde{X}_i}(x)dx}{\int x^{\rho^{(h)}-1} e^{-\frac{x}{\eta^{(h)}}} \mu_{\tilde{X}_i}(x)dx}$$

and the likelihood equations become:

$$\psi(\rho) = \frac{1}{n}\sum_{i=1}^{n} E_{1i} + \log\eta$$

$$\eta = \frac{\sum_{i=1}^{n} E_{2i}}{n\rho}.$$

(4) **M-step:** The M-step requires calculating the above equations and acquiring the next values $\rho^{(h+1)}, \eta^{(h+1)}$ of ρ and η as:

$$\psi(\rho^{(h+1)}) = \frac{1}{n}\sum_{i=1}^{n} E_{1i} + \log\eta^{(h+1)} \tag{2.5}$$

$$\eta^{(h+1)} = \frac{\sum_{i=1}^{n} E_{2i}}{n\rho^{(h+1)}}. \tag{2.6}$$

(5) **Convergence:** If convergence is achieved, the current values of $\rho^{(h+1)}, \eta^{(h+1)}$ are the maximum likelihood estimators (MLEs) of ρ and η. Otherwise, repeat steps 2 and 3.

2.2. Newton-Raphson method

The Newton-Raphson method is a dominant and popular iterative methodology for progressively improving approximations to the roots of a real-valued function. Dedicated to the memory of Joseph Raphson and Isaac Newton, this approach excels in resolving non-linear equations. Finding the relevant parameters in a likelihood function may be done directly using the NR method. An iterative process is used in this approach to get the solution of the likelihood equation.

Let $\theta = (\rho, \eta)^T$ denote the parameter vector. At step $(h + 1)$ of the iterative process, we acquire the parameters in the following manner,

$$\theta^{(h+1)} = \theta^{(h)} - \left[\left.\frac{\partial^2 L^*(\rho, \eta; \tilde{x})}{\partial\theta\partial\theta^T}\right|_{\theta=\theta^{(h)}}\right]^{-1} \cdot \left[\left.\frac{\partial L^*(\rho, \eta; \tilde{x})}{\partial\theta}\right|_{\theta=\theta^{(h)}}\right]$$

where

$$\frac{\partial L^*(\rho, \eta; \tilde{x})}{\partial\theta} = \begin{pmatrix} \frac{\partial L^*(\rho,\eta;\tilde{x})}{\partial\rho} \\ \frac{\partial L^*(\rho,\eta;\tilde{x})}{\partial\eta} \end{pmatrix}$$

$$\frac{\partial^2 L^*(\rho, \eta; \tilde{x})}{\partial\theta\partial\theta^T} = \begin{pmatrix} \frac{\partial^2 L^*(\rho,\eta;\tilde{x})}{\partial\rho^2} & \frac{\partial^2 L^*(\rho,\eta;\tilde{x})}{\partial\rho\partial\eta} \\ \frac{\partial^2 L^*(\rho,\eta;\tilde{x})}{\partial\rho\partial\eta} & \frac{\partial^2 L^*(\rho,\eta;\tilde{x})}{\partial\eta^2} \end{pmatrix}. \tag{2.7}$$

Here, the individual derivatives are given by:

$$\frac{\partial^2 L^*(\rho, \eta; \tilde{x})}{\partial\rho^2} = -n\psi'(\rho) + \sum_{i=1}^{n} \frac{\int_0^\infty x^{\rho-1}(\log x)^2 e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx}{\int_0^\infty x^{\rho-1}e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx} - \sum_{i=1}^{n}\left[\frac{\int_0^\infty x^{\rho-1}\log x e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx}{\int_0^\infty x^{\rho-1}e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx}\right]^2,$$

$$\frac{\partial^2 L^*(\rho, \eta; \tilde{x})}{\partial\eta^2} = \frac{n\rho}{\eta^2} - \sum_{i=1}^{n} \frac{\int_0^\infty \left(\frac{x-2\eta}{\eta^4}\right) x^\rho e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx}{\int_0^\infty x^{\rho-1}e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx} - \sum_{i=1}^{n}\left[\frac{\int_0^\infty x^{\rho-1}e^{-\frac{x}{\eta}}\left(\frac{x}{\eta^2}\right)\mu_{\tilde{x}_i}(x)dx}{\int_0^\infty x^{\rho-1}e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx}\right]^2,$$

$$\frac{\partial^2}{\partial\rho\partial\eta}L^*(\rho, \eta; \tilde{x}) = \frac{-n}{\eta} + \frac{\int_0^\infty \left(\frac{x^\rho}{\eta^2}\right)\log x e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx}{\int_0^\infty x^{\rho-1}e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx} - \frac{\int_0^\infty \left(\frac{x^\rho}{\eta^2}\right)e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx \,.\, \int_0^\infty x^{\rho-1} logx e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx}{\left(\int_0^\infty x^{\rho-1}e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx\right)^2}.$$

The iteration process will continue until convergence is achieved, which is defined as $\theta^{(h+1)} - \theta^{(h)} < \epsilon$, where ϵ is a predetermined positive value. In this study, the ML estimate of (ρ, η) obtained using the Newton-Raphson (NR) technique is denoted as $(\hat{\rho}_{NR}, \hat{\eta}_{NR})$. Once we have calculated the maximum likelihood estimate (MLE) of ρ and η, we can use the asymptotic normality of the MLEs to approximate the confidence interval for the parameters with a level of $100(1 - \alpha)\%$. The inverse of the Fisher information matrix is calculated as follows:

$$\rho_{ij} = \begin{pmatrix} Dn\frac{\rho}{\eta^2} + DI_{\eta\eta} & \frac{Dn}{\eta} - DI_{\rho\eta} \\ \frac{Dn}{\eta} - DI_{\rho\eta} & DI_{\rho\rho} - Dn\psi'(\rho) \end{pmatrix}.$$

Here,

$$D = -\frac{1}{(I_{\rho\rho} - n\psi'(\rho))(n\rho\eta^2 + I_{\eta\eta}) - (I_{\eta\rho} - n\frac{1}{\eta})^2},$$

$$I = \sum_{i=1}^{n} \log \int_0^{\infty} x^{\rho-1} e^{-\frac{x}{\eta}} \mu_{\tilde{x}_i}(x) dx$$

where the terms $I_{\eta\eta}$, $I_{\rho\eta}$, and $I_{\rho\rho}$ are derived and presented in the Appendix. With this information, we can readily calculate the asymptotic 95% confidence intervals for the parameters ρ and η. The 95% confidence interval for θ_j is obtained as

$$\theta_j \pm 1.96 \sqrt{(I(\theta)_{jj})^{-1}}.$$

The 95% confidence intervals for α and η are estimated, respectively, as

$$\hat{\rho}_{ML} \pm 1.96 \sqrt{\frac{Dn\rho}{\eta^2} + DI_{\eta\eta}}$$

$$\hat{\eta}_{ML} \pm 1.96 \sqrt{DI_{\rho\rho} - Dn\psi'(\rho)}.$$

3. Bayesian estimation

The Bayesian viewpoint has garnered much attention for statistical inference in recent decades as a strong and legitimate substitute for conventional statistical viewpoints. The Bayesian estimation is discussed in this part with the suppositions that ρ and η have independent gamma priors with the following probability density functions:

$$\pi_1(\rho) = \frac{d^c}{\Gamma(c)} \rho^{c-1} \exp(-\rho d), \quad \rho > 0$$

$$\pi_2(\eta) = \frac{b^a}{\Gamma(a)} \eta^{a-1} \exp(-\eta b), \quad \eta > 0$$

with the parameters $\rho \sim \Gamma(c, d)$ and $\eta \sim \Gamma(a, b)$. Based on the above priors, the joint posterior density function of ρ and η given the data can be written as follows:

$$\pi(\rho, \eta \mid \tilde{x}) = \frac{\pi_1(\rho)\pi_2(\eta)\ell(\rho, \eta; \tilde{x})}{\int_0^{\infty}\int_0^{\infty} \pi_1(\rho)\pi_2(\eta)\ell(\rho, \eta; \tilde{x}) d\rho d\eta}$$

where

$$\ell(\rho, \eta; \tilde{x}) = \rho^{n+c-1} \eta^{n+a-1} \exp(-\rho d) \exp(-\eta b) \prod_{i=1}^{n} \int x^{\rho-1} e^{-\frac{x}{\eta}} \mu_{\tilde{x}_i}(x) dx$$

is the likelihood function based on the fuzzy sample $\tilde{x}$. Then, under a squared error loss function, the Bayes estimate of any function of ρ and η, say $g(\rho, \eta)$, is

$$\begin{aligned} E(g(\rho, \eta) \mid \tilde{x}) &= \frac{\int_0^{\infty}\int_0^{\infty} g(\rho, \eta)\pi_1(\rho)\pi_2(\eta)\ell(\rho, \eta; \tilde{x}) d\rho d\eta}{\int_0^{\infty}\int_0^{\infty} \pi_1(\rho)\pi_2(\eta)\ell(\rho, \eta; \tilde{x}) d\rho d\eta} \\ &= \frac{\int_0^{\infty}\int_0^{\infty} g(\rho, \eta) Q(\rho, \eta) d\rho d\eta}{\int_0^{\infty}\int_0^{\infty} Q(\rho, \eta) d\rho d\eta} \end{aligned} \tag{3.1}$$

where $Q(\rho,\eta) = \ln[\pi_1(\rho)\pi_2(\eta)] + \ln \ell(\rho,\eta) \equiv H(\rho,\eta) + L(\rho,\eta)$. It is not easy to solve the above Eq (3.1), and so we use Tierney and Kadane's approximation for calculating the Bayes estimates. By replacing $H(\rho,\eta) = Q(\rho,\eta)/n$ and $H^*(\rho,\eta) = [\ln g(\rho,\eta) + Q(\rho,\eta)]/n$, the above expression (3.1) becomes

$$E(g(\rho,\eta) \mid \tilde{x}) \approx \frac{\int_0^\infty \int_0^\infty e^{nH^*(\rho,\eta)} d\rho d\eta}{\int_0^\infty \int_0^\infty e^{nH(\rho,\eta)} d\rho d\eta}. \tag{3.2}$$

Following Tierney and Kadane [23], the above Eq (3.2) can be approximated as follows.

$$\hat{g}_{BT}(\rho,\eta) = \left[\frac{\det\Sigma^*}{\det\Sigma}\right]^{1/2} \exp\{n[H^*(\tilde{\rho},\tilde{\eta}) - H(\hat{\rho},\hat{\eta})]\} \tag{3.3}$$

where $(\tilde{\rho},\tilde{\eta})$ and $(\hat{\rho},\hat{\eta})$ maximize $H^*(\rho,\eta)$ and $H(\rho,\eta)$, respectively, and Σ^* and Σ are the negatives of the inverse Hessians of $H^*(\rho,\eta)$ and $H(\rho,\eta)$ at $(\tilde{\rho},\tilde{\eta})$ and $(\hat{\rho},\hat{\eta})$, respectively.

In our case, we have

$$H(\rho,\eta) = \frac{1}{n}\{k + (n+c-1)\log\rho + (n+a-1)\log\eta - \rho d\} - \frac{1}{n}\left\{-\eta b + \sum_{i=1}^{n} \log \int x^{\rho-1} e^{-\frac{x}{\eta}} \mu_{\tilde{x}_i}(x)dx\right\}$$

where k is a constant and $(\hat{\rho},\hat{\eta})$ can be obtained by solving the following two equations.

$$\frac{\partial}{\partial\rho}H(\rho,\eta) = \frac{1}{n}\left\{\frac{n+c-1}{\rho} - d + \sum_{i=1}^{n} \frac{\int_0^\infty x^{\rho-1}\log x e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx}{\int_0^\infty x^{\rho-1}e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx}\right\},$$

$$\frac{\partial}{\partial\eta}H(\rho,\eta) = \frac{1}{n}\left\{\frac{n+a-1}{\eta} - b - \sum_{i=1}^{n} \frac{\int_0^\infty x^{\rho-1}e^{-\frac{x}{\eta}}\left(\frac{x}{\eta^2}\right)\mu_{\tilde{x}_i}(x)dx}{\int_0^\infty x^{\rho-1}e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx}\right\}.$$

From the second derivatives of $H(\rho,\eta)$, the determinant of the negative of the inverse Hessian of $H(\rho,\eta)$ at $(\hat{\rho},\hat{\eta})$ is given by

$$\det\Sigma = \left(H_{11}H_{22} - H_{12}^2\right)^{-1}$$

where

$$H_{11} = \frac{1}{n}\left\{\frac{-n+c-1}{\rho^2} + \sum_{i=1}^{n}\frac{\int_0^\infty x^{\rho-1}(\log x)^2 e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx}{\int_0^\infty x^{\rho-1}e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx}\right\} - \frac{1}{n}\sum_{i=1}^{n}\left[\frac{\int_0^\infty x^{\rho-1}\log x e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx}{\int_0^\infty x^{\rho-1}e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx}\right]^2,$$

$$H_{22} = \frac{1}{n}\left\{\frac{-n+a-1}{\eta^2} - \sum_{i=1}^{n}\frac{\int_0^\infty \left(\frac{x-2\eta}{\eta^4}\right) x^{\rho} e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx}{\int_0^\infty x^{\rho-1}e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx}\right\} - \frac{1}{n}\sum_{i=1}^{n}\left[\frac{\int_0^\infty x^{\rho-1}e^{-\frac{x}{\eta}}\left(\frac{x}{\eta^2}\right)\mu_{\tilde{x}_i}(x)dx}{\int_0^\infty x^{\rho-1}e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx}\right]^2,$$

$$H_{12} = \frac{\int_0^\infty \frac{x^\rho}{\eta^2}\log x e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx}{\int_0^\infty x^{\rho-1}e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx} - \frac{\int_0^\infty x^{\rho}e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx \int_0^\infty x^{\rho-1}\log x e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx}{\left(\int_0^\infty x^{\rho-1}e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx\right)^2}.$$

Following the same arguments with $g(\rho,\eta) = \rho$ and η, respectively, in $H^*(\rho,\eta)$, $\hat{\rho}_{BT}$ and $\hat{\eta}_{BT}$ in Eq (3.3) can then be obtained in a straightforward manner.

In this section, we explore the Bayesian estimation of the model parameters, applying the squared error loss function. We also introduce highest posterior density (HPD) credible intervals for the parameters, utilizing the approach outlined by Devroye [24], as well as samples drawn from the posterior distribution to form the HPD intervals.

The Bayes estimator for $g(\rho, \eta)$, under the squared error loss function, corresponds to the posterior mean and is expressed as follows:

$$\hat{g}(\rho, \eta) = E[g(\rho, \eta) \mid x, y]$$

In this section, we present an estimation of the shape parameter ρ and scale parameter η using approximate Bayes methods based on the assumptions of the priors:

$$\rho \sim \Gamma(l_1, m_1),$$

$$\eta \sim \Gamma(l_2, m_2).$$

The log-likelihood function for the observed data, which is based on the fuzzy data, is expressed as follows:

$$L^*(\theta) = L^*(\rho, \eta; \tilde{x}) = -n\rho \log \eta - n \log \Gamma(\rho) + \sum_{i=1}^{n} \log \int_0^{\infty} x^{\rho-1} e^{-\frac{x}{\eta}} \mu_{\tilde{x}_i}(x) dx.$$

The conditional posterior distributions for ρ and η are obtained as

$$\pi(\rho \mid \eta, X) \propto L^*(\rho \mid \eta, X)\pi(\rho), \tag{3.4}$$

$$\pi(\eta \mid \rho, X) \propto L^*(\rho \mid \eta, X)\pi(\eta). \tag{3.5}$$

We can estimate the parameters ρ and η iteratively by using the conditional posterior distributions above with the following procedure:

(1) Start with initial values for the parameters ρ_0 and η_0.
(2) For each iteration $i = 1, 2, 3, \ldots, h$, sample $\rho^{(i)}$ and $\eta^{(i)}$ from their conditional posterior distributions $\pi(\rho \mid \eta^{(i)}, X)$, $\pi(\eta \mid \rho^{(i)}, X)$, respectively.
(3) Using their individual posterior distributions, each parameter's sequence of samples is produced iteratively.
(4) Keep up the repeated sampling procedure until the Markov chain converges. Analyzing convergence may be done with diagnostic tools like trace plots.
(5) Draw conclusions about the parameters ρ and η based on the samples. The uncertainty in the parameter estimations can be summarized by computing summary statistics like the posterior mean and median.
(6) The parameters of the gamma distribution ρ and η may be estimated Bayesianly using the Gibbs sampling process described above. It gives the estimation theory a framework for integrating past knowledge and revising beliefs based on fuzzy evidence, which leads to more reliable parameter estimation.

4. Numerical study

In this section, various simulation results are recorded primarily to show the performance of Bayes estimation by Gibbs sampling, MLEs by the Newton-Raphson method, and the EM algorithm for different sample sizes, including an example to illustrate the methods of inference developed in this study. The estimation of the unknown parameters is obtained through the use of the three strategies described in the preceding sections. The calculations are performed using an open-source statistical computing and graphics software tool called R (version 4.3.2).

First, for a number of combinations of parameter values, $(\rho, \eta) = (1,1), (2,1), (2,2), (2,3), (3,3)$, with different sample sizes n, namely $n = 10, 20, 25, 50, 75, 100, 200$, for each n, we drawn an independent and identically distributed (i.i.d.) random sample from the gamma distribution. Fuzzy membership functions were used to fuzzify each instance of x.

$$\mu_{\tilde{x}_1}(x) = \begin{cases} 1 & \text{if } x \leq 0.05, \\ \frac{0.25-x}{0.2} & \text{if } 0.05 \leq x \leq 0.25, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mu_{\tilde{x}_2}(x) = \begin{cases} \frac{x-0.05}{0.2} & \text{if } 0.05 \leq x \leq 0.25, \\ \frac{0.5-x}{0.25} & \text{if } 0.25 \leq x \leq 0.5, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mu_{\tilde{x}_3}(x) = \begin{cases} \frac{x-0.25}{0.25} & \text{if } 0.25 \leq x \leq 0.5, \\ \frac{0.75-x}{0.25} & \text{if } 0.5 \leq x \leq 0.75, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mu_{\tilde{x}_4}(x) = \begin{cases} \frac{x-0.5}{0.25} & \text{if } 0.55 \leq x \leq 0.75, \\ \frac{1-x}{0.25} & \text{if } 0.75 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mu_{\tilde{x}_5}(x) = \begin{cases} \frac{x-0.75}{0.25} & \text{if } 0.75 \leq x \leq 1, \\ \frac{1.5-x}{0.5} & \text{if } 1 \leq x \leq 1.5, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mu_{\tilde{x}_6}(x) = \begin{cases} \frac{x-1}{0.5} & \text{if } 1 \leq x \leq 1.5, \\ \frac{2-x}{0.5} & \text{if } 1.5 \leq x \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mu_{\tilde{x}_7}(x) = \begin{cases} \frac{x-1.5}{0.5} & \text{if } 1.5 \leq x \leq 2, \\ 3-x & \text{if } 2 \leq x \leq 3, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mu_{\tilde{x}_8}(x) = \begin{cases} x-2 & \text{if } 2 \leq x \leq 3, \\ 1 & \text{if } x \geq 3, \\ 0 & \text{otherwise.} \end{cases}$$

- The simulation data shows that as the sample size increased, the mean squared error (MSE) of each estimator decreased (Figures 2–5).
- Larger sample sizes typically yielded more accurate parameter estimates with lower MSE, whereas smaller sample sizes led to higher variability and larger MSE in the MLE estimates.
- All parameter estimates fell within their respective confidence intervals, with the interval lengths shrinking as the sample size grew.

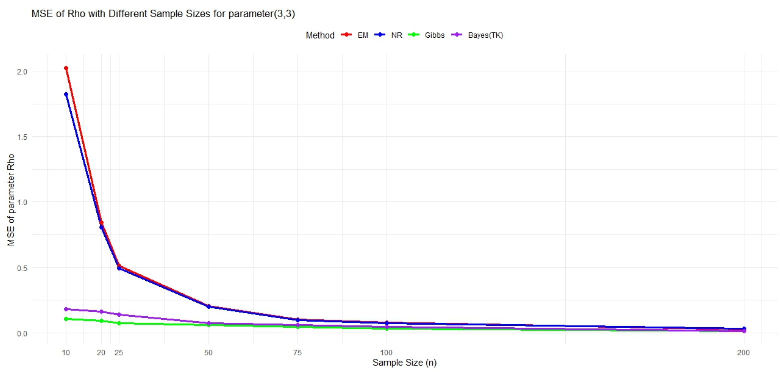

Figure 2. MSE of parameter ρ at various sample sizes.

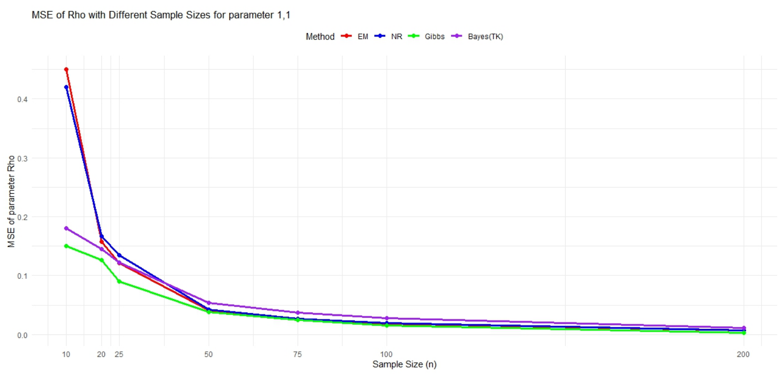

Figure 3. MSE of parameter ρ at various sample sizes.

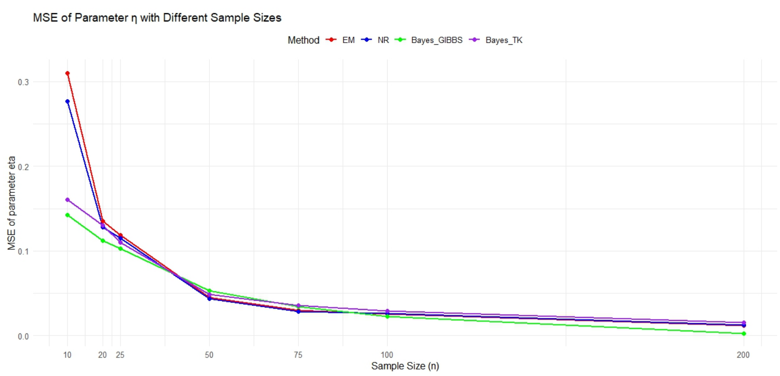

Figure 4. MSE of parameter η at various sample sizes.

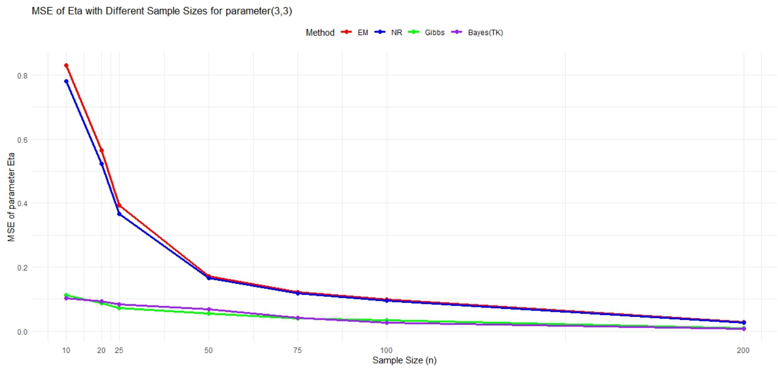

Figure 5. MSE of parameter η at various sample sizes.

- The width of the confidence intervals was influenced by both the sample size and the confidence level. As the sample size increased, the intervals became narrower, reflecting greater precision in the parameter estimates.
- Consistency can be seen in the parameter estimates produced by NR, EM, TK, and Gibbs sampling approaches when greater sample sizes converge to constant values.
- The efficacy of the strategy in striking a balance between computing practicality and estimate accuracy is demonstrated by the performance of Bayes estimators under the symmetric loss function. When compared to ML estimates and Bayes estimates using Gibbs sampling, the TK approximation Bayes estimators perform rather well.
- By analyzing density plots (Figure 6) and trace plots (Figure 7) , which display stable and well-mixed chains for the parameters, the convergence of Gibbs sampling was verified.
- The density plots (Figure 6) displays the predicted posterior distribution, which indicates convergence when the shape is unimodal, stable, and stays mostly the same throughout the iterations.
- The trace plots (Figure 7) iteratively display parameter values, suggesting convergence when it varies erratically without any discernible pattern and effectively explores the parameter space.
- In terms of bias and MSE for the parameters ρ and η, Gibbs estimates perform better than other methods for small sample sizes.

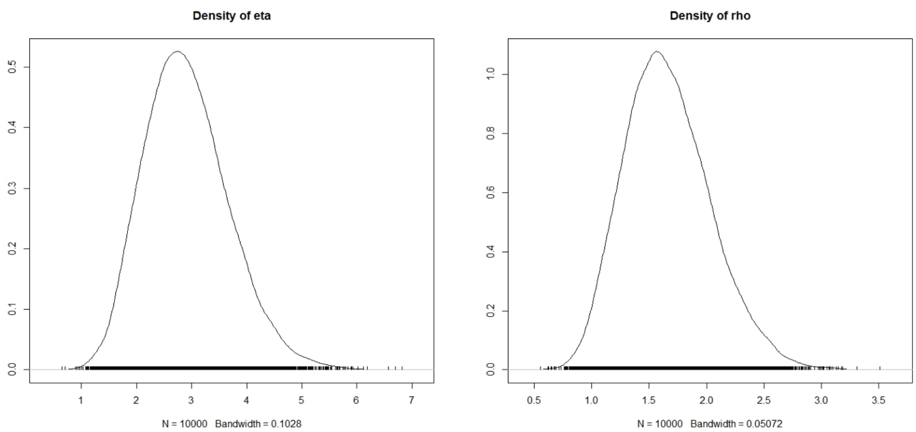

Figure 6. Illustration of the density plots of the parameters.

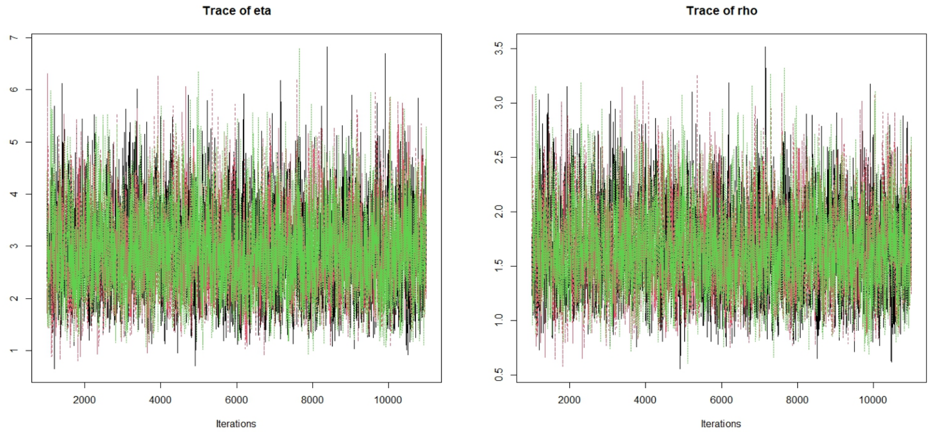

Figure 7. Illustration of the trace plots of the parameters.

- Gibbs sampling-derived highest posterior density (HPD) intervals are often more accurate than conventional asymptotic confidence intervals.

The distribution of the samples for each parameter is shown in the density plots. Based on the samples produced by the Gibbs sampler, these charts help visualize the estimated probability density function of the parameter. Density charts can provide insights into parameters, convergence, and distribution shape. The sampled values of each parameter are shown as a function of the iteration number in trace plots. You can see the sample's trajectory across the parameter space over time with the aid of these charts. Trace plots can show sample convergence, mixing, and autocorrelation.

Table 1. For various sample sizes, we compare estimates and mean squared errors of the parameters at $\rho = 1$ and $\eta = 1$ using EM, NR, Bayes (GIBBS), and Bayes (TK).

n	θ	EM		NR		Bayes (GIBBS)		Bayes (TK)	
		Estimate	MSE	Estimate	MSE	Estimate	MSE	Estimate	MSE
10	ρ	1.1205	0.4504	1.0922	0.4206	0.9803	0.1507	1.0453	0.1804
	η	0.9352	0.3103	0.9501	0.2766	0.9152	0.1423	1.0543	0.1608
20	ρ	1.0845	0.1579	1.0793	0.1667	0.9504	0.1265	1.0309	0.1453
	η	0.9752	0.135	0.9684	0.1283	0.9604	0.1123	1.0129	0.1302
25	ρ	1.0645	0.1211	1.0589	0.1352	0.9702	0.0903	1.0241	0.1223
	η	0.9799	0.1186	0.9857	0.1152	0.9704	0.1027	1.0103	0.1104
50	ρ	1.0551	0.0416	1.0429	0.042	0.9909	0.0382	1.0151	0.0539
	η	0.9896	0.0454	0.9912	0.0434	0.9855	0.0528	1.0088	0.0487
75	ρ	1.0298	0.025	1.0317	0.0268	0.9955	0.0241	1.0107	0.0366
	η	0.9921	0.0301	0.9954	0.0282	0.9902	0.0342	1.0045	0.0357
100	ρ	1.0199	0.0189	1.0215	0.0194	0.9984	0.0155	1.0073	0.0281
	η	0.9957	0.0254	0.9972	0.0264	0.9959	0.0226	1.0036	0.0292
200	ρ	1.0097	0.0067	1.0106	0.0068	0.9994	0.0024	1.0024	0.0111
	η	0.9973	0.0119	0.9981	0.0123	0.9979	0.0027	1.0019	0.0157

Table 2. For various sample sizes, we compare estimates and mean squared errors of the parameters at $\rho = 2$ and $\eta = 2$ using EM, NR, Bayes (GIBBS), and Bayes (TK).

n	θ	EM		NR		Bayes (GIBBS)		Bayes (TK)	
		Estimate	MSE	Estimate	MSE	Estimate	MSE	Estimate	MSE
10	ρ	2.3205	2.0254	2.2876	1.8258	1.9604	0.1045	2.0452	0.1804
	η	1.9252	0.8304	1.9408	0.7806	1.9522	0.1128	2.0842	0.1024
20	ρ	2.1854	0.8422	2.1502	0.8047	1.9753	0.0917	2.0643	0.1627
	η	1.9604	0.5654	1.9715	0.5223	1.9647	0.0872	2.0418	0.0927
25	ρ	2.1197	0.5126	2.1254	0.4928	1.9802	0.0724	2.0354	0.1403
	η	1.9682	0.3923	1.9768	0.3652	1.9741	0.0712	2.0237	0.0834
50	ρ	2.0788	0.2032	2.0852	0.1987	1.9924	0.0602	2.0204	0.0751
	η	1.9751	0.1721	1.9883	0.1648	1.9847	0.0547	2.0157	0.0672
75	ρ	2.0471	0.1022	2.0557	0.0967	1.9958	0.0452	2.0123	0.0581
	η	1.9907	0.1212	1.9942	0.1184	1.9891	0.0401	2.0102	0.0408
100	ρ	2.0354	0.0798	2.0412	0.0742	1.9984	0.0325	2.0083	0.0452
	η	1.9805	0.0987	1.9969	0.0943	1.9942	0.0328	2.0068	0.0254
200	ρ	2.0124	0.0246	2.0153	0.0297	1.9998	0.0103	2.0028	0.0141
	η	2.0047	0.0284	2.0032	0.0257	1.9988	0.0092	2.0023	0.0062

Table 3. For various sample sizes, we compare estimates and mean squared errors of the parameters at $\rho = 2$ and $\eta = 2$ using EM, NR, Bayes (GIBBS), and Bayes (TK).

n	θ	EM		NR		Bayes (GIBBS)		Bayes (TK)	
		Estimate	MSE	Estimate	MSE	Estimate	MSE	Estimate	MSE
10	ρ	2.9952	5.694	3.0676	5.8585	1.8524	0.1654	2.3778	0.2928
	η	1.7441	0.9879	1.7358	0.9267	1.8442	0.1398	2.1463	0.1324
20	ρ	2.3797	1.2802	2.4857	1.7924	1.8736	0.1427	2.3985	0.2814
	η	1.8827	0.6464	1.8677	0.6036	1.8563	0.1256	2.1284	0.1236
25	ρ	2.267	0.6316	2.3209	0.5831	1.885	0.1234	2.4078	0.2708
	η	1.9352	0.4171	1.8945	0.4111	1.8681	0.1152	2.1045	0.1174
50	ρ	2.1619	0.2543	2.1531	0.2504	1.8965	0.1089	2.2875	0.1025
	η	1.9347	0.2252	1.9449	0.2208	1.8794	0.0998	2.0847	0.0987
75	ρ	2.1213	0.1522	2.0967	0.1582	1.9172	0.0906	2.2105	0.0849
	η	1.9705	0.1635	1.9679	0.1615	1.8901	0.0846	2.0451	0.0479
100	ρ	2.0755	0.1169	2.0808	0.1224	1.9183	0.0754	2.1548	0.0685
	η	1.9889	0.1297	1.9827	0.1297	1.9035	0.0681	2.0247	0.0215
200	ρ	2.0246	0.0472	2.0309	0.0593	1.9294	0.0598	2.0956	0.0401
	η	2.0044	0.0593	1.9975	0.0531	1.9152	0.0467	2.0042	0.0021

Table 4. For various sample sizes, we compare estimates and mean squared errors of the parameters at $\rho = 2$ and $\eta = 3$ using EM, NR, Bayes (GIBBS), and Bayes (TK).

n	θ	EM		NR		Bayes (GIBBS)		Bayes (TK)	
		Estimate	MSE	Estimate	MSE	Estimate	MSE	Estimate	MSE
10	ρ	2.8457	4.8521	2.7126	4.1023	2.1154	0.0648	1.7309	0.1239
	η	2.7683	2.6124	2.8924	2.1752	3.0841	0.0147	2.8411	0.1967
20	ρ	2.4896	1.2392	2.3421	0.9624	2.0894	0.0524	1.7824	0.1197
	η	2.9235	1.1456	2.7921	1.0345	3.0451	0.0113	2.8624	0.1783
25	ρ	2.3952	0.9123	2.2548	0.7216	2.0351	0.0478	1.7594	1.1108
	η	2.9317	0.9845	2.8236	0.8873	3.0289	0.0097	2.8848	0.1654
50	ρ	2.2218	0.2912	2.1352	0.2518	2.0168	0.0375	1.8159	1.0893
	η	2.9846	0.4821	2.8973	0.4621	3.0102	0.0081	2.9282	0.1257
75	ρ	2.1825	0.1987	2.1012	0.1724	2.0113	0.0294	1.8839	0.9872
	η	2.9654	0.3543	2.8951	0.3256	3.0078	0.0045	2.9512	0.0865
100	ρ	2.0872	0.1083	2.0526	0.0994	2.0043	0.0147	1.9458	0.2455
	η	2.9927	0.2876	2.9674	0.2541	3.0059	0.0015	2.9813	0.0432
200	ρ	2.0368	0.0532	2.0123	0.0394	2.0027	0.0066	1.99015	0.0587
	η	3.0212	0.1492	3.0067	0.1145	3.0015	0.0012	2.9921	0.0044

Table 5. For various sample sizes, we compare estimates and mean squared errors of the parameters at $\rho = 3$ and $\eta = 3$ using EM, NR, Bayes (GIBBS), and Bayes (TK).

n	θ	EM		NR		Bayes (GIBBS)		Bayes (TK)	
		Estimate	MSE	Estimate	MSE	Estimate	MSE	Estimate	MSE
10	ρ	4.8142	12.8476	4.5231	10.3842	2.9652	0.1123	3.1457	0.2982
	η	2.4236	2.7189	2.6521	2.3451	2.9375	0.0298	2.8013	0.2035
20	ρ	3.7784	2.5438	3.5317	2.1825	2.9821	0.0954	3.2149	0.2657
	η	2.8345	1.0984	2.9128	1.0564	2.9743	0.0281	2.8432	0.1721
25	ρ	3.6547	1.8124	3.3986	1.5873	2.9152	0.0879	3.1316	0.1954
	η	2.8473	0.9742	2.9147	0.9041	2.9854	0.0243	2.8921	0.1564
50	ρ	3.3124	0.6275	3.1985	0.5821	2.9367	0.0743	3.1543	0.1589
	η	2.9217	0.4521	2.9482	0.4224	2.9897	0.0189	2.9452	0.1276
75	ρ	3.1782	0.3318	3.1125	0.2982	2.9548	0.0598	3.0912	0.1457
	η	2.9683	0.2845	2.9423	0.2981	2.9951	0.0157	2.9678	0.0982
100	ρ	3.1347	0.2118	3.1024	0.1983	2.9728	0.0475	3.0541	0.1148
	η	2.9821	0.2254	2.9634	0.2348	2.9984	0.0134	2.9872	0.0457
200	ρ	3.0764	0.1213	3.0432	0.1078	2.9912	0.0243	3.0251	0.0723
	η	2.9956	0.1012	2.9843	0.0943	2.9998	0.0081	2.9952	0.0089

Table 6. HPD by Gibbs and asymptotic confidence intervals of the parameters at $\rho = 1$ and $\eta = 1$.

n	θ	95% Confidence Intervals	HPD by Gibbs
10	ρ	[0.3784, 3.6273]	[0.5291, 3.3653]
	η	[0.1494, 2.4216]	[0.2126, 1.9875]
20	ρ	[0.5156, 2.1169]	[0.6199, 2.1847]
	η	[0.2531, 1.9123]	[0.3889, 1.7245]
25	ρ	[0.5954, 2.0548]	[0.6451, 1.9852]
	η	[0.3865, 1.8284]	[0.3914, 1.6871]
50	ρ	[0.6989, 1.6429]	[0.7089, 1.5045]
	η	[0.5612, 1.6324]	[0.5145, 1.5028]
75	ρ	[0.7107, 1.4855]	[0.7393, 1.4184]
	η	[0.5836, 1.4809]	[0.6099, 1.4108]
100	ρ	[0.7654, 1.3787]	[0.7830, 1.3447]
	η	[0.6269, 1.3745]	[0.6598, 1.3378]
200	ρ	[0.8140, 1.2793]	[0.8375, 1.2213]
	η	[0.7439, 1.2847]	[0.7482, 1.2450]

Table 7. HPD by Gibbs and asymptotic confidence intervals of the parameters at $\rho = 2$ and $\eta = 3$.

n	θ	95% Confidence Intervals	HPD by Gibbs
10	ρ	[0.7356, 8.7145]	[1.0432, 6.2321]
	η	[0.6216, 6.1453]	[0.6817, 5.4648]
20	ρ	[1.0084, 4.7084]	[1.1559, 4.1879]
	η	[1.0533, 4.7615]	[1.2123, 4.8684]
25	ρ	[1.0742, 4.4532]	[1.2914, 3.9314]
	η	[0.9401, 5.1564]	[1.4147, 4.7208]
50	ρ	[1.3604, 3.4585]	[1.3722, 3.0745]
	η	[1.6812, 4.4782]	[1.6059, 4.2970]
75	ρ	[1.4516, 3.1594]	[1.4628, 2.8832]
	η	[1.8302, 4.2518]	[1.8699, 4.1254]
100	ρ	[1.4763, 2.7345]	[1.5637, 2.6944]
	η	[2.0017, 4.1256]	[2.1124, 3.9542]
200	ρ	[1.6504, 2.4987]	[1.6578, 2.4827]
	η	[2.3393, 3.7546]	[2.3415, 3.6919]

Table 8. HPD by Gibbs and asymptotic confidence intervals of the parameters at $\rho = 3$ and $\eta = 1$.

n	θ	95% Confidence Intervals	HPD by Gibbs
10	ρ	[1.6139, 11.4124]	[1.7524, 8.6352]
	η	[0.1527, 2.1745]	[0.2145, 1.7425]
20	ρ	[1.3722, 7.1245]	[1.7548, 6.7183]
	η	[0.2983, 1.8254]	[0.3742, 1.7564]
25	ρ	[1.6623, 6.5423]	[1.7854, 5.6245]
	η	[0.4122, 1.7520]	[0.4152, 1.5487]
50	ρ	[2.0759, 4.9475]	[2.0784, 4.9421]
	η	[0.5716, 1.4884]	[0.5976, 1.4824]
75	ρ	[2.1537, 4.5625]	[2.2145, 4.3682]
	η	[0.6298, 1.3987]	[0.6354, 1.3756]
100	ρ	[2.3261, 4.3416]	[2.3134, 4.1325]
	η	[0.6897, 1.3647]	[0.6734, 1.3149]
200	ρ	[2.4579, 3.7854]	[2.4564, 3.6578]
	η	[0.7545, 1.2568]	[0.7827, 1.2147]

Estimating the lifetime in hours for a rechargeable battery output from the manufacturing process is scant. In fact it must be noted that a particular battery has a lifetime accrued for it even if it has been manufactured, which is subject to rough measurement and, hence, will allow only fuzzy triangular numbers as the representation in the records. All such lifetime estimations for different batches and instances of rechargeable batteries as produced in the plant look to be subjected to gamma distribution. Let us put aside all the assumptions in measuring the lifetime of the batteries. A sample of size 30 is taken from fictitious production batches. The average lifetime of the batteries is some fuzzy triangular number.

$$
\begin{aligned}
&(120, 150, 180), \quad (80, 110, 140), \quad (50, 90, 120), \quad (200, 250, 300), \quad (150, 180, 210),\\
&(75, 100, 130), \quad (60, 90, 120), \quad (110, 150, 200), \quad (95, 130, 160), \quad (100, 130, 160),\\
&(120, 160, 200), \quad (90, 120, 150), \quad (85, 120, 155), \quad (130, 170, 210), \quad (100, 140, 180),\\
&(175, 210, 245), \quad (150, 190, 230), \quad (125, 160, 195), \quad (70, 100, 140), \quad (90, 120, 160),\\
&(110, 150, 190), \quad (180, 230, 280), \quad (75, 110, 150), \quad (160, 200, 240), \quad (50, 90, 130),\\
&(210, 260, 310), \quad (95, 130, 165), \quad (80, 110, 150), \quad (200, 240, 280), \quad (50, 80, 120).
\end{aligned}
$$

A battery's lifespan is represented by each triplet, which is a fuzzy triangular number. We estimate the shape ρ and scale η of the gamma distribution since battery lifetimes follow a gamma distribution. The estimation of ρ and η, however, becomes a fuzzy estimation procedure because each observation is a fuzzy number. The reliability function becomes a fuzzy reliability function since ρ and η are estimated from fuzzy lifetime data. To do this, we establish fuzzy bounds for $R(t)$ using the triangular fuzzy estimates of ρ and η.

The reliability function $R(t)$, also known as the survival function, is the complement of the cumulative distribution function (CDF). For a gamma distribution with the shape parameter ρ and scale parameter η, the reliability function is expressed as: $R(t) = 1 - F(t)$, where $F(t)$ is the cumulative distribution function of the gamma distribution. Specifically:

$$\tilde{R}(t) = 1 - \int_0^t \frac{1}{\eta^\rho \Gamma(\rho)} x^{\rho-1} e^{-\frac{x}{\eta}}\, dx.$$

Here, $\Gamma(\rho)$ is the gamma function and $t > 0$.

After the creation of the randomized fuzzy values $\tilde{t}_i$ of the CDF function according to the size of the given samples and the default values of initial parameters according to the formula ($\tilde{R}(t_i)$), the values of t_i and the initial parameters were computed according to the functions of $\mu_{\tilde{t}_i}(t)$ for each fuzzy unit $\tilde{t}_i$. Then, we extract for each $\tilde{R}(t_i)$ and find the expectation of $\tilde{R}(t_i)$ as follows:

$$\tilde{R}(t) = \mathbb{E}(\tilde{R}(t_i)/\tilde{x}_i) = \frac{1}{K}\sum_{h=1}^{K} R^{(h)}(t) \tag{4.1}$$

The MSE is obtained by the following formula:

$$\text{MSE}(\tilde{R}) = \frac{1}{K}\sum_{i=1}^{K}(\tilde{R} - R)^2. \tag{4.2}$$

Table 9. Comparing biases and MSE of several estimation methods at $\rho = 3$ and $\eta = 2$ for the dataset.

		EM		NR		Bayes (Gibbs)		Bayes (TK)	
n	θ	Estimate	MSE	Estimate	MSE	Estimate	MSE	Estimate	MSE
30	ρ	3.2859	1.6759	2.8874	0.8397	2.9705	0.1263	2.9883	0.1045
	η	1.8768	0.3258	1.7091	0.3879	2.0834	0.1684	2.0653	0.1387
	$\tilde{R}$	0.6593	0.0018	0.6473	0.0021	0.6747	0.0012	0.6984	0.0001

In comparison to the EM and NR approaches, the Bayes (Gibbs) and Bayes (TK) methods have the lowest mean squared error (MSE) values, suggesting more accurate estimations for the considered data. The fuzzy reliability of the dataset is reported as 0.6984 with TK approximation. Overall, Bayes methods, especially TK, perform well for both the parameters.

5. Conclusions

The literature uses a variety of estimating techniques to determine the parameters of the gamma distribution, including complete and censored data. However, it is noted that previous research was limited to crisp and precise data. In spite of that, certain data may be imprecise or unclear in real-world scenarios and represented as fuzzy information. Unfortunately, there was no discussion of how to handle such types of imprecise data when it follows the gamma distribution. In this study, we investigated several approaches for parameter estimation of the gamma distribution when the relevant data are given as fuzzy information. The ML method makes use of the EM and NR algorithms, whereas the Bayesian strategy makes use of TK's approximation and Gibbs sampling. The effectiveness of these estimating techniques was then investigated using a simulated study. The outcomes of the simulation research clearly show that, although the EM approach has a slower processing speed, the ML estimates derived from the NR and EM methods behave very similarly. From the simulation findings, we can state that the Bayesian procedure with informative priors using the TK method gives smaller biases and MSEs when compared to other estimation methods discussed in the study.

Author contributions

Abbarapu Ashok: Conceptualization, Supervision, Funding acquisition, Formal analysis, Project administration; Nadiminti Nagamani: Writing-review & editing, Visualization, Supervision. Both authors evaluated and approved the final version.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interests in this paper.

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Appendix

The detailed explanation of the notations used in Section 2 are given below.

The observed log-likelihood function is 2.1 is

$$L^*(\theta) = L^*(\rho, \eta; \tilde{x}) = -n\rho \log \eta - n \log \Gamma\rho + \sum_{i=1}^{n} \log \int_0^\infty x^{\rho-1} e^{-\frac{x}{\eta}} \mu_{\tilde{x}_i}(x) dx$$

and the first- and second-order derivatives with respect to ρ are as follows:

$$\frac{\partial}{\partial\rho}L^*(\rho,\eta;\tilde{x}) = -n\log\eta - n\psi(\rho) + \sum_{i=1}^{n}\frac{\int_0^\infty x^{\rho-1}\log x\, e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx}{\int_0^\infty x^{\rho-1}\, e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx},$$

$$\begin{aligned}\frac{\partial^2}{\partial\rho^2}L^*(\rho,\eta;\tilde{x}) &= -n\psi'(\rho) + \sum_{i=1}^{n}\frac{\int_0^\infty x^{\rho-1}(\log x)^2\, e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx}{\int_0^\infty x^{\rho-1}\, e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx} - \sum_{i=1}^{n}\left(\frac{\int_0^\infty x^{\rho-1}\log x\, e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx}{\int_0^\infty x^{\rho-1}\, e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx}\right)^2\\ &= -n\psi'(\rho) + I_{\rho\rho}.\end{aligned}$$

The first- and second-order derivatives with respect to η are as follows:

$$\frac{\partial}{\partial\eta}L^*(\rho,\eta;\tilde{x}) = -\frac{n\rho}{\eta} - \sum_{i=1}^{n}\frac{\int_0^\infty x^{\rho-1}e^{-\frac{x}{\eta}}\left(\frac{x}{\eta^2}\right)\mu_{\tilde{x}_i}(x)\,dx}{\int_0^\infty x^{\rho-1}e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)\,dx}$$

$$\begin{aligned}\frac{\partial^2}{\partial\eta^2}L^*(\rho,\eta;\tilde{x}) &= \frac{n\rho}{\eta^2} - \sum_{i=1}^{n}\frac{\int_0^\infty\left(\frac{x-2\eta}{\eta^4}\right)x^{\rho}e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)\,dx}{\int_0^\infty x^{\rho-1}e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)\,dx} - \sum_{i=1}^{n}\left[\frac{\int_0^\infty x^{\rho}\eta^{-2}e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)\,dx}{\int_0^\infty x^{\rho-1}e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)\,dx}\right]^2\\ &= \frac{n\rho}{\eta^2} - I_{\eta\eta}\end{aligned}$$

and the second-order mixed derivative with respect to η and ρ is:

$$\begin{aligned}\frac{\partial^2}{\partial\eta\partial\rho}L^*(\rho,\eta;\tilde{x}) &= -n\frac{1}{\eta} + \frac{\int_0^\infty x^{\rho}\eta^{-2}\log x\, e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx}{\int_0^\infty x^{\rho-1}e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx} - \frac{\int_0^\infty x^{\rho}e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx\int_0^\infty x^{\rho-1}\log x\, e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx}{\left(\int_0^\infty x^{\rho-1}e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx\right)^2}\\ &= -n\frac{1}{\eta} + I_{\rho\eta}.\end{aligned}$$

Here the terms $I_{\rho\rho}$, $I_{\eta\eta}$, and $I_{\rho\eta}$ are denoted as follows:

$$I_{\rho\rho} = \sum_{i=1}^{n}\frac{\int_0^\infty x^{\rho-1}(\log x)^2 e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx}{\int_0^\infty x^{\rho-1}e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx} - \sum_{i=1}^{n}\left[\frac{\int_0^\infty x^{\rho-1}\log x\, e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx}{\int_0^\infty x^{\rho-1}e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx}\right]^2$$

$$I_{\eta\eta} = \sum_{i=1}^{n}\frac{\int_0^\infty\left(\frac{x-2\eta}{\eta^4}\right)x^{\rho}e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx}{\int_0^\infty x^{\rho-1}e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx} - \sum_{i=1}^{n}\left[\frac{\int_0^\infty x^{\rho}\eta^{-2}e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx}{\int_0^\infty x^{\rho-1}e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx}\right]^2$$

$$I_{\rho\eta} = \frac{\int_0^\infty x^{\rho}\eta^{-2}\log x\, e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx}{\int_0^\infty x^{\rho-1}e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx} - \frac{\int_0^\infty x^{\rho}e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx\int_0^\infty x^{\rho-1}\log x\, e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx}{\left(\int_0^\infty x^{\rho-1}e^{-\frac{x}{\eta}}\mu_{\tilde{x}_i}(x)dx\right)^2}.$$