Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 10(1): 403–419.
DOI: 10.3934/math.2025019
Received: 19 September 2024
Revised: 23 December 2024
Accepted: 30 December 2024
Published: 08 January 2025

Research article

On multi-valued generalized α-nonexpansive mappings and an application to two-point BVPs

Junaid Ahmad[1], Imen Ali Kallel[2,*], Ahmad Aloqaily[3,4] and Nabil Mlaiki[3,*]

[1] Department of Mathematics and Statistics, International Islamic University, H-10, Islamabad-44000, Pakistan

[2] Department of Mathematics, College of Science, Northern Border University, Arar, Saudi Arabia

[3] Department of Mathematics and Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia

[4] School of Computer, Data and Mathematical Sciences, Western Sydney University, Sydney 2150, Australia

* **Correspondence:** Email: imenkallel16@gmail.com; nmlaiki2012@gmail.com, nmlaiki@psu.edu.sa.

Abstract: In recent years, many researchers have studied the fixed points of generalized α-nonexpansive (GAN) mappings, yet there has been limited research on multi-valued GAN mappings. In this paper, we focused on the class of multi-valued GAN mappings in the context of Banach spaces. We introduced a novel iterative process to find fixed points of these mappings and established weak and strong convergence theorems under mild conditions. To demonstrate the numerical efficiency of our new approach, we presented a supportive example and compare the accuracy of our method with existing iteration processes. To show practical usability of the research, we considered a larger class of boundary value problems (BVPs) and proved a convergence result with a supportive example. Our results are new and unify several results of the current literature.

Keywords: iteration; approximate solution; fixed point; condition (I); Banach space
Mathematics Subject Classification: 47H09, 47H10

1. Introduction and preliminaries

Fixed point theory is rapidly advancing due to its versatility and wide-ranging applications across nearly every branch of science, including machine learning, artificial intelligence, applied physics, applied chemistry, and engineering [1–3]. The theory itself can be viewed as a combination of analysis and topology, where the solution to a given problem is interpreted as a fixed point of a mapping within a distance space setting. In this context, the existence of a solution is equivalent to the existence of a fixed point of the mapping, where a fixed point is an element in the domain that is mapped onto itself.

The first significant result concerning the existence and approximation of fixed points for a specific class of mappings was provided by the founder of functional analysis, Banach [4]. He introduced the concept of contractions, showing that if g is a contraction on a metric space S, that is, $d(gm, gn) \leq \theta d(m, n)$ for all $m, n \in S$ with $0 \leq \theta < 1$, then there exists a unique fixed point $w \in S$, provided that S is complete. Banach also proposed the Picard [5] iterative process, which generates a sequence $m_{i+1} = gm_i$ as a computational method to approach the fixed points of contractions. Although Banach's fixed point approach is a powerful technique for approximating and ensuring the existence of solutions to integral and other types of equations, it encounters limitations when the contraction constant equals one. Specifically, when the contraction constant θ in the contraction inequality is equal to one, the mapping is referred to as a nonexpansive mapping. Nonexpansive mappings play a crucial role in the study of inclusion and optimization problems, but the existence of fixed points is only guaranteed if the domain of the map is a closed, bounded, convex subset of a uniformly convex Banach space (UCBS) (see, e.g., [6–8] and others). Since Picard iterations do not always succeed in computing fixed points of nonexpansive maps (see, e.g., Ahmad et al. [9] and others), more general iterative processes are required to achieve this goal. For example Ullah et al. [10] introduced a modified iterative scheme for fixed points of generalized nonexpansive mappings. Agarwal et al. [11] introduced a two-step iterative scheme which is the modification of Ishikawa iterative scheme [12] for approximating fixed points of nonlinear mappings. Khan [13] intoduced a hybrid iterative scheme of Picard and Mann [14] called Picard-Mann iterative scheme for fixed points of contraction and nonexpansive mappings. Using idea of Noor [15], Abbas and Nazir [16] suggested a new three-step iterative scheme for fixed points of nonexpansive mappings. Thakur et al. [17] introduced a new type of iterative scheme for fixed points of generalized nonexpansive mappings. Recently, Ullah et al. [18] suggested M iterative scheme and proved that it converges faster to fixed points of generalized nonexpansive mappings as compared to the above mentioned iterative schemes.

As discussed earlier, the study of contractions and nonexpansive mappings has significant applications across various scientific fields. Therefore, it is natural to extend these concepts further to achieve broader applications. In 2008, Suzuki [19] introduced a generalization by weakening the concept of nonexpansive mappings, restricting the inequality to some elements. A self-map g on a subset D of a normed linear space X is called a Suzuki mapping on D if it satisfies the condition $\|gm - gn\| \leq \|m - n\|$ whenever $\frac{1}{2}\|m - gm\| \leq \|m - n\|$ for all $m, n \in D$. Suzuki demonstrated that every nonexpansive map, and therefore every contraction, satisfies this condition. However, through an example, he also proved that the converse is not true in general. These findings reveal that the class of Suzuki mappings properly includes all nonexpansive and contraction operators in normed linear spaces.

The concept of generalized maps under Suzuki's condition has inspired researchers to further generalize the ideas of contractions and nonexpansive mappings. In this vein, Aoyama and Kohsaka [20] introduced a new class of mappings in Banach spaces. A map g is termed α-nonexpansive (AN) on a set D if, for all $m, n \in D$, there exists a constant $\alpha \in [0, 1)$ such that

$$\|gm - gn\|^2 \leq \alpha\|m - gn\|^2 + \alpha\|n - gm\|^2 + (1 - 2\alpha)\|m - n\|^2.$$

It is evident that, nonexpansive nonlinear maps are 0-nonexpansive, but there exist discontinuous maps that are α-nonexpansive without being nonexpansive, that is, when G is nonexpansive, then it satisfies the above condition for $\alpha = 0$. The following example shows that the class of AN is larger than the class of nonexpansive mappings.

Example 1.1. *Suppose* $D = [0, 4]$ *and define* $G : D \to D$ *by*

$$Gm = \begin{cases} 0 & \textit{for } \; m \in [0, 4), \\ 2 & \textit{for } \; m = 4. \end{cases}$$

It follows that G is AN on D but not nonexpansive.

Subsequently, the authors in [21] proposed the notion of GAN maps. A map g is called GAN on D if, for all $m, n \in D$, there exists a constant $\alpha \in [0, 1)$ such that

$$\|gm - gn\| \leq \alpha\|m - gn\| + \alpha\|m - gn\| + (1 - 2\alpha)\|m - n\|,$$

provided that $\frac{1}{2}\|m - gm\| \leq \|m - n\|$. It was shown that the class of GAN maps is strictly larger than the class of Suzuki maps and partially includes the class of AN maps within a normed linear space setting using the following example in [21].

Example 1.2. *[21] Consider* $S = \mathbb{R}^2$ *with* $\|(m_1, m_2)\| = |m_1| + |m_2|$. *Suppose* $D = \{(0, 0), (2, 0), (0, 4), (4, 0), (4, 5), (5, 4)\}$ *and set* $g : D \to D$ *by*

$$gm = \begin{cases} (0, 0) & \textit{when } \; m = (0, 0), \\ (0, 0) & \textit{when } \; m = (2, 0), \\ (0, 0) & \textit{when } \; m = (0, 4), \\ (2, 0) & \textit{when } \; m = (4, 0), \\ (4, 0) & \textit{when } \; m = (4, 5), \\ (0, 4) & \textit{when } \; m = (5, 4). \end{cases}$$

Here, g is neither a Suzuki mapping nor GA on D. However, g is a GAN mapping on D with $\alpha = \frac{1}{5}$.

Given the numerous practical applications of fixed point theory in economics and other real-world problems, researchers have extended fixed point results from the context of single-valued maps to multi-valued maps. For instance, Nadler [22] explored a multi-valued version of contraction mappings and established an analog of Banach's result [4] within the setting of multi-valued maps. Inspired by this, Lim [23] subsequently proved fixed point theorems for multi-valued nonexpansive maps. Further developments in this area include the introduction of multi-valued versions of Suzuki maps by the authors in [24], and the notion of AN maps for multi-valued settings as presented in [25]. Additionally, Iqbal et al. [26] proposed a multi-valued version of GAN maps in normed linear spaces. While it is relatively straightforward to iterate a single-valued map using a fixed point iteration, doing so for multi-valued maps presents a greater challenge. The first significant result concerning the approximation of fixed points for multi-valued maps was provided by Sastry and Babu [27] within a Hilbert space framework. This result was later extended by Panyanak [28] to a uniformly convex Banach space (UCBS) setting. However, Song [29] offered critical remarks on the results in [27] and [28], correcting these results by introducing a strong condition known as the endpoint condition. Shahzad and Zegeye [30] further extended the work in [29] to multi-valued quasi-nonexpansive maps, utilizing the operator $P_G m = \{n \in Gm : \|m - n\| = d(m, Gm)\}$ as a multi-valued map. For more detailed studies on this topic, see [31], among others.

As previously discussed, many iterative processes have been developed for nonexpansive mappings and their generalizations in Banach spaces. To enhance the convergence rate of these processes in the case of multi-valued maps, we here provide a new effective iteration process. The iteration process obtains a faster speed under suitable assumptions.

Our new iteration process multi-valued case reads as:

$$\begin{cases} m_1 \in D, \\ l_i = (1-\mu)m_i + \mu a_i, \\ s_i = (1-\alpha_i)l_i + \alpha_i a_i', \\ n_i = a_i'', \\ m_{i+1} = a_i''', (i = 1, 2, 3, ...), \end{cases} \tag{1.1}$$

where the element $a_i \in P_G(m_i)$, $a_i' \in P_G(l_i)$, $a_i'' \in P_G(s_i)$, $a_i''' \in P_G(n_i)$, $0 < \alpha_i < 1$, and μ is constant between with $0 < \mu < 1$ but very close to 1. We shall prove some new results of the iterative process (1.1) in the setting of multi-valued GAN with a new application of BVPs.

Consider $S = (S, \|.\|)$, which is assumed to be a Banach space with D as a nonempty and possibly closed set in S. In such a case, the subset D is said to be a proximinal set when, for $m \in S$, one can find a $n \in D$ with the property that $d(m, n) = d(m, D)$, where the notation $d(m, D) = \inf\{\|m - s\| : s \in D\}$. In this research, we will use $P_{px}(D)$, $P_{cb}(D)$, and $P(D)$ respectively, for all proximinal subsets, closed and bounded subsets, and all subsets of the set D. In the domain D of a multi-valued map $G : D \to P(D)$, if there is a point $w \in Gw$, then w is known as a fixed point for G and F_G represents a set of all fixed points of G. If for all fixed points w, we have $\{w\} = Gw$, then G is called a map with an endpoint condition. The Hausdorff distance is defined as:

$$H(B, R) = \max\left\{\sup_{m\in B} d(m, R), \ \sup_{n\in R} (n, B)\right\}, \text{ for any choice of } B, R \in P_{cb}(D).$$

Definition 1.1. *Let G be a multi-valued mapping on a subset D of a Banach space and $\alpha \in [0, 1)$. Then G is called nonexpansive on D if $H(Gm, Gn) \le \|m - n\|$ for all $m, n \in D$. Also G is called quasi-nonexpansive on D if $H(Gm, Gw) \le \|m - w\|$ for each $w \in F_G$. The mapping G is called a Suzuki mapping if $\frac{1}{2}d(m, Gm) \le \|m - n\| \Rightarrow H(Gm, Gn) \le \|m - n\|$. G is known as GA if $H^2(Gm, Gn) \le \alpha d^2(m, Gn) + \alpha d^2(n, Gm) + (1 - 2\alpha)\|m - n\|^2$. Eventually, G is called GAN if $\frac{1}{2}d(m, Gm) \le \|m - n\| \Rightarrow H(Gm, Gn) \le \alpha d(m, Gn) + \alpha d(n, Gm) + (1 - 2\alpha)\|m - n\|$.*

The following example shows that the class of multi-valued GAN mappings is more general than the other classes of mappings in Definition 1.1.

Example 1.3. *Consider $S = \mathbb{R}^2$ with l_2 norm. Suppose $D = [0, 4] \times [0, 4]$ and set a multi-valued mapping G on D as follows:*

$$Gm = \begin{cases} [\frac{1}{2}, 2] \times [\frac{1}{2}, 2] & \text{when } m = (4, 4), \\ \{(0, 0)\} & \text{when } m \neq (4, 4). \end{cases}$$

Here, G is a neither multi-valued Suzuki mapping nor a multi-valued GA on D. However, G is a multi-valued GAN mapping on D with $\alpha = \frac{1}{2}$.

Definition 1.2. *[32] Suppose S represents a Banach space and for all $\{m_i\} \subseteq S$, we m_i converges weakly to $m \in S$. Then S is said to have Opial's condition if for any $n \neq m$, one has*

$$\limsup_{i\to\infty} \|m_i - m\| < \limsup_{i\to\infty} \|m_i - n\|.$$

Lemma 1.1. *[31] Suppose G is a multi-valued map on a subset D of a Banach space and $P_G m = \{n \in Gm : \|m - n\| = d(m, Gm)\}$. It follows that the following properties are equivalent.*

(w_1) $w \in F_G$;
(w_2) $P_G w = \{w\}$;
(w_3) $w \in F_{P_G}$.

Further, $F_G = F_{P_G}$.

We have the following result, which shows that the class of GAN is larger than the class of Suzuki maps.

Lemma 1.2. *[26] On a closed bounded subset D of a Banach space, if $G : D \to P_{cb}(D)$ is GAN, then G is also a Sukuki map on D.*

Lemma 1.3. *[26] On a closed bounded subset D of a Banach space, if $G : D \to P_{cb}(D)$ is GAN, then G is also quasi-nonexpansive on D.*

Lemma 1.4. *[26] On a closed bounded subset D of a Banach space, if $G : D \to P_{cb}(D)$ is GAN, then for any $m, n \in D$, one has*

$$d(m, Gn) \leq \left(\frac{3+\alpha}{1-\alpha}\right) d(m, Gm) + \|m - n\|.$$

The following crucial result is taken from [33].

Lemma 1.5. *For a sequence δ_i with $0 < \xi \leq \delta_i \leq \eta < 1$ and for any sequences $\{m_i\}$ and $\{n_i\}$ in a norm space S, it there is a constant $p \geq 0$ with $\limsup_{i\to\infty} \|m_i\| \leq p$, $\limsup_{i\to\infty} \|n_i\| \leq p$ and $\lim_{i\to\infty} \|\delta_i s_i + (1 - \delta_i) n_i\| = p$. Then $\lim_{i\to\infty} \|m_i - n_i\| = 0$ provided that S is a UCBS.*

2. Main results

This section proves some new findings for iteration process (1.1) using the notion of proximal GAN mappings. We now obtain the following result.

Lemma 2.1. *Suppose we have a multi-valued map $G : D \to P_{px}(D)$, where D is closed and convex in a UCBS S. If F_G is nonempty, P_G is GAN and $\{m_i\}$ is a sequence given by process (1.1). Then for arbitrary $w \in F_G$, one has that $\lim_{i\to\infty} \|m_i - w\|$ exists.*

Proof. Consider any $w \in F_G$. By Lemma 1.1, we have $P_G(w) = \{w\}$. By Lemma 1.3, we have

$$\begin{aligned} \|m_{i+1} - w\| &= \|a_i''' - w\| \leq H(P_G(n_i), P_G(w)) \leq \|n_i - w\| \\ &= \|a_i'' - w\| \leq H(P_G(s_i), P_G(w)) \leq \|s_i - w\| \\ &= \|(1-\alpha_i) l_i + \alpha_i a_i' - w\| \end{aligned}$$

$$\begin{aligned}
&\leq (1-\alpha_i)\|l_i - w\| + \alpha_i \|a_i' - w| \\
&\leq (1-\alpha_i)\|l_i - w\| + \alpha_i H(P_G(l_i), P_G(w)) \\
&\leq (1-\alpha_i)\|l_i - w\| + \alpha_i \|l_i - w\| \\
&= \|l_i - w\| = \|(1-\mu)m_i + \mu a_i - w\| \\
&\leq (1-\mu)\|m_i - w\| + \mu\|a_i - w\| \\
&\leq (1-\mu)\|m_i - w\| + \mu H(P_G m_i, w) \\
&\leq (1-\mu)\|m_i - w\| + \mu\|m_i - w\| \\
&= \|m_i - w\|.
\end{aligned}$$

We see from above that $\|m_{i+1}-w\| \leq \|m_i-w\|$ which suggests that $\{\|m_i-m\|\}$ must be a bounded sequence in a real-number set and also nonincreasing. By using these facts, we conclude that $\lim_{i\to\infty} \|m_i - w\|$ exists, $\forall w \in F_G$. □

The following result is also crucial for our main outcome.

Lemma 2.2. *Suppose we have a multi-valued map $G : D \to P_{px}(D)$, where D is closed and convex in a UCBS S. If F_G is nonempty, P_G is GAN and $\{m_i\}$ is a sequence given by process (1.1). Then* $\lim_{i\to\infty} d(m_i, P_G(m_i)) = 0$.

Proof. Since P_G is proximally GAN, for any $w \in F_G$, utilizing Lemma 2.1, we have that $\lim_{i\to\infty} \|m_i-w\|$ exists. We therefore put

$$\lim_{i\to\infty} \|m_i - w\| = p, \tag{2.1}$$

where $p \geq 0$. If $p = 0$, we have nothing to prove. We consider $p > 0$, and then it follows from the proof of Lemma 2.1 that $\|l_i - w\| \leq \|m_i - w\|$ and $\|m_{i+1} - w\| \leq \|l_i - w\|$. Hence using (2.1) with these, so we get

$$\limsup_{i\to\infty} \|l_i - w\| \leq p, \tag{2.2}$$

and

$$p \leq \liminf_{i\to\infty} \|l_i - w\|. \tag{2.3}$$

Combining (2.2) and (2.3), one has

$$\lim_{i\to\infty} \|l_i - w\| = p. \tag{2.4}$$

But P_G is quasi-nonexpansive and $a_i \in P_G(m_i)$, we get

$$\|a_i - w\| \leq H(P_G m_i, P_G w) \leq \|m_i - w\|. \tag{2.5}$$

Using (2.1) with (2.5), we have

$$\limsup_{i\to\infty} \|a_i - w\| \leq p. \tag{2.6}$$

From (2.4), one has

$$p = \lim_{i\to\infty} \|l_i - w\| = \lim_{i\to\infty} \|(1-\mu)(m_i - w) + \mu(a_i - w)\|. \tag{2.7}$$

Using (2.1), (2.6), and (2.7) with Lemma 1.5, we get

$$\lim_{i\to\infty} \|m_i - a_i\| = 0. \tag{2.8}$$

But $a_i \in P_G(m_i)$, so it follows that

$$\lim_{i\to\infty} d(m_i, P_G(m_i)) = 0. \tag{2.9}$$

□

Convergence of iterations on compact domains has its own importance. The following new result is obtained by the compactness condition on the domain.

Theorem 2.1. *Suppose we have a multi-valued map $G : D \to P_{px}(D)$, where D is closed and convex in a UCBS S. If F_G is nonempty, P_G is GAN and $\{m_i\}$ is a sequence given by process (1.1). Then $\{m_i\}$ converges strongly to any fixed point of G if D is compact in S.*

Proof. Since by Lemma 2.2 it is established that $\lim_{i\to\infty} d(m_i, P_G(m_i)) = 0$, it follows from the compactness of D that there is a point $v \in D$ and subsequence $\{m_{i_j}\}$ of $\{m_i\}$ such that $\{m_{i_j}\}$ converges to $v \in D$. Utilizing Lemma 1.4, one gets

$$\begin{aligned} d(v, P_G(v)) &\leq \|v - m_{i_j}\| + d(m_{i_j}, P_G(v)) \\ &\leq \|v - m_{i_j}\| + \left(\frac{3+\alpha}{1-\alpha}\right) d(m_{i_j}, P_G(m_{i_j})) + \|m_{i_j} - v\| \\ &= 2\|v - m_{i_j}\| + \left(\frac{3+\alpha}{1-\alpha}\right) d(m_{i_j}, P_G(m_{i_j})) \longrightarrow 0. \end{aligned}$$

As a whole, we see that $v \in P_G(v)$. Notice from Lemma 1.1 that $z \in F_{P_G} = F_G$. But $\lim_{i\to\infty} \|m_i - v\|$ exists in the view of Lemma 2.1. We conclude that v forms a strong limit point for $\{m_i\}$ and so the proof is complete. □

In the previous result, compactness played a crucial role. A natural question arises: Can we prove convergence in the strong sense without relying on the compactness of the domain? The following result demonstrates that compactness can, in fact, be replaced by a condition as $\liminf_{i\to\infty} d(m_i, F_G) = 0$.

Theorem 2.2. *Suppose we have a multi-valued map $G : D \to P_{px}(D)$, where D is closed and convex in a UCBS S. If F_G is nonempty, P_G is GAN and $\{m_i\}$ is a sequence given by process (1.1). Then $\{m_i\}$ converges strongly to any fixed point of G if* $\liminf_{i\to\infty} d(m_i, F_G) = 0$.

Proof. Since $F_G = F_{P_G}$ is nonempty, $\lim_{i\to\infty} \|m_i - w\|$ exists for any $w \in F_G = F_{P_G}$. Hence $\lim_{i\to\infty} d(m_i, F_{P_G})$ exists and by using the condition $\liminf_{i\to\infty} d(m_i, F_G) = 0$, we have

$$\lim_{i\to\infty} d(m_i, F_{P_G}) = 0.$$

An elementary calculation demonstrates that the sequence $\{m_i\}$ converges to a fixed point of $F_{P_G} = F_G$. □

We have the following definition from [34].

Definition 2.1. *Suppose we have a multi-valued map $G : D \to P_{px}(D)$, where D is closed and convex in a UCBS S. Then G is said to have condition (I) if one has a function ψ with $\psi(0) = 0$, $\psi(p) > 0$ if $p > 0$, and $d(m, Gm) \geq \psi(d(m, F_G))$ for all $m \in D$.*

Theorem 2.3. *Suppose we have a multi-valued map $G : D \to P_{px}(D)$, where D is closed and convex in a UCBS S. If F_G is nonempty, P_G is GAN and $\{m_i\}$ is a sequence given by process (1.1). Then $\{m_i\}$ converges strongly to any fixed point of G if G has condition (I).*

Proof. Notice from Lemma 2.1 that for any $w \in F_G$, we have that $\lim_{i\to\infty} \|m_i - w\|$ exists. We put $p = \lim_{i\to\infty} \|m_i - w\|$, where p is a constant with $0 \leq p < \infty$. Since the case when $p = 0$ is straigh forwardm, we assume that $p > 0$. Hence

$$\|m_{i+1} - w\| \leq \|m_i - w\|.$$

This gives

$$d(m_{i+1}, F_G) \leq d(m_i, F_G).$$

This shows that $\lim_{i\to\infty} d(m_i, F_G)$ exists. The aim is to prove that $\lim_{i\to\infty} d(m_i, F_G) = 0$. To prove this, we use Lemma 2.2 as $\lim_{i\to\infty} d(m_i, P_G(m_i)) = 0$. But $F(T) = F(P_T)$ due to Lemma 1.1. Therefore, we have, from condition (I) the following fact:

$$\lim_{i\to\infty} \psi(d(m_i, F_G) = 0.$$

By using $\psi(0) = 0$, we get $\lim_{i\to\infty} d(m_i, F_G) = 0$. It follows from Theorem 2.2 that $\{m_i\}$ converges to a fixed point of G. □

Using Opial's condition, we prove the following result of weak convergence.

Theorem 2.4. *Suppose we have multi-valued map $G : D \to P_{px}(D)$, where D is closed and convex in a UCBS S. If F_G is nonempty, P_G is GAN, $I - P_T$ is essentially demiclosed on zero, and $\{m_i\}$ is a sequence given by process (1.1). Then $\{m_i\}$ converges strongly to any fixed point of G if S has Opial's condition.*

Proof. As demonstrated in Lemma 2.1, the sequence u_m satisfies the condition $\lim_{i\to\infty} d(m_i, P_G(m_i)) = 0$. Additionally, by Milman-Pettis's Theorem, we know that S is reflexive. Applying Eberlein's Theorem, we can extract a subsequence m_{i_j} from the original sequence m_i that converges weakly to a limit point $w_1 \in S$. By utilizing the demicloseness property of $I - P_T$, we deduce that $w_1 \in F_{P_G} = F_G$. Our next objective is to establish that w_1 is the weak limit of the entire sequence m_i. To demonstrate this, we proceed by contradiction. Suppose there exists another point $w_2 \in S$, distinct from w_1, and a subsequence m_{i_k} of m_i that converges weakly to w_2. Consequently, w_2 must also belong to $F_{P_G} = F_G$. Invoking the Opial's condition of S together with Lemma 2.1, it leads to a contradiction as follows.

$$\begin{aligned} \lim_{i\to\infty} \|m_i - w_1\| &= \lim_{j\to\infty} \|m_{i_j} - w_1\| < \lim_{j\to\infty} \|m_{i_j} - w_2\| = \lim_{i\to\infty} \|m_i - w_2\| \\ &< \lim_{k\to\infty} \|m_{i_k} - w_2\| < \lim_{k\to\infty} \|m_{i_k} - w_1\| = \lim_{i\to\infty} \|m_i - w_1\|. \end{aligned}$$

Since we have $w_1 \neq w_2$, it follows that $\lim_{i\to\infty} \|m_i - w_1\| < \lim_{i\to\infty} \|m_i - w_2\|$, which is a contradiction. Therefore, we conclude that the sequence m_i must strongly converge to w_1. This completes the proof. □

3. Example

Now, we proceed to verify the convergence of the proposed iteration process numerically, which further demonstrates the practical applicability of our theoretical results. The primary example we consider here is only GAN but not Suzuki-type. This example goes beyond a wide range of nonexpansive and contraction mappings, highlighting the versatility of our approach. By employing this general example, we demonstrate that the newly introduced iteration scheme exhibits significantly improved convergence speed when compared to classical iterative methods commonly used for multi-valued maps in existing literature. This enhancement not only underscores the theoretical advantages of our process but also provides valuable insights into its practical efficiency in real-world applications.

Example 3.1. *Suppose $D = [6, 11]$ and set $G : D \to P_{px}(D)$ as follows.*

$$Gm = \begin{cases} [6, \frac{m+6}{2}] & \text{for } 6 \leq m < 11, \\ \{6\} & \text{for } m = 11. \end{cases}$$

Now assume that $m \in [6, 11)$, so we get $P_G(m) = \{\frac{m+6}{2}\}$ and for $m = 6$, we get $P_G(m) = \{6\}$. The aim is to prove that P_G is GAN on D but not a Suzuki-type.

Case (a). Suppose $m, n \in \{11\}$, and one has

$$\frac{1}{2}d(n, P_G(m)) + \frac{1}{2}d(m, P_G(n)) + (1 - 2(\frac{1}{2}))\|m - n\| \geq 0 = H(P_G(m), P_G(n)).$$

Case (b). Suppose $m, n \in [6, 11)$, and one has

$$\begin{aligned} \frac{1}{2}d(n, P_G(m)) + \frac{1}{2}d(m, P_G(n)) &= \frac{1}{2}d\left(n, \left\{\frac{m+6}{2}\right\}\right) + \frac{1}{2}d\left(m, \left\{\frac{n+6}{2}\right\}\right) \\ &\geq \frac{1}{2}\left(d\left(n, \left\{\frac{m+6}{2}\right\}\right) + d\left(m, \left\{\frac{n+6}{2}\right\}\right)\right) \\ &= \frac{1}{2}\left(|n - \frac{m+6}{2}| + |m - \frac{n+6}{2}|\right) \\ &\geq \frac{1}{2}|\frac{3m-3n}{2}| = \left|\frac{3m-3n}{2}\right| \\ &= \frac{3}{4}|m-n| \geq \frac{1}{2}|m-n| \\ &= H(P_G(m), P_G(n)). \end{aligned}$$

Case (c). Suppose $n = 11$ and $m \in [6, 11)$, and one has

$$\begin{aligned} \frac{1}{2}d(n, P_G(m)) + \frac{1}{2}d(m, P_G(n)) &= \frac{1}{2}d\left(n, \left\{\frac{m+6}{2}\right\}\right) + \frac{1}{2}d(m, \{6\}) \\ &\geq \frac{1}{2}d(m, \{6\}) = \frac{1}{2}|m-6| \\ &= |\frac{m-6}{2}| = H(P_G(m), P_G(n)). \end{aligned}$$

Case (d). Suppose $m = 11$ and $n \in [6, 11)$, and one has

$$\begin{aligned} \frac{1}{2}d(n, P_G(m)) + \frac{1}{2}d(m, P_G(n)) &= \frac{1}{2}d(n, \{6\}) + \frac{1}{2}d\left(m, \left\{\frac{n+6}{2}\right\}\right) \\ &\geq \frac{1}{2}d(n, \{6\}) = \frac{1}{2}|n-6| \\ &= |\frac{n-6}{2}| = H(P_G(m), P_G(n)). \end{aligned}$$

Hence P_G is GAN on D. But, for many points, P_G fails to satisfy the Suzuki condition. Hence, P_G belongs to a larger class of mappings. Using this map and $\alpha_i = 0.99$, $\beta_i = 0.80$, and $\gamma_i = 0.90$, we now provide some numerical results in Table 1. Also, 2D and 3D graphical analysis of the iterations is provided in Figures 1 and 2, respectively.

Table 1. Approximate values of different iterations.

i	New	M	Thakur	Abbas	Agarwal
1	4.6	4.6	4.6	4.6	4.6
2	6.0252500	6.0307000	6.0604000	6.1095400	6.1208000
3	6.0015939	6.0023560	6.0091204	6.0299975	6.0364816
4	6.0001006	6.0001800	6.0013772	6.0082148	6.0110174
5	6.0000064	6.0000130	6.0002080	6.0022496	6.0033273
6	6.0000004	6.0000011	6.0000314	6.0006161	6.0010048
7	6	6.0000001	6.0000047	6.0001687	6.0003035
8	6	6	6.0000007	6.0000460	6.0000916
9	6	6	6.0000001	6.0000127	6.0000277
10	6	6	6	6.0000035	6.0000084
11	6	6	6	6.0000009	6.0000025
12	6	6	6	6.0000003	6.0000008
13	6	6	6	6.0000001	6.0000002
14	6	6	6	6	6.0000001
15	6	6	6	6	6
16	6	6	6	6	6

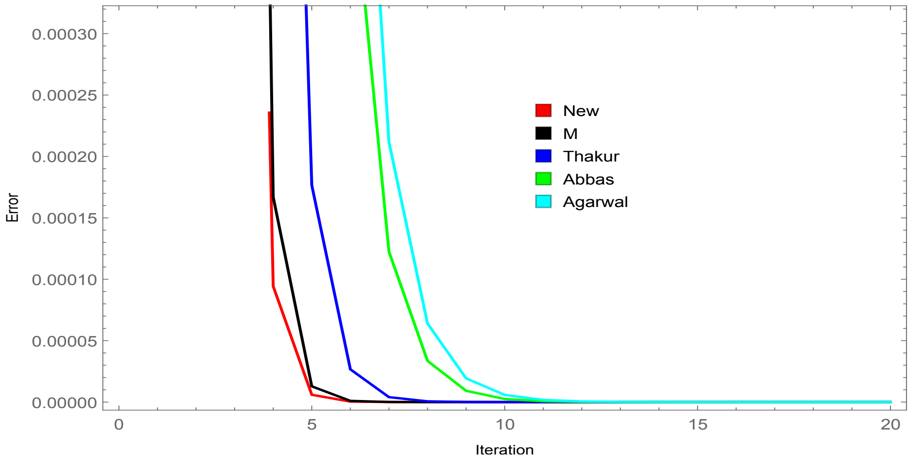

Figure 1. Comparison of graphical analysis.

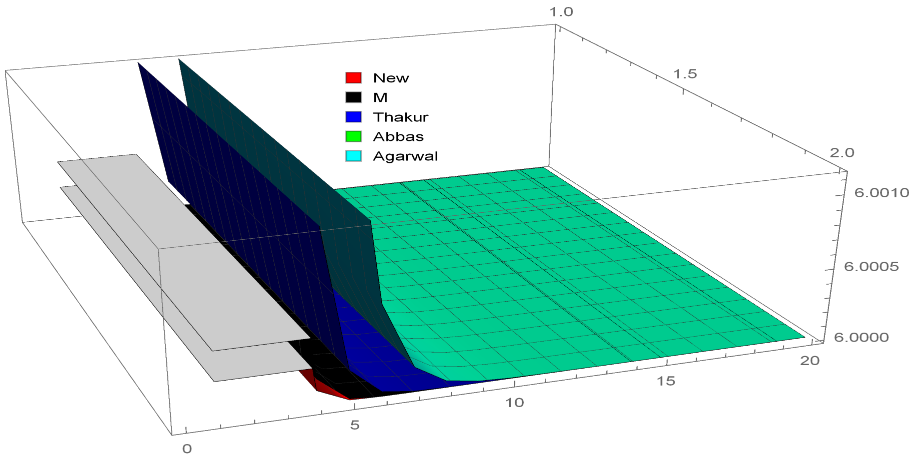

Figure 2. Comparison of graphical analysis.

Using the stopping criterion $|m_i - w| < 10^{-8}$ for the fixed point $w = 6$, we compare the number of iterations under different sets of points. Let $\alpha_i = \frac{2i}{(7i+11)^{\frac{1}{2}}}$ and $\beta_i = \frac{1}{(3i+9)^{\frac{1}{2}}}$. The observations are presented in Table 2. It is evident that our new approach significantly reduces the number of iterations needed to reach the fixed point $w = 6$ of G.

Table 2. Steps required to reach the fixed point under different starting points.

Average number of iterates to reach the fixed-point				
Starting point	Agarwal	Thakur	M	New
6.2	18	10	7	**6**
7.4	18	10	7	**6**
8.6	18	10	7	**6**
9.8	18	10	7	**6**
10.9	18	10	8	**7**

Again, we take $\alpha_i = 1 - \frac{1}{i+1}$ and $\beta_i = \sqrt{\frac{i}{i+1}}$, and obtain the following observations in Table 3.

Table 3. Steps required to reach fixed point under different starting points.

Average number of iterates to reach the fixed-point				
Starting point	Agarwal	Thakur	M	New
6.2	14	9	8	**5**
7.4	16	10	8	**5**
8.6	16	10	8	**6**
9.8	17	10	9	**6**
10.9	17	11	9	**8**

4. Application to boundary value problems

Many real-world problems can be modeled as boundary value problems (BVPs) of various orders. These BVPs are often complex, making exact solutions difficult or impossible to obtain. In such cases, fixed point theory provides effective tools to ensure the existence of solutions and to develop iterative methods for approximating them. The aim of this section is to explore the approximation of solutions for a broad class of BVPs, including those commonly encountered in fluid dynamics and other fields of applied sciences.

We solve the following class of BVPs:

$$\frac{d^2}{d\eta^2}m(\eta) = \phi(\eta, m(\eta), \frac{d}{d\eta}m(\eta)), \tag{4.1}$$

where the boundary conditions are:

$$m(0) = \alpha, \ m(1) = \beta. \tag{4.2}$$

It follows that the solution of (4.1) and (4.2) can be expressed as a fixed point of $G : S \to S$:

$$Gm = m + \int_0^1 U(\eta, \nu)(\frac{d^2}{d\eta^2}m(\nu) - \phi(\nu, m(\nu), \frac{d}{d\eta}m(\nu))d\nu, \tag{4.3}$$

where $S = C^1[0, 1]$ with its supremum norm and

$$U(\eta, \nu) = \begin{cases} \nu(1-\eta) & \text{if } \ 0 \le \nu\eta, \\ \eta(1-\nu) & \text{if } \ 0 \le \eta \le \nu \le 1. \end{cases}$$

Using (4.3), our iterative process takes the following form:

$$\begin{cases} m_1 \in C^1[0, 1], \\ l_i = Gm_i, \\ s_i = (1-\alpha_i)l_i + \alpha_i Gl_i, \\ n_i = Gs_i, \\ m_{i+1} = Gn_i, (i = 1, 2, 3, ...). \end{cases} \tag{4.4}$$

The following is our main result.

Theorem 4.1. *Suppose $\{m_i\}$ is a sequence generated by (4.4) and $\frac{1}{4\sqrt{3}} \times \sup_{[0,1]\times\mathbb{R}^3} \left|\frac{d\phi}{dm}\right| \le 1$. Then $\{m_i\}$ converges to the solutions of (4.1) and (4.2).*

Proof. Using integration on (4.3), we have

$$Gm = (\beta - \alpha)\eta + \alpha - \int_0^1 U(\eta, \nu)\phi(\nu, m, \frac{d}{d\nu}m)d\nu. \tag{4.5}$$

Hence by utilizing the well-known Mean Value Theorem on ϕ, one has

$$\begin{aligned}
|Gm - Gn| &= |[(\beta - \alpha)\eta + \alpha - \int_0^1 U(\eta, \nu)\phi(\nu, m, \frac{d}{d\nu}m)d\nu] - [(\beta - \alpha)\eta \\
&\quad +\alpha - \int_0^1 U(\eta, \nu)\phi(\nu, n, \frac{d}{d\nu}n)d\nu]| \\
&= |\int_0^1 U(\eta, \nu)\phi(\nu, m, \frac{d}{d\nu}m)d\nu - U(\eta, \nu)\phi(\nu, n, \frac{d}{d\nu}n)d\nu| \\
&= |\int_0^1 U(\eta, \nu)[\phi(\nu, m, \frac{d}{d\nu}m)d\nu - \phi(\nu, n, \frac{d}{d\nu}n)d\nu]d\nu| \\
&\leq \left(\int_0^1 |U(\eta, \nu)|^2 d\nu\right)^{\frac{1}{2}} \left(\int_0^1 |\phi(\nu, m, \frac{d}{d\nu}m)d\nu - \phi(\nu, n, \frac{d}{d\nu}n)d\nu|^{\frac{1}{2}} d\nu\right)^{\frac{1}{2}} \\
&\leq \frac{1}{4\sqrt{3}} \times \sup_{[0,1]\times\mathbb{R}^3} \left|\frac{d\phi}{dm}\right| \sup |m(\eta) - n(\eta)| \\
&\leq \|m(\eta) - n(\eta)\|.
\end{aligned}$$

Hence, G is nonexpansive and so GAN. Therefore, $\{m_i\}$ converges to a fixed point of G and also to the solutions (4.1) and (4.2). □

We support our outcome of this section by the following new example.

Example 4.1. *Assume that*

$$\frac{d^2}{d\eta^2}m(\eta) + m^2(\eta) - \eta^4 - 2 = 0, \tag{4.6}$$

where boundary conditions are:

$$m(0) = 0, \; m(1) = 1. \tag{4.7}$$

Clearly, the exact solution of (4.6) and (4.7) is $m(\eta) = \eta^2$. *Choose* $m_1(\eta) = \eta$ *and* $\alpha_i = \frac{1}{2} \in (0, 1)$, *and some numerical results are reported for our new iterative approach together with M iteration and Picard-Mann hybrid iteration in Table 4. This proves that our new approach is more effective than both of these iterative schemes.*

Table 4. Approximate solutions by our new approach for the problems (4.6) and (4.7).

η	Exact solution	Picard-Mann	M	New
0.1	0.01000	7.91352×10^{-8}	1.43000×10^{-12}	2.77556×10^{-17}
0.2	0.04000	1.57902×10^{-7}	2.86336×10^{-12}	5.55112×10^{-17}
0.3	0.09000	2.34468×10^{-7}	4.25193×10^{-12}	8.32667×10^{-17}
0.4	0.16000	3.04151×10^{-7}	5.51609×10^{-12}	1.11022×10^{-16}
0.5	0.25000	3.58385×10^{-7}	6.50080×10^{-12}	1.66533×10^{-16}
0.6	0.36000	3.84628×10^{-7}	6.97881×10^{-12}	1.11022×10^{-16}
0.7	0.49000	3.6816×10^{-7}	6.68277×10^{-12}	1.66533×10^{-16}
0.8	0.64000	2.96735×10^{-7}	5.38869×10^{-12}	1.11022×10^{-16}
0.9	0.81000	1.68305×10^{-7}	3.05744×10^{-12}	1.11022×10^{-16}

To analyze graphically our new approach for problems (4.6) and (4.7), we provided the observation in Figure 3 for different values of the parameters.

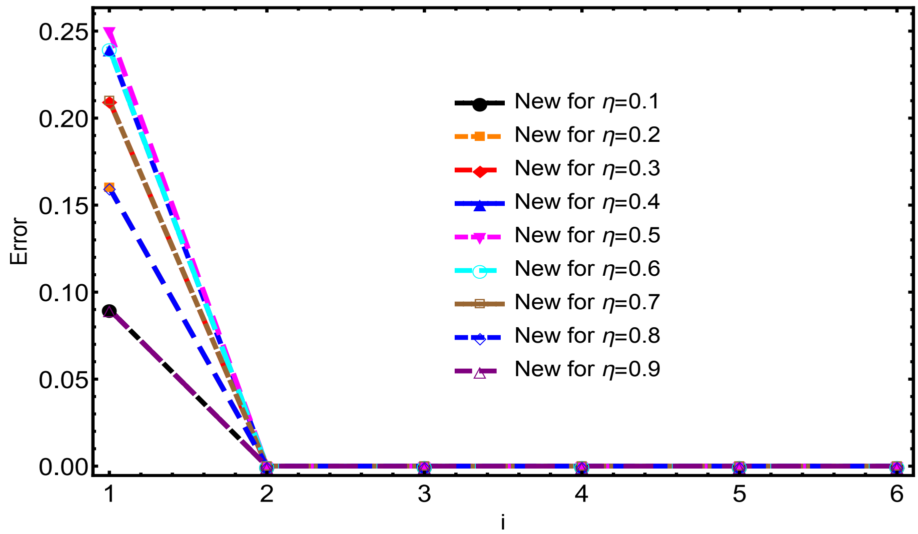

Figure 3. Comparison of graphical analysis.

5. Conclusions

In this paper, we introduced an iterative scheme for finding fixed points of proximally GAN maps and established several new results in the context of Banach spaces. Our main findings were derived under mild assumptions, enhancing the generality and applicability of our work. We demonstrated through a comprehensive example that proximally GAN maps exceeds other well-known classes in the literature, such as contraction mappings, nonexpansive mappings, and Suzuki-type maps. Additionally, we proposed various numerical and graphical experiments to illustrate the high convergence rates of our approach. For practical applications, we extended our method to a broader class of boundary value problems (BVPs), showcasing its potential utility in solving real-world problems. The results of this research improve and extend many results from the case of single-valued maps in [16–18] and other to the general setting of multi-valued maps.

Author contributions

Junaid Ahmad, Imen Ali Kallel: Conceptualization, methodology, writing-original draft; Ahmad Aloqaily, Nabil Mlaiki: Conceptualization, writing-original draft, formal Analysis, software, writing-review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Data availability

The data will be provided upon reasonable request from the corresponding author.

Acknowledgments

The authors, A. Aloqaily and N. Mlaiki, would like to thank Prince Sultan University for paying the APC and for the support through the TAS research lab.

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. S. Atallaoui, F. Hasan, W. Shatanawi, Z. Mustafa, A. Tallafha, On some fixed point results using CG simulation functions via w-distance with applications, *Heliyon*, **10** (2024), e29345. https://doi.org/10.1016/j.heliyon.2024.e29345c
2. N. Saleem, B. Iqbal, F. Hasan, W. Shatanawi, Existence results for Wardoski-type convex contractions and the theory of iterated function systems, *Symmetry*, **15** (2023), 1162. https://doi.org/10.3390/sym15061162
3. M. Zahid, F. Uddin, K. Shah, T. Abdeljawad, Fuzzy fixed point approach to study the existence of solution for Volterra type integral equations using fuzzy Sehgal contraction, *PLoS One*, **19** (2024), e0303642. https://doi.org/10.1371/journal.pone.0303642
4. S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations integrales, *Fund. Math.*, **3** (1922), 133–181.
5. E. M. Picard, Memorie sur la theorie des equations aux derivees partielles et la methode des approximation ssuccessives, *J. Math. Pure Appl.*, **6** (1890), 145–210.

6. F. E. Browder, Nonexpansive nonlinear operators in a Banach space, *Proc. Natl. Acad. Sci. U.S.A.*, **54** (1965), 1041–1044. https://doi.org/10.1073/pnas.54.4.1041
7. W. A. Kirk, A fixed point theorem for mappings which do not increase distance, *Amer. Math. Monthly*, **72** (1965), 1004–1006. https://doi.org/10.2307/2313345
8. K. Goebel, An elementary proof of the fixed point theorem of Browder and Kirk, *Michigan Math. J.*, **16** (1969), 381–383. https://doi.org/10.1307/mmj/1029000322
9. J. Ahmad, M. Arshad, A novel fixed point approach based on Green's function for solution of fourth order BVPs, *J. Appl. Math. Comput.*, **70** (2024), 2701–2719. https://doi.org/10.1007/s12190-024-02071-x
10. K. Ullah, J. Ahmad, M. S. U. Ullah, N. Muhamma, Convergence results of a faster iterative scheme including multi-valued mappings in Banach spaces, *Filomat*, **35** (2021), 1359–1368. https://doi.org/10.2298/FIL2104359U
11. R. P. Agarwal, D. O'Regon, D. R. Sahu, Iterative construction of fixed points of nearly asymptotically nonexpansive mappings, *J. Nonlinear Convex Anal.*, **8** (2007), 61–79.
12. S. Ishikawa, Fixed points by a new iteration method, *Proc. Amer. Math. Soc.*, **44** (1974), 147–150. https://doi.org/10.2307/2039245
13. S. H. Khan, A Picard-Mann hybrid iterative process, *Fixed Point Theory Appl.*, **2013** (2013), 69. https://doi.org/10.1186/1687-1812-2013-69
14. W. R. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.*, **4** (1953), 506–510.
15. M. A. Noor, New approximation schemes for general variational inequalities, *J. Math. Anal. Appl.*, **251** (2000), 217–229. https://doi.org/10.1006/jmaa.2000.7042
16. M. Abbas, T. Nazir, A new faster iteration process applied to constrained minimization and feasibility problems, *Mat. Vesnik*, **66** (2014), 223–234.
17. B. S. Thakur, D. Thakur, M. Postolache, A new iterative scheme for numerical reckoning fixed points of Suzuki's generalized nonexpansive mappings, *Appl. Math. Comput.*, **275** (2016), 147–155. https://doi.org/10.1016/j.amc.2015.11.065
18. K. Ullah, M. Arshad, Numerical reckoning fixed points for Suzuki's generalized nonexpansive mappings via new iteration process, *Filomat*, **32** (2018), 187–196. http://dx.doi.org/10.2298/FIL1801187U
19. T. Suzuki, Fixed point theorems and convergence theorems for some generalized non-expansive mapping, *J. Math. Anal. Appl.*, **340** (2008), 1088–1095. https://doi.org/10.1016/j.jmaa.2007.09.023
20. K. Aoyama, F. Kohsaka, Fixed point theorem for α-nonexpansive mappings in Banach spaces, *Nonlinear Anal.*, **74** (2011), 4387–4391. https://doi.org/10.1016/j.na.2011.03.057
21. R. Pant, R. Shukla, Approximating fixed points of generalized α-nonexpansive mappings in Banach spaces, *Numer. Funct. Anal. Optim.*, **38** (2017), 248–266. https://doi.org/10.1080/01630563.2016.1276075
22. S. B. Nadler, Multi-valued contraction mappings, *Pracific J. Math.*, **30** (1969), 745–488. http://dx.doi.org/10.2140/pjm.1969.30.475

23. T. C. Lim, A fixed point theorem for multivalued nonexpansive mappings in a uniformly convex Banach spaces, *Bull. Amer. Math. Soc.*, **80** (1974), 1123–1126.

24. A. Abkar, M. Eslamian, A fixed point theorem for generalized nonexpansive multivalued mappings, *Fixed Point Theory*, **12** (2011), 241–246.

25. H. R. Hajisharifi, On some generalization of multivalued nonexpansive mappings, *Bull. Iran. Math. Soc.*, **46** (2020), 557–571. https://doi.org/10.1007/s41980-019-00275-7

26. H. Iqbal, M. Abbas, S. M. Husnine, Existence and approximation of fixed points of multivalued generalized α-nonexpansive mappings in Banach spaces, *Numer. Algor.*, **85** (2020), 1029–1049. https://doi.org/10.1007/s11075-019-00854-z

27. K. P. R. Sastry, G. V. R. Babu, Convergence of Ishikawa iterates for a multi-valued mappings with a fixed point, *Czech. Math. J.*, **55** (2005), 817–826. https://doi.org/10.1007/s10587-005-0068-z

28. B. Panyanak, Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces, *Comput. Math. Appl.*, **54** (2007), 872–877. https://doi.org/10.1016/j.camwa.2007.03.012

29. Y. Song, H. Wang, Erratum to "Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces", *Comput. Math. Appl.*, **52** (2008), 2999–3002. https://doi.org/10.1016/j.camwa.2007.11.042

30. N. Shahzad, H. Zegeye, On Mann and Ishikawa iteration schemes for multi-valued maps in Banach spaces, *Nonlinear Anal.*, **71** (2009), 838–844. https://doi.org/10.1016/j.na.2008.10.112

31. Y. Song, Y. J. Cho, Some notes on Ishikawa iteration for multi-valued mappings, *Bull. Korean Math. Soc.*, **48** (2011), 575–584. https://doi.org/10.4134/BKMS.2011.48.3.575

32. Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, *Bull. Amer. Math. Soc.*, **73** (1967), 591–597.

33. J. Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, *Bull. Aust. Math. Soc.*, **43** (1991), 153–159. https://doi.org/10.1017/S0004972700028884

34. H. F. Senter, W. G. Dotson, Approximating fixed points of nonexpansive mappings, *Proc. Amer. Math. Soc.*, **44** (1974), 375–380.