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Research article

Development of novel distance measures for picture hesitant fuzzy sets and their application in medical diagnosis

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Abstract: The picture hesitant fuzzy set (PHFS) integrates elements of picture fuzzy sets and hesitant fuzzy sets, incorporating membership, abstinence, and non-membership degrees to provide a robust framework for addressing uncertainties and complex data in real-world scenarios. In this study, we introduce key characteristics of picture hesitant fuzzy elements, including average functions, variance functions, and hesitancy degrees, to enhance its descriptive capability. Based on these characteristics, we proposed novel distance measures for PHFS. Further, we investigated their properties and proved the triangle inequality of distance measure. These measures were systematically applied in a medical diagnostic context, where they demonstrated significant improvements in diagnostic accuracy by effectively distinguishing patient conditions. Sensitivity analyses and comparative evaluations further validated the practicality and robustness of the proposed methods, highlighting their potential for broader applications in decision-making under uncertainty.

Keywords: picture hesitant fuzzy set; characteristic; distance measure; hesitancy degree; medical diagnosis
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1. Introduction

The concept of a fuzzy set (FS), introduced by Zadeh [1], laid the foundation for handling uncertain and vague data across domains. A fuzzy set (FS) relies on a membership function that assigns a membership degree to each element in the discourse set X within the interval $[0, 1]$. Atanassov [2] further generalized this by introducing the intuitionistic fuzzy set (IFS), which includes a non-membership degree in addition to the membership degree. Notably, taking the non-membership degree as zero reduces an intuitionistic fuzzy set (IFS) to a standard fuzzy set (FS).

Later, Cuong [3] presented the picture fuzzy set (PFS), an extension of both FS and IFS, which incorporates an abstinence degree alongside membership and non-membership degrees. This additional degree enables a more nuanced representation of uncertainty, which enhances applications in fields such as medical diagnosis and pattern recognition.

To further improve the ability to capture precise fuzzy information, Torra and Narukawa [4, 5] introduced the hesitant fuzzy set (HFS), which permits multiple membership degrees for each element. This concept attracted significant scholarly attention. For instance, Xu [6] and Xia et al. [7] explored mathematical representations of HFS and developed robust aggregation operators. Xu and Xia also extended entropy and cross-entropy concepts to HFS [8], along with distance and similarity measures [9,10]. Additionally, Farhadinia [11] and Li et al. [12,13] contributed by exploring hesitancy degrees and distance measures in hesitant fuzzy contexts.

In an effort to broaden the applications of HFS theory, Zeng et al. [14] developed new distance measures and applied them to pattern recognition. Further advancements in HFS research include works by Rodriguez et al. [15], who introduced linguistic terms into HFS theory to enhance decision-making, and Chen et al. [16] and Wei et al. [17], who applied interval values within HFS. Such contributions demonstrate the effectiveness of hesitant fuzzy approaches in representing complex data.

In 2018, Wang et al. [18] proposed the concept of a picture hesitant fuzzy set (PHFS) and explored its use in multi-attribute decision-making (MADM). PHFS combines the advantages of PFS and HFS, providing a robust framework for addressing complex problems. In fields such as pattern recognition, approximate reasoning, image segmentation, and medical diagnosis, the distance and similarity measures play a pivotal role in fuzzy systems. In medical contexts, clinicians often encounter patients exhibiting symptoms accompanied by uncertainty. Accurately identifying diseases under such conditions is a crucial challenge in pattern recognition. Given the subjective nature of traditional diagnoses, there is a significant risk of misjudgment, potentially leading to severe consequences.

Consequently, various fuzzy systems have been applied in medical diagnosis. For example, Emanuel et al. [19] used interval type-2 fuzzy theory, Molla et al. [20] applied Pythagorean fuzzy theory, and Singh [21] proposed a dual hesitant fuzzy set (DHFS) distance measure for evaluating investment alternatives. Moreover, scholars have extended distance and similarity measures to PHFS, with works by Ahmad et al. [22] and Ali et al. [23] focusing on measures tailored for pattern recognition and multi-criteria decision-making.

Among fuzzy systems, PHFS stands out for its ability to incorporate diverse expert evaluations, making it particularly advantageous for decision-making and medical diagnosis. Here, motivated by the effectiveness of PHFS in managing uncertainty, we introduce novel characteristics for picture hesitant fuzzy elements (PHFE), such as the average function, variance function, and hesitancy degree. The average function enables us to capture central tendencies of membership values within PHFEs, offering a balanced view of the data, while the variance function provides insights into the consistency and stability of these values, ensuring that variability is accounted for in decision-making. Additionally, the hesitancy degree is vital in representing the ambiguity or indecision in data, enabling a more comprehensive analysis, especially when expert opinions are inconclusive. Based on these characteristics, we propose novel distance and weighted distance measures for PHFEs. To demonstrate the practical application of our approach, we apply our distance measure in a medical diagnosis example, showcasing its effectiveness in enhancing diagnostic accuracy and supporting reliable decision-making processes under uncertainty.

Considering the benefits of PHFS, our goals are as follows:

1. To introduce and analyze the essential characteristics of PHFEs, such as the average function, variance function, and hesitancy degree, providing a robust framework for describing fuzzy elements in complex environments.
2. To propose and explore various normalized distance measures, including Hamming, Euclidean, and generalized distance measures, tailored specifically for (PHFEs). These measures are developed to ensure compatibility with mathematical properties like the triangle inequality.
3. To demonstrate the feasibility and effectiveness of the proposed distance measures in distinguishing diseases and improving diagnostic accuracy. This is achieved through detailed medical diagnosis examples involving PHF data.
4. To validate the proposed distance measures by comparing them with existing methods, highlighting their superior performance and broader applicability in decision-making contexts.

The organization of this paper is as follows: In Section 2, we review fundamental representations of HFS and PHFS. In Section 3, we introduce characteristics such as the average function, variance function, and hesitancy degree for PHFS and propose new distance measures. In Section 4, we apply the proposed distance measures in a medical diagnosis example to illustrate their utility. Finally, the conclusion is presented in Section 5.

2. Preliminaries

The theory of intuitionistic fuzzy sets (IFSs) can effectively represent events involving two types of uncertainty. However, there are certain instances where the framework of IFSs is not applicable. For example, in a voting scenario with four possible options: Voting in favor, abstaining, voting against, and refusing to vote. This specific scenario cannot be represented using IFSs. Consequently, Cuong [3] introduced a novel theory called picture fuzzy sets (PFSs) to address this issue. The concept of PFS is discussed below.

Definition 1. *Cuong [3] Let X be a set. Then a (PFS) on X is described by*

$$\mathcal{P}_F = \{(M_{\mathcal{P}_F}(x), A_{\mathcal{P}_F}(x), N_{\mathcal{P}_F}(x)) : x \in X\}, \tag{2.1}$$

with $0 \leq M_{\mathcal{P}_F}(x) + A_{\mathcal{P}_F}(x) + N_{\mathcal{P}_F}(x) \leq 1$*, where* $M_{\mathcal{P}_F}(x), A_{\mathcal{P}_F}(x), N_{\mathcal{P}_F}(x) : X \to [0,1]$*. The mathematical structure of a picture fuzzy number is represented by (*$M_{\mathcal{P}_f}(x), A_{\mathcal{P}_f}(x), N_{\mathcal{P}_f}(x)$*) and*

$$R_{\mathcal{P}_f}(x) = 1 - \left(M_{\mathcal{P}_f}(x) + A_{\mathcal{P}_f}(x) + N_{\mathcal{P}_f}(x)\right), \tag{2.2}$$

referred to as the refusal degree.

In this study, let $X = \{x_1, x_2, \ldots, x_n\}$ denote the discourse set. Here, (HFS) and (PHFS) refer to the hesitant fuzzy set and the picture hesitant fuzzy set, respectively, while (HFE) and (PHFE) represent the hesitant fuzzy element and the picture hesitant fuzzy element, respectively.

Definition 2. *Xia and Xu [9] The mathematical expression that describes the (HFS) E on a given set X is as follows:*

$$E = \{\langle x, h_E(x)\rangle \mid x \in X\}, \tag{2.3}$$

where $h_E(x) \subseteq [0,1]$*, represents the set of possible membership degrees of the element* $x \in X$ *to the set E, and* $h(x) = h_E(x)$ *is called a hesitant fuzzy element (HFE).*

For given (HFEs) $h(x)$, $h_1(x)$ and $h_2(x)$, Torra [5], Xia and Xu [9], and Liao et al. [13] gave the following operations.

$$h^-(x) = \min h(x), \quad h^+(x) = \max h(x), \tag{2.4}$$

$$h^c(x) = \bigcup_{\gamma \in h(x)} \{1-\gamma\}, \tag{2.5}$$

$$h_1(x) \cup h_2(x) = \{\gamma \in h_1(x) \cup h_2(x) \mid \gamma \geq \max(h_1^-(x), h_2^-(x))\} = \cup_{\gamma_1 \in h_1(x), \gamma_2 \in h_2(x)} \max\{\gamma_1, \gamma_2\}, \tag{2.6}$$

$$h_1(x) \cap h_2(x) = \{\gamma \in h_1(x) \cap h_2(x) \mid \gamma \leq \min(h_1^+(x), h_2^+(x))\} = \cup_{\gamma_1 \in h_1(x), \gamma_2 \in h_2(x)} \min\{\gamma_1, \gamma_2\}, \tag{2.7}$$

$$h^\lambda(x) = \cup_{\gamma \in h(x)} \left\{\gamma^\lambda\right\}, \tag{2.8}$$

$$\lambda h(x) = \cup_{\gamma \in h(x)} \left\{1 - (1-\gamma)^\lambda\right\}, \quad \lambda > 0, \tag{2.9}$$

$$h_1(x) \oplus h_2(x) = \cup_{\gamma_1 \in h_1(x), \gamma_2 \in h_2(x)} \{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}, \tag{2.10}$$

$$h_1(x) \otimes h_2(x) = \cup_{\gamma_1 \in h_1(x), \gamma_2 \in h_2(x)} \{\gamma_1\gamma_2\}. \tag{2.11}$$

Furthermore, Xia and Xu [6] proposed a score function for (HFE) $h(x)$, defined as:

$$s(h(x)) = \frac{1}{l(h(x))} \sum_{\gamma \in h(x)} \gamma, \tag{2.12}$$

where $l(h(x))$ denotes the number of elements in $h(x)$. They also introduced a ranking rule as follows: if $s(h_1(x)) > s(h_2(x))$, then $h_1(x) > h_2(x)$; and if $s(h_1(x)) = s(h_2(x))$, then $h_1(x) = h_2(x)$.

Motivated by the concept of intuitionistic fuzzy sets, Wang and Li [18] introduced picture hesitant fuzzy sets (PHFS) as a means to effectively manage uncertain information in real-world scenarios. PHFS incorporates three functions: Membership, abstinence, and non-membership, enabling more precise handling of fuzzy information compared to HFS.

Definition 3. *Wang and Li [18] A (PHFS)* $\mathcal{P}_H$ *on X is of the shape:*

$$\mathcal{P}_H = \left\{\left(M_{ip_H}(x), A_{ip_H}(x), N_{ip_H}(x)\right) : x \in X\right\}, \quad i = 1, 2, 3, \ldots, z, \tag{2.13}$$

where $M_{ip_H}(x)$*,* $A_{ip_H}(x)$*, and* $N_{ip_H}(x)$ *are finite subsets of* $[0,1]$*. Furthermore, the mathematical expression of (PHFN) is designed by* $\left(M_{ip_h}(x), A_{ip_h}(x), N_{ip_h}(x)\right)$*, and the refusal degree is given by:*

$$R_{ip_h}(x) = 1 - \left(M_{ip_h}(x) + A_{ip_h}(x) + N_{ip_h}(x)\right).$$

Similarly, for two picture hesitant fuzzy numbers (PHFNs):

$$\mathcal{P}_{h_1} = \left\{M_{ip_{h_1}}(x), A_{ip_{h_1}}(x), N_{ip_{h_1}}(x)\right\}, \quad \mathcal{P}_{h_2} = \left\{M_{ip_{h_2}}(x), A_{ip_{h_2}}(x), N_{ip_{h_2}}(x)\right\},$$

we have:

$$\mathcal{P}_{h_1} \cup \mathcal{P}_{h_2} = \left\{\left\langle \begin{array}{c} x, \max\left(M_{ip_{h_1}}(x), M_{ip_{h_2}}(x)\right), \min\left(A_{ip_{h_1}}(x), A_{ip_{h_2}}(x)\right), \\ \min\left(N_{ip_{h_1}}(x), N_{ip_{h_2}}(x)\right) \end{array} \right\rangle \middle| \, x \in X\right\}, \tag{2.14}$$

$$\mathcal{P}_{\mathrm{h}_1} \cap \mathcal{P}_{\mathrm{h}_2} = \left\{ \left\langle \begin{array}{c} x, \min\left(M_{ip_{h_1}}(x), M_{ip_{h_2}}(x)\right), \max\left(A_{ip_{h_1}}(x), A_{ip_{h_2}}(x)\right), \\ \max\left(N_{ip_{h_1}}(x), N_{ip_{h_2}}(x)\right) \end{array} \right\rangle \middle| \; x \in X \right\}, \tag{2.15}$$

$$\mathcal{P}_{\mathrm{h}}^{c} = \left\{ \left(N_{i\mathcal{P}_{\mathrm{h}}}(x), A_{i\mathcal{P}_{\mathrm{h}}}(x), M_{i\mathcal{P}_{\mathrm{h}}}(x) \right) \right\}. \tag{2.16}$$

Definition 4. *Kifayat Ullah et al. [22] Let* $\mathcal{P}_{\mathrm{H}_1} = \left\{ \langle x, M_{ip_{H_1}}(x), A_{ip_{H_1}}(x), N_{ip_{H_1}}(x) \rangle \mid x \in X \right\}$ *and* $\mathcal{P}_{\mathrm{H}_2} = \left\{ \langle x, M_{ip_{H_2}}(x), A_{ip_{H_2}}(x), N_{ip_{H_2}}(x) \rangle \mid x \in X \right\}$ *be two PHFSs, then* $\mathrm{dis}(\mathcal{P}_{\mathrm{H}_1}, \mathcal{P}_{\mathrm{H}_2})$ *is called the distance measure of (PHFSs)* $\mathcal{P}_{\mathrm{H}_1}$ *and* $\mathcal{P}_{\mathrm{H}_2}$ *satisfying the following conditions:*

(1) $0 \le \mathrm{dis}(\mathcal{P}_{\mathrm{H}_1}, \mathcal{P}_{\mathrm{H}_2}) \le 1$;

(2) $\mathrm{dis}(\mathcal{P}_{\mathrm{H}_1}, \mathcal{P}_{\mathrm{H}_2}) = 0$ *if and only if* $\mathcal{P}_{\mathrm{H}_1} = \mathcal{P}_{\mathrm{H}_2}$;

(3) $\mathrm{dis}(\mathcal{P}_{\mathrm{H}_1}, \mathcal{P}_{\mathrm{H}_2}) = \mathrm{dis}(\mathcal{P}_{\mathrm{H}_2}, \mathcal{P}_{\mathrm{H}_1})$.

3. Novel distance measures of picture hesitant fuzzy set

In the following, we investigate some characteristics of (PHFE) to describe it and explore novel distance measures for (PHFSs) based on these characteristics.

Definition 5. *For a (PHFE)* $\mathcal{P}_{\mathrm{h}} = (M_i, A_i, N_i)$*, then*

$$\mathcal{P}_{\mathrm{h}}^{+} = \left\{ \mathcal{P}_{\mathrm{h}M_i^+}, \mathcal{P}_{\mathrm{h}A_i^+}, \mathcal{P}_{\mathrm{h}N_i^+} \right\} = \left\{ \max\{\gamma \mid \gamma \in M_i\}, \min\{\eta \mid \eta \in A_i\}, \min\{\nu \mid \nu \in N_i\} \right\}, \tag{3.1}$$

and

$$\mathcal{P}_{\mathrm{h}}^{-} = \left\{ \mathcal{P}_{\mathrm{h}M_i^-}, \mathcal{P}_{\mathrm{h}A_i^-}, \mathcal{P}_{\mathrm{h}N_i^-} \right\} = \left\{ \min\{\gamma \mid \gamma \in M_i\}, \max\{\eta \mid \eta \in A_i\}, \max\{\nu \mid \nu \in N_i\} \right\}, \tag{3.2}$$

are called the upper bound and the lower bound of (PHFE)$\mathcal{P}_{\mathrm{h}}$*, respectively.*

Definition 6. *For a (PHFE)* $\mathcal{P}_{\mathrm{h}} = (M_i, A_i, N_i)$*, the average function* $a(\mathcal{P}_{\mathrm{h}})$ *and the variance function* $v(\mathcal{P}_{\mathrm{h}})$ *are defined as follows:*

$$a(\mathcal{P}_{\mathrm{h}}) = \{a_{M_i}, a_{A_i}, a_{N_i}\} = \left\{ \frac{1}{l(M_i)} \sum_{\gamma \in M_i} \gamma, \frac{1}{l(A_i)} \sum_{\eta \in A_i} \eta, \frac{1}{l(N_i)} \sum_{\nu \in N_i} \nu \right\}, \tag{3.3}$$

and

$$v(\mathcal{P}_{\mathrm{h}}) = \{v_{M_i}, v_{A_i}, v_{N_i}\} = \left\{ \sqrt{\frac{1}{l(M_i)} \sum_{\gamma \in M_i} (\gamma - a_{M_i})^2}, \sqrt{\frac{1}{l(A_i)} \sum_{\eta \in A_i} (\eta - a_{A_i})^2}, \sqrt{\frac{1}{l(N_i)} \sum_{\nu \in N_i} (\nu - a_{N_i})^2} \right\}, \tag{3.4}$$

where $l(M_i)$, $l(A_i)$, *and* $l(N_i)$ *denote the number of elements in* M_i, A_i*, and* N_i*, respectively.*

Example 1. *For a given* $X = \{x\}$, $\mathcal{P}_{\mathrm{h}_1} = \{\{0.1, 0.2\}, \{0.6, 0.8\}, \{0.3, 0.4\}\}$ *and* $\mathcal{P}_{\mathrm{h}_2} = \{\{0.2, 0.3\}, \{0.4, 0.5\}, \{0.8, 0.9\}\}$ *are two (PHFEs). Then we have some related characteristics of* (PHFE) $\mathcal{P}_{\mathrm{h}_1}$ *and* (PHFE) $\mathcal{P}_{\mathrm{h}_2}$ *as follows:*

$$\mathcal{P}_{\mathrm{h}_1}^{+} = \left\{ \mathcal{P}_{\mathrm{h}_1 M_i^+}, \mathcal{P}_{\mathrm{h}_1 A_i^+}, \mathcal{P}_{\mathrm{h}_1 N_i^+} \right\} = \{0.2, 0.6, 0.3\}, \mathcal{P}_{\mathrm{h}_1}^{-} = \left\{ \mathcal{P}_{\mathrm{h}_1 M_i^-}, \mathcal{P}_{\mathrm{h}_1 A_i^-}, \mathcal{P}_{\mathrm{h}_1 N_i^-} \right\} = \{0.1, 0.8, 0.4\},$$

$$\mathcal{P}_{\mathrm{h}_2}^{+} = \left\{ \mathcal{P}_{\mathrm{h}_2 M_i^+}, \mathcal{P}_{\mathrm{h}_2 A_i^+}, \mathcal{P}_{\mathrm{h}_2 N_i^+} \right\} = \{0.3, 0.4, 0.8\}, \mathcal{P}_{\mathrm{h}_2}^{-} = \left\{ \mathcal{P}_{\mathrm{h}_2 M_i^-}, \mathcal{P}_{\mathrm{h}_2 A_i^-}, \mathcal{P}_{\mathrm{h}_2 N_i^-} \right\} = \{0.2, 0.5, 0.9\},$$

$$a(\mathcal{P}_{\mathrm{h}_1}) = \left\{a_{M_{i_1}}, a_{A_{i_1}}, a_{N_{i_1}}\right\} = \{0.15, 0.7, 0.35\}, a(\mathcal{P}_{\mathrm{h}_2}) \quad = \left\{a_{M_{i_2}}, a_{A_{i_2}}, a_{N_{i_2}}\right\} = \{0.25, 0.45, 0.85\},$$

$$v(\mathcal{P}_{\mathrm{h}_1}) = \left\{v_{M_{i_1}}, v_{A_{i_1}}, v_{N_{i_1}}\right\} = \{0.05, 0.10, 0.05\}, v(\mathcal{P}_{\mathrm{h}_2}) \quad = \left\{v_{M_{i_2}}, v_{A_{i_2}}, v_{N_{i_2}}\right\} = \{0.05, 0.05, 0.05\}.$$

Definition 7. *For a (PHFE)* $\mathcal{P}_h = \{M_i, A_i, N_i\}$*, then*

$$u(\mathcal{P}_h) = \{u_{M_i}, u_{A_i}, u_{N_i}\} = \left\{1 - \frac{1}{l(M_i)}, 1 - \frac{1}{l(A_i)}, 1 - \frac{1}{l(N_i)}\right\}, \tag{3.5}$$

is called the hesitancy degree of (PHFE) $\mathcal{P}_h$.

Following that, we will introduce novel distance measures for (PHFEs) $\mathcal{P}_{\mathrm{h}_1}$ and $\mathcal{P}_{\mathrm{h}_2}$, taking into account the previously discussed characteristics of (PHFE).

Definition 8. *For two (PHFEs)* $\mathcal{P}_{\mathrm{h}_1} and \mathcal{P}_{\mathrm{h}_2}$*, then*

$$\begin{aligned} d_{is_1}(\mathcal{P}_{\mathrm{h}_1}, \mathcal{P}_{\mathrm{h}_2}) = \frac{1}{15}\Big(&\left|\mathcal{P}^+_{\mathrm{h}_{1M_i}} - \mathcal{P}^+_{\mathrm{h}_{2M_i}}\right| + \left|\mathcal{P}^-_{\mathrm{h}_{1M_i}} - \mathcal{P}^-_{\mathrm{h}_{2M_i}}\right| \\ &+ \left|\mathcal{P}^+_{\mathrm{h}_{1A_i}} - \mathcal{P}^+_{\mathrm{h}_{2A_i}}\right| + \left|\mathcal{P}^-_{\mathrm{h}_{1A_i}} - \mathcal{P}^-_{\mathrm{h}_{2A_i}}\right| \\ &+ \left|\mathcal{P}^+_{\mathrm{h}_{1N_i}} - \mathcal{P}^+_{\mathrm{h}_{2N_i}}\right| + \left|\mathcal{P}^-_{\mathrm{h}_{1N_i}} - \mathcal{P}^-_{\mathrm{h}_{2N_i}}\right| \\ &+ \left|a_{1M_i} - a_{2M_i}\right| + \left|a_{1A_i} - a_{2A_i}\right| + \left|a_{1N_i} - a_{2N_i}\right| \\ &+ \left|v_{1M_i} - v_{2M_i}\right| + \left|v_{1A_i} - v_{2A_i}\right| + \left|v_{1N_i} - v_{2N_i}\right| \\ &+ \left|u_{1M_i} - u_{2M_i}\right| + \left|u_{1A_i} - u_{2A_i}\right| + \left|u_{1N_i} - u_{2N_i}\right| \Big), \end{aligned} \tag{3.6}$$

$$\begin{aligned} d_{is_2}(\mathcal{P}_{\mathrm{h}_1}, \mathcal{P}_{\mathrm{h}_2}) = \frac{1}{15}\Big(&\left|\mathcal{P}^+_{\mathrm{h}_{1M_i}} - \mathcal{P}^+_{\mathrm{h}_{2M_i}}\right|^2 + \left|\mathcal{P}^-_{\mathrm{h}_{1M_i}} - \mathcal{P}^-_{\mathrm{h}_{2M_i}}\right|^2 \\ &+ \left|\mathcal{P}^+_{\mathrm{h}_{1A_i}} - \mathcal{P}^+_{\mathrm{h}_{2A_i}}\right|^2 + \left|\mathcal{P}^-_{\mathrm{h}_{1A_i}} - \mathcal{P}^-_{\mathrm{h}_{2A_i}}\right|^2 \\ &+ \left|\mathcal{P}^+_{\mathrm{h}_{1N_i}} - \mathcal{P}^+_{\mathrm{h}_{2N_i}}\right|^2 + \left|\mathcal{P}^-_{\mathrm{h}_{1N_i}} - \mathcal{P}^-_{\mathrm{h}_{2N_i}}\right|^2 \\ &+ \left|a_{1M_i} - a_{2M_i}\right|^2 + \left|a_{1A_i} - a_{2A_i}\right|^2 + \left|a_{1N_i} - a_{2N_i}\right|^2 \\ &+ \left|v_{1M_i} - v_{2M_i}\right|^2 + \left|v_{1A_i} - v_{2A_i}\right|^2 + \left|v_{1N_i} - v_{2N_i}\right|^2 \\ &+ \left|u_{1M_i} - u_{2M_i}\right|^2 + \left|u_{1A_i} - u_{2A_i}\right|^2 + \left|u_{1N_i} - u_{2N_i}\right|^2 \Big)^{1/2}, \end{aligned} \tag{3.7}$$

$$\begin{aligned} d_{is_3}(\mathcal{P}_{\mathrm{h}_1}, \mathcal{P}_{\mathrm{h}_2}) = \frac{1}{15}\Big(&\left|\mathcal{P}^+_{\mathrm{h}_{1M_i}} - \mathcal{P}^+_{\mathrm{h}_{2M_i}}\right|^\lambda + \left|\mathcal{P}^-_{\mathrm{h}_{1M_i}} - \mathcal{P}^-_{\mathrm{h}_{2M_i}}\right|^\lambda \\ &+ \left|\mathcal{P}^+_{\mathrm{h}_{1A_i}} - \mathcal{P}^+_{\mathrm{h}_{2A_i}}\right|^\lambda + \left|\mathcal{P}^-_{\mathrm{h}_{1A_i}} - \mathcal{P}^-_{\mathrm{h}_{2A_i}}\right|^\lambda \\ &+ \left|\mathcal{P}^+_{\mathrm{h}_{1N_i}} - \mathcal{P}^+_{\mathrm{h}_{2N_i}}\right|^\lambda + \left|\mathcal{P}^-_{\mathrm{h}_{1N_i}} - \mathcal{P}^-_{\mathrm{h}_{2N_i}}\right|^\lambda \\ &+ \left|a_{1M_i} - a_{2M_i}\right|^\lambda + \left|a_{1A_i} - a_{2A_i}\right|^\lambda + \left|a_{1N_i} - a_{2N_i}\right|^\lambda \\ &+ \left|v_{1M_i} - v_{2M_i}\right|^\lambda + \left|v_{1A_i} - v_{2A_i}\right|^\lambda + \left|v_{1N_i} - v_{2N_i}\right|^\lambda \\ &+ \left|u_{1M_i} - u_{2M_i}\right|^\lambda + \left|u_{1A_i} - u_{2A_i}\right|^\lambda + \left|u_{1N_i} - u_{2N_i}\right|^\lambda \Big)^{1/\lambda}, \lambda > 0, \end{aligned} \tag{3.8}$$

are called the normalized Hamming distance measure, normalized Euclidean distance measure, and normalized generalized distance measure between (PHFEs) $\mathcal{P}_{h_1}$ *and* $\mathcal{P}_{h_2}$*, respectively.*

Example 2. *For a given* $X = \{x\}$*,* $\mathcal{P}_{h_1} = \{\{0.7, 0.8\}, \{0.1, 0.2\}, \{0.3, 0.4\}\}$ *and* $\mathcal{P}_{h_2} = \{\{0.5, 0.6, 0.7\}, \{0.1, 0.2, 0.3\}, \{0.3, 0.4, 0.5\}\}$ *are two (PHFEs) on X then we can obtain:*

$$\mathcal{P}^+_{h_1} = \left\{\mathcal{P}_{h_1 M_i^+}, \mathcal{P}_{h_1 A_i^+}, \mathcal{P}_{h_1 N_i^+}\right\} = \{0.8, 0.1, 0.3\}, \mathcal{P}^-_{h_1} = \left\{\mathcal{P}_{h_1 M_i^-}, \mathcal{P}_{h_1 A_i^-}, \mathcal{P}_{h_1 N_i^-}\right\} = \{0.7, 0.2, 0.4\},$$
$$\mathcal{P}^+_{h_2} = \left\{\mathcal{P}_{h_2 M_i^+}, \mathcal{P}_{h_2 A_i^+}, \mathcal{P}_{h_2 N_i^+}\right\} = \{0.7, 0.1, 0.3\}, \mathcal{P}^-_{h_2} = \left\{\mathcal{P}_{h_2 M_i^-}, \mathcal{P}_{h_2 A_i^-}, \mathcal{P}_{h_2 N_i^-}\right\} = \{0.5, 0.3, 0.5\},$$

$$a(\mathcal{P}_{h_1}) = \left\{a_{M_{i_1}}, a_{A_{i_1}}, a_{N_{i_1}}\right\} = \{0.75, 0.15, 0.35\}, a(\mathcal{P}_{h_2}) = \left\{a_{M_{i_2}}, a_{A_{i_2}}, a_{N_{i_2}}\right\} = \{0.6, 0.2, 0.4\},$$
$$v(\mathcal{P}_{h_1}) = \left\{v_{M_{i_1}}, v_{A_{i_1}}, v_{N_{i_1}}\right\} = \{0.05, 0.05, 0.05\}, v(\mathcal{P}_{h_2}) = \left\{v_{M_{i_2}}, v_{A_{i_2}}, v_{N_{i_2}}\right\} = \{0.08, 0.08, 0.08\},$$
$$u(\mathcal{P}_{h_1}) = \left\{u_{M_{i_1}}, u_{A_{i_1}}, u_{N_{i_1}}\right\} = \{0.5, 0.5, 0.5\}, u(\mathcal{P}_{h_2}) = \left\{u_{M_{i_2}}, u_{A_{i_2}}, u_{N_{i_2}}\right\} = \{0.67, 0.67, 0.67\}.$$

Thus, we have the normalized Hamming distance measure

$$d_{is_1}(\mathcal{P}_{h_1}, \mathcal{P}_{h_2}) = \frac{1}{15}\Big(0.1 + 0 + 0 + 0.2 + 0.1 + 0.1 + 0.15 + 0.05 + 0.05 + 0.03 + 0.03 + 0.03 + 0.17$$
$$+0.17 + 0.17\Big) = 0.09.$$

Theorem 1. *Let* $\mathcal{P}_{h_1}$ *and* $\mathcal{P}_{h_2}$ *be two (PHFEs), then the normalized Hamming distance measure* $d_{is_1}(\mathcal{P}_{h_1}, \mathcal{P}_{h_2})$ *satisfy the triangle inequality.*

proof.

1. It is established that the normalized Hamming distance measure d_{is_1} for two (PHFEs), $\mathcal{P}_{h_1}$ and $\mathcal{P}_{h_2}$, is always bounded within the interval $[0, 1]$. Therefore, we have

$$0 \le d_{is_1}(\mathcal{P}_{h_1}, \mathcal{P}_{h_2}) \le 1.$$

2. Let $\mathcal{P}_{h_1} = \mathcal{P}_{h_2}$, then we have:

$$\begin{array}{llll}
\mathcal{P}^+_{h_{1M_i}} = \mathcal{P}^+_{h_{2M_i}}, & \mathcal{P}^-_{h_{1M_i}} = \mathcal{P}^-_{h_{2M_i}}, & \mathcal{P}^+_{h_{1A_i}} = \mathcal{P}^+_{h_{2A_i}}, & \mathcal{P}^-_{h_{1A_i}} = \mathcal{P}^-_{h_{2A_i}}, \\
\mathcal{P}^+_{h_{1N_i}} = \mathcal{P}^+_{h_{2N_i}}, & \mathcal{P}^-_{h_{1N_i}} = \mathcal{P}^-_{h_{2N_i}}, & a_{1M_i} = a_{2M_i}, & a_{1A_i} = a_{2A_i}, \\
a_{1N_i} = a_{2N_i}, & v_{1M_i} = v_{2M_i}, & v_{1A_i} = v_{2A_i}, & v_{1N_i} = v_{2N_i}, \\
u_{1M_i} = u_{2M_i}, & u_{1A_i} = u_{2A_i}, & u_{1N_i} = u_{2N_i}. &
\end{array}$$

This implies that:

$$\begin{array}{lll}
\left|\mathcal{P}^+_{h_{1M_i}} - \mathcal{P}^+_{h_{2M_i}}\right| = 0, & \left|\mathcal{P}^-_{h_{1M_i}} - \mathcal{P}^-_{h_{2M_i}}\right| = 0, & \left|\mathcal{P}^+_{h_{1A_i}} - \mathcal{P}^+_{h_{2A_i}}\right| = 0, \\
\left|\mathcal{P}^-_{h_{1A_i}} - \mathcal{P}^-_{h_{2A_i}}\right| = 0, & \left|\mathcal{P}^+_{h_{1N_i}} - \mathcal{P}^+_{h_{2N_i}}\right| = 0, & \left|\mathcal{P}^-_{h_{1N_i}} - \mathcal{P}^-_{h_{2N_i}}\right| = 0, \\
\left|a_{1M_i} - a_{2M_i}\right| = 0, & \left|a_{1A_i} - a_{2A_i}\right| = 0, & \left|a_{1N_i} - a_{2N_i}\right| = 0, \\
\left|v_{1M_i} - v_{2M_i}\right| = 0, & \left|v_{1A_i} - v_{2A_i}\right| = 0, & \left|v_{1N_i} - v_{2N_i}\right| = 0, \\
\left|u_{1M_i} - u_{2M_i}\right| = 0, & \left|u_{1A_i} - u_{2A_i}\right| = 0, & \left|u_{1N_i} - u_{2N_i}\right| = 0.
\end{array}$$

Therefore, $d_{\mathrm{is}_1}(\mathcal{P}_{\mathrm{h}_1}, \mathcal{P}_{\mathrm{h}_2}) = \frac{1}{15}(0) = 0$.

Conversely, suppose that $d_{is_1}(\mathcal{P}_{\mathrm{h}_1}, \mathcal{P}_{\mathrm{h}_2}) = 0$, this implies that:

$$\begin{aligned}
\left|\mathcal{P}^{+}_{\mathrm{h}1M_i} - \mathcal{P}^{+}_{\mathrm{h}2M_i}\right| &= 0, & \left|\mathcal{P}^{-}_{\mathrm{h}1M_i} - \mathcal{P}^{-}_{\mathrm{h}2M_i}\right| &= 0, & \left|\mathcal{P}^{+}_{\mathrm{h}1A_i} - \mathcal{P}^{+}_{\mathrm{h}2A_i}\right| &= 0,\\
\left|\mathcal{P}^{-}_{\mathrm{h}1A_i} - \mathcal{P}^{-}_{\mathrm{h}2A_i}\right| &= 0, & \left|\mathcal{P}^{+}_{\mathrm{h}1N_i} - \mathcal{P}^{+}_{\mathrm{h}2N_i}\right| &= 0, & \left|\mathcal{P}^{-}_{\mathrm{h}1N_i} - \mathcal{P}^{-}_{\mathrm{h}2N_i}\right| &= 0,\\
\left|a_{1M_i} - a_{2M_i}\right| &= 0, & \left|a_{1A_i} - a_{2A_i}\right| &= 0, & \left|a_{1N_i} - a_{2N_i}\right| &= 0,\\
\left|v_{1M_i} - v_{2M_i}\right| &= 0, & \left|v_{1A_i} - v_{2A_i}\right| &= 0, & \left|v_{1N_i} - v_{2N_i}\right| &= 0,\\
\left|u_{1M_i} - u_{2M_i}\right| &= 0, & \left|u_{1A_i} - u_{2A_i}\right| &= 0, & \left|u_{1N_i} - u_{2N_i}\right| &= 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathcal{P}^{+}_{\mathrm{h}1M_i} &= \mathcal{P}^{+}_{\mathrm{h}2M_i}, & \mathcal{P}^{-}_{\mathrm{h}1M_i} &= \mathcal{P}^{-}_{\mathrm{h}2M_i}, & \mathcal{P}^{+}_{\mathrm{h}1A_i} &= \mathcal{P}^{+}_{\mathrm{h}2A_i}, & \mathcal{P}^{-}_{\mathrm{h}1A_i} &= \mathcal{P}^{-}_{\mathrm{h}2A_i},\\
\mathcal{P}^{+}_{\mathrm{h}1N_i} &= \mathcal{P}^{+}_{\mathrm{h}2N_i}, & \mathcal{P}^{-}_{\mathrm{h}1N_i} &= \mathcal{P}^{-}_{\mathrm{h}2N_i}, & a_{1M_i} &= a_{2M_i}, & a_{1A_i} &= a_{2A_i},\\
a_{1N_i} &= a_{2N_i}, & v_{1M_i} &= v_{2M_i}, & v_{1A_i} &= v_{2A_i}, & v_{1N_i} &= v_{2N_i},\\
u_{1M_i} &= u_{2M_i}, & u_{1A_i} &= u_{2A_i}, & u_{1N_i} &= u_{2N_i}.
\end{aligned}$$

Therefore, $\mathcal{P}_{\mathrm{h}_1} = \mathcal{P}_{\mathrm{h}_2}$.

3. From Definition 8, we have:

$$\begin{aligned}
d_{is_1}(\mathcal{P}_{\mathrm{h}_1}, \mathcal{P}_{\mathrm{h}_2}) = \frac{1}{15}\Big(&\left|\mathcal{P}^{+}_{\mathrm{h}1M_i} - \mathcal{P}^{+}_{\mathrm{h}2M_i}\right| + \left|\mathcal{P}^{-}_{\mathrm{h}1M_i} - \mathcal{P}^{-}_{\mathrm{h}2M_i}\right|\\
&+ \left|\mathcal{P}^{+}_{\mathrm{h}1A_i} - \mathcal{P}^{+}_{\mathrm{h}2A_i}\right| + \left|\mathcal{P}^{-}_{\mathrm{h}1A_i} - \mathcal{P}^{-}_{\mathrm{h}2A_i}\right|\\
&+ \left|\mathcal{P}^{+}_{\mathrm{h}1N_i} - \mathcal{P}^{+}_{\mathrm{h}2N_i}\right| + \left|\mathcal{P}^{-}_{\mathrm{h}1N_i} - \mathcal{P}^{-}_{\mathrm{h}2N_i}\right|\\
&+ \left|a_{1M_i} - a_{2M_i}\right| + \left|a_{1A_i} - a_{2A_i}\right| + \left|a_{1N_i} - a_{2N_i}\right|\\
&+ \left|v_{1M_i} - v_{2M_i}\right| + \left|v_{1A_i} - v_{2A_i}\right| + \left|v_{1N_i} - v_{2N_i}\right|\\
&+ \left|u_{1M_i} - u_{2M_i}\right| + \left|u_{1A_i} - u_{2A_i}\right| + \left|u_{1N_i} - u_{2N_i}\right|\Big)\\
= \frac{1}{15}\Big(&\left|\mathcal{P}^{+}_{\mathrm{h}2M_i} - \mathcal{P}^{+}_{\mathrm{h}1M_i}\right| + \left|\mathcal{P}^{-}_{\mathrm{h}2M_i} - \mathcal{P}^{-}_{\mathrm{h}1M_i}\right|\\
&+ \left|\mathcal{P}^{+}_{\mathrm{h}2A_i} - \mathcal{P}^{+}_{\mathrm{h}1A_i}\right| + \left|\mathcal{P}^{-}_{\mathrm{h}2A_i} - \mathcal{P}^{-}_{\mathrm{h}1A_i}\right|\\
&+ \left|\mathcal{P}^{+}_{\mathrm{h}2N_i} - \mathcal{P}^{+}_{\mathrm{h}1N_i}\right| + \left|\mathcal{P}^{-}_{\mathrm{h}2N_i} - \mathcal{P}^{-}_{\mathrm{h}1N_i}\right|\\
&+ \left|a_{2M_i} - a_{1M_i}\right| + \left|a_{2A_i} - a_{1A_i}\right| + \left|a_{2N_i} - a_{1N_i}\right|\\
&+ \left|v_{2M_i} - v_{1M_i}\right| + \left|v_{2A_i} - v_{1A_i}\right| + \left|v_{2N_i} - v_{1N_i}\right|\\
&+ \left|u_{2M_i} - u_{1M_i}\right| + \left|u_{2A_i} - u_{1A_i}\right| + \left|u_{2N_i} - u_{1N_i}\right|\Big) = d_{is_1}(\mathcal{P}_{\mathrm{h}_2}, \mathcal{P}_{\mathrm{h}_1}).
\end{aligned}$$

Therefore,

$$d_{is_1}(\mathcal{P}_{\mathrm{h}_1}, \mathcal{P}_{\mathrm{h}_2}) = d_{is_1}(\mathcal{P}_{\mathrm{h}_2}, \mathcal{P}_{\mathrm{h}_1}).$$

4.

$$
\begin{aligned}
d_{is_1}(\mathcal{P}_{h_1},\mathcal{P}_{h_3}) = \frac{1}{15}\Big(& \left|\mathcal{P}^{+}_{h_{1M_i}}-\mathcal{P}^{+}_{h_{3M_i}}\right| + \left|\mathcal{P}^{-}_{h_{1M_i}}-\mathcal{P}^{-}_{h_{3M_i}}\right| + \left|\mathcal{P}^{+}_{h_{1A_i}}-\mathcal{P}^{+}_{h_{3A_i}}\right| + \left|\mathcal{P}^{-}_{h_{1A_i}}-\mathcal{P}^{-}_{h_{3A_i}}\right| \\
& + \left|\mathcal{P}^{+}_{h_{1N_i}}-\mathcal{P}^{+}_{h_{3N_i}}\right| + \left|\mathcal{P}^{-}_{h_{1N_i}}-\mathcal{P}^{-}_{h_{3N_i}}\right| + \left|a_{1M_i}-a_{3M_i}\right| + \left|a_{1A_i}-a_{3A_i}\right| + \left|a_{1N_i}-a_{3N_i}\right| \\
& + \left|v_{1M_i}-v_{3M_i}\right| + \left|v_{1A_i}-v_{3A_i}\right| + \left|v_{1N_i}-v_{3N_i}\right| + \left|u_{1M_i}-u_{3M_i}\right| + \left|u_{1A_i}-u_{3A_i}\right| + \left|u_{1N_i}-u_{3N_i}\right|\Big) \\
\leq \frac{1}{15}\Big(& \left|\mathcal{P}^{+}_{h_{1M_i}}-\mathcal{P}^{+}_{h_{2M_i}}\right| + \left|\mathcal{P}^{-}_{h_{1M_i}}-\mathcal{P}^{-}_{h_{2M_i}}\right| + \left|\mathcal{P}^{+}_{h_{1A_i}}-\mathcal{P}^{+}_{h_{2A_i}}\right| + \left|\mathcal{P}^{-}_{h_{1A_i}}-\mathcal{P}^{-}_{h_{2A_i}}\right| \\
& + \left|\mathcal{P}^{+}_{h_{1N_i}}-\mathcal{P}^{+}_{h_{2N_i}}\right| + \left|\mathcal{P}^{-}_{h_{1N_i}}-\mathcal{P}^{-}_{h_{2N_i}}\right| + \left|a_{1M_i}-a_{2M_i}\right| + \left|a_{1A_i}-a_{2A_i}\right| + \left|a_{1N_i}-a_{2N_i}\right| \\
& + \left|v_{1M_i}-v_{2M_i}\right| + \left|v_{1A_i}-v_{2A_i}\right| + \left|v_{1N_i}-v_{2N_i}\right| + \left|u_{1M_i}-u_{2M_i}\right| + \left|u_{1A_i}-u_{2A_i}\right| + \left|u_{1N_i}-u_{2N_i}\right|\Big) \\
+ \frac{1}{15}\Big(& \left|\mathcal{P}^{+}_{h_{2M_i}}-\mathcal{P}^{+}_{h_{3M_i}}\right| + \left|\mathcal{P}^{-}_{h_{2M_i}}-\mathcal{P}^{-}_{h_{3M_i}}\right| + \left|\mathcal{P}^{+}_{h_{2A_i}}-\mathcal{P}^{+}_{h_{3A_i}}\right| + \left|\mathcal{P}^{-}_{h_{2A_i}}-\mathcal{P}^{-}_{h_{3A_i}}\right| \\
& + \left|\mathcal{P}^{+}_{h_{2N_i}}-\mathcal{P}^{+}_{h_{3N_i}}\right| + \left|\mathcal{P}^{-}_{h_{2N_i}}-\mathcal{P}^{-}_{h_{3N_i}}\right| + \left|a_{2M_i}-a_{3M_i}\right| + \left|a_{2A_i}-a_{3A_i}\right| + \left|a_{2N_i}-a_{3N_i}\right| \\
& + \left|v_{2M_i}-v_{3M_i}\right| + \left|v_{2A_i}-v_{3A_i}\right| + \left|v_{2N_i}-v_{3N_i}\right| + \left|u_{2M_i}-u_{3M_i}\right| + \left|u_{2A_i}-u_{3A_i}\right| + \left|u_{2N_i}-u_{3N_i}\right|\Big) \\
& = d_{is_1}(\mathcal{P}_{h_1},\mathcal{P}_{h_2}) + d_{is_1}(\mathcal{P}_{h_2},\mathcal{P}_{h_3}).
\end{aligned}
$$

Thus, Theorem 1 has been proven.

Theorem 2. *Let $\mathcal{P}_{h_1}$ and $\mathcal{P}_{h_2}$ be two (PHFEs), then the normalized Euclidean distance measure $d_{is_2}(\mathcal{P}_{h_1},\mathcal{P}_{h_2})$ and the normalized generalized distance measure $d_{is_3}(\mathcal{P}_{h_1},\mathcal{P}_{h_2})$ satisfy the triangle inequality.*

The proof is similar to Theorem 1.

In the context of practical applications, if each characteristic of (PHFE) is assigned different weights, we propose employing the normalized weighted Hamming distance measure, the normalized weighted Euclidean distance measure, and the normalized weighted generalized distance measure for (PHFEs).

$$
\begin{aligned}
\mathrm{dis}_{w_1}(\mathcal{P}_{h_1},\mathcal{P}_{h_2}) = {} & w_1 * \left|\mathcal{P}^{+}_{h_{1M_i}}-\mathcal{P}^{+}_{h_{2M_i}}\right| + w_2 * \left|\mathcal{P}^{-}_{h_{1M_i}}-\mathcal{P}^{-}_{h_{2M_i}}\right| + w_3 * \left|\mathcal{P}^{+}_{h_{1A_i}}-\mathcal{P}^{+}_{h_{2A_i}}\right| \\
& + w_4 * \left|\mathcal{P}^{-}_{h_{1A_i}}-\mathcal{P}^{-}_{h_{2A_i}}\right| + w_5 * \left|\mathcal{P}^{+}_{h_{1N_i}}-\mathcal{P}^{+}_{h_{2N_i}}\right| + w_6 * \left|\mathcal{P}^{-}_{h_{1N_i}}-\mathcal{P}^{-}_{h_{2N_i}}\right| \\
& + w_7 * \left|a_{1M_i}-a_{2M_i}\right| + w_8 * \left|a_{1A_i}-a_{2A_i}\right| + w_9 * \left|a_{1N_i}-a_{2N_i}\right| \\
& + w_{10} * \left|v_{1M_i}-v_{2M_i}\right| + w_{11} * \left|v_{1A_i}-v_{2A_i}\right| + w_{12} * \left|v_{1N_i}-v_{2N_i}\right| \\
& + w_{13} * \left|u_{1M_i}-u_{2M_i}\right| + w_{14} * \left|u_{1A_i}-u_{2A_i}\right| + w_{15} * \left|u_{1N_i}-u_{2N_i}\right|,
\end{aligned} \tag{3.9}
$$

$$\begin{aligned}\operatorname{dis}_{w_2}(\mathcal{P}_{h_1},\mathcal{P}_{h_2}) = \Big(& w_1 * \left(\left|\mathcal{P}^+_{h_{1M_i}} - \mathcal{P}^+_{h_{2M_i}}\right|\right)^2 + w_2 * \left(\left|\mathcal{P}^-_{h_{1M_i}} - \mathcal{P}^-_{h_{2M_i}}\right|\right)^2 + w_3 * \left(\left|\mathcal{P}^+_{h_{1A_i}} - \mathcal{P}^+_{h_{2A_i}}\right|\right)^2 \\ & + w_4 * \left(\left|\mathcal{P}^-_{h_{1A_i}} - \mathcal{P}^-_{h_{2A_i}}\right|\right)^2 + w_5 * \left(\left|\mathcal{P}^+_{h_{1N_i}} - \mathcal{P}^+_{h_{2N_i}}\right|\right)^2 + w_6 * \left(\left|\mathcal{P}^-_{h_{1N_i}} - \mathcal{P}^-_{h_{2N_i}}\right|\right)^2 \\ & + w_7 * \left(\left|a_{1M_i} - a_{2M_i}\right|\right)^2 + w_8 * \left(\left|a_{1A_i} - a_{2A_i}\right|\right)^2 + w_9 * \left(\left|a_{1N_i} - a_{2N_i}\right|\right)^2 \\ & + w_{10} * \left(\left|v_{1M_i} - v_{2M_i}\right|\right)^2 + w_{11} * \left(\left|v_{1A_i} - v_{2A_i}\right|\right)^2 + w_{12} * \left(\left|v_{1N_i} - v_{2N_i}\right|\right)^2 \\ & + w_{13} * \left(\left|u_{1M_i} - u_{2M_i}\right|\right)^2 + w_{14} * \left(\left|u_{1A_i} - u_{2A_i}\right|\right)^2 + w_{15} * \left(\left|u_{1N_i} - u_{2N_i}\right|\right)^2 \Big)^{1/2},\end{aligned} \tag{3.10}$$

$$\begin{aligned}\operatorname{dis}_{w_3}(\mathcal{P}_{h_1},\mathcal{P}_{h_2}) = \Big(& w_1 * \left(\left|\mathcal{P}^+_{h_{1M_i}} - \mathcal{P}^+_{h_{2M_i}}\right|\right)^\lambda + w_2 * \left(\left|\mathcal{P}^-_{h_{1M_i}} - \mathcal{P}^-_{h_{2M_i}}\right|\right)^\lambda + w_3 * \left(\left|\mathcal{P}^+_{h_{1A_i}} - \mathcal{P}^+_{h_{2A_i}}\right|\right)^\lambda \\ & + w_4 * \left(\left|\mathcal{P}^-_{h_{1A_i}} - \mathcal{P}^-_{h_{2A_i}}\right|\right)^\lambda + w_5 * \left(\left|\mathcal{P}^+_{h_{1N_i}} - \mathcal{P}^+_{h_{2N_i}}\right|\right)^\lambda + w_6 * \left(\left|\mathcal{P}^-_{h_{1N_i}} - \mathcal{P}^-_{h_{2N_i}}\right|\right)^\lambda \\ & + w_7 * \left(\left|a_{1M_i} - a_{2M_i}\right|\right)^\lambda + w_8 * \left(\left|a_{1A_i} - a_{2A_i}\right|\right)^\lambda + w_9 * \left(\left|a_{1N_i} - a_{2N_i}\right|\right)^\lambda \\ & + w_{10} * \left(\left|v_{1M_i} - v_{2M_i}\right|\right)^\lambda + w_{11} * \left(\left|v_{1A_i} - v_{2A_i}\right|\right)^\lambda + w_{12} * \left(\left|v_{1N_i} - v_{2N_i}\right|\right)^\lambda \\ & + w_{13} * \left(\left|u_{1M_i} - u_{2M_i}\right|\right)^\lambda + w_{14} * \left(\left|u_{1A_i} - u_{2A_i}\right|\right)^\lambda + w_{15} * \left(\left|u_{1N_i} - u_{2N_i}\right|\right)^\lambda \Big)^{1/\lambda}, \quad \lambda > 0,\end{aligned} \tag{3.11}$$

where $0 \le \omega_i \le 1 \quad (i = 1, 2, \ldots, 15)$ and $\sum_{i=1}^{15} \omega_i = 1$.

Theorem 3. *Let $\mathcal{P}_{h_1}$ and $\mathcal{P}_{h_2}$ be two (PHFEs). Then the normalized weighted distance measures, such as* $\operatorname{dis}_{w_1}(\mathcal{P}_{h_1},\mathcal{P}_{h_2})$, $\operatorname{dis}_{w_2}(\mathcal{P}_{h_1},\mathcal{P}_{h_2})$*, and* $\operatorname{dis}_{w_3}(\mathcal{P}_{h_1},\mathcal{P}_{h_2})$*, satisfy the triangle inequality.*

The proof is similar to Theorem 1.

Definition 9. *Let $X = (x_1, x_2, \ldots, x_n)$, and let $\mathcal{P}_{H_1}$ and $\mathcal{P}_{H_2}$ be PHFSs on X. Then*

$$\begin{aligned}d_{is_1}(\mathcal{P}_{H_1},\mathcal{P}_{H_2}) = \frac{1}{15n}\sum_{i=1}^{n}\Big(& \left|\mathcal{P}^+_{H_{1M_i}}(x_i) - \mathcal{P}^+_{H_{2M_i}}(x_i)\right| + \left|\mathcal{P}^-_{H_{1M_i}}(x_i) - \mathcal{P}^-_{H_{2M_i}}(x_i)\right| \\ & + \left|\mathcal{P}^+_{H_{1A_i}}(x_i) - \mathcal{P}^+_{H_{2A_i}}(x_i)\right| + \left|\mathcal{P}^-_{H_{1A_i}}(x_i) - \mathcal{P}^-_{H_{2A_i}}(x_i)\right| \\ & + \left|\mathcal{P}^+_{H_{1N_i}}(x_i) - \mathcal{P}^+_{H_{2N_i}}(x_i)\right| + \left|\mathcal{P}^-_{H_{1N_i}}(x_i) - \mathcal{P}^-_{H_{2N_i}}(x_i)\right| \\ & + \left|a_{1M_i}(x_i) - a_{2M_i}(x_i)\right| + \left|a_{1A_i}(x_i) - a_{2A_i}(x_i)\right| + \left|a_{1N_i}(x_i) - a_{2N_i}(x_i)\right| \\ & + \left|v_{1M_i}(x_i) - v_{2M_i}(x_i)\right| + \left|v_{1A_i}(x_i) - v_{2A_i}(x_i)\right| + \left|v_{1N_i}(x_i) - v_{2N_i}(x_i)\right| \\ & + \left|u_{1M_i}(x_i) - u_{2M_i}(x_i)\right| + \left|u_{1A_i}(x_i) - u_{2A_i}(x_i)\right| + \left|u_{1N_i}(x_i) - u_{2N_i}(x_i)\right| \Big),\end{aligned} \tag{3.12}$$

$$\begin{aligned}d_{is_2}(\mathcal{P}_{H_1},\mathcal{P}_{H_2}) = \Big(\frac{1}{15n}\sum_{i=1}^{n}\Big(& \left|\mathcal{P}^+_{H_{1M_i}}(x_i) - \mathcal{P}^+_{H_{2M_i}}(x_i)\right|^2 + \left|\mathcal{P}^-_{H_{1M_i}}(x_i) - \mathcal{P}^-_{H_{2M_i}}(x_i)\right|^2 \\ & + \left|\mathcal{P}^+_{H_{1A_i}}(x_i) - \mathcal{P}^+_{H_{2A_i}}(x_i)\right|^2 + \left|\mathcal{P}^-_{H_{1A_i}}(x_i) - \mathcal{P}^-_{H_{2A_i}}(x_i)\right|^2\end{aligned}$$

$$+\left|\mathcal{P}^{+}_{\mathrm{H}_{1N_i}}(x_i)-\mathcal{P}^{+}_{\mathrm{H}_{2N_i}}(x_i)\right|^2+\left|\mathcal{P}^{-}_{\mathrm{H}_{1N_i}}(x_i)-\mathcal{P}^{-}_{\mathrm{H}_{2N_i}}(x_i)\right|^2$$
$$+\left|a_{1M_i}(x_i)-a_{2M_i}(x_i)\right|^2+\left|a_{1A_i}(x_i)-a_{2A_i}(x_i)\right|^2+\left|a_{1N_i}(x_i)-a_{2N_i}(x_i)\right|^2$$
$$+\left|v_{1M_i}(x_i)-v_{2M_i}(x_i)\right|^2+\left|v_{1A_i}(x_i)-v_{2A_i}(x_i)\right|^2+\left|v_{1N_i}(x_i)-v_{2N_i}(x_i)\right|^2$$
$$\left.\left.+\left|u_{1M_i}(x_i)-u_{2M_i}(x_i)\right|^2+\left|u_{1A_i}(x_i)-u_{2A_i}(x_i)\right|^2+\left|u_{1N_i}(x_i)-u_{2N_i}(x_i)\right|^2\right)\right)^{1/2}, \quad (3.13)$$

$$d_{is_3}(\mathcal{P}_{\mathrm{H}_1},\mathcal{P}_{\mathrm{H}_2})=\left(\frac{1}{15n}\sum_{i=1}^{n}\left(\left|\mathcal{P}^{+}_{\mathrm{H}_{1M_i}}(x_i)-\mathcal{P}^{+}_{\mathrm{H}_{2M_i}}(x_i)\right|^\lambda+\left|\mathcal{P}^{-}_{\mathrm{H}_{1M_i}}(x_i)-\mathcal{P}^{-}_{\mathrm{H}_{2M_i}}(x_i)\right|^\lambda\right.\right.$$
$$+\left|\mathcal{P}^{+}_{\mathrm{H}_{1A_i}}(x_i)-\mathcal{P}^{+}_{\mathrm{H}_{2A_i}}(x_i)\right|^\lambda+\left|\mathcal{P}^{-}_{\mathrm{H}_{1A_i}}(x_i)-\mathcal{P}^{-}_{\mathrm{H}_{2A_i}}(x_i)\right|^\lambda$$
$$+\left|\mathcal{P}^{+}_{\mathrm{H}_{1N_i}}(x_i)-\mathcal{P}^{+}_{\mathrm{H}_{2N_i}}(x_i)\right|^\lambda+\left|\mathcal{P}^{-}_{\mathrm{H}_{1N_i}}(x_i)-\mathcal{P}^{-}_{\mathrm{H}_{2N_i}}(x_i)\right|^\lambda$$
$$+\left|a_{1M_i}(x_i)-a_{2M_i}(x_i)\right|^\lambda+\left|a_{1A_i}(x_i)-a_{2A_i}(x_i)\right|^\lambda+\left|a_{1N_i}(x_i)-a_{2N_i}(x_i)\right|^\lambda$$
$$+\left|v_{1M_i}(x_i)-v_{2M_i}(x_i)\right|^\lambda+\left|v_{1A_i}(x_i)-v_{2A_i}(x_i)\right|^\lambda+\left|v_{1N_i}(x_i)-v_{2N_i}(x_i)\right|^\lambda$$
$$\left.\left.+\left|u_{1M_i}(x_i)-u_{2M_i}(x_i)\right|^\lambda+\left|u_{1A_i}(x_i)-u_{2A_i}(x_i)\right|^\lambda+\left|u_{1N_i}(x_i)-u_{2N_i}(x_i)\right|^\lambda\right)\right)^{1/\lambda}, \quad (3.14)$$

where $\lambda > 0$.
Moreover, if every characteristic of (PHFE) has a distinct weight, then

$$d_{is_1}(\mathcal{P}_{\mathrm{H}_1},\mathcal{P}_{\mathrm{H}_2})=\frac{1}{n}\sum_{i=1}^{n}\left(\omega_1\left|\mathcal{P}^{+}_{\mathrm{H}_{1M_i}}(x_i)-\mathcal{P}^{+}_{\mathrm{H}_{2M_i}}(x_i)\right|+\omega_2\left|\mathcal{P}^{-}_{\mathrm{H}_{1M_i}}(x_i)-\mathcal{P}^{-}_{\mathrm{H}_{2M_i}}(x_i)\right|\right.$$
$$+\omega_3\left|\mathcal{P}^{+}_{\mathrm{H}_{1A_i}}(x_i)-\mathcal{P}^{+}_{\mathrm{H}_{2A_i}}(x_i)\right|+\omega_4\left|\mathcal{P}^{-}_{\mathrm{H}_{1A_i}}(x_i)-\mathcal{P}^{-}_{\mathrm{H}_{2A_i}}(x_i)\right|$$
$$+\omega_5\left|\mathcal{P}^{+}_{\mathrm{H}_{1N_i}}(x_i)-\mathcal{P}^{+}_{\mathrm{H}_{2N_i}}(x_i)\right|+\omega_6\left|\mathcal{P}^{-}_{\mathrm{H}_{1N_i}}(x_i)-\mathcal{P}^{-}_{\mathrm{H}_{2N_i}}(x_i)\right|$$
$$+\omega_7\left|a_{1M_i}(x_i)-a_{2M_i}(x_i)\right|+\omega_8\left|a_{1A_i}(x_i)-a_{2A_i}(x_i)\right|+\omega_9\left|a_{1N_i}(x_i)-a_{2N_i}(x_i)\right|$$
$$+\omega_{10}\left|v_{1M_i}(x_i)-v_{2M_i}(x_i)\right|+\omega_{11}\left|v_{1A_i}(x_i)-v_{2A_i}(x_i)\right|+\omega_{12}\left|v_{1N_i}(x_i)-v_{2N_i}(x_i)\right|$$
$$+\omega_{13}\left|u_{1M_i}(x_i)-u_{2M_i}(x_i)\right|+\omega_{14}\left|u_{1A_i}(x_i)-u_{2A_i}(x_i)\right|$$
$$\left.+\omega_{15}\left|u_{1N_i}(x_i)-u_{2N_i}(x_i)\right|\right), \quad (3.15)$$

$$d_{is_2}(\mathcal{P}_{\mathrm{H}_1},\mathcal{P}_{\mathrm{H}_2}) = \Bigg(\frac{1}{n}\sum_{i=1}^{n}\Bigg(\omega_1\left|\mathcal{P}^{+}_{\mathrm{H}_{1M_i}}(x_i)-\mathcal{P}^{+}_{\mathrm{H}_{2M_i}}(x_i)\right|^2+\omega_2\left|\mathcal{P}^{-}_{\mathrm{H}_{1M_i}}(x_i)-\mathcal{P}^{-}_{\mathrm{H}_{2M_i}}(x_i)\right|^2$$
$$+\omega_3\left|\mathcal{P}^{+}_{\mathrm{H}_{1A_i}}(x_i)-\mathcal{P}^{+}_{\mathrm{H}_{2A_i}}(x_i)\right|^2+\omega_4\left|\mathcal{P}^{-}_{\mathrm{H}_{1A_i}}(x_i)-\mathcal{P}^{-}_{\mathrm{H}_{2A_i}}(x_i)\right|^2$$
$$+\omega_5\left|\mathcal{P}^{+}_{\mathrm{H}_{1N_i}}(x_i)-\mathcal{P}^{+}_{\mathrm{H}_{2N_i}}(x_i)\right|^2+\omega_6\left|\mathcal{P}^{-}_{\mathrm{H}_{1N_i}}(x_i)-\mathcal{P}^{-}_{\mathrm{H}_{2N_i}}(x_i)\right|^2$$
$$+\omega_7\left|a_{1M_i}(x_i)-a_{2M_i}(x_i)\right|^2+\omega_8\left|a_{1A_i}(x_i)-a_{2A_i}(x_i)\right|^2+\omega_9\left|a_{1N_i}(x_i)-a_{2N_i}(x_i)\right|^2$$
$$+\omega_{10}\left|v_{1M_i}(x_i)-v_{2M_i}(x_i)\right|^2+\omega_{11}\left|v_{1A_i}(x_i)-v_{2A_i}(x_i)\right|^2+\omega_{12}\left|v_{1N_i}(x_i)-v_{2N_i}(x_i)\right|^2$$
$$+\omega_{13}\left|u_{1M_i}(x_i)-u_{2M_i}(x_i)\right|^2+\omega_{14}\left|u_{1A_i}(x_i)-u_{2A_i}(x_i)\right|^2$$
$$+\omega_{15}\left|u_{1N_i}(x_i)-u_{2N_i}(x_i)\right|^2\Bigg)\Bigg)^{1/2}, \tag{3.16}$$

$$d_{is_3}(\mathcal{P}_{\mathrm{H}_1},\mathcal{P}_{\mathrm{H}_2}) = \Bigg(\frac{1}{n}\sum_{i=1}^{n}\Bigg(\omega_1\left|\mathcal{P}^{+}_{\mathrm{H}_{1M_i}}(x_i)-\mathcal{P}^{+}_{\mathrm{H}_{2M_i}}(x_i)\right|^\lambda+\omega_2\left|\mathcal{P}^{-}_{\mathrm{H}_{1M_i}}(x_i)-\mathcal{P}^{-}_{\mathrm{H}_{2M_i}}(x_i)\right|^\lambda$$
$$+\omega_3\left|\mathcal{P}^{+}_{\mathrm{H}_{1A_i}}(x_i)-\mathcal{P}^{+}_{\mathrm{H}_{2A_i}}(x_i)\right|^\lambda+\omega_4\left|\mathcal{P}^{-}_{\mathrm{H}_{1A_i}}(x_i)-\mathcal{P}^{-}_{\mathrm{H}_{2A_i}}(x_i)\right|^\lambda$$
$$+\omega_5\left|\mathcal{P}^{+}_{\mathrm{H}_{1N_i}}(x_i)-\mathcal{P}^{+}_{\mathrm{H}_{2N_i}}(x_i)\right|^\lambda+\omega_6\left|\mathcal{P}^{-}_{\mathrm{H}_{1N_i}}(x_i)-\mathcal{P}^{-}_{\mathrm{H}_{2N_i}}(x_i)\right|^\lambda$$
$$+\omega_7\left|a_{1M_i}(x_i)-a_{2M_i}(x_i)\right|^\lambda+\omega_8\left|a_{1A_i}(x_i)-a_{2A_i}(x_i)\right|^\lambda+\omega_9\left|a_{1N_i}(x_i)-a_{2N_i}(x_i)\right|^\lambda$$
$$+\omega_{10}\left|v_{1M_i}(x_i)-v_{2M_i}(x_i)\right|^\lambda+\omega_{11}\left|v_{1A_i}(x_i)-v_{2A_i}(x_i)\right|^\lambda+\omega_{12}\left|v_{1N_i}(x_i)-v_{2N_i}(x_i)\right|^\lambda$$
$$+\omega_{13}\left|u_{1M_i}(x_i)-u_{2M_i}(x_i)\right|^\lambda+\omega_{14}\left|u_{1A_i}(x_i)-u_{2A_i}(x_i)\right|^\lambda$$
$$+\omega_{15}\left|u_{1N_i}(x_i)-u_{2N_i}(x_i)\right|^\lambda\Bigg)\Bigg)^{1/\lambda}, \quad \lambda > 0, \tag{3.17}$$

are called the normalized weighted Hamming distance measure, normalized weighted Euclidean distance measure, and normalized weighted generalized distance measure between (PHFSs) $\mathcal{P}_{\mathrm{H}_1}$ *and* $\mathcal{P}_{\mathrm{H}_2}$ *based on the characteristics of (PHFE). Where* $0 \le \omega_i \le 1$, $i = 1, 2, \ldots, 15$ *and* $\sum_{i=1}^{15}\omega_i = 1$.

4. Application

The distance measure is a critical tool widely employed in both theoretical research and practical applications. It plays a pivotal role in various fields, including cluster analysis, pattern recognition, medical diagnosis, and decision-making.

Example 3. Medical diagnosis represents a vital element of computer-aided systems, emphasizing the assessment of a patient's clinical symptoms and diagnostic tests to determine the most likely disease. In this study, five diseases are examined: P_1 (Viral fever), P_2 (Malaria), P_3 (Typhoid), P_4 (Stomach problem), and P_5 (Chest problem). These diseases are associated with five corresponding symptoms: S_1 (Body temperature), S_2 (Headache), S_3 (Cough), S_4 (Stomachache pain), and S_5 (Chest pain). The relationships between these diseases and their symptoms are summarized in Table 1, while Table 2 illustrates the relationships between patients and their symptoms. The main objective is to identify the most probable disease for each patient based on their symptoms, ensuring precise diagnosis and facilitating effective treatment strategies.

Each (PHFE) corresponds to the precise extent of the association between disease data and symptoms or between patient data and symptoms, as illustrated in Tables 1 and 2. Let $X=\{S_1, S_2, S_3, S_4, S_5\}$, where a (PHFS) on X represents either a patient A_j $(j = 1, 2, 3, 4)$ or a disease P_i $(i = 1, 2, 3, 4, 5)$. The normalized distance measure between a disease P_i and a patient A_j is calculated using the proposed methodology. A smaller distance measure indicates a higher probability of diagnosing the patient with the respective disease.

The process of calculating medical diagnoses and analyzing the results is presented in Tables 3–6. Based on the proposed distance measures and the data provided in these tables, patient A_1 (Al) is diagnosed with P_1 (Viral fever) for $\lambda = 0.5, 1, 2, 3$, and 4, and with P_4 (Stomach problem) for $\lambda = 10$. Patient A_2 (Bob) is diagnosed with P_4 (Stomach problem) for $\lambda = 0.5$ and 1, and with P_1 (Viral fever) for $\lambda = 2, 3, 4$, and 10. Patient A_3 (Joe) is consistently diagnosed with P_1 (Viral fever), and patient A_4 (Ted) is also consistently diagnosed with P_1 (Viral fever).

These results demonstrate the robustness and effectiveness of the proposed distance measures in accurately diagnosing diseases based on clinical symptoms.

Table 1. Disease data for symptoms.

	S_1	S_2	S_3	S_4	S_5
P_1	{{0.6, 0.4, 0.3}, {0.2, 0.0}, {0.8, 0.7, 0.4}}	{{0.7, 0.5, 0.3, 0.2}, {0.3, 0.1}, {0.5, 0.3, 0.2, 0.1}}	{{0.5, 0.3}, {0.5, 0.4, 0.2}, {0.2, 0.1}}	{{0.5, 0.4, 0.3, 0.2, 0.1}, {0.5, 0.3}, {0.6, 0.5, 0.3, 0.2, 0.1}}	{{0.5, 0.4, 0.2, 0.1}, {0.5, 0.4, 0.3}, {0.4, 0.3, 0.2, 0.1}}
P_2	{{0.9, 0.8, 0.7}, {0.1, 0.0}, {0.6, 0.3, 0.1}}	{{0.5, 0.3, 0.2, 0.1}, {0.4, 0.3}, {0.9, 0.8, 0.7, 0.6}}	{{0.2, 0.1}, {0.7, 0.6, 0.5}, {0.5, 0.3}}	{{0.6, 0.5, 0.3, 0.2, 0.1}, {0.3, 0.2}, {0.5, 0.4, 0.3, 0.2, 0.1}}	{{0.4, 0.3, 0.2, 0.1}, {0.6, 0.5, 0.4}, {0.6, 0.4, 0.3, 0.2}}
P_3	{{0.6, 0.3, 0.1}, {0.3, 0.2}, {0.5, 0.4, 0.2}}	{{0.9, 0.8, 0.7, 0.6}, {0.1, 0.0}, {0.4, 0.3, 0.2, 0.1}}	{{0.5, 0.3}, {0.5, 0.4, 0.3}, {0.4, 0.3}}	{{0.5, 0.4, 0.3, 0.2, 0.1}, {0.5, 0.4}, {0.9, 0.8, 0.7, 0.6, 0.5}}	{{0.6, 0.4, 0.3, 0.2}, {0.4, 0.3, 0.2}, {0.5, 0.4, 0.2, 0.1}}
P_4	{{0.5, 0.4, 0.2}, {0.5, 0.3}, {0.3, 0.2, 0.1}}	{{0.4, 0.3, 0.2, 0.1}, {0.4, 0.3}, {0.5, 0.3, 0.2, 0.1}}	{{0.4, 0.3}, {0.6, 0.5, 0.4}, {0.3, 0.2}}	{{0.9, 0.8, 0.7, 0.6, 0.5}, {0.1, 0.0}, {0.7, 0.6, 0.5, 0.3, 0.2}}	{{0.5, 0.4, 0.2, 0.1}, {0.5, 0.4, 0.3}, {0.8, 0.7, 0.6, 0.5}}
P_5	{{0.3, 0.2, 0.1}, {0.7, 0.6}, {0.5, 0.4, 0.2}}	{{0.5, 0.3, 0.2, 0.1}, {0.5, 0.3}, {0.4, 0.3, 0.2, 0.1}}	{{0.3, 0.2}, {0.6, 0.4, 0.3}, {0.4, 0.3}}	{{0.7, 0.6, 0.5, 0.3, 0.2}, {0.2, 0.1}, {0.9, 0.8, 0.7, 0.6, 0.5}}	{{0.8, 0.7, 0.6, 0.5}, {0.2, 0.1, 0.0}, {0.4, 0.3, 0.2, 0.1}}

Table 2. Patient data for symptoms.

	S_1	S_2	S_3	S_4	S_5
A_1	$\{\{0.9, 0.7, 0.5\}, \{0.1, 0.0\}, \{0.5, 0.4, 0.2\}\}$	$\{\{0.4, 0.3, 0.2, 0.1\}, \{0.5, 0.4\}, \{0.5, 0.4, 0.3, 0.1\}\}$	$\{\{0.4, 0.3\}, \{0.5, 0.4, 0.2\}, \{0.2, 0.1\}\}$	$\{\{0.6, 0.5, 0.4, 0.2, 0.1\}, \{0.3, 0.2\}, \{0.9, 0.8, 0.6, 0.5, 0.4\}\}$	$\{\{0.4, 0.3, 0.2, 0.1\}, \{0.5, 0.4, 0.3\}, \{0.5, 0.4, 0.3, 0.2\}\}$
A_2	$\{\{0.5, 0.4, 0.2\}, \{0.5, 0.3\}, \{0.9, 0.7, 0.6\}\}$	$\{\{0.5, 0.4, 0.3, 0.1\}, \{0.4, 0.3\}, \{0.7, 0.4, 0.3, 0.1\}\}$	$\{\{0.2, 0.1\}, \{0.7, 0.6, 0.5\}, \{0.3, 0.2\}\}$	$\{\{0.9, 0.8, 0.6, 0.5, 0.4\}, \{0.1, 0.0\}, \{0.6, 0.4, 0.3, 0.2, 0.1\}\}$	$\{\{0.5, 0.4, 0.3, 0.2\}, \{0.5, 0.4, 0.3\}, \{0.6, 0.3, 0.2, 0.1\}\}$
A_3	$\{\{0.9, 0.7, 0.6\}, \{0.1, 0.0\}, \{0.8, 0.7, 0.5\}\}$	$\{\{0.7, 0.4, 0.3, 0.1\}, \{0.2, 0.1\}, \{0.6, 0.5, 0.4, 0.2\}\}$	$\{\{0.3, 0.2\}, \{0.5, 0.4, 0.3\}, \{0.5, 0.3\}\}$	$\{\{0.6, 0.4, 0.3, 0.2, 0.1\}, \{0.4, 0.3\}, \{0.6, 0.4, 0.3, 0.2, 0.1\}\}$	$\{\{0.6, 0.3, 0.2, 0.1\}, \{0.4, 0.3, 0.2\}, \{0.5, 0.4, 0.2, 0.1\}\}$
A_4	$\{\{0.8, 0.7, 0.5\}, \{0.2, 0.1\}, \{0.9, 0.7, 0.5\}\}$	$\{\{0.6, 0.5, 0.4, 0.2\}, \{0.4, 0.3\}, \{0.4, 0.3, 0.2, 0.1\}\}$	$\{\{0.5, 0.3\}, \{0.5, 0.4, 0.3\}, \{0.4, 0.3\}\}$	$\{\{0.6, 0.4, 0.3, 0.2, 0.1\}, \{0.4, 0.3\}, \{0.6, 0.5, 0.4, 0.2, 0.1\}\}$	$\{\{0.5, 0.4, 0.2, 0.1\}, \{0.5, 0.4, 0.3\}, \{0.4, 0.3, 0.2, 0.1\}\}$

Table 3. Normalized generalized distance measures between A_l and diseases P_i, $i = 1, 2, 3, 4, 5$.

A_l	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 2$	$\lambda = 3$	$\lambda = 4$	$\lambda = 10$
P_1	**0.0215917**	**0.0792890**	**0.1372727**	**0.1590846**	**0.1811565**	0.3680563
P_2	0.0362257	0.0831933	0.1473736	0.1929749	0.2277158	0.3370483
P_3	0.0516143	0.1024718	0.1708586	0.2206327	0.2583607	0.3678254
P_4	0.0463843	0.0949614	0.1562820	0.1944856	0.2214545	**0.2981283**
P_5	0.0530238	0.1109942	0.1976700	0.2633571	0.3123987	0.4501101

Table 4. Normalized generalized distance measures between Bob and diseases P_i, $i = 1, 2, 3, 4, 5$.

Bob	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 2$	$\lambda = 3$	$\lambda = 4$	$\lambda = 10$
P_1	0.0358908	0.0778457	**0.1342093**	**0.1740103**	**0.2033980**	**0.2873545**
P_2	0.0476015	0.1060223	0.1817962	0.2317905	0.2675776	0.3698351
P_3	0.0809893	0.1394174	0.2057908	0.2462847	0.2740206	0.3994317
P_4	**0.0219417**	**0.0658023**	0.1421767	0.2057392	0.2561777	0.4055136
P_5	0.0636871	0.1130900	0.1684717	0.2037327	0.2290297	0.3021722

Table 5. Normalized generalized distance measures between Joe and diseases P_i, $i = 1, 2, 3, 4, 5$.

Joe	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 2$	$\lambda = 3$	$\lambda = 4$	$\lambda = 10$
P_1	**0.0311657**	**0.0598506**	**0.0989361**	**0.1295466**	**0.1539220**	**0.2248739**
P_2	0.0371389	0.0779744	0.1291708	0.1647957	0.1926287	0.2829736
P_3	0.0457718	0.0966266	0.1630490	0.2096375	0.2448786	0.3538900
P_4	0.0792490	0.1352725	0.1997921	0.2409857	0.2700455	0.3537134
P_5	0.0743473	0.1375754	0.2198130	0.2781398	0.3225091	0.4531066

Table 6. Normalized generalized distance measures between Ted and diseases P_i, $i = 1, 2, 3, 4, 5$.

Ted	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 2$	$\lambda = 3$	$\lambda = 4$	$\lambda = 10$
P_1	**0.0152562**	**0.0407857**	**0.0759397**	**0.0996926**	**0.1169566**	**0.1646771**
P_2	0.0496966	0.0990685	0.1745871	0.2389096	0.2929653	0.4793828
P_3	0.0383785	0.0888440	0.1556240	0.1986697	0.2281489	0.3050077
P_4	0.0508915	0.1101930	0.1856829	0.2356880	0.2727491	0.3989416
P_5	0.0591256	0.1232246	0.2019790	0.2511524	0.2854115	0.3816457

4.1. Discussion and comparative analysis of distance measures

The comparative analysis of distance measures, as illustrated in Table 7, demonstrates the significant advancements introduced in this study for handling (PHFS). By addressing the limitations of methods like the generalized picture hesitant normalized distance measure GPHNDM [24] and the generalized picture hesitant Hausdorff distance measure GPHHDM [25], the proposed measures prove to be more reliable and practical, particularly in medical diagnostic contexts.

Table 7. Comparative analysis of distance measures.

Measure	**Al**	**Bob**	**Joe**	**Ted**
GPHNDM [24]	P_1	$P_1(\lambda = 0.5, 1, 3, 4)$ $P_4(\lambda = 2)$ $P_5(\lambda = 10)$	P_1	P_1
GPHHDM [25]	P_1	$P_4(\lambda = 0.5, 1, 2)$ $P_5(\lambda = 3, 4, 10)$	P_1	P_1
Our proposed distance measures	$P_1(\lambda = 0.5, 1, 2, 3, 4)$ $P_4(\lambda = 10)$	$P_4(\lambda = 0.5, 1)$ $P_1(\lambda = 2, 3, 4, 10)$	P_1	P_1

For example, in Bob's diagnosis, traditional approaches often yield ambiguous outcomes across λ values, complicating the interpretation of results, particularly in complex cases. Sensitivity analyses reveal that these methods frequently produce multiple diagnostic results, introducing significant uncertainty. In contrast, the proposed measures consolidate these outcomes into stable and interpretable conclusions, showcasing superior reliability. Similarly, in Al's case, the proposed measures demonstrate adaptability by providing a broader yet stable diagnostic range, outperforming existing methods. In simpler cases like Joe and Ted, while all methods yield consistent results, the proposed measures excel in detecting subtle differences in hesitancy and variance.

From a theoretical perspective, the incorporation of advanced features such as the average function, variance function, and degree of hesitancy enhances the ability to model and evaluate uncertain data with greater precision. Adherence to mathematical principles, such as the triangle inequality, ensures robustness and reliability, marking a substantial contribution to the theoretical framework of PHFS.

Practically, the proposed measures significantly improve medical diagnostics by enabling physicians to assess patient symptoms with higher precision and reduced ambiguity, leading to better diagnostic outcomes. Beyond medical diagnosis, these measures demonstrate broad applicability in domains such as pattern recognition, clustering, and multi-criteria decision-making, addressing real-world challenges characterized by uncertainty and complexity.

While the proposed measures offer significant theoretical and practical advancements, further validation using diverse real-world datasets is essential to assess their scalability and robustness. Future research could explore extending these measures to dynamic systems, enabling real-time decision-making under evolving conditions of uncertainty.

4.2. Advantages

The advantages of the proposed normalized distance measure can be summarized as follows:

1. The proposed distance measure eliminates the need to equalize the lengths of membership, neutral, and non-membership degrees for any two PHFEs, thereby simplifying computations.
2. PHFEs are characterized by fundamental properties, including upper bound, lower bound, average function, variance function, and hesitancy degree, which collectively enhance their descriptive power.
3. The proposed distance measure satisfies the triangle inequality, ensuring mathematical validity and consistency.
4. The normalized weighted distance measure is highly adaptable to diverse application requirements, accommodating various characteristics of PHFEs.

Additionally, other distance measures, such as the Hausdorff distance and hybrid distance, can be applied using the feature vector representation of PHFEs, further enriching the analysis.

5. Conclusions

In this research, we have introduced innovative characteristics to enhance the descriptive capabilities of (PHFSs), including the average function, variance function, and hesitancy degree. Utilizing these characteristics, we developed novel distance measures for PHFSs, demonstrating their compliance with fundamental mathematical properties such as the triangle inequality and achieving significant improvements in diagnostic accuracy for medical applications.

While the results are promising, the study has certain limitations. The evaluation primarily focuses on medical diagnostic data, which may limit its applicability to other domains. Additionally, the computational complexity of the proposed measures increases with high-dimensional datasets, requiring further optimization for scalability. Despite these challenges, the proposed measures offer broad potential for applications in pattern recognition, clustering, and decision-making under uncertainty.

Researchers will address these limitations by validating the methods across diverse datasets, enhancing computational efficiency, and exploring broader applications in dynamic systems and real-time decision-making.

Author contributions

Noura Omair Alshehri: Conceptualization, validation; Rania Saeed Alghamdi: Writing-review, validation; Noura Awad Al Qarni: Methodology development, writing-original draft, validation. All authors contributed equally to the manuscript. All authors have read and approved the final version of the manuscript for publication. of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest about the publication of the research article.

References

1. L. A. Zadeh, Fuzzy sets, *Inf. Control*, **8** (1965), 338–353. http://dx.doi.org/10.1016/S0019-9958(65)90241-X
2. K. T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Set. Syst.*, **20** (1986), 87–96. http://dx.doi.org/10.1016/S0165-0114(86)90034-3
3. B. C. Cuong, *Picture fuzzy sets—first results, Part 1, Seminar neuro-fuzzy systems with applications*, Institute of Mathematics, Hanoi, 2013.
4. V. Torra, Y. Narukawa, *On hesitant fuzzy sets and decision*, In 2009 IEEE International Conference on Fuzzy Systems, IEEE, 2009, 1378–1382. https://doi.org/10.1109/FUZZY.2009.5276884
5. V. Torra, Hesitant fuzzy sets, *Int. J. Intell. Syst.*, **25** (2010), 529–539. https://doi.org/10.1002/int.20418
6. M. Xia, Z. Xu, Hesitant fuzzy information aggregation in decision making, *Int. J. Approx. Reason.*, **52** (2011), 395–407. https://doi.org/10.1016/j.ijar.2010.09.002
7. M. Xia, Z. Xu, N. Chen, Some hesitant fuzzy aggregation operators with their application in group decision making, *Group Decis. Negot.*, **22** (2013), 259–279. https://doi.org/10.1007/s10726-011-9261-7
8. Z. Xu, M. Xia, Hesitant fuzzy entropy and cross-entropy and their use in multiattribute decision-making, *Int. J. Intell. Syst.*, **27** (2012), 799–822. https://doi.org/10.1002/int.21548

9. Z. Xu, M. Xia, Distance and similarity measures for hesitant fuzzy sets, *Infor. Sciences*, **181** (2012), 2128–2138. https://doi.org/10.1016/j.ins.2010.08.004

10. Z. Xu, M. Xia, On distance and correlation measures of hesitant fuzzy information, *Int. J. Intell. Syst.*, **26** (2011), 410–425. https://doi.org/10.1002/int.20535

11. B. Farhadinia, Information measures for hesitant fuzzy sets and interval-valued hesitant fuzzy sets, *Infor. Sciences*, **240** (2013), 129–144. https://doi.org/10.1016/j.ins.2013.02.029

12. D. Li, W. Zeng, J. Li, New distance and similarity measures on hesitant fuzzy sets and their applications in multiple criteria decision making, *Eng. Appl. Artif. Intel.*, **40** (2015), 11–16. https://doi.org/10.1016/j.engappai.2015.01.004

13. D. Li, W. Zeng, Y. Zhao, Note on distance measure of hesitant fuzzy sets, *Infor. Sciences*, **321** (2015), 103–115. https://doi.org/10.1016/j.ins.2015.03.029

14. W. Zeng, D. Li, Q. Yin, Distance and similarity measures between hesitant fuzzy sets and their application in pattern recognition, *Pattern Recogn. Lett.*, **84** (2016), 267–271. https://doi.org/10.1016/j.patrec.2016.01.010

15. R. M. Rodriguez, L. Martinez, F. Herrera, Hesitant fuzzy linguistic term sets for decision making, *IEEE T. Fuzzy Syst.*, **20** (2012), 109–119. https://doi.org/10.1109/TFUZZ.2011.2170076

16. N. Chen, Z. Xu, M. Xia, Interval-valued hesitant preference relations and their applications to group decision making, *Knowl.-Based Syst.*, **37** (2013), 528–540. https://doi.org/10.1016/j.knosys.2012.09.010

17. G. Wei, X. Zhao, R. Lin, Some hesitant interval-valued fuzzy aggregation operators and their applications to multiple attribute decision making, *Knowl.-Based Syst.*, **46** (2013), 43–53. https://doi.org/10.1016/j.knosys.2013.03.020

18. R. Wang, Y. Li, Picture hesitant fuzzy set and its application to multiple criteria decision-making, *Symmetry*, **10** (2018), 295. https://doi.org/10.3390/sym10070295

19. E. O. Robles, O. Castillo, P. Melin, Towards asymmetric uncertainty modeling in designing general type-2 fuzzy classifiers for medical diagnosis, *Expert Syst. Appl.*, **183** (2021), 115370. https://doi.org/10.1016/j.eswa.2021.115370

20. M. U. Molla, B. C. Giri, P. Biswas, Extended PROMETHEE method with Pythagorean fuzzy sets for medical diagnosis problems, *Soft Comput.*, **25** (2021), 4503–4512. https://doi.org/10.1007/s00500-020-05425-0

21. P. Singh, Distance and similarity measures for multiple-attribute decision making with dual hesitant fuzzy sets, *Comput. Appl. Math.*, **36** (2017), 111–126. https://doi.org/10.1007/s40314-015-0219-2

22. Z. Ahmad, T. Mahmood, M. Saad, N. Jan, K. Ullah, Similarity measures for picture hesitant fuzzy sets and their applications in pattern recognition, *J. Prime Res. Math.*, **15** (2019), 81–100. Available from: `https://jprm.sms.edu.pk/media/pdf/jprm/volume_15/06.pdf`.

23. Z. Ali, T. Mahmood, H. AlSalman, B. F. Alkhamees, S. M. M. Rahman, Analysis of medical diagnosis based on variation co-efficient similarity measures under picture hesitant fuzzy sets and their application, *Math. Biosci. Eng.*, **19** (2022), 855–872. https://doi.org/10.3934/mbe.2022039

24. Z. Ali, T. Mahmood, K. Ullah, Picture hesitant fuzzy clustering based on generalized picture hesitant fuzzy distance measures, *Knowledge*, **1** (2021), 40–51. https://doi.org/10.3390/knowledge1010004

25. N. Jan, Z. Ali, K. Ullah, T. Mahmood, Some generalized distance and similarity measures for picture hesitant fuzzy sets and their applications in building material recognition and multi-attribute decision making, *Punjab Univ. J. Math.*, **51** (2020). Available from: `http://pu.edu.pk/images/journal/maths/PDF/Paper-7_51_7_2020.pdf`.

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