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Research article

Orderings of the second-largest order statistic with modified proportional reversed hazard rate samples

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Abstract: Order statistics is a significant research topic within probability and statistics, particularly due to its widespread application in areas such as reliability and actuarial science. Extensive research has been conducted on extreme order statistics, and this paper focused on the second-order statistics. Specifically, the study investigated the second-largest order statistics derived from dependent heterogeneous modified proportional reversed hazard rate samples, utilizing the stochastic properties of the Archimedean copula. This paper first examined the usual stochastic order of the second-largest order statistic between two groups of dependent heterogeneous random variables. These variables were analyzed under conditions involving the same tilt parameters with different proportional reversed hazard rate parameters, and different tilt parameters with the same proportional reversed hazard rate parameters. The study derived the sufficient conditions required for establishing the usual stochastic order in these cases. Next, the paper addressed the reversed hazard rate order relationship for the second- largest order statistic between two groups of independent heterogeneous random variables. This analysis was conducted under various conditions: the same tilt parameters with different proportional reversed hazard rate parameters, different tilt parameters with the same proportional reversed hazard rate parameters, and different sample sizes with the same parameters. The sufficient conditions for establishing the reversed hazard rate order were also derived. Finally, the theoretical findings were substantiated through numerical examples, confirming the main conclusions of the paper.

Keywords: Archimedean copula; modified proportional reversed hazard rate; majorization orders; second largest order statistics; stochastic orders
Mathematics Subject Classification: Primary 90B25, Secondary 60E15, 60K10

1. Introduction

Order statistics have a wide range of application backgrounds and have been widely studied by researchers in fields such as reliability analysis and auction theory. Let $X_1, \ldots, X_n$ be a set of stochastic variables, where the k-th order statistic is $X_{n-k+1:n}$. With respect to reliability analysis, $X_{1:n}, X_{n:n}, X_{2:n}$, and $X_{k:n}$ represent the lifetimes of a parallel system, series system, fail-safe system, and k-out-of-n system, respectively, wherein a k-out-of-n system means that if n components in the system are composed, the system will stop working when the number of failed components is more than $n - k + 1$. For a detailed discussion and introduction of reliability analysis, reference can be made to literature (for example, [1]). In auction theory, the famous second-price reverse sealed auction winner's transaction price and the second-price sealed auction winner's transaction price can be represented by order statistics $X_{1:n}$ and $X_{n-1:n}$, respectively, where the final price paid by the second-price sealed auction winner is the second-highest bid. The second-price sealed auction can be applied to the trading of many bulk commodities such as foreign exchange auctions and treasury bond insurance, which is conducive to improving the efficiency of resource allocation. For a detailed discussion and introduction of auction theory, reference can be made to literature, such as [2].

The investigations of order statistics mostly involve stochastic comparisons between the maximum and minimum order statistics in independent situations. In the independent case, the research results obtained from the same distribution of components in the sample are the most prominent, which can be referred to in references such as [3, 4]. For situations with independent and different distributions, outstanding results have also been achieved, which can be referred to in references such as [5, 6]. In recent years, researchers have begun to study a more widely existing type of problem in practical situations—the stochastic comparison problem of dependent sample order statistics. The most commonly studied is the Archimedean copula under the numerous existing copula. Yan et al. [7] investigated the stochastic comparisons of the largest order statistics with two heterogeneous exponential samples. Mesfioui et al. [8] investigated stochastic comparisons of order statistics from heterogeneous random variables with Archimedean copula. Hazra et al. [9] obtained the stochastic comparisons of maximum order statistics from the location-scale family of distributions. With the further deepening of mathematical research, the study is no longer limited to the stochastic comparison problem between extreme order statistics but further investigates the stochastic comparison problem between the second largest order statistics and the second smallest order statistics. Cai et al. [10] compared the hazard rate functions of two independent multivariate outlier samples under the proportional hazard rate model, and obtained the hazard rate order of the second-order order statistic. Zhao et al. [11] established the orderings of the extreme order statistics from heterogeneous beta distributions with applications. Zhang and Yan [12] obtained stochastic comparison at component level and system level series system with two proportional hazards rate components. Zhang and Yan [13] obtained the stochastic comparisons of parallel and series systems with type II half logistic-resilience scale components. Panja et al. [14] considered stochastic comparisons of lifetimes of series and parallel systems with dependent and heterogeneous components having lifetimes following the proportional odds model. Liu and Yan [15] obtained the orderings of extreme claim amounts from heterogeneous and dependent Weibull-G insurance portfolios. Zhang and Yan [16] considered reliability optimization of parallel-series and series-parallel systems with statistically dependent components. Das et al. [17] investigated the case in which the marginal distributions can have arbitrary distribution functions depending on some parameter, and the extreme order statistics arising from the dependent modified proportional hazard rate scale (MPHRS) and modified proportional reversed hazard rate scale (MPRHRS) models were compared in the sense of the reversed hazard rate order and the hazard rate order. Samanta et al. [18] considered two sets of dependent variables, in terms of the usual stochastic, star, Lorenz, hazard rate, reversed hazard rate, and dispersve orders. Several examples and counterexamples are presented for illustrating all the results established there. Yan and Niu [19] investigated the stochastic comparisons of second-order statistics from dependent and heterogeneous modified proportional hazard rate observations. Zhang et al. [20] studied the orderings of fail-safe systems with heterogeneous and dependent components subject to stochastic shocks. Barmalzan et al. [21] presented a joint distribution of two fail-safe systems with different life distributions, and randomly compared the fail-safe systems of two multivariate outlier models with independent components, obtaining the ranking relationship of hazard rate order. Biplab et al. [22] studied the stochastic comparison problem of two fail-safe systems with dependent and heterogeneous components under stochastic shocks, and obtained a general stochastic-order ranking relationship and sufficient conditions for obtaining the ranking relationship. Wang et al. [23] investigated large sample properties of maximum likelihood estimator using moving extremes ranked set sampling. Hazra et al. [24] studied the stochastic comparison of the second-largest and second-smallest order statistics of samples using an Archimedean copula in a semi-parametric family. For more investigations of stochastic orders and their applications, readers can refer to [25–28, 39].

However, in practical situations, life data usually has different hazard rate shapes. Therefore, in order to reflect some of the features and shapes, the distribution should have considerable flexibility. To address this issue, a parameter can be added to expand the distribution family and improve its flexibility. Balakrishnan et al. [29] addressed this issue by proposing a modified proportional reserved hazard rate model (MPRHR) as follows. Let a system be composed of n components $X_1, X_2, \ldots, X_n$ with independent lifetimes, and the distribution functions of the components $X_1, X_2, \ldots, X_n$ are $F_1, F_2, \ldots, F_n$, respectively, and then $X_1, X_2, \ldots, X_n$ are called the modified proportional reversed hazard rate model that follows a skewed parameter α, a modified proportional reversed hazard rate parameter $\beta_1, \beta_2, \ldots, \beta_n$, a basis distribution function F (represented as modified proportional reversed hazard rate $(\alpha; \beta_1, \beta_2, \ldots, \beta_n; F)$) if and only if

$$F_i(x; \beta_i) = \frac{\alpha(F(x))^{\beta_i}}{1-\bar{\alpha}(F(x))^{\beta_i}}, \text{ for all } i = 1, 2, \ldots, n,$$

wherein $\alpha > 0$, $\bar{\alpha} = 1-\alpha$, and $\beta > 0$, $i = 1, 2, \ldots, n$. Balakrishnan et al. [29] established some stochastic comparisons between the corresponding order statistics based on the modified proportional reserved hazard rate model. Zhang et al. [30] studied the stochastic comparison problem of dependent and heterogeneous samples following the modified proportional reversed hazard rate model, and obtained the usual stochastic order and reserved hazard rate order of extreme order statistics. Barmalzan et al. [31] studied orderings of extremes-dependent modified proportional hazard and modified proportional reversed hazard variables under an Archimedean copula. Zhang et al. [32] established stochastic comparisons of the largest claim amount from heterogeneous and dependent insurance portfolios. Shrahili et al. [33] obtained relative orderings of modified proportional hazard rate and modified proportional reversed hazard rate models. Barmalzan et al. [21] obtained the relationship between hazard rate order and reserved hazard rate order between extreme order statistics with modified proportional hazard rate samples under an Archimedean copula. Zhang and Zhang [34] investigated the allocation problem of multiple minimal repairs carried out for any two components in coherent systems. Guo et al. [35] investigated optimal redundancy allocations for series systems under hierarchical dependence structures. Lv et al. [36] investigated the stochastic comparisons of the second-order statistics from dependent and heterogeneous general semi-parametric family of distributions observations. Seresht et al. [37] studied the stochastic comparison problem of extreme order statistics of two systems with an Archimedean copula and dependent heterogeneous stochastic variables under stochastic shocks, and obtained the normal stochastic order relationship between the two systems. Song et al. [38] studied dispersive and star orders on extreme order statistics from location-scale samples. Zhang et al. [40] investigated the increasing convex order of capital allocation with dependent assets under threshold model. Guo et al. [41] studied sufficient conditions of the second-largest claim amounts arising from two sets of dependent and heterogeneous individual risk models according to various stochastic orders.

Therefore, inspired by the above articles, this paper will investigate the ordering properties of the second-largest order statistic composed of dependent heterogeneous modified proportional reversed hazard rate samples. The study focuses on the Archimedean copula and dependent heterogeneous modified proportional reversed hazard rate samples. Under conditions with the same tilt parameters but different proportional reversed hazard rate parameters, and under conditions with the same proportional reversed hazard rate parameters but different tilt parameters, we obtain the usual stochastic order of the second-largest order statistic for two groups of dependent heterogeneous stochastic variables. Additionally, we establish sufficient conditions for the usual stochastic order. Meanwhile, based on the independent heterogeneous modified proportional reversed hazard rate samples, under the conditions of the same tilt parameters and different proportional reversed hazard rate parameters and different tilt parameters, the same proportional reversed hazard rate parameters, and different sample sizes and the same parameters, we obtain the reversed hazard rate order relationship of the second-largest order statistic of two groups of independent heterogeneous stochastic variables and the sufficient conditions for the establishment of the reversed hazard rate order. These findings extend the results of [21, 30] on extreme order statistics to the second-largest order statistic of dependent samples.

The remainder of this article is structured as follows: In Section 2, we provide a concise review of key concepts and two important lemmas related to stochastic order, optimization order, Archimedean copulas, and modified proportional reversed hazard rate models discussed in this paper. Section 3 investigates the usual stochastic ordering relationship and the sufficient conditions for obtaining the second-largest order statistic under dependent heterogeneous modified proportional reversed hazard rate samples using an Archimedean copula. Numerical examples are presented to validate the proposed theorem. In Section 4, we examine the reversed order relationship for the second-largest order statistic under independent heterogeneous modified proportional reversed hazard rate samples with an Archimedean copula, and provide the sufficient conditions necessary for establishing this relationship. Numerical examples are also included to demonstrate the validity of the theorem.

2. Preliminaries

2.1. Stochastic order

In this section, we will introduce some famous concepts and two important lemmas related to stochastic order, majorization order, Archimedean copulas, and modified proportional reversed hazard rate models. In this article, "increasing" represents non-decreasing, and "decreasing" represents nonincreasing. Let $\mathcal{D}_+ = \{\boldsymbol{a} : a_1 \geq a_2 \geq \cdots \geq a_n\}$, $\mathcal{I}_+ = \{\boldsymbol{a} : a_1 \leq a_2 \leq \cdots \leq a_n\}$, and $\mathcal{N} = 1, 2, \ldots, n$. Meanwhile, for the sake of simplicity, $a \overset{\text{sgn}}{=} b$ is used to indicate that the symbols on both sides of the equal sign are the same. Stochastic order is a very useful tool for comparing stochastic variables. Let X be a stochastic variable, and denote the distribution function, survival function, probability density function, hazard rate function, and reversed hazard rate function by $F_X(t)$, $\bar{F}_X(t) = 1 - F_X(t)$, $f_X(t)$, $h_X(t) = f_X(t)/\bar{F}_X(t)$, and $\tilde{r}_X(t) = f_X(t)/F_X(t)$, respectively.

Stochastic orderes are a very useful tool to compare random variables arising from reliability theory, operations research, actuarial science, economics, finance, and so on.

Definition 1. *Let X and Y be two absolutely continuous stochastic variables.*

(i) *The usual stochastic order: If for all $x \in \mathbb{R}$, $\bar{F}_X(x) \leq \bar{F}_Y(x)$ is established, it is said that the usual stochastic order of X is less than Y (denoted by $X \leq_{\text{st}} Y$);*

(ii) *the hazard rate order: If for all $x \in \mathbb{R}$, $h_X(x) \geq h_Y(x)$ or $\bar{F}_Y(x)/\bar{F}_X(x)$ is increasing in $x \in \mathbb{R}$ is established, it is said that the hazard rate order of X is less than Y (denoted by $X \leq_{\text{hr}} Y$);*

(iii) *the reversed hazard rate order: If for all $x \in \mathbb{R}$, $\tilde{r}_X(x) \leq \tilde{r}_Y(x)$ or $F_Y(x)/F_X(x)$ is increasing in $x \in \mathbb{R}$ is established, it is said that the reversed hazard rate order of X is less than Y (denoted by $X \leq_{\text{rh}} Y$).*

Definition 2. *If X and Y are discrete random variables, let the distribution columns of X and Y be $p_i = P\{X = i\}$ and $q_i = P\{Y = i\}$, $i = 1, 2, \ldots, n$.*

(i) *If for any $i = 1, 2, \ldots, n$, $\sum_{i=1}^{j} p_{i:n} \geq \sum_{i=1}^{j} q_{i:n}$, it is said that the usual stochastic order of Y is less than X (denoted by $X \geq_{\text{st}} Y$);*

(ii) *if for any $i = 1, 2, \ldots, n$, $\sum_{i=1}^{j} p_{i:n} / \sum_{i=1}^{j} q_{i:n}$ is increasing in i, it is said that the reversed hazard rate order of X is less than Y (denoted by $X \leq_{\text{rh}} Y$);*

(iii) *if for any $i = 1, 2, \ldots, n$, $\sum_{i=1}^{j} p_{i:n} / \sum_{i=1}^{j} q_{i:n}$ is increasing in i, it is said that the hazard rate order of Y is less than X (denoted by $X \geq_{\text{hr}} Y$).*

For more detailed discussion and introduction of stochastic orders and their applications, readers can refer to the works of [41, 42]. The following will introduce the majorization order, which is an important tool for research in many fields.

2.2. Majorization order

Definition 3. *If vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$ are arrangement increasing, then $x_{(1)} \leq x_{(2)} \cdots \leq x_{(n)}$ and $y_{(1)} \leq y_{(2)} \cdots \leq y_{(n)}$.*

(i) *If for any $i = 1, 2, \ldots, n$, there are $\sum_{i=1}^{n} x_{(j)} = \sum_{i=1}^{n} y_{(j)}$, and $\sum_{i=1}^{n} x_{(j)} \leq \sum_{i=1}^{n} y_{(j)}$, then $\boldsymbol{x}$ is said to majorize $\boldsymbol{y}$ (denoted by $\boldsymbol{x} \overset{\text{m}}{\geq} \boldsymbol{y}$) ;*

(ii) *if for any $i = 1, 2, \ldots, n$, there are $\sum_{i=1}^{n} x_{(j)} \leq \sum_{i=1}^{n} y_{(j)}$, then $\boldsymbol{x}$ is said to weak super majorized $\boldsymbol{y}$ (denoted by $\boldsymbol{x} \overset{\text{w}}{\geq} \boldsymbol{y}$);*

(iii) *if for any $i = 1, 2, \ldots, n$, there are $\sum_{i=1}^{n} x_{(j)} \geq \sum_{i=1}^{n} y_{(j)}$, then $\boldsymbol{x}$ is said to majorizated $\boldsymbol{y}$ (denoted by $\boldsymbol{x} \geq_{\text{w}} \boldsymbol{y}$).*

According to [44], for any two real-valued vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, the following relationship holds

$$\boldsymbol{x} \succeq_{\mathrm{w}} \boldsymbol{y} \Leftarrow \boldsymbol{x} \overset{\mathrm{m}}{\geq} \boldsymbol{y} \Rightarrow \boldsymbol{x} \overset{\mathrm{w}}{\geq} \boldsymbol{y},$$

note that the opposite sign does not hold true. The concept of majorization is used to characterize the discreteness of vectors, that is, in the sense of optimization order, larger variables mean more non-uniformity, while smaller vectors mean more uniformity. For more information on optimizing sequences, please refer to [44].

2.3. Archimedean copula

First, let us review the concept of a copula. For a random vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ with the joint distribution function K and respective marginal distribution functions $F_1(t), F_2(t), \ldots, F_n(t)$, the *copula* of $X_1, X_2, \ldots, X_n$ is a distribution function $C : [0,1]^n \mapsto [0,1]$ satisfying

$$K(\boldsymbol{x}) = \mathbb{P}(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n) = C(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)).$$

Similarly, a *survival copula* of $X_1, X_2, \ldots, X_n$ is a survival function $\hat{C} : [0,1]^n \mapsto [0,1]$ satisfying

$$K(\boldsymbol{x}) = \mathbb{P}(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n) = \hat{C}(\bar{F}_1(x_1), \bar{F}_2(x_2), \ldots, \bar{F}_n(x_n)),$$

where $K(\boldsymbol{x})$ is the joint survival function. Next we will introduce the Archimedes copula that will be used in this chapter.

Definition 4. *[45] For a decreasing and continuous function $\phi : [0,1] \mapsto [0,+\infty]$ such that $\phi(0) = +\infty$ and $\phi(1) = 0$, let $\psi = \phi^{-1}$ be the pseudo-inverse of ϕ. If for all $k = 0, 1, \ldots, n-2$, $(-1)^k \phi^{(k)}(x) \geq 0$ and $(-1)^{n-2}\phi^{(n-2)}(x)$ is decreasing and convex, Then*

$$C_\phi(u_1, u_2, \ldots, u_n) = \psi\Big(\sum_{i=1}^{n} \phi(u_i)\Big), \text{ for all } u_i \in [0,1], \quad i = 1, 2, \ldots, n,$$

is said to be an Archimedean copula with the generator.

The Archimedean copula can be applied to fields such as reliability analysis, risk assessment, and hazard management. For more information about Archimedes copulas, please refer to relevant literature such as [45].

2.4. Lemmas

The following two lemmas play an important role in establishing the inequality relationship of weak majorization order.

Lemma 1. *[44] Assume $\phi : I \to R$ is a real-valued function, continuously differentiable within I, that*

(i) *if for all $\boldsymbol{x}, \boldsymbol{y} \in I$, $\boldsymbol{x} \overset{\mathrm{m}}{\preceq} \boldsymbol{y}$, if and only if $\phi(x) \leq \phi(y)$ in $k = 1, 2, \ldots, n$ is increasing, where $\phi_{(k)}(\boldsymbol{z}) = \partial\phi(\boldsymbol{z})/\partial \boldsymbol{z}_{(k)}$ represents the partial derivative of ϕ with respect to its the k-th parameter*;

(ii) *if for all $\boldsymbol{x}, \boldsymbol{y} \in I$, $\boldsymbol{x} \overset{\mathrm{m}}{\preceq} \boldsymbol{y}$, if and only if $\phi(x) \geq \phi(y)$ in $k = 1, 2, \ldots, n$ is decreasing, where $\phi_{(k)}(\boldsymbol{z}) = \partial\phi(\boldsymbol{z})/\partial \boldsymbol{z}_{(k)}$ represents the partial derivative of ϕ with respect to its the k-th parameter.*

Lemma 2. *[44] Assume that ϕ is a real-valued function, continuously differentiable within D_n, and $\phi_{(k)}(\boldsymbol{Z}) = \partial\phi(\boldsymbol{Z})/\partial z_{(k)}$ represents the partial derivative of ϕ with respect to the k-th parameter, $k = 1, 2, \ldots, n$, then*

(i) *if for all $\boldsymbol{x}, \boldsymbol{y} \in D_n$, $\boldsymbol{x} \overset{\mathrm{w}}{\preceq} \boldsymbol{y}$, if and only if $0 \geq \phi_{(1)}(\boldsymbol{z}) \geq \phi_{(2)}(\boldsymbol{z}) \cdots \geq \phi_{(n)}(\boldsymbol{z})$;*

(ii) *if for all $\boldsymbol{x}, \boldsymbol{y} \in D_n$, $\boldsymbol{x} \preceq_{\mathrm{w}} \boldsymbol{y}$, if and only if $\phi_{(1)}(\boldsymbol{z}) \geq \phi_{(2)}(\boldsymbol{z}) \cdots \geq \phi_{(n)}(\boldsymbol{z}) \geq 0$.*

3. Usual stochastic order of dependent heterogeneous samples

This chapter will investigate the usual stochastic order of the second-largest order statistic from dependent heterogeneous observations. Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ and $\boldsymbol{X}^* = (X_1^*, \ldots, X_n^*)$ be two sets of n -dimensional stochastic variables under dependent heterogeneous observations, following $X_i \sim MPRHR(\alpha, \lambda_i; F, \psi)$ and $X_i \sim MPRHR(\alpha, \lambda_i^*; F, \psi)$, where, $i = 1, 2, \ldots, n$, F is the baseline distribution function, and ψ is an Archimedean copula generator. Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\boldsymbol{\alpha}^* = (\alpha_1^*, \ldots, \alpha_n^*)$, $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$, and $\boldsymbol{\lambda}^* = (\lambda_1^*, \ldots, \lambda_n^*)$.

Theorem 1 establishes the usual stochastic order of the second-largest order statistic under identical skew parameters but different modified proportional reversed hazard rate parameters.

3.1. Usual stochastic order of MPRHR parameters with heterogeneous variables

Theorem 1. *Let $X_1, X_2, \ldots, X_n$ be dependent heterogeneous stochastic variables of n dimensions following $X_i \sim MPRHR(\alpha, \lambda_i; F, \psi)$, and $X_1^*, X_2^*, \ldots, X_n^*$ are the other set of n-dimensional dependent heterogeneous stochastic variables following $X_i^* \sim MPRHR(\alpha, \lambda_i^*; F, \psi)$, where $0 < \alpha \leq 1$, $i = 1, 2, \ldots, n$. Let N_1 and N_2 be two positive real-valued stochastic variables each independently distributed with $X_i^{'}s$ and $X_i^{*'}s$, respectively, and both values are not less than 2. If $\boldsymbol{\lambda}, \boldsymbol{\lambda}^* \in \mathcal{D}_+$, $N_1 \geq_{st} N_2$, and ψ is concave in the logarithm, then*

$$\boldsymbol{\lambda} \overset{w}{\preceq} \boldsymbol{\lambda}^* \Rightarrow X_{n-1:N_1} \geq_{st} X^*_{n-1:N_2}.$$

Proof. The distribution function of $X_{n-1:n}$ can be given by

$$F_{X_{n-1:n}}(x) = \sum_{i=1}^{n} \psi\left(\sum_{j\neq i}^{n} \phi\left(\frac{\alpha F^{\lambda_j}(x)}{1 - \bar{\alpha} F^{\lambda_j}(x)}\right)\right) - (n-1)\psi\left(\sum_{i=1}^{n} \phi\left(\frac{\alpha F^{\lambda_i}(x)}{1 - \bar{\alpha} F^{\lambda_i}(x)}\right)\right).$$

Because $N_1 \geq_{st} N_2$, we have

$$\begin{aligned} F_{X_{n-1:N_1}}(x) &= 1 - \bar{F}_{X_{n-1:N_1}}(X) \\ &= 1 - \sum_{m=2}^{n} P(X_{n-1:N_1} > x | N_1 = m) P(N_1 = m) \\ &= 1 - \sum_{m=2}^{n} P(X_{n-1:m} > x) P(N_1 = m) \\ &\leq 1 - \sum_{m=2}^{n} P(X_{n-1:m} > x) P(N_2 = m) = F_{X_{n-1:N_2}}(x). \end{aligned}$$

To prove the result, we need to demonstrate $F_{X_{n-1:m}}(x) \le F_{Y_{n-1:m}}(x)$, $m = 2, 3, \ldots, n$. First, for any $k = 1, 2, \ldots, m$, take the partial derivative of $F_{X_{n-1:m}}(x)$ with respect to λ_k, since ψ is decreasing and convex,

$$
\begin{aligned}
&\frac{\partial F_{X_{n-1:m}}(x)}{\partial \lambda_k} \\
&= \phi'\left(\frac{\alpha F^{\lambda_k}(x)}{1-\bar{\alpha}F^{\lambda_k}(x)}\right)\frac{\alpha \ln F(x) F^{\lambda_k}(x)}{[1-\bar{\alpha}F^{\lambda_k}(x)]^2} \\
&\times\left[\sum_{i\ne k}^{m}\psi'\left(\sum_{j\ne i}^{m}\phi\left(\frac{\alpha F^{\lambda_j}(x)}{1-\bar{\alpha}F^{\lambda_j}(x)}\right)\right)-(m-1)\psi'\left(\sum_{i=1}^{m}\phi\left(\frac{\alpha F^{\lambda_i}(x)}{1-\bar{\alpha}F^{\lambda_i}(x)}\right)\right)\right] \\
&= \frac{\psi\left(\phi\left(\frac{\alpha F^{\lambda_k}(x)}{1-\bar{\alpha}F^{\lambda_k}(x)}\right)\right)}{\psi'\left(\phi\left(\frac{\alpha F^{\lambda_k}(x)}{1-\bar{\alpha}F^{\lambda_k}(x)}\right)\right)}\frac{\ln F(x)}{1-\bar{\alpha}F^{\lambda_k}(x)} \\
&\times\left[\sum_{i\ne k}^{m}\psi'\left(\sum_{j\ne i}^{m}\phi\left(\frac{\alpha F^{\lambda_j}(x)}{1-\bar{\alpha}F^{\lambda_j}(x)}\right)\right)-(m-1)\psi'\left(\sum_{i=1}^{m}\phi\left(\frac{\alpha F^{\lambda_i}(x)}{1-\bar{\alpha}F^{\lambda_i}(x)}\right)\right)\right] \le 0.
\end{aligned}
$$

Moreover, since $\ln F(x) \le 0$, then

$$
\begin{aligned}
&\frac{\partial F_{X_{n-1:m}}(x)}{\partial \lambda_k} - \frac{\partial F_{X_{n-1:m}}(x)}{\partial \lambda_t} \\
&= \ln F(x)\left\{\frac{\psi\left(\phi\left(\frac{\alpha F^{\lambda_k}(x)}{1-\bar{\alpha}F^{\lambda_k}(x)}\right)\right)}{\psi'\left(\phi\left(\frac{\alpha F^{\lambda_k}(x)}{1-\bar{\alpha}F^{\lambda_k}(x)}\right)\right)}\frac{1}{1-\bar{\alpha}F^{\lambda_k}}(x)\right. \\
&\times\left[\sum_{i\ne k}^{m}\psi'\left(\sum_{j\ne i}^{m}\phi\left(\frac{\alpha F^{\lambda_j}(x)}{1-\bar{\alpha}F^{\lambda_j}(x)}\right)\right)-(m-1)\psi'\left(\sum_{i=1}^{m}\phi\left(\frac{\alpha F^{\lambda_i}(x)}{1-\bar{\alpha}F^{\lambda_i}(x)}\right)\right)\right] \\
&- \frac{\psi\left(\phi\left(\frac{\alpha F^{\lambda_t}(x)}{1-\bar{\alpha}F^{\lambda_t}(x)}\right)\right)}{\psi'\left(\phi\left(\frac{\alpha F^{\lambda_t}(x)}{1-\bar{\alpha}F^{\lambda_t}(x)}\right)\right)}\frac{1}{1-\bar{\alpha}F^{\lambda_t}}(x) \\
&\left.\times\left[\textstyle\sum_{i\ne t}^{m}\psi'\left(\sum_{j\ne i}^{m}\phi\left(\frac{\alpha F^{\lambda_j}(x)}{1-\bar{\alpha}F^{\lambda_j}(x)}\right)\right)-(m-1)\psi'\left(\sum_{i=1}^{m}\phi\left(\frac{\alpha F^{\lambda_i}(x)}{1-\bar{\alpha}F^{\lambda_i}(x)}\right)\right)\right]\right\} \\
&\overset{\text{sgn}}{=} \frac{\psi\left(\phi\left(\frac{\alpha F^{\lambda_t}(x)}{1-\bar{\alpha}F^{\lambda_t}(x)}\right)\right)}{\psi'\left(\phi\left(\frac{\alpha F^{\lambda_t}(x)}{1-\bar{\alpha}F^{\lambda_t}(x)}\right)\right)}\frac{1}{1-\bar{\alpha}F^{\lambda_t}(x)} \\
&\times\left[\sum_{i\ne t}^{m}\psi'\left(\sum_{j\ne i}^{m}\phi\left(\frac{\alpha F^{\lambda_j}(x)}{1-\bar{\alpha}F^{\lambda_j}(x)}\right)\right)-(m-1)\psi'\left(\sum_{i=1}^{m}\phi\left(\frac{\alpha F^{\lambda_i}(x)}{1-\bar{\alpha}F^{\lambda_i}(x)}\right)\right)\right] \\
&- \frac{\psi\left(\phi\left(\frac{\alpha F^{\lambda_k}(x)}{1-\bar{\alpha}F^{\lambda_k}(x)}\right)\right)}{\psi'\left(\phi\left(\frac{\alpha F^{\lambda_k}(x)}{1-\bar{\alpha}F^{\lambda_k}(x)}\right)\right)}\frac{1}{1-\bar{\alpha}F^{\lambda_k}(x)} \\
&\times\left[\sum_{i\ne k}^{m}\psi'\left(\sum_{j\ne i}^{m}\phi\left(\frac{\alpha F^{\lambda_j}(x)}{1-\bar{\alpha}F^{\lambda_j}(x)}\right)\right)-(m-1)\psi'\left(\sum_{i=1}^{m}\phi\left(\frac{\alpha F^{\lambda_i}(x)}{1-\bar{\alpha}F^{\lambda_i}(x)}\right)\right)\right] \\
&=: P_1Q_1 - U_1V_1 = P_1(Q_1 - V_1) + (P_1 - U_1)V_1,
\end{aligned}
$$

where

$$Q_1 = \left[\sum_{i\neq t}^{m}\psi^{'}\left(\sum_{j\neq i}^{m}\phi\left(\frac{\alpha F^{\lambda_j}(x)}{1-\bar{\alpha}F^{\lambda_j}(x)}\right)\right)-(m-1)\psi^{'}\left(\sum_{i=1}^{m}\phi\left(\frac{\alpha F^{\lambda_i}(x)}{1-\bar{\alpha}F^{\lambda_i}(x)}\right)\right)\right],$$

$$V_1 = \left[\sum_{i\neq k}^{m}\psi^{'}\left(\sum_{j\neq i}^{m}\phi\left(\frac{\alpha F^{\lambda_j}(x)}{1-\bar{\alpha}F^{\lambda_j}(x)}\right)\right)-(m-1)\psi^{'}\left(\sum_{i=1}^{m}\phi\left(\frac{\alpha F^{\lambda_i}(x)}{1-\bar{\alpha}F^{\lambda_i}(x)}\right)\right)\right],$$

$$P_1 = \frac{\psi\left(\phi\left(\frac{\alpha F^{\lambda_t}(x)}{1-\bar{\alpha}F^{\lambda_t}(x)}\right)\right)}{\psi'\left(\phi\left(\frac{\alpha F^{\lambda_t}(x)}{1-\bar{\alpha}F^{\lambda_t}(x)}\right)\right)}\frac{1}{1-\bar{\alpha}F^{\lambda_t}(x)},$$

and

$$U_1 = \frac{\psi\left(\phi\left(\frac{\alpha F^{\lambda_k}(x)}{1-\bar{\alpha}F^{\lambda_k}(x)}\right)\right)}{\psi'\left(\phi\left(\frac{\alpha F^{\lambda_k}(x)}{1-\bar{\alpha}F^{\lambda_k}(x)}\right)\right)}\frac{1}{1-\bar{\alpha}F^{\lambda_k}(x)}.$$

For any $1 \le k < t \le m$, $\lambda_k \ge \lambda_t$, since ϕ is decreasing and ψ concave in the logarithm,

$$\frac{\psi\left(\phi\left(\frac{\alpha F^{\lambda_k}(x)}{1-\bar{\alpha}F^{\lambda_k}(x)}\right)\right)}{\psi'\left(\phi\left(\frac{\alpha F^{\lambda_k}(x)}{1-\bar{\alpha}F^{\lambda_k}(x)}\right)\right)} \geqslant \frac{\psi\left(\phi\left(\frac{\alpha F^{\lambda_k}(x)}{1-\bar{\alpha}F^{\lambda_k}(x)}\right)\right)}{\psi'\left(\phi\left(\frac{\alpha F^{\lambda_k}(x)}{1-\bar{\alpha}F^{\lambda_k}(x)}\right)\right)}.$$

Therefore,

$$\begin{aligned}
&(P_1 - U_1)V_1\\
&= \left[\frac{\psi\left(\phi\left(\frac{\alpha F^{\lambda_t}(x)}{1-\bar{\alpha}F^{\lambda_t}(x)}\right)\right)}{\psi'\left(\phi\left(\frac{\alpha F^{\lambda_t}(x)}{1-\bar{\alpha}F^{\lambda_t}(x)}\right)\right)}\frac{1}{1-\bar{\alpha}F^{\lambda_t}(x)} - \frac{\psi\left(\phi\left(\frac{\alpha F^{\lambda_k}(x)}{1-\bar{\alpha}F^{\lambda_k}(x)}\right)\right)}{\psi'\left(\phi\left(\frac{\alpha F^{\lambda_k}(x)}{1-\bar{\alpha}F^{\lambda_k}(x)}\right)\right)}\frac{1}{1-\bar{\alpha}F^{\lambda_k}(x)}\right]\\
&\times\left[\sum_{i\neq k}^{m}\psi^{'}\left(\sum_{j\neq i}^{m}\phi\left(\frac{\alpha F^{\lambda_j}(x)}{1-\bar{\alpha}F^{\lambda_j}(x)}\right)\right)-(m-1)\psi^{'}\left(\sum_{i=1}^{m}\phi\left(\frac{\alpha F^{\lambda_i}(x)}{1-\bar{\alpha}F^{\lambda_i}(x)}\right)\right)\right]\\
&\overset{\text{sgn}}{=} \frac{\psi\left(\phi\left(\frac{\alpha F^{\lambda_k}(x)}{1-\bar{\alpha}F^{\lambda_k}(x)}\right)\right)}{\psi'\left(\phi\left(\frac{\alpha F^{\lambda_k}(x)}{1-\bar{\alpha}F^{\lambda_k}(x)}\right)\right)}\left(\frac{1}{1-\bar{\alpha}F^{\lambda_k}(x)} - \frac{1}{1-\bar{\alpha}F^{\lambda_t}(x)}\right)\\
&\quad + \frac{1}{1-\bar{\alpha}F^{\lambda_t}(x)}\left(\frac{\psi\left(\phi\left(\frac{\alpha F^{\lambda_k}(x)}{1-\bar{\alpha}F^{\lambda_k}(x)}\right)\right)}{\psi'\left(\phi\left(\frac{\alpha F^{\lambda_k}(x)}{1-\bar{\alpha}F^{\lambda_k}(x)}\right)\right)} - \frac{\psi\left(\phi\left(\frac{\alpha F^{\lambda_t}(x)}{1-\bar{\alpha}F^{\lambda_t}(x)}\right)\right)}{\psi'\left(\phi\left(\frac{\alpha F^{\lambda_t}(x)}{1-\bar{\alpha}F^{\lambda_t}(x)}\right)\right)}\right) \ge 0.
\end{aligned}$$

For any $\lambda_k \ge \lambda_t$, because ϕ is decreasing and convex, we can obtain

$$\psi^{'}\left(\sum_{j\neq k}^{m}\phi\left(\frac{\alpha F^{\lambda_j}(x)}{1-\bar{\alpha}F^{\lambda_j}(x)}\right)\right) \leqslant \psi^{'}\left(\sum_{j\neq t}^{m}\phi\left(\frac{\alpha F^{\lambda_j}(x)}{1-\bar{\alpha}F^{\lambda_j}(x)}\right)\right).$$

Hence,

$$
\begin{aligned}
&P_1(Q_1 - V_1) \\
&= \frac{\psi\left(\phi\left(\frac{\alpha F^{\lambda_t}(x)}{1-\bar{\alpha}F^{\lambda_t}(x)}\right)\right)}{\psi'\left(\phi\left(\frac{\alpha F^{\lambda_t}(x)}{1-\bar{\alpha}F^{\lambda_t}(x)}\right)\right)} \frac{1}{1-\bar{\alpha}F^{\lambda_t}(x)} \\
&\times \left[\sum_{i\neq t}^{m} \psi'\left(\sum_{j\neq i}^{m} \phi\left(\frac{\alpha F^{\lambda_j}(x)}{1-\bar{\alpha}F^{\lambda_j}(x)}\right)\right) - \sum_{i\neq k}^{m} \psi'\left(\sum_{j\neq i}^{m} \phi\left(\frac{\alpha F^{\lambda_j}(x)}{1-\bar{\alpha}F^{\lambda_j}(x)}\right)\right)\right] \\
&\overset{\text{sgn}}{=} \left[\sum_{i\neq k}^{m} \psi'\left(\sum_{j\neq i}^{m} \phi\left(\frac{\alpha F^{\lambda_j}(x)}{1-\bar{\alpha}F^{\lambda_j}(x)}\right)\right) - \sum_{i\neq t}^{m} \psi'\left(\sum_{j\neq i}^{m} \phi\left(\frac{\alpha F^{\lambda_j}(x)}{1-\bar{\alpha}F^{\lambda_j}(x)}\right)\right)\right] \\
&= \psi'\left(\sum_{j\neq t}^{m} \phi\left(\frac{\alpha F^{\lambda_j}(x)}{1-\bar{\alpha}F^{\lambda_j}(x)}\right)\right) - \psi'\left(\sum_{j\neq k}^{m} \phi\left(\frac{\alpha F^{\lambda_j}(x)}{1-\bar{\alpha}F^{\lambda_j}(x)}\right)\right) \geq 0.
\end{aligned}
$$

Combining $P_1(Q_1 - V_1) + (P_1 - U_1)V_1 \geq 0$ with Lemma 2, the conclusion is proved.

□

Next, we provide a numerical example to demonstrate the result of Theorem 1.

Example 1. *Consider the case when* $n = 4$. *Let the distribution function* $F(x) = 1 - e^{-(ax)^b}$, $a > 0$, $b > 0$, *generating element* $\psi(x) = \exp\{(1 - e^x)/\theta\}$, $0 < \theta \leq 1$, $a = 1.4$, $b = 0.6$, $\theta = 0.1$, *and* $\boldsymbol{\lambda} = (1.7, 1.6, 0.5, 0.3) \overset{w}{\preceq} (1.4, 0.7, 0.3, 0.2) = \boldsymbol{\lambda}^*$. *Suppose* N_1 *is a positive real value with the probability distribution* $P(N_1 = 2) = 0.15, P(N_1 = 3) = 0.35$, $P(N_1 = 4) = 0.5$, *and* N_2 *is positive real value with the probability distribution* $P(N_2 = 2) = 0.2$, $P(N_2 = 3) = 0.4$, $P(N_2 = 4) = 0.4$. *It is easy to see that all conditions of Theorem 1 are satisfied.* $X_{3:N_1}$ *and* $X^*_{3:N_2}$*'s distribution functions* $F_{X_{3:N_1}}(x; \boldsymbol{\lambda})$ *and* $F_{X^*_{3:N_2}}(x; \boldsymbol{\lambda}^*)$ *are shown in Figure 1, where* $x = -\ln\mu, \mu \in (0, 1]$. *According to Figure 1, we know* $F_{X_{3:N_1}}(x; \boldsymbol{\lambda}) \leq F_{X^*_{3:N_2}}(x; \boldsymbol{\lambda}^*)$. *Therefore, the validity of Theorem 1 has been verified.*

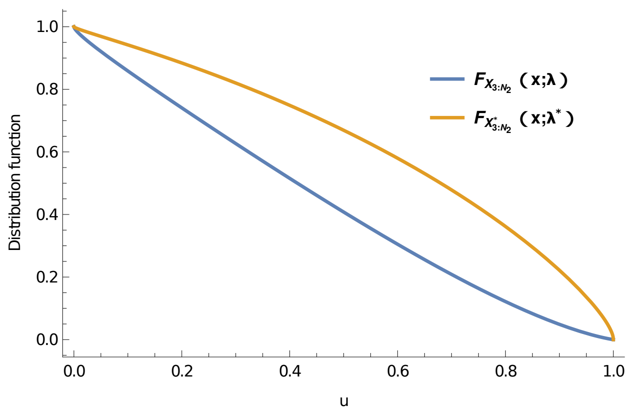

Figure 1. Curves of distribution function $F_{X_{3:N_1}}(x; \boldsymbol{\lambda})$ and $F_{X^*_{3:N_2}}(x; \boldsymbol{\lambda}^*)$, for all $x = -\ln\mu, \mu \in (0, 1]$.

Theorem 1 indicates that in reliability theory, the results under the weak majorization order of the modified proportional reversed hazard rate parameter vector with multiple heterogeneity are more reliable than those under the usual stochastic order. Next, Theorem 2 provides the usual stochastic order of the second-largest order statistic with different skew parameters and identical modified proportional reversed hazard rate parameters.

3.2. Usual stochastic order of the MPRHR from heterogeneous skew parameters

Theorem 2. *Let $X_1, X_2, ..., X_n$ be dependent heterogeneous stochastic variables of n dimensions following $X_i \sim MPRHR(\alpha_i, \lambda; F, \psi)$, with $0 < \alpha_i \leq 1 (i = 1, 2, \ldots, n)$. $X_1^*, X_2^*, \ldots, X_n^*$ are the other set of n-dimensional dependent heterogeneous stochastic variables following $X_i^* \sim MPRHR(\alpha_i^*, \lambda; F, \psi)$, with $0 < \alpha_i^* \leq 1 (i = 1, 2, \ldots, n)$. Let N_1 and N_2 be two positive real-valued stochastic variables each independently distributed with $X_i^{'}s$ and $X_i^{*'}s$, respectively, and both values are not less than 2. If $\boldsymbol{\alpha}, \boldsymbol{\alpha}^* \in \mathcal{I}_+$, $N_1 \geq_{st} N_2$, and ψ is concave in the logarithm, then*

$$\frac{1}{\boldsymbol{\alpha}} \overset{w}{\preceq} \frac{1}{\boldsymbol{\alpha}^*} \Rightarrow X_{n-1:N_1} \geq_{st} X^*_{n-1:N_2}.$$

Proof. The distribution function of $X_{n-1:n}$ can be given by

$$F_{X_{n-1:n}}(x) = \sum_{i=1}^{n} \psi\left(\sum_{j\neq i}^{n} \phi\left(\frac{\alpha_j F^\lambda(x)}{1 - \bar{\alpha}_j F^\lambda(x)}\right)\right) - (n-1)\psi\left(\sum_{i=1}^{n} \phi\left(\frac{\alpha_i F^\lambda(x)}{1 - \bar{\alpha}_i F^\lambda(x)}\right)\right).$$

Because $N_1 \geq_{st} N_2$, we have

$$\begin{aligned} F_{X_{n-1:N_1}}(x) &= 1 - \bar{F}_{X_{n-1:N_1}}(X) \\ &= 1 - \sum_{m=2}^{n} P(X_{n-1:N_1} > x | N_1 = m) P(N_1 = m) \\ &= 1 - \sum_{m=2}^{n} P(X_{n-1:m} > x) P(N_1 = m) \\ &\leq 1 - \sum_{m=2}^{n} P(X_{n-1:m} > x) P(N_2 = m) = F_{X_{n-1:N_2}}(x). \end{aligned}$$

To prove the result, we need to demonstrate $F_{X_{n-1:m}}(x) \leq F_{Y_{n-1:m}}(x)$, $m = 2, 3, \ldots, n$. Suppose $\alpha_k = 1/\alpha_k, k = 1, 2, \ldots, m$. Regarding α_k, the partial derivative can be obtained as follows:

$$\begin{aligned} &\frac{\partial F_{X_{n-1:m}}(x)}{\partial \alpha_k} \\ &= \frac{\psi\left(\phi\left(\frac{\frac{1}{\alpha_k} F^\lambda(x)}{1-(1-\frac{1}{\alpha_k})F^\lambda(x)}\right)\right)}{\psi'\left(\phi\left(\frac{\frac{1}{\alpha_k} F^\lambda(x)}{1-(1-\frac{1}{\alpha_k})F^\lambda(x)}\right)\right)} \frac{-\frac{1}{\alpha_k}[1 - F^\lambda(x)]}{1 - (1 - \frac{1}{\alpha_k})F^\lambda(x)} \\ &\times \left\{ \sum_{i\neq k}^{m} \psi^{'}\left(\sum_{j\neq i}^{m} \phi\left(\frac{\frac{1}{\alpha_j} F^\lambda(x)}{1 - (1 - \frac{1}{\alpha_j})F^\lambda(x)}\right)\right) \right. \\ &\left. - (m-1)\psi^{'}\left(\sum_{i=1}^{m} \phi\left(\frac{\frac{1}{\alpha_i} F^\lambda(x)}{1 - (1 - \frac{1}{\alpha_i})F^\lambda(x)}\right)\right) \right\}. \end{aligned}$$

Since ψ is decreasing and convex, we have the following

$$\frac{\psi\left(\phi\left(\frac{\frac{1}{\alpha_k}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_k})F^{\lambda}(x)}\right)\right)}{\psi'\left(\phi\left(\frac{\frac{1}{\alpha_k}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_k})F^{\lambda}(x)}\right)\right)} \leq 0,$$

and

$$\psi'\left(\sum_{j\neq i}^{m}\phi\left(\frac{\frac{1}{\alpha_j}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_j})F^{\lambda}(x)}\right)\right) \leq \psi'\left(\sum_{i=1}^{m}\phi\left(\frac{\frac{1}{\alpha_i}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_i})F^{\lambda}(x)}\right)\right).$$

Because $1/\alpha_k \in (0,1]$, $\left\{-1/\alpha_k[1-F^{\lambda}(x)]\right\}/\left\{1-(1-1/\alpha_k)F^{\lambda}(x)\right\} \leq 0$. Then, we have $\partial F_{X_{n-1:m}}(x)/\partial\alpha_k \leq 0$.

$$\begin{aligned}
&\frac{\partial F_{X_{n-1:m}}(x)}{\partial\lambda_k} - \frac{\partial F_{X_{n-1:m}}(x)}{\partial\lambda_t} \\
&\overset{\text{sgn}}{=} \frac{\psi\left(\phi\left(\frac{\frac{1}{\alpha_k}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_k})F^{\lambda}(x)}\right)\right)}{\psi'\left(\phi\left(\frac{\frac{1}{\alpha_k}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_k})F^{\lambda}(x)}\right)\right)} \frac{-\frac{1}{\alpha_k}[1-F^{\lambda}(x)]}{1-(1-\frac{1}{\alpha_k})F^{\lambda}(x)} \\
&\quad \times \left\{\sum_{i\neq k}^{m}\psi'\left(\sum_{j\neq i}^{m}\phi\left(\frac{\frac{1}{\alpha_j}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_j})F^{\lambda}(x)}\right)\right)\right. \\
&\quad \left. -(m-1)\psi'\left(\sum_{i=1}^{m}\phi\left(\frac{\frac{1}{\alpha_i}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_i})F^{\lambda}(x)}\right)\right)\right\} \\
&\quad - \frac{\psi\left(\phi\left(\frac{\frac{1}{\alpha_t}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_t})F^{\lambda}(x)}\right)\right)}{\psi'\left(\phi\left(\frac{\frac{1}{\alpha_t}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_t})F^{\lambda}(x)}\right)\right)} \frac{-\frac{1}{\alpha_t}[1-F^{\lambda}(x)]}{1-(1-\frac{1}{\alpha_t})F^{\lambda}(x)} \\
&\quad \times \left\{\sum_{i\neq t}^{m}\psi'\left(\sum_{j\neq i}^{m}\phi\left(\frac{\frac{1}{\alpha_j}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_j})F^{\lambda}(x)}\right)\right)\right. \\
&\quad \left. -(m-1)\psi'\left(\sum_{i=1}^{m}\phi\left(\frac{\frac{1}{\alpha_i}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_i})F^{\lambda}(x)}\right)\right)\right\} \\
&=: P_2Q_2 - U_2V_2 = P_2(Q_2-V_2) + (P_2-U_2)V_2,
\end{aligned}$$

where

$$P_2 = \frac{\psi\left(\phi\left(\frac{\frac{1}{\alpha_k}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_k})F^{\lambda}(x)}\right)\right)}{\psi'\left(\phi\left(\frac{\frac{1}{\alpha_k}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_k})F^{\lambda}(x)}\right)\right)} \frac{-\frac{1}{\alpha_k}[1-F^{\lambda}(x)]}{1-(1-\frac{1}{\alpha_k})F^{\lambda}(x)},$$

$$Q_2 = \left\{ \sum_{i \neq k}^{m} \psi' \left(\sum_{j \neq i}^{m} \phi \left(\frac{\frac{1}{\alpha_j} F^{\lambda}(x)}{1 - (1 - \frac{1}{\alpha_j}) F^{\lambda}(x)} \right) \right) \right.$$
$$\left. -(m-1)\psi' \left(\sum_{i=1}^{m} \phi \left(\frac{\frac{1}{\alpha_i} F^{\lambda}(x)}{1 - (1 - \frac{1}{\alpha_i}) F^{\lambda}(x)} \right) \right) \right\},$$

and

$$V_2 = \left\{ \sum_{i \neq t}^{m} \psi' \left(\sum_{j \neq i}^{m} \phi \left(\frac{\frac{1}{\alpha_j} F^{\lambda}(x)}{1 - (1 - \frac{1}{\alpha_j}) F^{\lambda}(x)} \right) \right) \right.$$
$$\left. -(m-1)\psi' \left(\sum_{i=1}^{m} \phi \left(\frac{\frac{1}{\alpha_i} F^{\lambda}(x)}{1 - (1 - \frac{1}{\alpha_i}) F^{\lambda}(x)} \right) \right) \right\}.$$

For any $1 \leq k < t \leq m, \alpha_k \geq \alpha_t$, and furthermore, since ϕ is decreasing and ψ is log-concave, then Therefore,

$$\frac{\psi\left(\phi\left(\frac{\frac{1}{\alpha_k}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_k})F^{\lambda}(x)}\right)\right)}{\psi'\left(\phi\left(\frac{\frac{1}{\alpha_k}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_k})F^{\lambda}(x)}\right)\right)} \geq \frac{\psi\left(\phi\left(\frac{\frac{1}{\alpha_t}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_t})F^{\lambda}(x)}\right)\right)}{\psi'\left(\phi\left(\frac{\frac{1}{\alpha_t}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_t})F^{\lambda}(x)}\right)\right)}.$$

Because $V_2 \leq 0$ and $-[1 - F^{\lambda}(x)] \leq 0$, then

$$\begin{aligned}
&(P_2 - U_2)V_2 \\
&= \left\{ \frac{\psi\left(\phi\left(\frac{\frac{1}{\alpha_k}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_k})F^{\lambda}(x)}\right)\right)}{\psi'\left(\phi\left(\frac{\frac{1}{\alpha_k}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_k})F^{\lambda}(x)}\right)\right)} \frac{-\frac{1}{\alpha_k}[1 - F^{\lambda}(x)]}{1 - (1 - \frac{1}{\alpha_k})F^{\lambda}(x)} \right. \\
&\quad \left. - \frac{\psi\left(\phi\left(\frac{\frac{1}{\alpha_t}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_t})F^{\lambda}(x)}\right)\right)}{\psi'\left(\phi\left(\frac{\frac{1}{\alpha_t}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_t})F^{\lambda}(x)}\right)\right)} \frac{-\frac{1}{\alpha_t}[1 - F^{\lambda}(x)]}{1 - (1 - \frac{1}{\alpha_t})F^{\lambda}(x)} \right\} \\
&\quad \times \left\{ \sum_{i \neq t}^{m} \psi' \left(\sum_{j \neq i}^{m} \phi \left(\frac{\frac{1}{\alpha_j} F^{\lambda}(x)}{1 - (1 - \frac{1}{\alpha_j}) F^{\lambda}(x)} \right) \right) \right. \\
&\quad \left. -(m-1)\psi' \left(\sum_{i=1}^{m} \phi \left(\frac{\frac{1}{\alpha_i} F^{\lambda}(x)}{1 - (1 - \frac{1}{\alpha_i}) F^{\lambda}(x)} \right) \right) \right\} \\
&= -[1 - F^{\lambda}(x)]
\end{aligned}$$

$$\times\left\{\frac{\psi\left(\phi\left(\frac{\frac{1}{\alpha_k}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_k})F^{\lambda}(x)}\right)\right)}{\psi'\left(\phi\left(\frac{\frac{1}{\alpha_k}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_k})F^{\lambda}(x)}\right)\right)}\frac{\frac{1}{\alpha_k}}{1-(1-\frac{1}{\alpha_k})F^{\lambda}(x)}\right.$$

$$\left.-\frac{\psi\left(\phi\left(\frac{\frac{1}{\alpha_t}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_t})F^{\lambda}(x)}\right)\right)}{\psi'\left(\phi\left(\frac{\frac{1}{\alpha_t}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_t})F^{\lambda}(x)}\right)\right)}\frac{\frac{1}{\alpha_t}}{1-(1-\frac{1}{\alpha_t})F^{\lambda}(x)}\right\}$$

$$\times\left\{\sum_{i\neq t}^{m}\psi'\left(\sum_{j\neq i}^{m}\phi\left(\frac{\frac{1}{\alpha_j}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_j})F^{\lambda}(x)}\right)\right)\right.$$

$$\left.-(m-1)\psi'\left(\sum_{i=1}^{m}\phi\left(\frac{\frac{1}{\alpha_i}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_i})F^{\lambda}(x)}\right)\right)\right\}$$

$$\overset{\text{sgn}}{=}\frac{\frac{1}{\alpha_k}}{1-(1-\frac{1}{\alpha_k})F^{\lambda}(x)}$$

$$\times\frac{\psi\left(\phi\left(\frac{\frac{1}{\alpha_k}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_k})F^{\lambda}(x)}\right)\right)}{\psi'\left(\phi\left(\frac{\frac{1}{\alpha_k}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_k})F^{\lambda}(x)}\right)\right)}-\frac{\frac{1}{\alpha_t}}{1-(1-\frac{1}{\alpha_t})F^{\lambda}(x)}\frac{\psi\left(\phi\left(\frac{\frac{1}{\alpha_t}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_t})F^{\lambda}(x)}\right)\right)}{\psi'\left(\phi\left(\frac{\frac{1}{\alpha_t}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_t})F^{\lambda}(x)}\right)\right)}$$

$$=\frac{\psi\left(\phi\left(\frac{\frac{1}{\alpha_k}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_k})F^{\lambda}(x)}\right)\right)}{\psi'\left(\phi\left(\frac{\frac{1}{\alpha_k}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_k})F^{\lambda}(x)}\right)\right)}\left(\frac{\frac{1}{\alpha_k}}{1-(1-\frac{1}{\alpha_k})F^{\lambda}(x)}-\frac{\frac{1}{\alpha_t}}{1-(1-\frac{1}{\alpha_t})F^{\lambda}(x)}\right)$$

$$+\frac{\frac{1}{\alpha_t}}{1-(1-\frac{1}{\alpha_t})F^{\lambda}(x)}\left\{\frac{\psi\left(\phi\left(\frac{\frac{1}{\alpha_k}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_k})F^{\lambda}(x)}\right)\right)}{\psi'\left(\phi\left(\frac{\frac{1}{\alpha_k}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_k})F^{\lambda}(x)}\right)\right)}\right.$$

$$\left.-\frac{\psi\left(\phi\left(\frac{\frac{1}{\alpha_t}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_t})F^{\lambda}(x)}\right)\right)}{\psi'\left(\phi\left(\frac{\frac{1}{\alpha_t}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_t})F^{\lambda}(x)}\right)\right)}\right\}$$

$$\geq 0.$$

For any $\alpha_k \geq \alpha_t$, since ϕ is decreasing and convex, we obtain

$$\psi'\left(\sum_{j\neq k}^{m}\phi\left(\frac{\frac{1}{\alpha_k}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_k})F^{\lambda}(x)}\right)\right)\leq\psi'\left(\sum_{j\neq t}^{m}\phi\left(\frac{\frac{1}{\alpha_t}F^{\lambda}(x)}{1-(1-\frac{1}{\alpha_t})F^{\lambda}(x)}\right)\right).$$

Thus,

$$P_2(Q_2 - V_2)$$
$$\overset{\text{sgn}}{=} \sum_{i \neq k}^{m} \psi' \left(\sum_{j \neq i}^{m} \phi \left(\frac{\frac{1}{\alpha_j} F^{\lambda}(x)}{1 - (1 - \frac{1}{\alpha_j}) F^{\lambda}(x)} \right) \right) - \sum_{i \neq t}^{m} \psi' \left(\sum_{j \neq i}^{m} \phi \left(\frac{\frac{1}{\alpha_j} F^{\lambda}(x)}{1 - (1 - \frac{1}{\alpha_j}) F^{\lambda}(x)} \right) \right)$$
$$= \psi' \left(\sum_{j \neq t}^{m} \phi \left(\frac{\frac{1}{\alpha_j} F^{\lambda}(x)}{1 - (1 - \frac{1}{\alpha_j}) F^{\lambda}(x)} \right) \right) - \psi' \left(\sum_{j \neq k}^{m} \phi \left(\frac{\frac{1}{\alpha_j} F^{\lambda}(x)}{1 - (1 - \frac{1}{\alpha_j}) F^{\lambda}(x)} \right) \right) \geq 0.$$

Combining the above, $P_2(Q_2 - V_2) + (P_2 - U_2)V_2 \geq 0$. By Lemma 2, the conclusion is proved.

□

Next, this paper will provide a numerical example to demonstrate the results of Theorem 2.

Example 2. *Consider the case of* $n = 4$. *Let the distribution function* $F(x) = 1 - e^{-(ax)^b}$, $a > 0$, $b > 0$, *generating element* $\psi(x) = \exp\left\{1 - (1 + x)^{\theta}\right\}$, *and* $0 < \theta \leq 1$. *Suppose* $\lambda = 0.6$, $a = 0.8$, $b = 0.6$, $\theta = 7$, $\boldsymbol{\alpha} = (1/9, 1/8, 1/7, 1/6)$, $\boldsymbol{\alpha}^* = (1/8, 1/6, 1/5, 1/4)$, *and thus* $1/\boldsymbol{\alpha} \overset{w}{\leq} 1/\boldsymbol{\alpha}^*$. *Suppose* N_1 *is a positive real value with the probability distribution* $P(N_1 = 2) = 0.15$, $P(N_1 = 3) = 0.35$, $P(N_1 = 4) = 0.5$, *and* N_2 *is positive real value with the probability distribution* $P(N_2 = 2) = 0.2$, $P(N_2 = 3) = 0.4$, $P(N_2 = 4) = 0.4$. *Obviously* $N_1 \geq_{st} N_2$. $X_{3:N_1}$ *and* $X^*_{3:N_2}$*'s distribution functions* $F_{X_{3:N_1}}(x; \boldsymbol{\alpha})$ *and* $F_{X^*_{3:N_2}}(x; \boldsymbol{\alpha}^*)$ *are shown in Figure 2, where* $x = -\ln \mu$, $\mu \in (0, 1]$. *According to Figure 2, we know* $F_{X_{3:N_1}}(x; \boldsymbol{\alpha}) \leq F_{X^*_{3:N_2}}(x; \boldsymbol{\alpha}^*)$. *Therefore, the validity of Theorem 1 has been verified.*

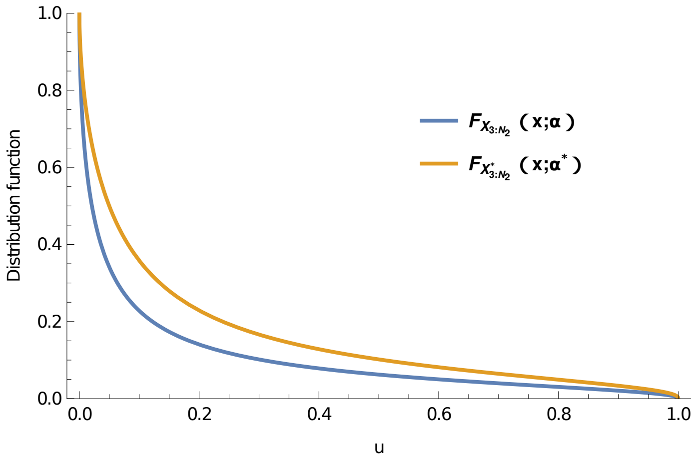

Figure 2. Curves of distribution function $F_{X_{3:N_1}}(x; \boldsymbol{\alpha})$ and $F_{X^*_{3:N_2}}(x; \boldsymbol{\alpha}^*)$, for all $x = -\ln \mu, \mu \in (0, 1]$.

Remark 1. *The combination of Theorems 1 and 2 readily leads to the following conclusion. Let* $X_1, X_2, \ldots, X_n$ *be dependent heterogeneous random variables of n dimensions following* $X_i \sim MPRHR(\alpha_i, \lambda_i; F, \psi)$, *with* $0 < \alpha_i \leq 1 (i = 1, 2, \ldots, n)$. $X^*_1, X^*_2, \ldots, X^*_n$ *are the other set of n-dimensional dependently heterogeneous stochastic variables following* $X^*_i \sim MPRHR(\alpha^*_i, \lambda_i; F, \psi)$, *with* $0 < \alpha^*_i \leq 1 (i = 1, 2, \ldots, n)$. *Let* N_1 *and* N_2 *be two positive real-valued stochastic variables each independently distributed with* X_i'*s and* $X_i^{*'}$*s, respectively, and both values are not less than 2. If* $\boldsymbol{\lambda}, \boldsymbol{\lambda}^* \in \mathcal{D}_+$, $\boldsymbol{\alpha}, \boldsymbol{\alpha}^* \in \mathcal{I}_+$, $N_1 \geq_{st} N_2$, *and* ψ *is concave in the logarithm, then*

$$\boldsymbol{\lambda} \overset{w}{\leq} \boldsymbol{\lambda}^*, \frac{1}{\boldsymbol{\alpha}} \overset{w}{\leq} \frac{1}{\boldsymbol{\alpha}^*} \Rightarrow X_{n-1:N_1} \geq_{st} X^*_{n-1:N_2}.$$

However, to verify the validity of the conclusion, further clarification will be provided in Example 3 as follows.

Example 3. *Consider the case when* $n = 4$. *Let the distribution function* $F(x) = 1 - e^{-(ax)^b}$, $a > 0$, $b > 0$, *generating element* $\psi(x) = \exp^{1-(1+x)^\theta}$, *and* $\theta > 0$. *Suppose* $\alpha = 0.8$, $b = 0.6$, $\theta = 1$, $\boldsymbol{\alpha} = (1/9, 1/6, 1/6, 1/5)$, *and* $\boldsymbol{\alpha}^* = (1/7, 1/5, 1/4, 1/4)$, *then* $1/\boldsymbol{\alpha} \overset{w}{\preceq} \boldsymbol{\alpha}^*$, *and we can know* $\boldsymbol{\lambda} = (1.8, 1.4, 0.4, 0.3) \overset{w}{\preceq} (1.5, 0.9, 0.4, 0.2) = \boldsymbol{\lambda}^*$. *Suppose* N_1 *is a positive real value with the probability distribution* $P(N_1 = 2) = 0.15, P(N_1 = 3) = 0.35$, $P(N_1 = 4) = 0.5$, *and* N_2 *is a positive real values with the probability distribution* $P(N_2 = 2) = 0.3$, $P(N_2 = 3) = 0.35$, $P(N_2 = 4) = 0.35$. $X_{3:N_1}$ *and* $X^*_{3:N_2}$*'s distribution functions are* $F_{X_{3:N_1}}(x; \boldsymbol{\lambda}, \boldsymbol{\alpha})$ *and* $F_{X^*_{3:N_2}}(x; \boldsymbol{\lambda}^*, \boldsymbol{\alpha}^*)$. *Plot the graph of* $F_{X_{3:N_1}}(x; \boldsymbol{\lambda}, \boldsymbol{\alpha}) - F_{X^*_{3:N_2}}(x; \boldsymbol{\lambda}^*, \boldsymbol{\alpha}^*)$, *as shown in Figure 3, where* $x = -\ln\mu$, $\mu \in (0, 1]$. *According to Figure 3, the graph intersects the x-axis under the conditions of Theorem 2. Therefore, the conclusion does not hold.*

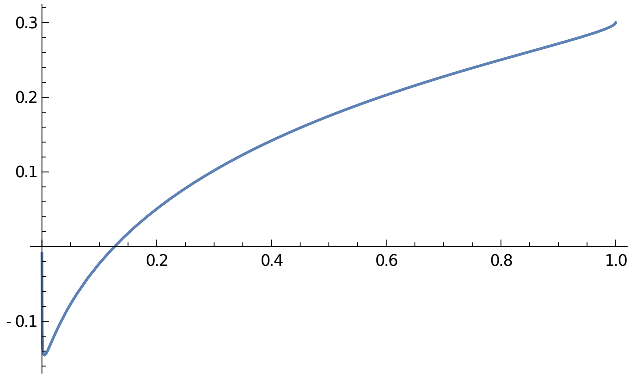

Figure 3. Curve of distribution function $F_{X_{3:N_1}}(x; \boldsymbol{\lambda}, \boldsymbol{\alpha}) - F_{X^*_{3:N_2}}(x; \boldsymbol{\lambda}^*, \boldsymbol{\alpha}^*)$, for all $x = -\ln\mu, \mu \in (0, 1]$.

4. Reversed hazard rate orders of independent heterogeneous samples

In this section, we will investigate the reversed hazard order of the second-largest order statistic under independent heterogeneous observation samples. Obviously, through the study of the above theorems, a natural question arises: If we strengthen the conditions of Theorems 1 and 2, turning $N_1 \geq_{st} N_2$ into $N_1 \geq_{rh} N_2$, can the corresponding conclusion be strengthened from the usual stochastic order to reversed hazard rate order? The answer is no. Take Theorem 2 as an example, and this article will provide an example to illustrate. The gamma distribution is widely used in many fields such as engineering, science, and business. For a stochastic variable X that follows a Gamma distribution, with shape parameter $\alpha > 0$, and scale parameter $\beta > 0$ (denoted by $X \sim \Gamma(\alpha, \beta)$), the probability density function is

$$f(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \; x \in \mathrm{R}_+.$$

Example 4. *Assume the base distribution function* $F(x)$ *to be* $\Gamma(5, 0.6)$, *and generate* $\psi(x) = e^{1-(1+x)^\theta}$, $\theta > 0$. *Let* $n = 4$, $\lambda = 0.4$, $\theta = 5$, *and* $\mathbf{1/\boldsymbol{\alpha}} = (8, 4, 3, 1) \overset{m}{\succeq} (6, 4, 4, 2) = \mathbf{1/\boldsymbol{\alpha}^*}$. *The probability distribution of positive real value* N_1 *is* $P(N_1 = 2) = 0.01$, $P(N_1 = 3) = 0.3$, $P(N_1 = 4) = 0.699$. *The probability distribution of positive real value* N_2 *is* $P(N_2 = 2) = 0.5$, $P(N_2 = 3) = 0.15$, $P(N_2 = 4) =$

0.35, *obviously,* $N_1 \geq_{rh} N_2$. *And the graph of* $F_{X^*_{3:N_2}}(x;\boldsymbol{\alpha}^*)/F_{X_{3:N_1}}(x;\boldsymbol{\alpha})$ *is the ratio of the distribution function of* $X^*_{3:N_2}$ *and* $X_{3:N_2}$, *as shown in the Figure 4, where* $x \in [8,10]$. *By observing Figure 4, the curve is neither monotonically increasing nor monotonically decreasing. We can see that neither* $X_{3:N_1} \leq_{rh} X^*_{3:N_2}$ *nor* $X_{3:N_1} \leq_{rh} X^*_{3:N_2}$ *are true.*

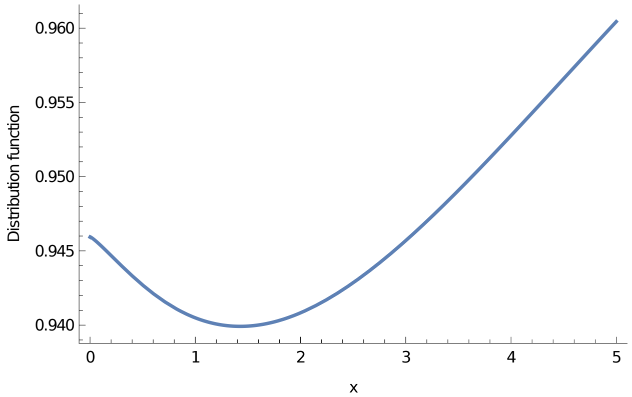

Figure 4. Curve of distribution function $F_{X_{3:N_2}}(x;\boldsymbol{\alpha}^*)/F_{X_{3:N_1}}(x;\boldsymbol{\alpha})$, for all $x \in [8,10]$.

4.1. Reversed hazard rate order of the heterogeneous tilt parameter

In the following part, under the condition of the same sample size, the reversed hazard efficiency order of the second-largest order statistic is studied when the modified proportional reversed hazard rate parameters are the same but the tilt parameters are different.

Theorem 3. *Suppose that* $X_1, X_2, \ldots, X_n$ *are independent heterogeneous stochastic variables* $X_i \sim MPRHR(\alpha_i, \lambda; F)$, *where* $0 < \alpha_i \leq 1 (i = 1, 2, \ldots, n)$. $X_1^*, X_2^*, \cdots, X_n^*$ *is another set of independent heterogeneous stochastic variables* $X_i^* \sim MPRHR(\alpha_i^*, \lambda; F)$, *where* $0 < \alpha_i^* \leq 1 (i = 1, 2, \ldots, n)$. *If* $\boldsymbol{\alpha}, \boldsymbol{\alpha}^* \in \mathcal{D}_+$, *then*

$$\frac{\mathbf{1}}{\boldsymbol{\alpha}} \overset{m}{\geq} \frac{\mathbf{1}}{\boldsymbol{\alpha}^*} \Rightarrow X_{n-1:n} \leq_{rh} X^*_{n-1:n}.$$

Proof. The distribution function of $X_{n-1:n}$ is

$$\begin{aligned} F_{X_{n-1:n}}(x) &= \sum_{i=1}^{n} \prod_{j \neq i}^{n} \frac{\alpha_j F^\lambda(x)}{1 - \overline{\alpha}_j F^\lambda(x)} - (n-1) \prod_{i=1}^{n} \frac{\alpha_j F^\lambda(x)}{1 - \overline{\alpha}_j F^\lambda(x)} \\ &= \prod_{i=1}^{n} \frac{\alpha_i F^\lambda(x)}{1 - \overline{\alpha}_i F^\lambda(x)} \left[\sum_{i=1}^{n} \frac{1 - \overline{\alpha}_i F^\lambda(x)}{\alpha_i F^\lambda(x)} - (n-1) \right] \\ &= \prod_{i=1}^{n} \frac{\alpha_i F^\lambda(x)}{1 - \overline{\alpha}_i F^\lambda(x)} \left[\sum_{i=1}^{n} \frac{1 - F^\lambda(x)}{\alpha_i F^\lambda(x)} + 1 \right]. \end{aligned}$$

Therefore, the reversed hazard rate function is

$$\begin{aligned} \tilde{r}_{X_{n-1:n}}(x) &= \prod_{i=1}^{n} \frac{\alpha_j F^\lambda(x)}{1 - \overline{\alpha}_j F^\lambda(x)} \left[\sum_{i=1}^{n} \frac{1 - \overline{\alpha}_j F^\lambda(x)}{\alpha_j F^\lambda(x)} + 1 \right] \\ &= \sum_{i=1}^{n} \frac{\lambda \tilde{r}(x)}{1 - \overline{\alpha}_j F^\lambda(x)} - \frac{\sum_{i=1}^{n} \frac{\lambda \tilde{r}(x)}{\alpha_i F^\lambda(x)}}{\sum_{i=1}^{n} \frac{1 - \tilde{r}^2(x)}{\alpha_i F^\lambda(x)} + 1}, \end{aligned}$$

where $\tilde{r}(x)$ is $F(x)$, the reversed hazard rate function. Let $a_i = 1/\alpha_i, i = 1, 2, \ldots, n$. The partial derivative of $\tilde{r}_{X_{n-1:n}}$ with respect to a_k is

$$\frac{\partial \tilde{r}_{n-1:n}(x)}{\partial a_k} = -\left(\left(\frac{\lambda \tilde{r}(x)}{F^{\lambda}(x)}\right)\Big/\left(\sum_{i=1}^{n}\frac{a_i(1-F^{\lambda}(x))}{F^{\lambda}(x)}+1\right)\right) + \frac{\frac{1}{a_k^2}F^{\lambda}(x)\lambda\tilde{r}(x)}{\left[1-(1-\frac{1}{a_k})F^{\lambda}(x)\right]^2} + \frac{\sum_{i=1}^{n} \frac{a_i\lambda\tilde{r}(x)}{F^{\lambda}(x)}\frac{1-F^{\lambda}(x)}{F^{\lambda}(x)}}{\left[\sum_{i=1}^{n} \frac{a_i(1-F^{\lambda}(x))}{F^{\lambda}(x)}+1\right]^2}.$$

In order to prove the result, it is also necessary to prove that $\partial r_{n-1:n}(x)/\partial a_k$, $k = 1, 2, \ldots, n$, is decreasing. By theorem conditions, $\boldsymbol{a} \in \mathcal{I}_+$. One knows, for any $1 \le k < t \le n$, $a_k \le a_t$.

$$\frac{\partial \tilde{r}_{X_{x-1:n}}(x)}{\partial a_k} - \frac{\partial \tilde{r}_{X_{x-1:n}}(x)}{\partial a_t} \overset{\text{sgn}}{=} \frac{\frac{1}{a_k^2}}{\left[1-(1-\frac{1}{a_k})F^{\lambda}(x)\right]^2} - \frac{\frac{1}{a_t^2}}{\left[1-(1-\frac{1}{a_t})F^{\lambda}(x)\right]^2} \ge 0.$$

Therefore, from Lemma 1, we know $\tilde{r}_{X_{n-1:n}}(x) \le \tilde{r}_{Y_{n-1:n}}(x)$. Theorem 3 is proved. □

In the following part, this paper will give a numerical example to demonstrate the result of Theorem 3.

Example 5. *Assume the base distribution function* $F(x) = 1 - e^{-(ax)^b}$, $a > 0$, $b > 0$, *and let* $n = 4$, $\lambda = 0.4$, $a = 0.3$, $b = 1.5$, *and* $\mathbf{1}/\boldsymbol{\alpha} = (8, 4, 3, 1) \overset{m}{\ge} (6, 4, 4, 2) = \mathbf{1}/\boldsymbol{\alpha}^*$. *It is easy to know all the conditions of Theorem 3.* $X_{3:4}$ *and* $X^*_{3:4}$ *are the reversed hazard rate functions of the curves of* $\tilde{r}_{X_{3:4}}(x; \boldsymbol{\lambda})$ *and* $\tilde{r}_{X^*_{3:4}}(x; \boldsymbol{\lambda}^*)$, *as shown in Figure 5, where for all* $x = -\ln u$, $u \in (0, 1]$. *By observing Figure 5, it can be seen that* $\tilde{r}_{X_{3:4}}(x; \boldsymbol{\lambda}) \le \tilde{r}_{X^*_{3:4}}(x; \boldsymbol{\lambda}^*)$, *and therefore,* $X_{3:4} \le_{rh} X^*_{3:4}$.

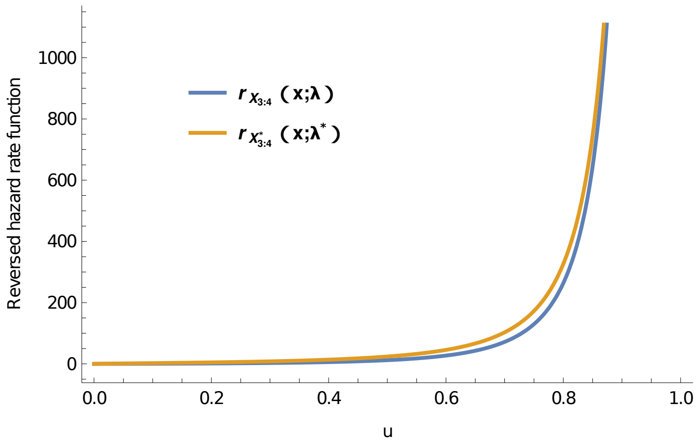

Figure 5. Curves of reversed hazard rate function $\tilde{r}_{X_{3:4}}(x; \boldsymbol{\lambda})$ and $\tilde{r}_{X_{3:4}}(x; \boldsymbol{\lambda}^*)$, for all $x = -\ln u$, $u \in (0, 1]$.

4.2. Reversed hazard rate order of the heterogeneous parameter

Next, under the condition of the same sample size, the reversed hazard rate order of the second-order statistic is established when the tilt parameter is the same and the modified proportional reversed hazard rate parameter is different.

Theorem 4. *Suppose that $X_1, X_2, \ldots, X_n$ are independent stochastic variables with multivariate outlier modified proportional reversed hazard rate distribution $(\frac{\alpha F^{\lambda_1}(x)}{1-\overline{\alpha}F^{\lambda_1}(x)}I_p, \frac{\alpha F^{\lambda}(x)}{1-\overline{\alpha}F^{\lambda}(x)}I_q)$. $X_1^*, X_2^*, \ldots, X_n^*$ is another set of independent stochastic variables with multivariate outlier modified proportional reversed hazard rate distribution $(\frac{\alpha F^{\lambda_2}(x)}{1-\overline{\alpha}F^{\lambda_2}(x)}I_p, \frac{\alpha F^{\lambda}(x)}{1-\overline{\alpha}F^{\lambda}(x)}I_q)$, where $p, q \geq 1$, and $p+q=n$. If $\lambda \geq \lambda_2 \geq \lambda_1$, then $X_{n-1:n} \leq_{rh} X_{n-1:n}^*$.*

Proof. The distribution functions of $X_{n-1:n}$ and $X_{n-1:n}^*$ are

$$F_{X_{n-1:n}}(x) = p\left[\frac{\alpha F^{\lambda_1}(x)}{1-\overline{\alpha}F^{\lambda_1}(x)}\right]^{p-1}\left[\frac{\alpha F^{\lambda}(x)}{1-\overline{\alpha}F^{\lambda}(x)}\right]^{q} + q\left[\frac{\alpha F^{\lambda_1}(x)}{1-\overline{\alpha}F^{\lambda_1}(x)}\right]^{p} \times\left[\frac{\alpha F^{\lambda}(x)}{1-\overline{\alpha}F^{\lambda}(x)}\right]^{q-1} - (n-1)\left[\frac{\alpha F^{\lambda_1}(x)}{1-\overline{\alpha}F^{\lambda_1}(x)}\right]^{p}\left[\frac{\alpha F^{\lambda}(x)}{1-\overline{\alpha}F^{\lambda}(x)}\right]^{q},$$

and

$$F^*_{X_{n-1}}(x) = p\left[\frac{\alpha F^{\lambda_2}(x)}{1-\overline{\alpha}F^{\lambda_2}(x)}\right]^{p-1}\left[\frac{\alpha F^{\lambda}(x)}{1-\overline{\alpha}F^{\lambda}(x)}\right]^{q} + q\left[\frac{\alpha F^{\lambda_2}(x)}{1-\overline{\alpha}F^{\lambda_2}(x)}\right]^{p} \times\left[\frac{\alpha F^{\lambda}(x)}{1-\overline{\alpha}F^{\lambda}(x)}\right]^{q-1} - (n-1)\left[\frac{\alpha F^{\lambda_2}(x)}{1-\overline{\alpha}F^{\lambda_2}(x)}\right]^{p}\left[\frac{\alpha F^{\lambda}(x)}{1-\overline{\alpha}F^{\lambda}(x)}\right]^{q},$$

where $\overline{\alpha} = 1-\alpha$ and $p+q=n$. Let $F^{\lambda}(x) = e^{-\lambda(-\ln F(x))}$ and $t = -\ln F(x)$. The distribution function of $X_{n-1:n}$ is

$$F_{X_{n-1:n}}(x) = p\left[\frac{\alpha}{e^{\lambda_1 t}-\overline{\alpha}}\right]^{p-1}\left[\frac{\alpha}{e^{\lambda t}-\overline{\alpha}}\right]^{q} + q\left[\frac{\alpha}{e^{\lambda_1 t}-\overline{\alpha}}\right]^{p}\left[\frac{\alpha}{e^{\lambda t}-\overline{\alpha}}\right]^{q-1} - (n-1)\left[\frac{\alpha}{e^{\lambda_1 t}-\overline{\alpha}}\right]^{p}\left[\frac{\alpha}{e^{\lambda t}-\overline{\alpha}}\right]^{q}, t \geq 0.$$

For convenience, this article is set

$$\begin{aligned} A_i &= (p-1)\frac{\lambda_i e^{\lambda_i t}}{e^{\lambda_i t}-\overline{\alpha}} + q\frac{\lambda e^{\lambda t}}{e^{\lambda t}-\overline{\alpha}},\\ B_i &= p\frac{\lambda_i e^{\lambda t}}{e^{\lambda t}-\overline{\alpha}} + (q-1)\frac{\lambda e^{\lambda t}}{e^{\lambda t}-\overline{\alpha}},\\ C_i &= p\frac{\lambda_i e^{\lambda_i t}}{e^{\lambda_i t}-\overline{\alpha}} + q\frac{\lambda e^{\lambda t}}{e^{\lambda t}-\overline{\alpha}}, \quad for\ all\ i = 1, 2. \end{aligned}$$

Therefore,

$$\tilde{r}_{X_{n-1:n}}(t) = \frac{\mathrm{d}\ln F_{X_{n-1:n}}(t)}{\mathrm{d}t} = \frac{pA_1(\frac{e^{\lambda_1 t}-\overline{\alpha}}{\alpha}) + qB_1(\frac{e^{\lambda}-\overline{\alpha}}{\alpha}) - (n-1)C_1}{p(\frac{e^{\lambda_1 t}-\overline{\alpha}}{\alpha}) + q(\frac{e^{\lambda}-\overline{\alpha}}{\alpha}) - (n-1)}.$$

To prove the result, we need to prove $\tilde{r}_{X_{n-1:n}}(t) \leq \tilde{r}_{X_{n-1:n}^*}(t)$, for any $\lambda \geq \lambda_2 \geq \lambda_1 > 0$,

$$\frac{pA_1(\frac{e^{\lambda_1 t}-\overline{\alpha}}{\alpha}) + qB_1(\frac{e^{w}-\overline{\alpha}}{\alpha}) - (n-1)C_1}{p(\frac{e^{\lambda_1 t}-\overline{\alpha}}{\alpha}) + q(\frac{e^{\lambda t}-\overline{\alpha}}{\alpha}) - (n-1)} \leq \frac{pA_2(\frac{e^{\lambda_1 t}-\overline{\alpha}}{\alpha}) + qB_2(\frac{e^{\lambda t}-\overline{\alpha}}{\alpha}) - (n-1)C_2}{p(\frac{e^{\lambda_1 t}-\overline{\alpha}}{\alpha}) + q(\frac{e^{\lambda t}-\overline{\alpha}}{\alpha}) - (n-1)}.$$

Hence,

$$\left\{p\left[(p-1)\frac{\lambda_1 e^{\lambda_i t}}{e^{\lambda_i t}-\overline{\alpha}}+q\frac{\lambda e^{\lambda t}}{e^{\lambda t}-\overline{\alpha}}\right](\frac{e^{\lambda t}-\overline{\alpha}}{\alpha})+q\left[p\frac{\lambda_1 e^{\lambda_i t}}{e^{\lambda_i t}-\overline{\alpha}}+(q-1)\frac{\lambda e^{\lambda t}}{e^{\lambda t}-\overline{\alpha}}\right](\frac{e^{\lambda t}-\overline{\alpha}}{\alpha})\right\}$$
$$-(n-1)\left[p\frac{\lambda_1 e^{\lambda_i t}}{e^{\lambda_i t}-\overline{\alpha}}+q\frac{\lambda e^{\lambda t}}{e^{\lambda t}-\overline{\alpha}}\right]\times\frac{1}{p(\frac{e^{\lambda t}-\overline{\alpha}}{\alpha})+q(\frac{e^{\lambda t}-\overline{\alpha}}{\alpha})-(n-1)}$$
$$\leq\left\{p\left[(p-1)\frac{\lambda_2 e^{\lambda_i t}}{e^{\lambda_i t}-\overline{\alpha}}+q\frac{\lambda e^{\lambda t}}{e^{\lambda t}-\overline{\alpha}}\right](\frac{e^{\lambda t}-\overline{\alpha}}{\alpha})+q\left[p\frac{\lambda_2 e^{\lambda_i t}}{e^{\lambda_i t}-\overline{\alpha}}+(q-1)\frac{\lambda e^{\lambda t}}{e^{\lambda t}-\overline{\alpha}}\right](\frac{e^{\lambda t}-\overline{\alpha}}{\alpha})\right\}$$
$$-(n-1)\left[p\frac{\lambda_1 e^{\lambda_i t}}{e^{\lambda_i t}-\overline{\alpha}}+q\frac{\lambda e^{\lambda t}}{e^{\lambda t}-\overline{\alpha}}\right]\times\frac{2}{p(\frac{e^{\lambda t}-\overline{\alpha}}{\alpha})+q(\frac{e^{\lambda t}-\overline{\alpha}}{\alpha})-(n-2)}.$$

For simplicity, set

$$M_1=\frac{1}{p(\frac{e^{\lambda t}-\overline{\alpha}}{\alpha})+q(\frac{e^{\lambda t}-\overline{\alpha}}{\alpha})-(n-1)}\times p\left[(p-1)\frac{\lambda_1 e^{\lambda t}}{e^{\lambda t}-\overline{\alpha}}+q\frac{\lambda e^{\lambda t}}{e^{\lambda t}-\overline{\alpha}}\right](\frac{e^{\lambda t}-\overline{\alpha}}{\alpha})$$
$$+q\left[p\frac{\lambda_1 e^{\lambda_1 t}}{e^{\lambda_1 t}-\overline{\alpha}}+(q-1)\frac{\lambda e^{\lambda t}}{e^{\lambda t}-\overline{\alpha}}\right](\frac{e^{\lambda t}-\overline{\alpha}}{\alpha})-(n-1)\left[p\frac{\lambda_1 e^{\lambda_1 t}}{e^{\lambda_1 t}-\overline{\alpha}}+q\frac{\lambda e^{\lambda t}}{e^{\lambda t}-\overline{\alpha}}\right],$$

$$M_2=\frac{1}{p(\frac{e^{\lambda t}-\overline{\alpha}}{\alpha})+q(\frac{e^{\lambda t}-\overline{\alpha}}{\alpha})-(n-1)}\times p\left[(p-1)\frac{\lambda_1 e^{\lambda t}}{e^{\lambda t}-\overline{\alpha}}+q\frac{\lambda e^{\lambda t}}{e^{\lambda t}-\overline{\alpha}}\right](\frac{e^{\lambda t}-\overline{\alpha}}{\alpha})$$
$$+q\left[p\frac{\lambda_1 e^{\lambda_1 t}}{e^{\lambda_1 t}-\overline{\alpha}}+(q-1)\frac{\lambda e^{\lambda t}}{e^{\lambda t}-\overline{\alpha}}\right](\frac{e^{\lambda t}-\overline{\alpha}}{\alpha})-(n-1)\left[p\frac{\lambda_1 e^{\lambda_1 t}}{e^{\lambda_1 t}-\overline{\alpha}}+q\frac{\lambda e^{\lambda t}}{e^{\lambda t}-\overline{\alpha}}\right],$$

and

$$M_3=\frac{1}{p(\frac{e^{\lambda t}-\overline{\alpha}}{\alpha})+q(\frac{e^{\lambda t}-\overline{\alpha}}{\alpha})-(n-1)}\times p\left[(p-1)\frac{\lambda_1 e^{\lambda_1 t}}{e^{\lambda t}-\overline{\alpha}}+q\frac{\lambda e^{\lambda t}}{e^{\lambda t}-\overline{\alpha}}\right](\frac{e^{\lambda t}-\overline{\alpha}}{\alpha})$$
$$+q\left[p\frac{\lambda_1 e^{\lambda_1 t}}{e^{\lambda_1 t}-\overline{\alpha}}+(q-1)\frac{\lambda e^{\lambda t}}{e^{\lambda t}-\overline{\alpha}}\right](\frac{e^{\lambda t}-\overline{\alpha}}{\alpha})-(n-1)\left[p\frac{\lambda_1 e^{\lambda_1 t}}{e^{\lambda_1 t}-\overline{\alpha}}+q\frac{\lambda e^{\lambda t}}{e^{\lambda t}-\overline{\alpha}}\right].$$

For any $\lambda\geq\lambda_2\geq\lambda_1>0$, $\lambda e^{\lambda t}/(\lambda e^{\lambda t}-\overline{\alpha})$ is increasing in λ:

$$\frac{\lambda_2 e^{\lambda_2 t}}{e^{\lambda_2 t}-\overline{\alpha}}\geq\frac{\lambda_1 e^{\lambda_1 t}}{e^{\lambda_1 t}-\overline{\alpha}}.$$

Furthermore,

$$M_3-M_2\overset{\text{sgn}}{=}p(p-1)(\frac{e^{\lambda_2 t}-\overline{\alpha}}{a})\left[\frac{\lambda_2 e^{\lambda_2 t}}{e^{\lambda_2 t}-\overline{\alpha}}-\frac{\lambda e^{\lambda t}}{e^{\lambda t}-\overline{\alpha}}\right]+pq(\frac{e^{\lambda t}-\overline{\alpha}}{\alpha})$$
$$\times\left[\frac{\lambda_2 e^{\lambda_2 t}}{e^{\lambda_2 t}-\overline{\alpha}}-\frac{\lambda_1 e^{\lambda_1 t}}{e^{\lambda_1 t}-\overline{\alpha}}\right]-p(n-1)\left[\frac{\lambda_2 e^{\lambda_2 t}}{e^{\lambda_2 t}-\overline{\alpha}}-\frac{\lambda_1 e^{\lambda_i t}}{e^{\lambda_i t}-\overline{\alpha}}\right]$$
$$=p\left[\frac{\lambda_2 e^{\lambda_2 t}}{e^{\lambda_2 t}-\overline{\alpha}}-\frac{\lambda_1 e^{\lambda_1 t}}{e^{\lambda_1 t}-\overline{\alpha}}\right]\left[(p-1)(\frac{e^{\lambda_2 t}-\overline{\alpha}}{\alpha})+q(\frac{e^{\lambda t}-\overline{\alpha}}{\alpha})-(n-1)\right]$$

$$\geq p\left[\frac{\lambda_2 e^{\lambda_2 t}}{e^{\lambda_2 t}-\overline{\alpha}}-\frac{\lambda_1 e^{\lambda_1 t}}{e^{\lambda_1 t}-\overline{\alpha}}\right][p+q-1-(n-1)]=0.$$

Let

$$\begin{aligned}
P &= q\left[p\frac{\lambda_1 e^{\lambda_i t}}{e^{\lambda_i t}-\overline{\alpha}}+(q-1)\frac{\lambda e^{\lambda t}}{e^{\lambda t}-\overline{\alpha}}\right](\frac{e^{\lambda t}-\overline{\alpha}}{\alpha})-(n-1)\left[p\frac{\lambda_1 e^{\lambda t}}{e^{\lambda t}-\overline{\alpha}}+q\frac{\lambda e^{\lambda t}}{e^{\lambda t}-\overline{\alpha}}\right],\\
Q &= p\left[(p-1)\frac{\lambda_1 e^{\lambda_1 t}}{e^{\lambda_1 t}-\overline{\alpha}}+q\frac{\lambda e^{\lambda t}}{e^{\lambda t}-\overline{\alpha}}\right],\\
U &= q(\frac{e^{\lambda_2 t}-\overline{\alpha}}{\alpha})-(n-1),\\
V &= p.
\end{aligned}$$

Therefore,

$$QU-PV=pq(\frac{e^{\lambda t}-\overline{\alpha}}{\alpha})\left[\frac{\lambda e^{\lambda t}}{e^{\lambda t}-\overline{\alpha}}-\frac{\lambda_1 e^{\lambda_1 t}}{e^{\lambda_1 t}-\overline{\alpha}}\right]+p(n-1)(\frac{\lambda_1 e^{\lambda_2 t}}{e^{\lambda_1 t}-\overline{\alpha}})\geq 0.$$

Hence,

$$\begin{aligned}
M_2-M_1 &= \frac{Q(\frac{e^{\lambda t}-\overline{\alpha}}{\alpha})+P}{V(\frac{e^{\lambda t}-\overline{\alpha}}{\alpha})+U}-\frac{Q(\frac{e^{\lambda t}-\overline{\alpha}}{\overline{\alpha}})+P}{V(\frac{e^{\lambda t}-\overline{\alpha}}{\alpha})+U}\\
&= \frac{\left[Q(\frac{e^{\lambda t}-\overline{\alpha}}{\alpha})+P\right]\left[V(\frac{e^{\lambda t}-\overline{\alpha}}{\alpha})+U\right]-\left[Q(\frac{e^{\lambda t}-\overline{\alpha}}{\alpha})+P\right]\left[V(\frac{e^{\lambda t}-\overline{\alpha}}{\alpha})+U\right]}{\left[V(\frac{e^{\lambda_2 t}-\overline{\alpha}}{\alpha})+U\right]\left[V(\frac{e^{\lambda_1 t}-\overline{\alpha}}{\alpha})+U\right]}\\
&\overset{\text{sgn}}{=} (QU-PV)(\frac{e^{\lambda_2 t}-\overline{\alpha}}{\alpha}-\frac{e^{\lambda_i t}-\overline{\alpha}}{\alpha})\geq 0.
\end{aligned}$$

Combining $M_1 \leq M_2$ with $M_2 \leq M_3$, we have $M_1 \leq M_3$. Therefore, the conclusion is proved. □

4.3. Reversed hazard rate order of different sample sizes

Next, Theorem 5 establishes the reversed hazard rate order of the second-largest order statistic with the same tilt parameter and modified proportional reversed hazard rate parameter under the condition of different sample sizes.

Theorem 5. *Suppose that $X_1, X_2, \ldots, X_n$ is a set of independent stochastic variables with multivariate outlier modified proportional reversed hazard rate distribution $(\frac{\alpha F^{\lambda_1}(x)}{1-\overline{\alpha}F^{\lambda_1}(x)}I_p, \frac{\alpha F^{\lambda_2}(x)}{1-\overline{\alpha}F^{\lambda_2}(x)}I_q)$, where $p, q \leq 1$, and $p+q=n$. $X_1^*, X_2^*, \cdots, X_n^*$ is another set of independent stochastic variables with multivariate outlier modified proportional reversed hazard rate distribution $(\frac{\alpha F^{\lambda_1}(x)}{1-\overline{\alpha}F^{\lambda_1}(x)}I_p, \frac{\alpha F^{\lambda_2}(x)}{1-\overline{\alpha}F^{\lambda_2}(x)}I_q)$, where $p, q \leq 1$, and $p+q=n$. If $p^* \leq p \leq q \leq q^*$ and $\lambda_2 \geq \lambda_1$, then*

$$(p,q) \leq_w (p^*,q^*) \Rightarrow X_{n-1:n} \leq_{rh} X^*_{n-1:n}.$$

Proof. The reversed hazard rate function of $X_{n-1:n}$ is

$$\tilde{r}_{n-1:n}(t)=\frac{pT_1(\frac{e^{\lambda t}-\overline{\alpha}}{\alpha})+qT_2(\frac{e^{\lambda t}-\overline{\alpha}}{\alpha})-(n-1)T}{p(\frac{e^{\lambda t}-\overline{\alpha}}{\alpha})+q(\frac{e^{\lambda t}-\overline{\alpha}}{\alpha})-(n-1)},$$

where $T_1 = (p-1)\frac{\lambda_1 e^{\lambda_1 t}}{e^{\lambda_1 t}-\overline{\alpha}} + q\frac{\lambda_2 e^{\lambda_2 t}}{e^{\lambda_2 t}-\overline{\alpha}}$, $T_2 = p\frac{\lambda_1 e^{\lambda_1 t}}{e^{\lambda_1 t}-\overline{\alpha}} + (q-1)\frac{\lambda_2 e^{\lambda_2 t}}{e^{\lambda_2 t}}$, and $T = p\frac{\lambda_1 e^{\lambda_1 t}}{e^{\lambda_1 t}-\overline{\alpha}} + q\frac{\lambda_2 e^{\lambda_2 t}}{e^{\lambda_2 t}-\overline{\alpha}}$. In order to prove the result, it is necessary to prove $\tilde{r}_{X_{n-1:n}}(t) \leq \tilde{r}_{X^*_{n-1:n}}(t)$, for any $\lambda_2 \geq \lambda_1 > 0$:

$$\frac{pT_1(\frac{e^{\lambda t}-\overline{\alpha}}{\alpha}) + qT_2(\frac{e^{\lambda t}-\overline{\alpha}}{\alpha}) - (n-1)T}{p(\frac{e^{\lambda t}-\overline{\alpha}}{\alpha}) + q(\frac{e^{\lambda t}-\overline{\alpha}}{\alpha}) - (n-1)}$$

$$\leq \frac{p^*T_1^*(\frac{e^{\lambda 4}-\overline{\alpha}}{\alpha}) + q^*T_2(\frac{e^{\lambda t}-\overline{\alpha}}{\alpha}) - (n^*-1)T^*}{p^*(\frac{e^{\lambda t}-\overline{\alpha}}{\alpha}) + q^*(\frac{e^{\lambda t}-\overline{\alpha}}{\alpha}) - (n^*-1)},$$

where $T_1^* = (p^*-1)\frac{\lambda_1 e^{\lambda_1 t}}{e^{\lambda_1 t}-\overline{\alpha}} + q^*\frac{\lambda_2 e^{\lambda_2 t}}{e^{\lambda_2 t}-\overline{\alpha}}$, $T_2^* = p^*\frac{\lambda_1 e^{\lambda_1 t}}{e^{\lambda_1 t}-\overline{\alpha}}$, and $T^* = p^*\frac{\lambda_1 e^{\lambda_t t}}{e^{\lambda_1 t}-\overline{\alpha}} + q^*\frac{\lambda_2 e^{\lambda_2 t}}{e^{\lambda_2 t}-\overline{\alpha}}$.

Let $a_i = \frac{\lambda_i e^{\lambda_i t}}{e^{\lambda_i t}-\overline{\alpha}}$, $b_i = \frac{e^{\lambda_i t}-\overline{\alpha}}{\alpha}$, and $c_{ij} = a_i b_j$, $i, j = 1, 2$. Hence, $\phi(p, q)$ can be expressed as the following formula:

$$\begin{aligned}\phi(p,q) &= \frac{pT_1(\frac{e^{\lambda t}-\overline{\alpha}}{\alpha}) + qT_2(\frac{e^{\lambda t}-\overline{\alpha}}{\alpha}) - (n-1)T}{p(\frac{e^{\lambda t}-\overline{\alpha}}{\alpha}) + q(\frac{e^{\lambda t}-\overline{\alpha}}{\alpha}) - (n-1)} \\ &= \frac{p(p-1)c_{11} + pqc_{21} + pqc_{12} + q(q-1)c_{22} - (n-1)(pa_1 + qa_2)}{pb_1 + qb_2 - (n-1)}.\end{aligned}$$

The partial derivative of $\phi(p, q)$ about p can be obtained as follows:

$$\begin{aligned}\frac{\partial\phi(p,q)}{\partial p} &\overset{\text{sgn}}{=} [(2p-1)c_{11} + q(c_{21} + c_{12}) - (n-1)a_1 - pa_1 - qa_2]\,[pb_1 + qb_2 - (n-1)] \\ &\quad - [p(p-1)c_{11} + pqc_{21} + pqc_{12} + q(q-1)c_{22} - (n-1)(pa_1 + qa_2)]\,(b_1 - 1) \\ &= a_1\,[pb_1 + qb_2 - (n-1)]\,[(p-1)b_1 + qb_2 - (n-1)] + (pc_{11} + qc_{22})(b_1 - 1) \\ &\geq 0.\end{aligned}$$

In the same way, in order to take $\phi(p, q)$, the partial derivative about p, we get

$$\begin{aligned}\frac{\partial\phi(p,q)}{\partial q} &\overset{\text{sgn}}{=} a_2\,[pb_1 + qb_2 - (n-1)]\,[pb_1 + (q-1)b_2 - (n-1)] + (pc_{11} + qc_{22})(b_2 - 1) \\ &\geq 0.\end{aligned}$$

Also, for $a_2 \geq a_1 \geq 0$, and $b_2 \geq b_1 \geq 1$, there is

$$\begin{aligned}\frac{\partial\phi(p,q)}{\partial p} - \frac{\partial\phi(p,q)}{\partial q} &\overset{\text{sgn}}{=} [pc_{21} + (q-1)c_{22} - (p-1)c_{11} - qc_{12} - (n-1)(a_2 - a_1)] \\ &\quad \times [pb_1 + qb_2 - (n-1)] + (pc_{11} + qc_{22})(b_2 - b_1) \\ &\geq [pc_{21} + (q-1)c_{22} - (p-1)c_{11} - qc_{12} - (n-1)(a_2 - a_1)] \\ &\quad \times [pb_1 + qb_2 - (n-1)] + (pb_1 + qb_2)(a_1b_2 - a_1b_1) \\ &= [pc_{21} + (q-1)c_{22} - pc_{11} - (q-1)c_{12} - (n-1)(a_2 - a_1)] \\ &\quad \times [pb_1 + qb_2 - (n-1)] + (n-1)(c_{12} - c_{11}).\end{aligned}$$

It is proved by Lemma 2. □

Example 6. *Assume the base distribution function* $F(x) = 1 - e^{-(ax)^b}$, $a > 0$, $b > 0$, *Let* $n = 6$, $n^* = 8$, $a = 1.5$, $b = 0.4$, $\alpha = 0.04$, $\lambda_1 = 0.2$, $\lambda_2 = 0.4$, $p = 2$, $q = 4$, $p^* = 1$, *and* $q^* = 7$. *Thus,* $\lambda_1 \leq \lambda_2$, $p^* \leq p \leq q \leq q^*$, *and* $(p, q) \leq_w (p^*, q^*)$ *satisfy Theorem 5.* $X_{5:6}$ *and* $X^*_{7:8}$ *are the inverse hazard rate functions of the curves of* $\tilde{r}_{X_{5:6}}(x; \boldsymbol{\lambda})$ *and* $\tilde{r}_{X_{7:8}}(x; \boldsymbol{\lambda}^*)$, *as shown in Figure 6, for all* $x = -\ln u$, $u \in (0, 1]$. *By observing Figure 6, it can be seen that* $\tilde{r}_{X_{3:4}}(x; \boldsymbol{\lambda}) \leq \tilde{r}_{X^*_{3:4}}(x; \boldsymbol{\lambda}^*)$, *and therefore,* $X_{5:6} \leq_{rh} X^*_{7:8}$.

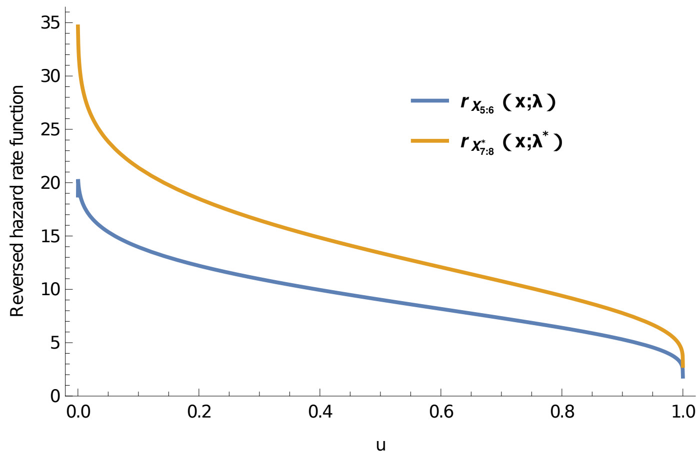

Figure 6. Curves of reversed hazard rate functions $\tilde{r}_{X_{5:6}}(x; \boldsymbol{\lambda})$ and $\tilde{r}_{X_{7:8}}(x; \boldsymbol{\lambda}^*)$, for all $x = -\ln u$ and $u \in (0, 1]$.

5. Conclusions

This article investigates that stochastic comparison problem of the second-largest order statistic in both dependent heterogeneous and independent heterogeneous modified proportional reversed hazard rate samples. First, for the dependent heterogeneous modified proportional reversed hazard rate samples, the usual stochastic order of the second-largest order statistic of two sets of dependent heterogeneous stochastic variables was obtained under the conditions of the same tilt parameter but different modified proportional reversed hazard rates, and different tilt parameters but the same modified proportional reversed hazard rate. Second, a study was conducted on independent heterogeneous modified proportional reversed hazard rate samples, and the reversed hazard rate order of the second-largest order statistic of two independent heterogeneous stochastic variables was obtained under the conditions of the same tilt parameter but different modified proportional reversed hazard rate, different tilt paramaters but the same modified proportional reversed hazard rate, and different sample sizes and parameters.

Due to the complexity of dependent statistics, many issues are still unresolved and worthy of further discussion. In future research, the results can be extended to the second statistic under dependent heterogeneous and independent heterogeneous modified proportional reversed hazard rate observations. Meanwhile, further research will be conducted on the convex, star-shaped, and dispersion order variances of second-order order statistics under dependent conditions.

Use of Generative-AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that there is no conflict of interest.

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