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Research article

The solitary wave phenomena of the fractional Calogero-Bogoyavlenskii-Schiff equation

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Abstract: The Riemann waves in two spatial dimensions are described by the fractional Calogero-Bogoyavlenskii-Schiff equation, which has been used to explain numerous physical phenomena including magneto-sound waves in plasmas, tsunamis, and flows in rivers and internal oceans. This work concerned itself with obtaining new analytic soliton solutions for the fractional Calogero-Bogoyavlenskii-Schiff model based on the fractional conformable. By solving the model equation with the Riccati-Bernoulli sub-ODE technique in association with the Bäcklund transformation, the solution was found in terms of trigonometric, hyperbolic, and rational functions. To analyze the detailed features of the wave structures as well as the pattern of dynamics of these solutions, 3D and contour diagrams were plotted by using Wolfram Mathematica. A great advantage of these types of visualizations is that they demonstrate amplitude, shape, and propagation characteristics of the selected soliton solutions. The results reveal that the proposed approach is accurate, universal, and fast for the investigation of the different aspects of the Riemann problem and the related phenomena concerning the propagation of waves.

Keywords: fractional partial differential equations; fractional Calogero-Bogoyavlenskii-Schiff equation; Bäcklund transformation; Riccatti equation; solitary wave solutions
Mathematics Subject Classification: 34G20, 35A20, 35A22, 35R11

1. Introduction

Nonlinear partial differential equations (NPDEs) have been of great importance in the development of numerous branches of applied science during the last few decades [1, 2]. Since these equations are crucial for capturing subtle nonlinear effects in complex systems, they have led to the creation of various nonlinear models. Numerous methods have been suggested for obtaining the numerical, analytical, and semi-analytical solutions which include the spline scheme [3] and the finite difference method [4], as well as others such as Adomian decomposition, variational iteration methods [5], and expansion methods [6] including methods based on sine cosine expansions [7] and the modified simple equation method [8]. Over the past few years, nonlinear dynamic systems and solitons have captured much attention because of their importance in optics, fluid dynamics, and material science [9, 10]. Generalized systems, algebraic constraints, and fractional models with nonlinear differential equations are used for the modeling of physical events like wave propagation, fluid dynamics, and electromagnetic vibrations. The study of fractional differential equations has garnered significant attention due to their ability to model complex real-world phenomena more accurately. Recent works, such as the fractional series solutions for nonlinear reaction-diffusion models [12], soliton solutions for perturbed equations [13], and fractional analysis of coupled systems [14], underscore the advancements in this field. This study builds upon these developments by applying innovative techniques to analyze the fractional-order damped Burgers' equation, expanding its applicability to fluid dynamics and nonlinear wave studies [15, 16].

Nonlinear fractional differential equations (NLFDEs) were first proposed in 1695 and have developed into a very active and rapidly developing branch of current science [17]. These equations have been found highly useful in discovering new phenomena of highly fluxional activities in various domains such as nuclear physics, geo-optical filament optical physics, solid state physics, fluid mechanics, mechanical engineering, plasma physics, mathematical physics, and astrophysics. This flexibility is the result of the fractional calculus being able to capture a number of processes that display memory mechanisms and anomalous diffusion, that is, characteristics that can be found in many practical situations. For instance, fractional calculus has been used in fluid mechanics describing non- Newtonian fluids, quantum mechanics for path integration, and in material science describing viscoelasticity. Such applications demonstrate its potential to expand the normative repertoire and thereby incrementally modify the approaches to the study of natural, physical, and engineering phenomena in the context of moving between the modern and postmodern models. To meet the different needs of fractional calculus, a number of definitions of fractional derivatives have been given. The most popular are the beta-derivative [18], the conformable derivative [19], the Laplace definition [20], and the Caputo-Fabrizio derivative [21]. These definitions give rise to the mathematical background needed to analyze the fractional-order processes efficiently. Fractional partial differential equations (FPDEs) are needed to describe many natural phenomena in biology, physics, chemistry, and economics. Such equations are useful for the analysis of waves and material characteristics in such domains, requiring the application of NLFPDEs. Such models are the extended Zakharov-Kuznetsov equation [22], the fifth-order Lax equation [23], the Fokas equation [24], the Clannish Random Walker parabolic equation [25], the Oskolkov equation [26], and Schrödinger dynamical systems [27], as well as the dispersive long-wave equations [28]. To obtain analytical soliton solutions for FPDEs, more complicated techniques, including Lie symmetry analysis [29], the Jacobi elliptic function method [30], the Khater method [31], and the $(\psi\text{–}\varphi)$-expansion method [32], have been used. These techniques provide exact soliton solutions and provide significant information regarding the dynamics of the system involving fractional integers. Fractional differential equations have emerged as powerful tools to model complex phenomena across various scientific disciplines [33, 34]. Recent advances have highlighted their utility in fields such as electrical engineering, fluid dynamics, and plasma physics, offering innovative methods to obtain soliton and solitary wave solutions [35]. The studies here employ novel analytical and approximate approaches to address fractional-order models, demonstrating their relevance and applicability in diverse real-world scenarios [36, 37].

To gain insights into various phenomena in different fields including fluids, waves, and engineering sciences, it is therefore important to seek forms of solutions especially in terms of solitons. For such studies, the theories of nonlinear fractional partial differential equations (NFPDEs) must be explored with soliton solutions having a central role. These constant phase solutions, which retain their form and speed, bring order and structure to chaos, are important for theoretical and application purposes [38–40]. By looking at the function of solitons, one can pay attention to the fact that solitons, are important information carriers in various situations. To find out new soliton solutions, different types of modern approaches have been constructed. The Sine-Gordon approach [41] and the exp-function method [42] have been well-established, as well as the Sardar sub-equation method [43], the extended direct algebraic method (EDAM) [44], and the Hirota bilinear approach [45]. These techniques, which are still being improved, extend and enrich the methodological arsenal of approaches to constructing solutions of NPDEs and new solutions.

The Riccati-Bernoulli sub-ODE method in conjunction with Bäcklund transformations appears to be a versatile and efficient means for obtaining rich new solutions of the NLFDEs [46–48]. This method enables straightforward investigation of analytical solutions of some NLFDEs such as those involving fractional derivatives. However, as far as the authors are aware, the application of the Riccati-Bernoulli sub-ODE technique has been performed previously but the $(2 + 1)$-dimensional time-fractional Calogero-Bogoyavlenskii-Schiff (CBS) equation has not been investigated through the current strategy. Furthermore, the Calogero-Bogoyavlenskii-Schiff (CBS) equation as a significant nonlinear partial differential equation has been studied due to its applications in the mathematical physics and other applied sciences. The CBS equation has been investigated in relation to its soliton solutions and integrability, and its relevancy was discovered in areas of shallow water waves, nonlinear optics, and plasma. In particular, the extension of the CBS equation with the fractional calculus offers the opportunity for researchers to study the function that cannot be described in terms of integer-order mathematics. This capability improves the analysis of various effects, for instance, wave interactions with heterogeneous media and inefficiency trends in various engineering and physical systems. The present work shall continue from these enhanced developments by developing and discussing new fractional solutions of the CBS equation to underscore the versatility and efficiency of fractional calculus in solving practical problems. It is therefore the intention of this article to present a more extensive range of generic and general closed-form wave solutions for the said model. By applying the Bäcklund transformation procedure [49], we obtain a number of other solutions in terms of trigonometric, hyperbolic, and rational functions. Moreover, the physical realization of these solutions is illustrated in terms of three-dimensional (3D) plots as well as contour plots where the amplitude, the propagation characteristics, and the structural behavior of the response can be further understood. The results of this research help to investigate the physical aspects associated with the processes described by the nonlinear fractional equations, with special reference to the conformable derivative. These results may have implications for practical use in some related areas to wave mechanics, fluid dynamics, and other fields of science and engineering.

Among all definitions, the Riemann-Liouville, Caputo, Grunwald-Letnikov, Weyl, and Riesz methods are defined frequently since they provide useful equipment for analyzing and modeling systems. However, these classical fractional derivatives are much different from the Newton-Leibnitz calculus where the normal definitions are not respected, notably, the quotient rule or the chain rule during differentiation. Such differences inhibit their usability in specific scenarios and cause problems while comparing them with traditional derivatives. In order to solve these problems, Khalil et al. [19]. presented a new idea of a conformable fractional derivative (CFD) in 2014. The use of the CFD eliminates the majority of the drawbacks of previous definitions by matching the derivative more closely with Newton while, at the same time, affording sufficient leeway to achieve good modeling of fractional-order systems. This is due to its better performance and a way of providing a more accurate description of fractional models, which has made it the common method in most of the current nonlinear dynamics and fractional calculus studies. The conformable fractional derivative is defined for a function $f(x)$ as:

$$T^{\alpha}(f)(x) = \lim_{\epsilon\to 0} \frac{f(x+\epsilon x^{1-\alpha}) - f(x)}{\epsilon}, \quad 0 < \alpha \leq 1. \tag{1.1}$$

$$\begin{cases} T^{\alpha}(af+bg) = aT^{\alpha}(f) + bT^{\alpha}(g). \\ T^{\alpha}(fg)(x) = f(x)T^{\alpha}(g)(x) + g(x)T^{\alpha}(f)(x). \\ T^{\alpha}\left(\dfrac{f}{g}\right)(x) = \dfrac{g(x)T^{\alpha}(f)(x) - f(x)T^{\alpha}(g)(x)}{g(x)^2}. \\ T^{\alpha}(f(g(x))) = f'(g(x))T^{\alpha}(g)(x). \end{cases} \tag{1.2}$$

Such properties enable us to come up with straightforward physical explanations and improve the effectiveness of projecting existing circumstances in various disciplines.

2. Methodology

In this section, the details of the overall approach followed by the Riccati-Bernoulli sub-ODE method are described. Consider the general fractional partial differential equation (FPDE) expressed as:

$$F\left(f, D_t^{\alpha}f, D_x^{\alpha}f, D_t^{2\alpha}f, D_x^{2\alpha}f, \ldots\right) = 0. \tag{2.1}$$

For convenience in the integration process, Eq (2.1) is transformed into the form of a nonlinear ordinary differential equation, with the help of the transformation, $F(\zeta) = f(t, x_1, x_2, x_3, \ldots, x_m)$,

$$F\left(f, \frac{df}{d\zeta}, \frac{d^2f}{d\zeta^2}, \frac{d^3f}{d\zeta^3}, \ldots\right) = 0. \tag{2.2}$$

The variable $\zeta = \zeta(t, x_1, x_2, x_3, \ldots, x_m)$ can be parameterized into different forms depending on the specifications of the problem. We assume the formal solution form for Eq (2.2):

$$U(x,y,t) = u(\psi) = \sum_{i=-m}^{m} b_i T(\psi)^i, \tag{2.3}$$

where constants (b_i) are decided with $b_m \neq 0$ or $b_{-m} \neq 0$, and $T(\psi)$ is derived from a Bäcklund transformation:

$$T(\psi) = \frac{-\varsigma P_2 + P_1\phi(\psi)}{P_1 + P_2\phi(\psi)}, \tag{2.4}$$

where ς, P_1, and P_2 are constants with ($P_2 \neq 0$), and where $\phi(\psi)$ satisfies the Riccati equation:

$$\frac{d\phi}{d\psi} = \varsigma + \phi(\psi)^2. \tag{2.5}$$

The balance parameter (m) in Eq (2.3) is the balance number and is calculated when the highest-order derivative is balanced with the dominant nonlinear term as indicated in Eq (2.2). Substitution of Eq (2.3) into Eq (2.2) or into its integrated form yields a set of terms that involves $\phi(\psi)$ of different order. By comparing the coefficients of the $\phi(\psi)$ terms, a system of algebraic equations in (b_i) and other parameters is obtained using the comparison coefficients method. We use Maple software solve this system of algebraic equations which gives the actual value of unknown coefficients and parameters. By replacing (b_i) and the other parameters in Eq (2.3) and putting the general solution $\phi(\psi)$ of Eq (2.3) into the original FPDE (2.1), soliton solutions are derived. The emerging families of soliton solutions are listed below [50]:

$$\begin{aligned}
&\phi(\psi) = \begin{cases} -\sqrt{-\varsigma}\tanh(\sqrt{-\varsigma}\psi), & \text{as } \varsigma < 0, \\ -\sqrt{-\varsigma}\coth(\sqrt{-\varsigma}\psi), & \text{as } \varsigma < 0, \end{cases}\\
&\phi(\psi) = -\frac{1}{\psi}, \quad \text{as } \varsigma = 0,\\
&\phi(\psi) = \begin{cases} \sqrt{\varsigma}\tan(\sqrt{\varsigma}\psi), & \text{as } \varsigma > 0, \\ -\sqrt{\varsigma}\cot(\sqrt{\varsigma}\psi), & \text{as } \varsigma > 0. \end{cases}
\end{aligned} \tag{2.6}$$

The technique of the sub-ODE Riccati-Bernoulli equation integrated with a Bäcklund transformation provides a powerful framework for the systematic generation of exact solutions of FPDEs. In its essence, the method reduces the examined FPDE to a nonlinear ODE, an approach that takes advantage of the fractional calculus being capable of capturing procedures manifesting memory and heredity characteristics. The application of the Riccati equation in this method is significant; this equation can be used to describe nonlinear wave structures by simple functions from mathematical analysis, and by hyperbolic and trigonometric functions that naturally enter into physical processes. The Bäcklund transformation on the other hand can map between solutions and finds new solutions from old solutions. This change allows the addition of a parameter into the existing solution space, making it parameterized and more capable of further reacting to the underlying dynamics of the problem. Combined, these approaches offer a versatile arsenal for analyzing the rich interplay of FPDEs to lend themselves well to a diverse range of disciplines including non-linear optics, fluid mechanics, and plasmas.

3. Execution of the problem

The fractional CBS model deals with Riemann waves in two dimensions and can help to explain physical phenomena such as magneto-sound waves in a plasma, tsunamis, river tides, and internal ocean waves [51]. First implemented by Bogoyavlenskii as an extension of the Lax formalism, the CBS model was also derived by Schiff in terms of the reduction of the self-dual Yang-Mills equation [52]. In terms of the conformable fractional derivative, we have the space-time fractional form of the (2 + 1)-dimensional fractional CBS equation given below [53]:

$$(D_x^\alpha F)(D_t^\alpha F) + 4(D_x^\alpha F)\left(D_x^\alpha D_y^\alpha F\right) + 2\left(D_x^{2\alpha} F\right)\left(D_y^\alpha F\right) + \left(D_x^{3\alpha} D_y^\alpha F\right) = 0. \tag{3.1}$$

We begin by presenting a variable transformation in the way shown below:

$$F(x,y,t)=F(\zeta),\\ \zeta=\frac{\mu x^{\alpha}}{\alpha}+\frac{\lambda y^{\alpha}}{\alpha}+\frac{\omega t^{\alpha}}{\alpha}. \tag{3.2}$$

This transformation changes Eq (3.3) into a nonlinear ordinary differential equation (NODE). Integrating the result with respect to (ζ), and assuming the constant of integration is zero, gives the following result with respect to (ζ).

$$(\omega\mu)\frac{df}{d\zeta}+\mu^3\lambda\frac{d^3f}{d\zeta^3}+3\mu^2\lambda\left(\frac{df}{d\zeta}\right)^2=0. \tag{3.3}$$

To find the value of the balance number m we use the concept of balanced homogeneity of Eq (3.3) with (f''') and $(f')^2$. Solving the equation $m+3=2m+2$ entails getting an ideology value of ($m=1$). Substituting ($m=1$) into Eq (2.3) leads to the following closed-form series solution for Eq (3.1):

$$f(\zeta)=\sum_{m=-1}^{1}b_i(T(\psi))^i=b_{-1}(T(\psi))^{-1}+b_0+b_1(T(\psi))^1. \tag{3.4}$$

By substituting Eq (3.2) into Eq (3.3) and grouping together the coefficients of the various powers of $\phi(\psi)$, the result is an expression in $\phi(\psi)$. By equalizing the coefficients of the obtained terms to zero, we obtain a system of nonlinear algebraic equations. Utilizing Maple, we solve this system and identify four distinct cases of solutions as follows:

Case 1.

$$b_1=0,\ \ b_{-1}=b_{-1},\ \ \mu=1/2\,\frac{b_{-1}}{\varsigma},\ \ \lambda=\lambda,\omega=\frac{{b_{-1}}^2\lambda}{\varsigma},\ \ \varsigma=\varsigma. \tag{3.5}$$

Case 2.

$$b_1=b_1,\ \ b_{-1}=0,\ \ \mu=-1/2\,b_1,\ \ \lambda=\lambda,\ \ \omega=\varsigma\,{b_1}^2\lambda,\ \ \varsigma=\varsigma. \tag{3.6}$$

Case 3.

$$b_1=\frac{ib_{-1}{P_2}^2}{{P_1}^2},\ \ b_{-1}=b_{-1},\ \ \mu=\frac{-1/2\,ib_{-1}{P_2}^2}{{P_1}^2},\ \ \lambda=\lambda,\ \ \omega=\frac{-4\,i{P_2}^2{b_{-1}}^2\lambda}{{P_1}^2},\ \ \varsigma=\frac{i{P_1}^2}{{P_2}^2}. \tag{3.7}$$

Case 4.

$$b_1=-\frac{b_{-1}}{\varsigma},\ \ b_{-1}=b_{-1},\ \ \mu=1/2\,\frac{b_{-1}}{\varsigma},\ \ \lambda=\lambda,\ \ \omega=4\,\frac{{b_{-1}}^2\lambda}{\varsigma},\ \ \varsigma=\varsigma. \tag{3.8}$$

In this regard, the four cases obtained by using the algebraic system offer an approach to distinguish different types of families of solutions with reference to (ς). Systems of such families behave differently in other situations as can be seen from the above cases. These solution behaviors reveal the temporal nature of the system and the associated physical processes for various fractional-order controls and initial conditions. In addition, these solution families include all the interactions that are incorporated in the system, presenting the role of (ς) on the behavior and steadiness of solitons. This analysis has profound potential in enhancing the knowledge on flow physics and wave structure-borne nonlinear processes.

Family 1. The following solitary wave solutions are obtained for Case 1 and $\varsigma < 0$.

$$F_1(x,y,t) = \frac{b_{-1}\left(P_1 - P_2\sqrt{-\varsigma}\tanh\left(\sqrt{-\varsigma}\left(1/2\,\frac{b_{-1}x^{\alpha}}{\varsigma\alpha} + \frac{\lambda y^{\alpha}}{\alpha} + \frac{b_{-1}{}^{2}\lambda t^{\alpha}}{\varsigma\alpha}\right)\right)\right)}{\left(-\varsigma P_2 - P_1\sqrt{-\varsigma}\tanh\left(\sqrt{-\varsigma}\left(1/2\,\frac{b_{-1}x^{\alpha}}{\varsigma\alpha} + \frac{\lambda y^{\alpha}}{\alpha} + \frac{b_{-1}{}^{2}\lambda t^{\alpha}}{\varsigma\alpha}\right)\right)\right)} + b_0 \tag{3.9}$$

or

$$F_2(x,y,t) = \frac{b_{-1}\left(P_1 - P_2\sqrt{-\varsigma}\coth\left(\sqrt{-\varsigma}\left(1/2\,\frac{b_{-1}x^{\alpha}}{\varsigma\alpha} + \frac{\lambda y^{\alpha}}{\alpha} + \frac{b_{-1}{}^{2}\lambda t^{\alpha}}{\varsigma\alpha}\right)\right)\right)}{\left(-\varsigma P_2 - P_1\sqrt{-\varsigma}\coth\left(\sqrt{-\varsigma}\left(1/2\,\frac{b_{-1}x^{\alpha}}{\varsigma\alpha} + \frac{\lambda y^{\alpha}}{\alpha} + \frac{b_{-1}{}^{2}\lambda t^{\alpha}}{\varsigma\alpha}\right)\right)\right)} + b_0. \tag{3.10}$$

Family 2. The following solitary wave solutions are obtained for Case 1 and $\varsigma > 0$.

$$F_3(x,y,t) = \frac{b_{-1}\left(P_1 + P_2\sqrt{\varsigma}\tan\left(\sqrt{\varsigma}\left(1/2\,\frac{b_{-1}x^{\alpha}}{\varsigma\alpha} + \frac{\lambda y^{\alpha}}{\alpha} + \frac{b_{-1}{}^{2}\lambda t^{\alpha}}{\varsigma\alpha}\right)\right)\right)}{\left(-\varsigma P_2 + P_1\sqrt{\varsigma}\tan\left(\sqrt{\varsigma}\left(1/2\,\frac{b_{-1}x^{\alpha}}{\varsigma\alpha} + \frac{\lambda y^{\alpha}}{\alpha} + \frac{b_{-1}{}^{2}\lambda t^{\alpha}}{\varsigma\alpha}\right)\right)\right)} + b_0 \tag{3.11}$$

or

$$F_4(x,y,t) = \frac{b_{-1}\left(P_1 - P_2\sqrt{\varsigma}\cot\left(\sqrt{\varsigma}\left(1/2\,\frac{b_{-1}x^{\alpha}}{\varsigma\alpha} + \frac{\lambda y^{\alpha}}{\alpha} + \frac{b_{-1}{}^{2}\lambda t^{\alpha}}{\varsigma\alpha}\right)\right)\right)}{\left(-\varsigma P_2 - P_1\sqrt{\varsigma}\cot\left(\sqrt{\varsigma}\left(1/2\,\frac{b_{-1}x^{\alpha}}{\varsigma\alpha} + \frac{\lambda y^{\alpha}}{\alpha} + \frac{b_{-1}{}^{2}\lambda t^{\alpha}}{\varsigma\alpha}\right)\right)\right)} + b_0. \tag{3.12}$$

Family 3. The following solitary wave solutions are obtained for Case 2 and $\varsigma < 0$.

$$F_5(x,y,t) = b_0 + \frac{b_1\left(-\varsigma P_2 - P_1\sqrt{-\varsigma}\tanh\left(\sqrt{-\varsigma}\left(-1/2\,\frac{b_1x^{\alpha}}{\alpha} + \frac{\lambda y^{\alpha}}{\alpha} + \frac{\varsigma b_1{}^{2}\lambda t^{\alpha}}{\alpha}\right)\right)\right)}{\left(P_1 - P_2\sqrt{-\varsigma}\tanh\left(\sqrt{-\varsigma}\left(-1/2\,\frac{b_1x^{\alpha}}{\alpha} + \frac{\lambda y^{\alpha}}{\alpha} + \frac{\varsigma b_1{}^{2}\lambda t^{\alpha}}{\alpha}\right)\right)\right)} \tag{3.13}$$

or

$$F_6(x,y,t) = b_0 + \frac{b_1\left(-\varsigma P_2 - P_1\sqrt{-\varsigma}\coth\left(\sqrt{-\varsigma}\left(-1/2\,\frac{b_1x^{\alpha}}{\alpha} + \frac{\lambda y^{\alpha}}{\alpha} + \frac{\varsigma b_1{}^{2}\lambda t^{\alpha}}{\alpha}\right)\right)\right)}{\left(P_1 - P_2\sqrt{-\varsigma}\coth\left(\sqrt{-\varsigma}\left(-1/2\,\frac{b_1x^{\alpha}}{\alpha} + \frac{\lambda y^{\alpha}}{\alpha} + \frac{\varsigma b_1{}^{2}\lambda t^{\alpha}}{\alpha}\right)\right)\right)}. \tag{3.14}$$

Family 4. The following solitary wave solutions are obtained for Case 2 and $\varsigma > 0$.

$$F_7(x,y,t) = b_0 + \frac{b_1\left(-\varsigma P_2 + P_1\sqrt{\varsigma}\tan\left(\sqrt{\varsigma}\left(-1/2\,\frac{b_1x^{\alpha}}{\alpha} + \frac{\lambda y^{\alpha}}{\alpha} + \frac{\varsigma b_1{}^{2}\lambda t^{\alpha}}{\alpha}\right)\right)\right)}{\left(P_1 + P_2\sqrt{\varsigma}\tan\left(\sqrt{\varsigma}\left(-1/2\,\frac{b_1x^{\alpha}}{\alpha} + \frac{\lambda y^{\alpha}}{\alpha} + \frac{\varsigma b_1{}^{2}\lambda t^{\alpha}}{\alpha}\right)\right)\right)} \tag{3.15}$$

or

$$F_8(x,y,t) = b_0 + \frac{b_1\left(-\varsigma P_2 - P_1\sqrt{\varsigma}\cot\left(\sqrt{\varsigma}\left(-1/2\,\frac{b_1x^{\alpha}}{\alpha} + \frac{\lambda y^{\alpha}}{\alpha} + \frac{\varsigma b_1{}^{2}\lambda t^{\alpha}}{\alpha}\right)\right)\right)}{\left(P_1 - P_2\sqrt{\varsigma}\cot\left(\sqrt{\varsigma}\left(-1/2\,\frac{b_1x^{\alpha}}{\alpha} + \frac{\lambda y^{\alpha}}{\alpha} + \frac{\varsigma b_1{}^{2}\lambda t^{\alpha}}{\alpha}\right)\right)\right)}. \tag{3.16}$$

Family 5. The following solitary wave solutions are obtained for Case 3, $\varsigma < 0$, and

$$\psi = \frac{-1/2\, ib_{-1}{P_2}^2 x^{\alpha}}{{P_1}^2\alpha} + \frac{\lambda\, y^{\alpha}}{\alpha} - \frac{4\, i{P_2}^2{b_{-1}}^2\lambda\, t^{\alpha}}{{P_1}^2\alpha}.$$

$$F_9(x,y,t) = \frac{b_{-1}\left(P_1 - P_2\sqrt{\frac{-i{P_1}^2}{{P_2}^2}}\tanh\left(\sqrt{\frac{-i{P_1}^2}{{P_2}^2}}\psi\right)\right)}{\left(\frac{-i{P_1}^2}{P_2} - P_1\sqrt{\frac{-i{P_1}^2}{{P_2}^2}}\tanh\left(\sqrt{\frac{-i{P_1}^2}{{P_2}^2}}\psi\right)\right)} + b_0 + \frac{ib_{-1}{P_2}^2\left(\frac{-i{P_1}^2}{P_2} - P_1\sqrt{\frac{-i{P_1}^2}{{P_2}^2}}\tanh\left(\sqrt{\frac{-i{P_1}^2}{{P_2}^2}}\psi\right)\right)}{{P_1}^2\left(P_1 - P_2\sqrt{\frac{-i{P_1}^2}{{P_2}^2}}\tanh\left(\sqrt{\frac{-i{P_1}^2}{{P_2}^2}}\psi\right)\right)} \tag{3.17}$$

or

$$F_{10}(x,y,t) = \frac{b_{-1}\left(P_1 - P_2\sqrt{\frac{-i{P_1}^2}{{P_2}^2}}\coth\left(\sqrt{\frac{-i{P_1}^2}{{P_2}^2}}\psi\right)\right)}{\left(\frac{-i{P_1}^2}{P_2} - P_1\sqrt{\frac{-i{P_1}^2}{{P_2}^2}}\coth\left(\sqrt{\frac{-i{P_1}^2}{{P_2}^2}}\psi\right)\right)} + b_0 + \frac{ib_{-1}{P_2}^2\left(\frac{-i{P_1}^2}{P_2} - P_1\sqrt{\frac{-i{P_1}^2}{{P_2}^2}}\coth\left(\sqrt{\frac{-i{P_1}^2}{{P_2}^2}}\psi\right)\right)}{{P_1}^2\left(P_1 - P_2\sqrt{\frac{-i{P_1}^2}{{P_2}^2}}\coth\left(\sqrt{\frac{-i{P_1}^2}{{P_2}^2}}\psi\right)\right)}. \tag{3.18}$$

Family 6. The following solitary wave solutions are obtained for Case 3 and $\varsigma > 0$.

$$F_{11}(x,y,t) = \frac{b_{-1}\left(P_1 + P_2\sqrt{\frac{i{P_1}^2}{{P_2}^2}}\tan\left(\sqrt{\frac{i{P_1}^2}{{P_2}^2}}\psi\right)\right)}{\left(\frac{-i{P_1}^2}{P_2} + P_1\sqrt{\frac{i{P_1}^2}{{P_2}^2}}\tan\left(\sqrt{\frac{i{P_1}^2}{{P_2}^2}}\psi\right)\right)} + b_0 + \frac{ib_{-1}{P_2}^2\left(\frac{-i{P_1}^2}{P_2} + P_1\sqrt{\frac{i{P_1}^2}{{P_2}^2}}\tan\left(\sqrt{\frac{i{P_1}^2}{{P_2}^2}}\psi\right)\right)}{{P_1}^2\left(P_1 + P_2\sqrt{\frac{i{P_1}^2}{{P_2}^2}}\tan\left(\sqrt{\frac{i{P_1}^2}{{P_2}^2}}\psi\right)\right)} \tag{3.19}$$

or

$$F_{12}(x,y,t) = \frac{b_{-1}\left(P_1 - P_2\sqrt{\frac{i{P_1}^2}{{P_2}^2}}\cot\left(\sqrt{\frac{i{P_1}^2}{{P_2}^2}}\psi\right)\right)}{\left(\frac{-i{P_1}^2}{P_2} - P_1\sqrt{\frac{i{P_1}^2}{{P_2}^2}}\cot\left(\sqrt{\frac{i{P_1}^2}{{P_2}^2}}\psi\right)\right)} + b_0 + \frac{ib_{-1}{P_2}^2\left(\frac{-i{P_1}^2}{P_2} - P_1\sqrt{\frac{i{P_1}^2}{{P_2}^2}}\cot\left(\sqrt{\frac{i{P_1}^2}{{P_2}^2}}\psi\right)\right)}{{P_1}^2\left(P_1 - P_2\sqrt{\frac{i{P_1}^2}{{P_2}^2}}\cot\left(\sqrt{\frac{i{P_1}^2}{{P_2}^2}}\psi\right)\right)}. \tag{3.20}$$

Family 7. The following solitary wave solutions are obtained for Case 3 and $\varsigma = 0$.

$$F_{13}(x,y,t) = \frac{b_{-1}\left(P_1 - \frac{P_2}{\psi}\right)}{\left(\frac{-i{P_1}^2}{P_2} - \frac{P_1}{\psi}\right)} + b_0 + \frac{ib_{-1}{P_2}^2\left(\frac{-i{P_1}^2}{P_2} - \frac{P_1}{\psi}\right)}{{P_1}^2\left(P_1 - \frac{P_2}{\psi}\right)}. \tag{3.21}$$

Family 8. The following solitary wave solutions are obtained for Case 4, $\varsigma < 0$, and

$$\psi = 1/2\,\frac{b_{-1}x^{\alpha}}{\varsigma\,\alpha} + \frac{\lambda\, y^{\alpha}}{\alpha} + 4\,\frac{{b_{-1}}^2\lambda\, t^{\alpha}}{\varsigma\,\alpha}.$$

$$F_{14}(x,y,t) = \frac{b_{-1}\,(P_1 - P_2\sqrt{-\varsigma}\tanh(\sqrt{-\varsigma}\psi))}{-\varsigma\, P_2 - P_1\sqrt{-\varsigma}\tanh(\sqrt{-\varsigma}\psi)} + b_0 - \frac{b_{-1}\,(-\varsigma\, P_2 - P_1\sqrt{-\varsigma}\tanh(\sqrt{-\varsigma}\psi))}{\varsigma\,(P_1 - P_2\sqrt{-\varsigma}\tanh(\sqrt{-\varsigma}\psi))} \tag{3.22}$$

or

$$F_{15}(x,y,t)=\frac{b_{-1}\,(P_1-P_2\sqrt{-\varsigma}\coth(\sqrt{-\varsigma}\psi))}{-\varsigma P_2-P_1\sqrt{-\varsigma}\coth(\sqrt{-\varsigma}\psi)}+b_0-\frac{b_{-1}\,(-\varsigma P_2-P_1\sqrt{-\varsigma}\coth(\sqrt{-\varsigma}\psi))}{\varsigma\,(P_1-P_2\sqrt{-\varsigma}\coth(\sqrt{-\varsigma}\psi))}. \quad (3.23)$$

Family 9. The following solitary wave solutions are obtained for Case 4 and $\varsigma > 0$.

$$F_{16}(x,y,t)=\frac{b_{-1}\left(P_1+P_2\sqrt{\varsigma}\tan\left(\sqrt{\varsigma}\psi\right)\right)}{-\varsigma P_2+P_1\sqrt{\varsigma}\tan\left(\sqrt{\varsigma}\psi\right)}+b_0-\frac{b_{-1}\left(-\varsigma P_2+P_1\sqrt{\varsigma}\tan\left(\sqrt{\varsigma}\psi\right)\right)}{\varsigma\left(P_1+P_2\sqrt{\varsigma}\tan\left(\sqrt{\varsigma}\psi\right)\right)} \quad (3.24)$$

or

$$F_{17}(x,y,t)=\frac{b_{-1}\left(P_1-P_2\sqrt{\varsigma}\cot\left(\sqrt{\varsigma}\psi\right)\right)}{-\varsigma P_2-P_1\sqrt{\varsigma}\cot\left(\sqrt{\varsigma}\psi\right)}+b_0-\frac{b_{-1}\left(-\varsigma P_2-P_1\sqrt{\varsigma}\cot\left(\sqrt{\varsigma}\psi\right)\right)}{\varsigma\left(P_1-P_2\sqrt{\varsigma}\cot\left(\sqrt{\varsigma}\psi\right)\right)}. \quad (3.25)$$

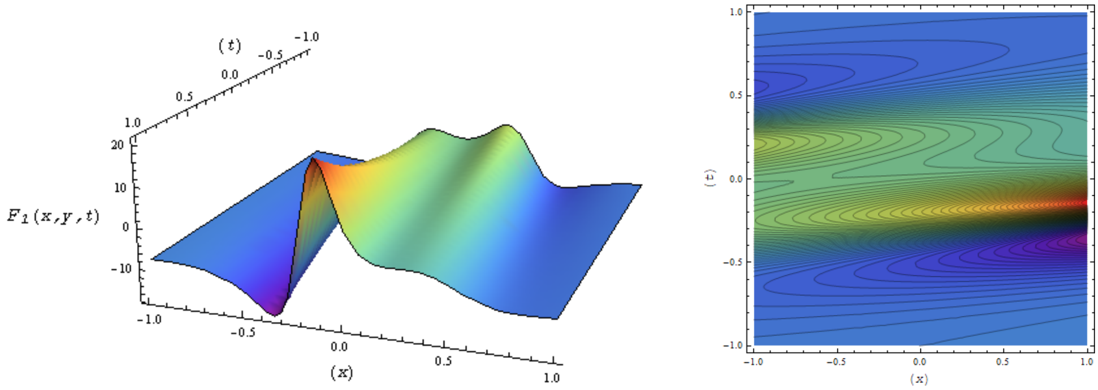
The soliton solutions arrived at by means of the method applied are very useful in realism applications. For example, in the realm of nonlinear optics, such solutions can give information about the evolution of optical pulses in fibers with negative dispersion. These solutions are also employed in fluid mechanics for tracking the behavior of surface and internal waves of a stratified liquid. The obtained sub-ODEs provide means for the identification of wave features, including its amplitude, width, and velocity, which define energy transport and dissipation in such systems.

Furthermore, the nature of the method allows for its implementation in other systems with fractional dynamics including viscoelastic materials and anomalous diffusion. For instance, the fractional form of the CBS equation can simulate small waves in shallow water in the heterogeneous medium several times or perturbations of the magnetic field in a plasma multiple of times. This approach gives the connection between mathematical solutions and physical observables and is effectively applicable for solving problems in applied sciences related to engineering and physics.

4. Results and discussion

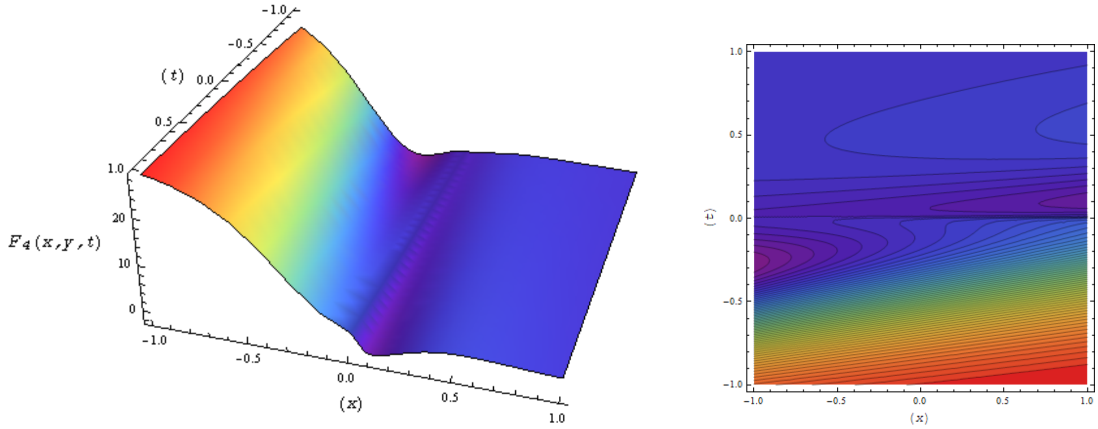
Herein, a new technique for obtaining solutions for space-time fractional-order, nonlinear PDEs applicable to wave mechanics in magnetoacoustic waves, tsunamis, and internal ocean currents is introduced. This technique produces a bypass in terms of exclusively different soliton structures, including periodic cross kink waves, shock waves, bright kink waves, M-shaped waves, breather-shaped waves, and dark kink waves using the Bäcklund transformation with the help of the Riccati-Bernoulli sub-ODE method providing closed-form solutions. This enables the direct derivation of the exact solution of PDEs in a simple form of ODEs as well as the explication of the dynamic and periodic features of these solitons expressed in hyperbolic, rational, and trigonometric function forms. By doing so, the research contributes both toward soliton theory and enables the development of a strong foundation of stability and propagation of waves in fractional systems. It can be concluded that there are definite implications for the understanding of material science, fluid dynamics, and wave mechanics in this work.

Figure 1 shows cross kink waves for the fractional-order parameter $\alpha = 1$. This wave is portrayed in a 3-dimensional plot where it reveals the construction of the wave, the direction it affords, as well as the amplitude to support its periodicity. Further, the contour plot is likely to provide a higher-resolution view of the fractal-like features and would be useful for understanding wave behavior. Such representations are also of great importance in realistic applications such as modeling of magneto-sound waves in plasmas, the dynamics of tsunamis, and the behavior of fluid flow in which it is crucial to analyze the characteristics of wave propagation for such a system. Figure 2 shown below illustrates the shock wave pattern when ($\alpha = 0.6$). In the 3D plot, it can clearly be noted that there is a very rapid variation of the amplitude of the wave, which gives it the shock wave sudden-type characteristic. In particular, the steep gradients of the wave are shown in the contour plot, which can be important for investigating various systems in which parameters sharply change: hydrodynamics, magneto-sound waves, and tsunamis. The presence of this shock wave solution underlines the test and utilization of the L-shape shock wave solution of the fractional Calogero-Bogoyavlenskii-Schiff equation. Figure 3 is a bright kink wave in relation to a fractional order which is illustrated as 0.1 here. In the 3D plot, the localized characteristic of the wave is clearly seen, where the wave amplitude rises from a small value and forms a peak. The contour plot also serves to explain the structure of the bright kink wave, and stresses the smooth nature of the emerging propagation trends. This solution is most appropriate when studying wave processes in isolated systems of material points, and in plasmas and hydrodynamic systems in which localized perturbations are essential for controlling wave behavior and energy exchange. Figure 4 illustrates the M-shaped wave solution of the fractional-order parameter ($\alpha = 1$). As seen earlier, the 3D plot displays an M pattern and analyzing the wave and the plot shows that the wave has two dips with a hump in between resembling a capital letter M. The wave continues to grow and shrink and decrease and increase as it moves, and this makes it a dynamic solution. The contour plot extends the depiction of wave amplitude change over space and time by graphically highlighting periodic oscillations in the M-shaped wave. This solution is important for modeling and analyzing constitutive equations of phenomena exhibiting oscillated behavior like wave and wave interaction in nonlinear fluid dynamics and plasma. Figure 5 shows the breather-shaped wave solution for the fractional-order parameter ($\alpha = 1$). The $3D$ plot shows the waves type of oscillation, with the amplitude increasing and reducing at intervals, somewhat similar to a breather. The wave demonstrates localized pulsating fringes that can be observed as constructive wave pulses emerging and collapsing, described in solutions of breather-types nonlinear wave equations. By making a contour plot of the solution, the true nature of the wave propagating system is well-depicted particularly in terms of localization. This solution is applicable for analyzing behaviors such as wave pulses in specific nonlinear media including plasma physics and fluid dynamics. In Figure 6, the dark kink wave solution is shown for the value of the fractional-order parameter ($\alpha = 0.1$). The dark kinks 3D plot illustrates the model's overall dark kink structure, which includes low points at the wave center and rises to a flat base at the edges, depicting a localized attenuation of waves. The contours enhance the visualization of the wave, identifying the narrow and single valley of the dark kink. This solution is important in a number of physical situations, for example, dark solitons in non-linear optical fibers, and quantum field theory, where the amplitude of the wave is a solitary wave and may be taken to represent a soliton, a localized disturbance in an otherwise smooth and perhaps uniform medium. Table 1 shows the comparison of the current approach with the modified extended tanh-function method [54]. Table 2 shows the comparison of the Riccati-Bernoulli sub-ODE method with other methods.



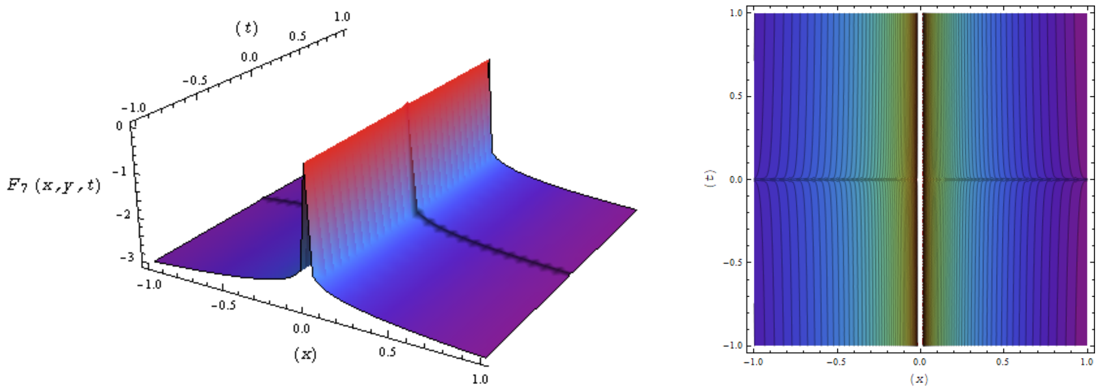
(a) **(b)**

Figure 1. Visualization of the solution $F_1(x, y, t)$, showing $3D$ and contour plots, for fractional-order parameter $\alpha = 1$.



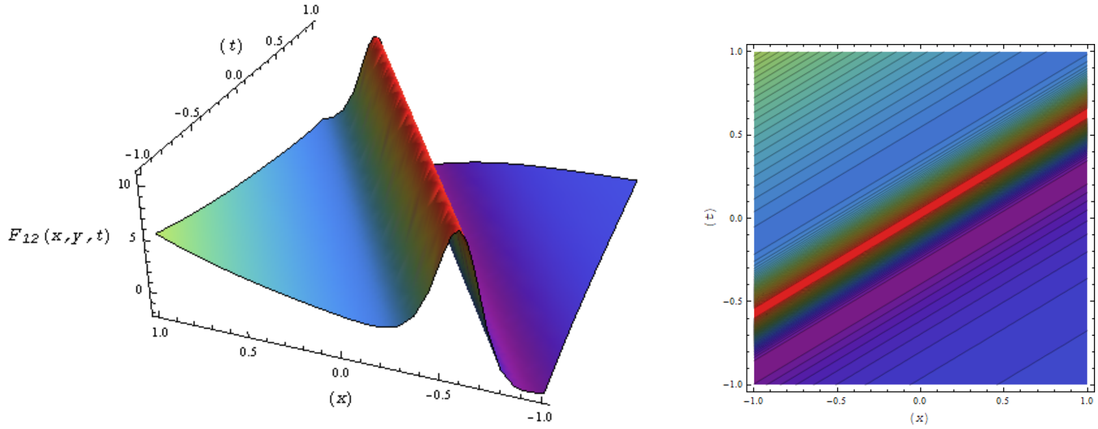
(a) **(b)**

Figure 2. Visualization of the solution $F_4(x, y, t)$, showing $3D$ and contour plots, for fractional-order parameter $\alpha = 0.6$.



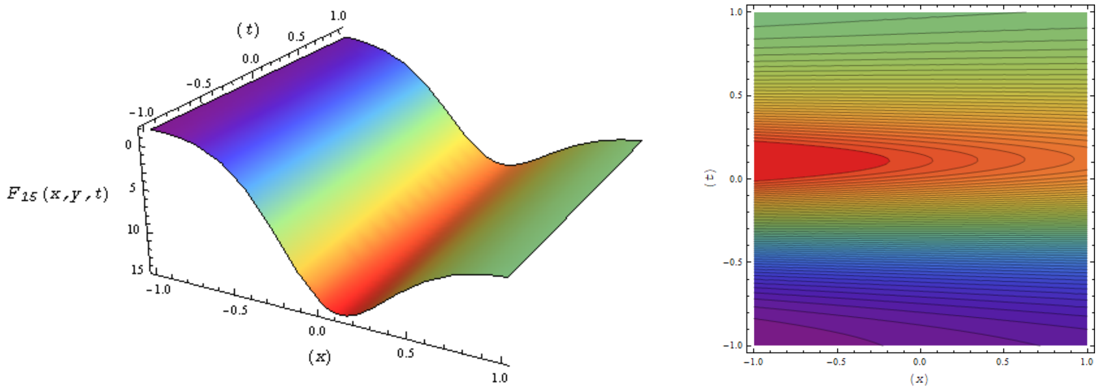
(a) **(b)**

Figure 3. Visualization of the solution $F_7(x, y, t)$, showing $3D$ and contour plots, for fractional-order parameter $\alpha = 0.1$.



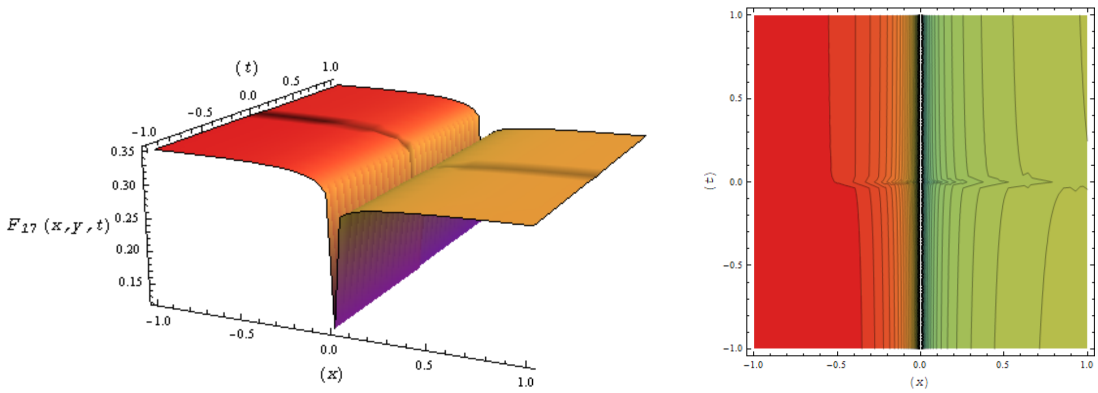
(a) **(b)**

Figure 4. Visualization of the solution $F_{12}(x, y, t)$, showing 3D and contour plots, for fractional-order parameter $\alpha = 1$.



(a) **(b)**

Figure 5. Visualization of the solution $F_{15}(x, y, t)$, showing 3D and contour plots, for fractional-order parameter $\alpha = 1$.



(a) **(b)**

Figure 6. Visualization of the solution $F_{17}(x, y, t)$, showing 3D and contour plots, for fractional-order parameter $\alpha = 0.1$.

Table 1. Comparison of the current approach with the modified extended tanh-function method [54].

Present method	Case I: $\varsigma < 0$	$F(x,y,t) = \frac{b_{-1}\left(P_1 - P_2\sqrt{-\varsigma}\tanh\left(\sqrt{-\varsigma}\left(1/2\,\frac{b_{-1}x^{\alpha}}{\varsigma\alpha} + \frac{\lambda y^{\alpha}}{\alpha} + \frac{{b_{-1}}^{2}\lambda t^{\alpha}}{\varsigma\alpha}\right)\right)\right)}{\left(-\varsigma P_2 - P_1\sqrt{-\varsigma}\tanh\left(\sqrt{-\varsigma}\left(1/2\,\frac{b_{-1}x^{\alpha}}{\varsigma\alpha} + \frac{\lambda y^{\alpha}}{\alpha} + \frac{{b_{-1}}^{2}\lambda t^{\alpha}}{\varsigma\alpha}\right)\right)\right)} + b_0$
	Case II: $\varsigma > 0$, and $\psi = 1/2\,\frac{b_{-1}x^{\alpha}}{\varsigma\alpha} + \frac{\lambda y^{\alpha}}{\alpha} + 4\,\frac{{b_{-1}}^{2}\lambda t^{\alpha}}{\varsigma\alpha}$	$F(x,y,t) = \frac{b_{-1}(P_1 + P_2\sqrt{\varsigma}\tan(\sqrt{\varsigma}\psi))}{-\varsigma P_2 + P_1\sqrt{\varsigma}\tan(\sqrt{\varsigma}\psi)} + b_0 - \frac{b_{-1}(-\varsigma P_2 + P_1\sqrt{\varsigma}\tan(\sqrt{\varsigma}\psi))}{\varsigma(P_1 + P_2\sqrt{\varsigma}\tan(\sqrt{\varsigma}\psi))}$
	Case III: $\varsigma = 0$, and $\psi = \frac{-1/2\,ib_{-1}{P_2}^{2}x^{\alpha}}{{P_1}^{2}\alpha} + \frac{\lambda y^{\alpha}}{\alpha} - \frac{4\,i{P_2}^{2}{b_{-1}}^{2}\lambda t^{\alpha}}{{P_1}^{2}\alpha}$	$F(x,y,t) = \frac{b_{-1}\left(P_1 - \frac{P_2}{\psi}\right)}{\left(\frac{-i{P_1}^{2}}{P_2} - \frac{P_1}{\psi}\right)} + b_0 + \frac{ib_{-1}{P_2}^{2}\left(\frac{-i{P_1}^{2}}{P_2} - \frac{P_1}{\psi}\right)}{{P_1}^{2}\left(P_1 - \frac{P_2}{\psi}\right)}$
Tanh-function method	Case I: $\xi = x + y - k\frac{t^{\alpha}}{\alpha}$	$F(\xi) = \frac{12\sqrt{-b}\tanh[\sqrt{-b}(\xi)]}{\beta+\gamma}$
	Case II: $\xi = x + y - k\frac{t^{\alpha}}{\alpha}$	$F(\xi) = \frac{12\sqrt{b}\cot[\sqrt{b}(\xi)]}{\beta+\gamma}$
	Case III: $\xi = x + y - k\frac{t^{\alpha}}{\alpha}$	$F(\xi) = \frac{12}{(\beta+\gamma)(\xi)}$

Table 2. Comparison of the Riccati-Bernoulli sub-ODE method with other methods.

Method	**Advantages**	**Limitations**
Riccati-Bernoulli sub-ODE method with Bäcklund transformation	- Systematic approach for deriving exact soliton solutions. - Capable of handling fractional-order systems. - Provides a rich variety of solutions including periodic, kink, and breather waves.	- Requires complex algebraic manipulation for solving the resulting equations. - May rely on computational tools for parameter determination.
Modified extended tanh-function method	- Straightforward and efficient for finding exact solutions. - Particularly effective for hyperbolic and trigonometric function-based solutions.	- Limited to specific types of nonlinear equations. - May not be suitable for equations with fractional derivatives.
Exp-function method	- General and flexible approach for solving nonlinear differential equations. - Suitable for equations with higher-order nonlinearities.	- Involves trial-and-error in selecting the ansatz. - May result in complex expressions that are difficult to interpret physically.
Sine-cosine method	- Simple implementation for equations with periodic solutions. - Effective for equations with trigonometric solutions.	- Limited to equations that inherently possess periodic or oscillatory behavior. - Unsuitable for fractional-order systems.
Homotopy perturbation method	- Combines numerical and analytical techniques for approximate solutions. - Applicable to a wide variety of nonlinear problems.	- Generally provides approximate rather than exact solutions. - Accuracy depends on the number of perturbation terms considered.

5. Conclusions

In this work, we first presented and implemented the Riccati-Bernoulli sub-ODE method along with the Bäcklund transformation to derive analytical soliton solutions for the fractional Calogero-Bogoyavlenskii-Schiff (CBS) equation. This new approach provided a large number of correct soliton solutions such as periodic cross kink waves, shocks waves, bright kinks, M-formed, breather-shaped, and dark kinks, of which each possesses a different physical meaning in terms of propagation. The level of fractional order (α) was found to decide the dynamics and waveform of the solitons that exhibited different features when the value of (α) changed.

The work's main advantage is that it offers a means to obtain closed-form solutions and analyze wave motions and disturbances in fractional systems. These solutions not only enrich the theory for soliton science and technology but also constructe a strong foundation for the future investigation on all the nonlinear wave phenomena, material science, and topological solitons. The above presented idea seems to be very useful and efficient in order to have a detailed thorough and comprehensive investigation of the given types of FPDEs together with interactive visualization of various aspects of the Riemann problem and other phenomena.

In conclusion, the Riccati-Bernoulli sub-ODE method with Bäcklund transformation has considerable prospects for the further development of theoretical and applied developments in a number of scientific and technical disciplines. Because of its flexibility and substantial performance, there appear to be a wide range of possibilities for future work using this algorithm, including with larger and more complicated systems, and incorporating other fields and methods.

Author contributions

H.G.: Conceptualization, Visualization, Funding, Data curation, Resources, Writing—review and editing; A.A.H.A.: Formal analysis, Project administration, Data curation, Validation; A.H.H.: Investigation, Validation, Resources, Software. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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