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Research article

Spatial decay estimates for the coupled system of wave-plate type with thermal effect

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Abstract: In this article, we investigate the spatial decay estimates for the biharmonic conduction equations within a coupled wave-plate system incorporating thermal effects in a two-dimensional cylindrical domain. Using the method of a second-order differential inequality, we can obtain the spatial decay estimates result for these equations. When the distance tends to infinity, the energy can decay exponentially. This result shows us that the Saint-Venant principle is also valid for the hyperbolic-parabolic coupled system.

Keywords: spatial decay estimates; wave-plate type with thermal effect; hyperbolic-parabolic coupled equations; Saint-Venant principle
Mathematics Subject Classification: 35B44, 35B33

1. Introduction

Over the last fifty years, many authors studied the Saint-Venant principle in both applied mathematics and mechanics. The results of the classical Saint-Venant principle have been greatly expanded by a large number of investigations. In order to track the results about the Saint-Venant's principle, one could see articles by Horgan [1, 2], Horgan and Knowles [3]. The Saint-Venant type theorem states that the energy expression can decay exponentially when the axial distance from the near end to infinity along a semi-infinite strip or cylinder for different type of equations. For example, in [4], the authors studied the spatial behavior for the high order equation. In [5], the the authors studied the spatial behavior for the transient heat conduction. In [6], the partial behavior for the primitive equations was studied. In [7,8], the authors studied the spatial behavior for the fluid flow in porous medium. In order to obtain the decay results, they must impose a priori assumptions that the solutions algebraically decay to zero at infinity.

In recent years, the biharmonic equation is used to describe the behavior of a two-dimensional physical field within a plane. It can represent many different physical phenomenas, including sound waves, electric fields, and magnetic fields. Many important applications are studied in applied mathematics and mechanics. In order to obtain the Saint-Venant type result for the biharmonic equations, many studies and various methods have been proposed for researching the spatial behavior for the solutions of the biharmonic equations in a semi-infinite strip in R^2. We mention the studies by Knowles [9, 10], Flavin [11], Flavin and Knops [12], and Horgan [13]. We note that some time-dependent problems concerning the biharmonic operator were considered in the literature. We mention the papers by Liu and Lin [14], Knops and Lupoli [15], and Song [16, 17] in connection with the spatial behavior of solutions for a fourth-order transformed problem associated with the slow flow of an incompressible viscous fluid along a semi-infinite strip. Other results for the Saint-Venant principle may be found in [18–22].

In [23], the authors studied the properties of solutions for the wave plate type. The equations were a coupled system with a thermal effect. They obtained the analytic property. The exponential stability was also obtained by using the method of a C_0-semigroup. The equations have the following form:

$$\begin{cases}\rho_1 u_{,tt} - \triangle u - \mu\triangle u_{,t} + a\triangle v = 0,\\ \rho_2 v_{,tt} + \gamma\triangle^2 v + a\triangle u + m\triangle\theta = 0,\\ \tau\theta_{,t} - k\triangle\theta - m\triangle v_{,t} = 0.\end{cases} \tag{1.1}$$

The above system was used to describe the system constituted by an elastic membrane and an elastic plate that are subject to a thermal effect (see [24]). Here, u represents the vertical deflection of the membrane, v represents the vertical deflection of the plate, and θ represents the difference of temperature. $\rho_1, \rho_2, \mu, a, \gamma, m, \tau$, and k are all nonnegative coefficients.

In the present paper, we take $a = 0$. This is to say, the vertical deflection of the membrane does not have any effect on the system. Equation (1.1) turns to

$$\begin{cases}\rho v_{,tt} + \gamma\Delta^2 v + m\Delta\theta = 0,\\ \tau\theta_{,t} - k\Delta\theta - m\Delta v_{,t} = 0.\end{cases} \tag{1.2}$$

Here $\triangle$ is the harmonic operator, and $\triangle^2$ is the biharmonic operator. The comma is used to indicate partial differentiation and the differentiation, with respect to the direction x_k is denoted as $,k$, thus $u_{,\alpha}$ denotes $\frac{\partial u}{\partial x_\alpha}$, and $u_{,t}$ denotes $\frac{\partial u}{\partial t}$.

Our problem is considered on the domain Ω_0 which is an unbounded region defined by

$$\Omega_0 := \{(x_1, x_2) \mid x_1 > 0, 0 < x_2 < h\}, \tag{1.3}$$

with h being a fixed positive constant. We denote the notation

$$L_z = \{(x_1, x_2) \mid x_1 = z \geq 0, 0 \leq x_2 \leq h\}. \tag{1.4}$$

The problem is considered in the time interval $[0, T]$, where T is a fixed positive constant.

We must add some a priori asymptotic decay assumptions for solutions at infinity.

$$\begin{aligned}&v(x_1, x_2, t), \dot{v}(x_1, x_2, t), v_{,\alpha}(x_1, x_2, t), \theta(x_1, x_2, t), \theta_{,1}(x_1, x_2, t) \to 0,\\ &v_{,\alpha t}(x_1, x_2, t), v_{,\alpha\beta}(x_1, x_2, t), v_{,\alpha\beta\beta}(x_1, x_2, t) \to 0,\\ &(uniformly \quad in \quad x_2) \quad as \quad x_1 \to \infty.\end{aligned} \tag{1.5}$$

In this paper, we study the spatial decay estimates for the system (1.2). Since equations of (1.2) are hyperbolic-parabolic coupled equations, it is difficult to construct the energy function. How to control the energy function by its own differentiation will be the main difficulty of this article. We have never seen any results about the Saint-Venant principle for system (1.2). If we follow the previous method that the energy function is controlled by its own derivative, we cannot obtain the desired result for this system of equations. The weighted energy method will be used, and a second-order differential inequality will be derived. This method is firstly used in current research on the Saint-Venant principle. We think this method is applicable to the study of other biharmonic operators. From these points, the result obtained in this paper is new and interesting.

In this paper, we are concerned with the spatial decay estimates for the coupled system of wave-plate type with a thermal effect. We formulate some energy expressions in section 2. In section 3, we derive some important inequalities and formulate a second-order differential inequality. We derive our main spatial decay estimates for the solutions in section 4. The usual summation convention is employed with repeated Greek subscripts α summed from 1 to 2. Hence,

$$u_{,\alpha\alpha} = \sum_{\alpha=1}^{2} \frac{\partial^2 u}{\partial x_\alpha^2}.$$

A is an area element on the $x_1 - x_2$ plane, $dA = dx_2 d\xi$.

2. Definitions of the energy functions

In the following, we will define some energy functions that will be used in deriving our result. Multiplying both sides of $(1.2)_1$ by $\exp(-\omega\eta)v_{,\eta}(\xi - z)$ and integrating, we have

$$\begin{aligned}
0 &= \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)(\xi - z)v_{,\eta}(\rho v_{,\eta\eta} + \gamma v_{,\alpha\alpha\beta\beta} + m\theta_{,\alpha\alpha})\mathrm{d}A\mathrm{d}\eta \\
&= \frac{\omega}{2}\rho \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)(\xi - z)v_{,\eta}^2 \mathrm{d}A\mathrm{d}\eta + \frac{\rho}{2}\int_z^\infty \int_{L_\xi} \exp(-\omega t)(\xi - z)v_{,t}^2 \mathrm{d}A \\
&\quad - \gamma \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)(\xi - z)v_{,\alpha\eta}v_{,\alpha\beta\beta}\mathrm{d}A\mathrm{d}\eta - \gamma \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)v_{,\eta}v_{,1\beta\beta}\mathrm{d}A\mathrm{d}\eta \\
&\quad - m \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)(\xi - z)v_{,\alpha\eta}\theta_{,\alpha}\mathrm{d}A\mathrm{d}\eta - \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)v_{,\eta}\theta_{,1}\mathrm{d}A\mathrm{d}\eta \\
&= \frac{\omega\rho}{2} \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)(\xi - z)v_{,\eta}^2 \mathrm{d}A\mathrm{d}\eta + \frac{\rho}{2}\int_z^\infty \int_{L_\xi} \exp(-\omega t)(\xi - z)v_{,t}^2 \mathrm{d}A \\
&\quad + \frac{\gamma\omega}{2} \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)(\xi - z)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta + \gamma \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha 1}\mathrm{d}A\mathrm{d}\eta \\
&\quad - \gamma \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)v_{,\eta}v_{,1\beta\beta}\mathrm{d}A\mathrm{d}\eta - m \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)(\xi - z)v_{,\alpha\eta}\theta_{,\alpha}\mathrm{d}A\mathrm{d}\eta \\
&\quad - m \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)v_{,\eta}\theta_{,1}\mathrm{d}A\mathrm{d}\eta + \frac{\gamma}{2}\int_z^\infty \int_{L_\xi} \exp(-\omega t)(\xi - z)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A.
\end{aligned} \tag{2.1}$$

We define a function

$$
\begin{aligned}
F_1(z,t) = {} & \frac{\omega\rho}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\eta}^2\mathrm{d}A\mathrm{d}\eta + \frac{\rho}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)(\xi-z)v_{,t}^2\mathrm{d}A \\
& + \frac{\gamma\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta + \frac{\gamma}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)(\xi-z)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A \\
& - m\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\alpha\eta}\theta_{,\alpha}\mathrm{d}A\mathrm{d}\eta.
\end{aligned}
\tag{2.2}
$$

Inserting (2.2) into (2.1), we have

$$
\begin{aligned}
F_1(z,t) = {} & -\gamma\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha 1}\mathrm{d}A\mathrm{d}\eta + \gamma\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}v_{,1\beta\beta}\mathrm{d}A\mathrm{d}\eta \\
& + m\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}\theta_{,1}\mathrm{d}A\mathrm{d}\eta.
\end{aligned}
\tag{2.3}
$$

Multiplying both sides of $(1.2)_2$ by $\exp(-\omega\eta)(\xi-z)v_{,\eta}$ and integrating, we have

$$
\begin{aligned}
0 = {} & \int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\eta}(\tau\theta_{,\eta} - k\theta_{,\alpha\alpha} - mv_{,\alpha\alpha\eta})\mathrm{d}A\mathrm{d}\eta \\
= {} & \tau\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\eta}\theta_{,\eta}\mathrm{d}A\mathrm{d}\eta + k\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\alpha\eta}\theta_{,\alpha}\mathrm{d}A\mathrm{d}\eta \\
& + k\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}\theta_{,1}\mathrm{d}A\mathrm{d}\eta + m\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\alpha\eta}v_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta \\
& - m\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}v_{,1\eta}\mathrm{d}A\mathrm{d}\eta \\
= {} & \tau\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\eta}\theta_{,\eta}\mathrm{d}A\mathrm{d}\eta + k\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\alpha\eta}\theta_{,\alpha}\mathrm{d}A\mathrm{d}\eta \\
& + k\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}\theta_{,1}\mathrm{d}A\mathrm{d}\eta + m\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\alpha\eta}v_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta \\
& + \frac{m}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned}
\tag{2.4}
$$

We define a function

$$
\begin{aligned}
F_2(z,t) = {} & \tau\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\alpha\eta}\theta_{,\alpha}\mathrm{d}A\mathrm{d}\eta + k\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\eta}\theta_{,\eta}\mathrm{d}A\mathrm{d}\eta \\
& + m\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\alpha\eta}v_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta.
\end{aligned}
\tag{2.5}
$$

Inserting (2.5) into (2.4), we can also obtain

$$
F_2(z,t) = -k\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}\theta_{,1}\mathrm{d}A\mathrm{d}\eta - \frac{m}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta. \tag{2.6}
$$

Multiplying both sides of $(1.2)_1$ by $\exp(-\omega\eta)(\xi-z)\theta$ and integrating,

$$\begin{aligned}
0 =& \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)(\xi-z)\theta(\rho v_{,\eta\eta} + \gamma v_{\alpha\alpha\beta\beta} + m\theta_{\alpha\alpha})\mathrm{d}A\mathrm{d}\eta \\
=& -\rho \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)(\xi-z)\theta_{,\eta} v_{,\eta}\mathrm{d}A\mathrm{d}\eta + \rho\omega \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)(\xi-z)\theta v_{,\eta}\mathrm{d}A\mathrm{d}\eta \\
&+ \rho \int_z^\infty \int_{L_\xi} \exp(-\omega t)(\xi-z)\theta v_{,t}\mathrm{d}A - \gamma \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)(\xi-z)\theta_{,\alpha} v_{,\alpha\beta\beta}\mathrm{d}A\mathrm{d}\eta \\
&- \gamma \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)\theta v_{,1\beta\beta}\mathrm{d}A\mathrm{d}\eta - m \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)(\xi-z)\theta_{,\alpha}\theta_{,\alpha}\mathrm{d}A\mathrm{d}\eta \\
&- m \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)\theta\theta_{,1}\mathrm{d}A\mathrm{d}\eta.
\end{aligned} \tag{2.7}$$

We define a function

$$\begin{aligned}
F_3(z,t) =& -\rho \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)(\xi-z)\theta_{,\eta} v_{,\eta}\mathrm{d}A\mathrm{d}\eta + \rho \int_z^\infty \int_{L_\xi} \exp(-\omega t)(\xi-z)\theta v_{,t}\mathrm{d}A \\
&+ \omega\rho \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)(\xi-z)\theta v_{,\eta}\mathrm{d}A\mathrm{d}\eta - m \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)(\xi-z)\theta_{,\alpha}\theta_{,\alpha}\mathrm{d}A\mathrm{d}\eta \\
&- \gamma \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)(\xi-z)\theta_{,\alpha} v_{,\alpha\beta\beta}\mathrm{d}A\mathrm{d}\eta.
\end{aligned} \tag{2.8}$$

Inserting (2.8) into (2.7), we can also obtain another expression of $F_3(z,t)$.

$$F_3(z,t) = \gamma \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)\theta v_{,1\beta\beta}\mathrm{d}A\mathrm{d}\eta + \frac{m}{2} \int_0^t \int_{L_z} \exp(-\omega\eta)\theta^2 \mathrm{d}x_2\mathrm{d}\eta. \tag{2.9}$$

Multiplying both sides of $(1.2)_2$ by $\exp(-\omega\eta)(\xi-z)\theta$ and integrating, we have

$$\begin{aligned}
0 =& \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)(\xi-z)\theta(\tau\theta_{,\eta} - k\theta_{,\alpha\alpha} - m v_{,\alpha\alpha\eta})\mathrm{d}A\mathrm{d}\eta \\
=& \frac{\tau\omega}{2} \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)(\xi-z)\theta^2\mathrm{d}A\mathrm{d}\eta + \frac{\tau}{2} \int_z^\infty \int_{L_\xi} \exp(-\omega t)(\xi-z)\theta^2\mathrm{d}A \\
&+ k \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)(\xi-z)\theta_{,\alpha}\theta_{,\alpha}\mathrm{d}A\mathrm{d}\eta + k \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)\theta\theta_{,1}\mathrm{d}x_2\mathrm{d}\eta \\
&+ m \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)(\xi-z)\theta_{,\alpha} v_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta + m \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)\theta v_{,1\eta}\mathrm{d}A\mathrm{d}\eta \\
=& \frac{\tau\omega}{2} \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)(\xi-z)\theta^2\mathrm{d}A\mathrm{d}\eta + \frac{\tau}{2} \int_z^\infty \int_{L_\xi} \exp(-\omega t)(\xi-z)\theta^2\mathrm{d}A \\
&+ k \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)(\xi-z)\theta_{,\alpha}\theta_{,\alpha}\mathrm{d}A\mathrm{d}\eta - \frac{k}{2} \int_0^t \int_{L_z} \exp(-\omega\eta)\theta^2\mathrm{d}x_2\mathrm{d}\eta \\
&+ m \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)(\xi-z)\theta_{,\alpha} v_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta + m \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)\theta v_{,1\eta}\mathrm{d}A\mathrm{d}\eta.
\end{aligned} \tag{2.10}$$

We define a function

$$
\begin{aligned}
F_4(z,t) = {} & \frac{\tau\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)\theta^2\mathrm{d}A\mathrm{d}\eta + \frac{\tau}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)(\xi-z)\theta^2\mathrm{d}A \\
& + k\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)\theta_{,\alpha}\theta_{,\alpha}\mathrm{d}A\mathrm{d}\eta + m\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta v_{,1\eta}\mathrm{d}A\mathrm{d}\eta \\
& + m\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)\theta_{,\alpha}v_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta.
\end{aligned}
\tag{2.11}
$$

Inserting (2.11) into (2.10), we can obtain another expression of $F_4(z,t)$

$$
F_4(z,t) = \frac{k}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)\theta^2\mathrm{d}x_2\mathrm{d}\eta. \tag{2.12}
$$

Multiplying both sides of $(1.2)_1$ by $\exp(-\omega\eta)(\xi-z)v_{,\alpha\alpha}$ and integrating, we have

$$
\begin{aligned}
0 = {} & \int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\alpha\alpha}(\rho v_{,\eta\eta}+\gamma v_{\alpha\alpha\beta\beta}+m\theta_{\beta\beta})\mathrm{d}A\mathrm{d}\eta \\
= {} & -\rho\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\alpha\alpha\eta}v_{,\eta}\mathrm{d}A\mathrm{d}\eta - m\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\alpha}\theta_{,\beta}\mathrm{d}A\mathrm{d}\eta \\
& + \rho\int_z^\infty\int_{L_\xi}\exp(-\omega t)(\xi-z)v_{,\alpha\alpha}v_{,\eta}\mathrm{d}A - \gamma\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\alpha\alpha\beta}v_{,\alpha\alpha\beta}\mathrm{d}A\mathrm{d}\eta \\
& - \gamma\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\alpha}v_{,1\alpha\alpha}\mathrm{d}A\mathrm{d}\eta - m\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\alpha\alpha\beta}\theta_{,\beta}\mathrm{d}A\mathrm{d}\eta \\
& + \rho\omega\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\alpha\alpha}v_{,\eta}\mathrm{d}A\mathrm{d}\eta \qquad (2.13) \\
= {} & \rho\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\alpha\eta}v_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta + \rho\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,1\eta}v_{,\eta}\mathrm{d}A\mathrm{d}\eta \\
& + \rho\omega\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\alpha\alpha}v_{,\eta}\mathrm{d}A\mathrm{d}\eta + \rho\int_z^\infty\int_{L_\xi}\exp(-\omega t)(\xi-z)v_{,\alpha\alpha}v_{,\eta}\mathrm{d}A \\
& - \gamma\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\alpha\alpha\beta}v_{,\alpha\alpha\beta}\mathrm{d}A\mathrm{d}\eta + \frac{\gamma}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\alpha\alpha}v_{,\beta\beta}\mathrm{d}x_2\mathrm{d}\eta \\
& - m\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\alpha\alpha\beta}\theta_{,\beta}\mathrm{d}A\mathrm{d}\eta - m\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\alpha}\theta_{,\beta}\mathrm{d}A\mathrm{d}\eta.
\end{aligned}
$$

We define a function

$$
\begin{aligned}
F_5(z,t) = {} & -\rho\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\alpha\eta}v_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta - \rho\int_z^\infty\int_{L_\xi}\exp(-\omega t)(\xi-z)v_{,\alpha\alpha}v_{,\eta}\mathrm{d}A \\
& + \gamma\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\alpha\alpha\beta}v_{,\alpha\alpha\beta}\mathrm{d}A\mathrm{d}\eta \\
& + m\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\alpha\alpha\beta}\theta_{,\beta}\mathrm{d}A\mathrm{d}\eta
\end{aligned}
$$

$$- \rho\omega \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)(\xi - z)v_{,\alpha\alpha}v_{,\eta}\mathrm{d}A\mathrm{d}\eta. \tag{2.14}$$

Inserting (2.14) into (2.13), we also have

$$\begin{aligned} F_5(z,t) = &-\frac{\rho}{2}\int_0^t \int_{L_z} \exp(-\omega\eta)v_{,\eta}^2 \mathrm{d}x_2\mathrm{d}\eta + \frac{\gamma}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\alpha\alpha}v_{,\beta\beta}\mathrm{d}x_2\mathrm{d}\eta \\ &- m\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\alpha}\theta_{,\beta}\mathrm{d}A\mathrm{d}\eta. \end{aligned} \tag{2.15}$$

We now define a new function

$$F(z,t) = F_1(z,t) + k_1F_2(z,t) + k_1\frac{\tau}{\rho}F_3(z,t) + k_2F_4(z,t) + k_3F_5(z,t), \tag{2.16}$$

with k_1, k_2, and k_3 being positive constants that will be determined later.

A combination of (2.2), (2.5), (2.8), (2.11), (2.14), and (2.16) gives

$$\begin{aligned} F(z,t) = &\frac{\omega\rho}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi - z)v_{,\eta}^2\mathrm{d}A\mathrm{d}\eta + \frac{\gamma\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi - z)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta \\ &+ \frac{\omega\rho}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)(\xi - z)v_{,t}^2\mathrm{d}A + (k_1k + m)\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi - z)v_{,\alpha\eta}\theta_{,\alpha}\mathrm{d}A\mathrm{d}\eta \\ &+ \tilde{k_1}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi - z)v_{,\alpha\eta}v_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta + \tilde{k_2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)(\xi - z)\theta v_{,t}\mathrm{d}A \\ &+ \tilde{k_3}\omega\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi - z)\theta v_{,\eta}\mathrm{d}A\mathrm{d}\eta - \tilde{k_4}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi - z)\theta_{,\alpha}v_{,\alpha\beta\beta}\mathrm{d}A\mathrm{d}\eta \\ &+ \tilde{k_5}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi - z)\theta_{,\alpha}\theta_{,\alpha}\mathrm{d}A\mathrm{d}\eta + k_2\frac{\tau}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)(\xi - z)\theta^2\mathrm{d}A \\ &+ \tilde{k_6}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi - z)\theta^2\mathrm{d}A\mathrm{d}\eta + \tilde{k_7}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi - z)\theta_{,\alpha}v_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta \\ &+ k_2m\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta v_{,1\eta}\mathrm{d}A\mathrm{d}\eta - k_3\rho\omega\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi - z)v_{,\alpha\alpha}v_{,\eta}\mathrm{d}A\mathrm{d}\eta \\ &- k_3\rho\int_z^\infty\int_{L_\xi}\exp(-\omega t)(\xi - z)v_{,\alpha\alpha}v_{,\eta}\mathrm{d}A + k_3\gamma\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi - z)v_{,\alpha\alpha\beta}v_{,\alpha\alpha\beta}\mathrm{d}A\mathrm{d}\eta \\ &+ k_3m\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi - z)v_{,\alpha\alpha\beta}\theta_{,\beta}\mathrm{d}A\mathrm{d}\eta + \frac{\gamma}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)(\xi - z)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A, \end{aligned} \tag{2.17}$$

with

$$\tilde{k_1} = (k_1m - k_3\rho),\ \ \tilde{k_2} = k_1\frac{\tau}{\rho},\ \ \tilde{k_3} = k_1\frac{\tau}{\rho},\ \ \tilde{k_4} = k_1\frac{\tau}{\rho}\gamma,$$

$$\tilde{k_5} = \left(k_2k - k_1\frac{\tau}{\rho}m\right),\ \ \tilde{k_6} = k_2\frac{\tau\omega}{2},\ \ \tilde{k_7} = k_2m.$$

From the definition of $F(z,t)$ in (2.16), we can also get another expression of $F(z,t)$ by combining Eqs (2.3), (2.6), (2.9), (2.12), and (2.15)

$$\begin{aligned} F(z,t) =& -\gamma \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta) v_{,\alpha\eta} v_{,\alpha 1} \mathrm{d}A\mathrm{d}\eta + \gamma \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta) v_{,\eta} v_{,1\beta\beta} \mathrm{d}A\mathrm{d}\eta \\ & - (m+k) \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta) v_{,\eta} \theta_{,1} \mathrm{d}A\mathrm{d}\eta + \frac{k}{2} \int_0^t \int_{L_z} \exp(-\omega\eta) \theta^2 \mathrm{d}x_2\mathrm{d}\eta \\ & - \frac{(m+\rho)}{2} \int_0^t \int_{L_z} \exp(-\omega\eta) v_{,\eta}^2 \mathrm{d}x_2\mathrm{d}\eta + \gamma \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta) \theta v_{,1\beta\beta} \mathrm{d}A\mathrm{d}\eta \\ & + \frac{m}{2} \int_0^t \int_{L_z} \exp(-\omega\eta) \theta^2 \mathrm{d}x_2\mathrm{d}\eta - m \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta) v_{,\alpha\alpha} \theta_{,\beta} \mathrm{d}A\mathrm{d}\eta \\ & + \frac{\gamma}{2} \int_0^t \int_{L_z} \exp(-\omega\eta) v_{,\alpha\alpha} v_{,\beta\beta} \mathrm{d}x_2\mathrm{d}\eta. \end{aligned} \tag{2.18}$$

Equalities (2.17) and (2.18) will play important roles in deriving the main result of this paper in the next section.

3. Some basic inequalities

We now begin to bound $F(z,t)$ in (2.17).

Using the Schwarz inequality, the fourth term on the right side of (2.17) can be bounded by

$$\begin{aligned} &\left| (k_1 k + m) \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)(\xi - z) v_{,\alpha\eta} \theta_{,\alpha} \mathrm{d}A\mathrm{d}\eta \right| \\ \le& \frac{k_1 m}{8} \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)(\xi - z) v_{,\alpha\eta} v_{,\alpha\eta} \mathrm{d}A\mathrm{d}\eta \\ & + \frac{2(k_1 k + m)^2}{k_1 m} \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)(\xi - z) \theta_{,\alpha} \theta_{,\alpha} \mathrm{d}A\mathrm{d}\eta, \end{aligned} \tag{3.1}$$

the sixth term on the right side of (2.17) can be bounded by

$$\begin{aligned} &\left| k_1 \frac{\tau}{\rho} \omega \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)(\xi - z) \theta v_{,\eta} \mathrm{d}A\mathrm{d}\eta \right| \\ \le& k_1 \frac{\tau}{2\rho} \omega \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)(\xi - z) \theta^2 \mathrm{d}A\mathrm{d}\eta \\ & + k_1 \frac{\tau}{2\rho} \omega \int_0^t \int_z^\infty \int_{L_\xi} \exp(-\omega\eta)(\xi - z) v_{,\eta}^2 \mathrm{d}A\mathrm{d}\eta, \end{aligned} \tag{3.2}$$

the seventh term on the right side of (2.17) can be bounded by

$$\begin{aligned} &\left| k_1 \frac{\tau}{\rho} \int_z^\infty \int_{L_\xi} \exp(-\omega t)(\xi - z) \theta v_{,t} \mathrm{d}A \right| \\ \le& k_1 \frac{\tau}{2\rho} \int_z^\infty \int_{L_\xi} \exp(-\omega t)(\xi - z) \theta^2 \mathrm{d}A + k_1 \frac{\tau}{2\rho} \int_z^\infty \int_{L_\xi} \exp(-\omega t)(\xi - z) v_{,t}^2 \mathrm{d}A, \end{aligned} \tag{3.3}$$

the eighth term on the right side of (2.17) can be bounded by

$$\begin{aligned}
&\left|-\frac{k_1\tau}{\rho}\gamma\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)\theta_{,\alpha}v_{,\alpha\beta\beta}\mathrm{d}A\mathrm{d}\eta\right|\\
\leq&\frac{k_1^2\tau^2}{\rho^2k_3\gamma}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)\theta_{,\alpha}\theta_{,\alpha}\mathrm{d}A\mathrm{d}\eta\\
&+\frac{k_3\gamma}{4}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\alpha\beta\beta}v_{,\alpha\beta\beta}\mathrm{d}A\mathrm{d}\eta,
\end{aligned}\tag{3.4}$$

the tenth term on the right side of (2.17) can be bounded by

$$\begin{aligned}
&\left|k_2m\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)\theta_{,\alpha}v_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta\right|\\
\leq&\frac{2k_2^2m}{k_1}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)\theta_{,\alpha}\theta_{,\alpha}\mathrm{d}A\mathrm{d}\eta\\
&+\frac{k_1m}{8}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\alpha\eta}v_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta,
\end{aligned}\tag{3.5}$$

the eleventh term on the right side of (2.17) can be bounded by

$$\begin{aligned}
&\left|k_2m\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta v_{,1\eta}\mathrm{d}A\mathrm{d}\eta\right|\\
\leq&\frac{2(k_2m)^2}{k_1m}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta^2\mathrm{d}A\mathrm{d}\eta\\
&+\frac{k_1m}{8}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,1\eta}^2\mathrm{d}A\mathrm{d}\eta,
\end{aligned}\tag{3.6}$$

the twelfth term on the right side of (2.17) can be bounded by

$$\begin{aligned}
&\left|k_3\rho\omega\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\alpha\alpha}v_{,\eta}\mathrm{d}A\mathrm{d}\eta\right|\\
\leq&\frac{k_3\rho\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\alpha\alpha}v_{,\beta\beta}\mathrm{d}A\mathrm{d}\eta\\
&+\frac{k_3\rho\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\eta}^2\mathrm{d}A\mathrm{d}\eta,
\end{aligned}\tag{3.7}$$

the thirteenth term on the right side of (2.17) can be bounded by

$$\begin{aligned}
&\left|k_3\rho\int_z^\infty\int_{L_\xi}\exp(-\omega t)(\xi-z)v_{,\alpha\alpha}v_{,t}\mathrm{d}A\right|\\
\leq&\frac{k_3\rho}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)(\xi-z)v_{,\alpha\alpha}v_{,\beta\beta}\mathrm{d}A\\
&+\frac{k_3\rho}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)(\xi-z)v_{,t}^2\mathrm{d}A,
\end{aligned}\tag{3.8}$$

and the fifteenth term on the right side of (2.17) can be bounded by

$$\begin{aligned}
&\left|k_3 m\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\alpha\alpha\beta}\theta_{,\beta}\mathrm{d}A\mathrm{d}\eta\right|\\
\leq&\frac{k_3\gamma}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\alpha\alpha\beta}v_{,\alpha\alpha\beta}\mathrm{d}A\mathrm{d}\eta\\
&+\frac{k_3 m^2}{2\gamma}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)\theta_{,\beta}\theta_{,\beta}\mathrm{d}A\mathrm{d}\eta.
\end{aligned}\tag{3.9}$$

Combining (2.17) and (3.1)–(3.9), we obtain

$$\begin{aligned}
F(z,t)\geq&\left(\frac{\omega\rho}{2}-\frac{k_3\omega\rho}{2}-\frac{k_1\tau\omega}{2\rho}\right)\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\eta}^2\mathrm{d}A\mathrm{d}\eta\\
&+\left(\frac{\rho}{2}-\frac{\tau k_1}{2\rho}-\frac{k_3\rho}{2}\right)\int_z^\infty\int_{L_\xi}\exp(-\omega t)(\xi-z)v_{,t}^2\mathrm{d}A\\
&+\left(\frac{\gamma\omega}{2}-\frac{k_3\rho\omega}{2}\right)\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta\\
&+\left(\frac{\gamma}{2}-\frac{k_3\rho}{2}\right)\int_z^\infty\int_{L_\xi}\exp(-\omega t)(\xi-z)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A\\
&+\left(\frac{3k_1 m}{4}-k_3\rho\right)\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\alpha\eta}v_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta\\
&+\left(\frac{k_2\tau\omega}{2}-\frac{k_1\tau\omega}{2\rho}-\frac{2(k_2 m)^2}{k_1 m}\right)\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)\theta^2\mathrm{d}A\mathrm{d}\eta\\
&+\left(\frac{k_2\tau}{2}-\frac{k_1\tau}{2\rho}\right)\int_z^\infty\int_{L_\xi}\exp(-\omega t)(\xi-z)\theta^2\mathrm{d}A\\
&+\tilde{k}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)\theta_{,\beta}\theta_{,\beta}\mathrm{d}A\mathrm{d}\eta\\
&+\frac{k_3\gamma}{4}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\alpha\alpha\beta}v_{,\alpha\alpha\beta}\mathrm{d}A\mathrm{d}\eta,
\end{aligned}\tag{3.10}$$

with

$$\tilde{k}=\left(k_2 k-\frac{k_1\tau m}{\rho}-\frac{2(k_1 k+m)^2}{k_1 m}-\frac{(k_1\tau)^2}{\rho^2 k_3\gamma}-\frac{2k_2^2 m}{k_1}-\frac{k_3 m^2}{2\gamma}\right).$$

If we suggest

$$k>\frac{2\left(\frac{k_1\tau m}{\rho}+\frac{2(k_1 k+m)^2}{k_1 m}+\frac{(k_1\tau)^2}{\rho^2 k_3\gamma}+\frac{2k_2^2 m}{k_1}+\frac{k_3 m^2}{2\gamma}\right)}{k_2}$$

and choose

$$k_1=\frac{\rho^2}{4\tau},\ \ k_2=\frac{2k_1}{\rho},\ \ k_3=min\left\{\frac{\gamma}{2\rho},\frac{1}{4},\frac{mk_1}{4\rho}\right\},\ \ \omega=\frac{16k_2 m}{k_1\tau},$$

we have

$$
\begin{aligned}
F(z,t) \geq\ & \frac{\omega\rho}{4}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\eta}^2\mathrm{d}A\mathrm{d}\eta+\frac{\rho}{4}\int_z^\infty\int_{L_\xi}\exp(-\omega t)(\xi-z)v_{,t}^2\mathrm{d}A \\
&+\frac{\gamma\omega}{4}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta+\frac{\gamma}{4}\int_z^\infty\int_{L_\xi}\exp(-\omega t)(\xi-z)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A \\
&+\frac{k_1 m}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\alpha\eta}v_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta+\frac{k_2\tau}{4}\int_z^\infty\int_{L_\xi}\exp(-\omega t)(\xi-z)\theta^2\mathrm{d}A \\
&+\frac{k_2\tau\omega}{8}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)\theta^2\mathrm{d}A\mathrm{d}\eta \\
&+\frac{k_1 k}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)\theta_{,\beta}\theta_{,\beta}\mathrm{d}A\mathrm{d}\eta \\
&+\frac{k_3\gamma}{4}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\alpha\alpha\beta}v_{,\alpha\alpha\beta}\mathrm{d}A\mathrm{d}\eta \\
&= G(z,t).
\end{aligned}
\tag{3.11}
$$

In this part we will derive a second-order differential inequality to obtain our result.

Differentiating (2.17) with respect to z and using the same method as deriving (3.11), we have

$$
\begin{aligned}
-\frac{\partial F(z,t)}{\partial z} \geq\ & \frac{\omega\rho}{4}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\eta}^2\mathrm{d}A\mathrm{d}\eta+\frac{\rho}{4}\int_z^\infty\int_{L_\xi}\exp(-\omega t)v_{,t}^2\mathrm{d}A \\
&+\frac{\gamma\omega}{4}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta+\frac{\gamma}{4}\int_z^\infty\int_{L_\xi}\exp(-\omega t)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A \\
&+\frac{k_1 m}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta+\frac{k_2\tau\omega}{8}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta^2\mathrm{d}A\mathrm{d}\eta \\
&+\frac{k_2\tau}{4}\int_z^\infty\int_{L_\xi}\exp(-\omega t)\theta^2\mathrm{d}A+\frac{k_1 k}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta_{,\beta}\theta_{,\beta}\mathrm{d}A\mathrm{d}\eta \\
&+\frac{k_3\gamma}{4}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)v_{,\alpha\alpha\beta}v_{,\alpha\alpha\beta}\mathrm{d}A\mathrm{d}\eta.
\end{aligned}
\tag{3.12}
$$

Differentiating (2.17) again with respect to z, we also obtain

$$
\begin{aligned}
\frac{\partial^2 F(z,t)}{\partial z^2} \geq\ & \frac{\omega\rho}{4}\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta+\frac{\rho}{4}\int_{L_z}\exp(-\omega t)v_{,t}^2\mathrm{d}x_2 \\
&+\frac{\gamma\omega}{4}\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}x_2\mathrm{d}\eta+\frac{\gamma}{4}\int_{L_z}\exp(-\omega t)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}x_2 \\
&+\frac{k_1 m}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\alpha\eta}v_{,\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta+\frac{k_2\tau\omega}{8}\int_0^t\int_{L_z}\exp(-\omega\eta)\theta^2\mathrm{d}x_2\mathrm{d}\eta \\
&+\frac{k_2\tau}{4}\int_{L_z}\exp(-\omega t)\theta^2\mathrm{d}x_2+\frac{k_1 k}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)\theta_{,\beta}\theta_{,\beta}\mathrm{d}x_2\mathrm{d}\eta \\
&+\frac{k_3\gamma}{4}\int_0^t\int_{L_z}\exp(-\omega\eta)v_{,\alpha\alpha\beta}v_{,\alpha\alpha\beta}\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned}
\tag{3.13}
$$

Using the Schwarz inequality in (2.18) and combining (3.12) and (3.13), we can obtain

$$F(z,t) \leq k_4\left(\frac{-\partial F(z,t)}{\partial z}\right) + k_5\frac{\partial^2 F(z,t)}{\partial z^2}, \tag{3.14}$$

with k_4 and k_5 being computable positive constants.

Inequality (3.14) is the key inequality that will be used in deriving our main result.

4. Spatial decay estimates for the solutions

We will obtain the following theory in this paper.

Theorem 4.1. *Let* (u, v) *be a classical solution (the solution is smooth and differentiable) of the initial boundary value problems* (1.2)–(1.5)*. For the energy expression* $G(z,t)$ *defined in* (3.11)*, we can obtain the decay estimates*

$$\begin{aligned}
&\frac{\omega\rho}{4}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\eta}^2\mathrm{d}A\mathrm{d}\eta + \frac{\rho}{4}\int_z^\infty\int_{L_\xi}\exp(-\omega t)(\xi-z)v_{,t}^2\mathrm{d}A\\
&+\frac{\gamma\omega}{4}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta + \frac{\gamma}{4}\int_z^\infty\int_{L_\xi}\exp(-\omega t)(\xi-z)v_{,\alpha\beta}v_{,\alpha\beta}\mathrm{d}A\\
&+\frac{k_1 m}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\alpha\eta}v_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta + \frac{k_2\tau}{4}\int_z^\infty\int_{L_\xi}\exp(-\omega t)(\xi-z)\theta^2\mathrm{d}A\\
&+\frac{k_2\tau\omega}{8}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)\theta^2\mathrm{d}A\mathrm{d}\eta + \frac{k_1 k}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)\theta_{,\beta}\theta_{,\beta}\mathrm{d}A\mathrm{d}\eta\\
&+\frac{k_3\gamma}{4}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(\xi-z)v_{,\alpha\alpha\beta}v_{,\alpha\alpha\beta}\mathrm{d}A\mathrm{d}\eta\\
&\leq F(0,t)\mathrm{e}^{-k_7 z},
\end{aligned}\tag{4.1}$$

where k_7 *is a positive constant that will be defined later.*

Proof. We now rewrite (3.14) as the following inequality:

$$\frac{\partial^2 F}{\partial z^2} - \frac{k_4}{k_5}\frac{\partial F}{\partial z} - \frac{1}{k_5} \geq 0. \tag{4.2}$$

Inequality (4.2) can be rewritten as

$$\left(\frac{\partial}{\partial z} - k_6\right)\left[\frac{\partial\varphi(z,t)}{\partial z} + k_7\varphi(z,t)\right] \geq 0, \tag{4.3}$$

where k_6 and k_7 satisfy

$$k_7 - k_6 = -\frac{k_4}{k_5}, \quad k_6 k_7 = \frac{1}{k_5}. \tag{4.4}$$

Solving (4.4), we have

$$k_6 = \frac{1}{2}\left(\sqrt{\left(\frac{k_4}{k_5}\right)^2 + \frac{4}{k_5}} + \frac{k_4}{k_5}\right), \ \ k_7 = \frac{1}{2}\left(\sqrt{\left(\frac{k_4}{k_5}\right)^2 + \frac{4}{k_5}} - \frac{k_4}{k_5}\right).$$

Inequality (4.3) can be rewritten as

$$\frac{\partial}{\partial z}\left[\exp(-k_6 z)\left(\frac{\partial \varphi(z,t)}{\partial z}+k_7\varphi(z,t)\right)\right]\geq 0. \tag{4.5}$$

Integrating (4.5) from z to ∞, we have

$$\frac{\partial \varphi(z,t)}{\partial z}+k_7\varphi(z,t)\leq 0. \tag{4.6}$$

Solving (4.6) and using (3.11), we can obtain the desired result (4.1). □

5. Conclusions

Inequality (4.1) shows the spatial decay estimates result that the solutions can decay exponentially as the distance from the entry section tends to infinity. The result can be viewed as a version of Saint-Venant principle. Using the result (4.1), we can also obtain point-wise decay estimates for the solutions. This is the specific property for the biharmonic equation. Next, we will give a numerical simulation of solutions for these equations. What is more, the structural stability for these equations in an unbounded domain would be interesting. We will study it in another paper.

Author contributions

All authors of this article have contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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