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Research article

Proportional grey picture fuzzy sets and their application in multi-criteria decision-making with high-dimensional data

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Abstract: Picture fuzzy sets (PFS) extend intuitionistic fuzzy sets by incorporating positive, neutral, and negative memberships to capture richer information. A notable challenge of PFS and its derivatives is the need to specify these degrees using decimals, thus limiting their practical applicability. To address this issue, we utilize proportional picture fuzzy sets (PPFS) to define these parameters through proportional relationships. Our approach selects a PFS as the unit fuzzy set, while the newly formulated proportional grey picture fuzzy sets (PGPFS) exploits the proportionality between the individual and the unit fuzzy set parameters. Additionally, we introduce the concept of a fuzzy tensor entropy measures and aggregation operators for PGPFS. Additionally, we develop an aggregation decision-making method based on PGPFS, thereby, accommodating the inherent ambiguity and uncertainty of the data. The feasibility of the PGPFS approach in addressing multi-criteria decision-making (MCDM) scenarios with uncertain criteria and expert weights is verified through an application of haze management scheme selection. The reasonableness and effectiveness of the method are further confirmed through sensitivity and comparative analyses.

Keywords: proportional picture grey fuzzy set; entropy measures; aggregation operators; fuzzy tensor; multi-criteria decision making
Mathematics Subject Classification: 03E72, 90B50, 03B52

Abbreviations

CIFS	Circular intuitionistic fuzzy sets
CPFS	Complex Pythagorean fuzzy sets
DFS	Decomposition fuzzy sets
DPFS	Decomposed Pythagorean fuzzy sets
ELECTRE	Elimination et choice translating reality
FS	Fuzzy sets
GGN	General grey number
GRP	Grey correlation projection
IFS	Intuitionistic fuzzy sets
IGFS	Interval graphical fuzzy sets
IIFS	Interval intuitionistic fuzzy sets
ISLFS	Interval spherical linguistic fuzzy sets
IVPFN	Interval-valued picture fuzzy number
IVPFOWIA	Interval-valued picture fuzzy ordered weighted interactive averaging
IVPFS	Interval-valued picture fuzzy sets
MCDM	Multi-criteria decision-making
OPFWGO	Optimized picture fuzzy weighted geometric operator
PFN	Picture fuzzy numbers
PFPPtHMq DST	Picture fuzzy power partitioned Hamymean
PFS	Picture fuzzy sets
PFW PPtHMq DST	Picture fuzzy weighted power partitioned Hamymean
PFWGO	Picture fuzzy-weighted geometric aggregation operators
PFWIA	Picture fuzzy weighted interactive average
PGPFE	Proportional grey picture fuzzy entropy
PGPFN	Proportional grey picture fuzzy number
PGPFS	Proportional grey picture fuzzy set
PGPFWA	Proportional grey picture fuzzy weighted aggregation
PPFN	Proportional picture fuzzy number
PPFS	Proportional picture fuzzy sets
PPFWA	Proportional picture fuzzy-weighted average
PPFWG	Proportional picture fuzzy-weighted geometric
PtFS	Pythagorean fuzzy sets
TODIM	Portuguese acronym for interactive multicriteria decision-making

1. Introduction

The inherent ambiguity of much real-world information makes precise numerical expressions challenging. To address this, Zadeh [1] proposed fuzzy sets (FS), which use a number between 0 and 1 to represent the membership degree of information. However, FS cannot entirely eliminate uncertainty in decision-making problems [2]. Consequently, researchers have developed various FS extensions. Atanassov [3] introduced intuitionistic fuzzy sets (IFS), which use two dimensions-membership degrees and non-membership degrees to manage uncertain information. Due to its advantages, IFS has attracted significant research attention. For instance, interval intuitionistic fuzzy sets (IIFS), where membership and non-membership degrees are expressed as interval numbers, make fuzzy representations more comprehensive [4]. Yager [5] proposed Pythagorean fuzzy sets (PtFS), which also feature membership and non-membership degrees, but require their sum of squares to be no more than 1, thus allowing for a broader range of expression compared to IFS. Additional extensions of IFS are the addition of dimensions to express information, such as Picture fuzzy sets (PFS) by Cuong and Kreinovich [6] and Spherical fuzzy sets (SFS) by Mahmood et al. [7], which include positive, neutral, and negative degrees. This approach enables more effective handling of certain facts, events, and the data that other sets (e.g., IFS and PtFS) cannot manage. Additionally, circular intuitionistic fuzzy sets (CIFS) [8], interval graphical fuzzy sets (IGFS) [9], decomposition fuzzy sets (DFS) [10], decomposed Pythagorean fuzzy sets (DPFS) [2], interval graphical fuzzy sets (IGFS) [11], and interval spherical linguistic fuzzy sets (ISLFS) [12] further extend the range of information that FS can express.

The newly introduced fuzzy set, called the proportional picture fuzzy set (PPFS) [13], departs from the traditional method of determining the parameters by decimals and proposes a proportional representation approach. Based on the picture fuzzy set, it captures the experts' proportional representation of the degrees of positive and negative memberships in relation to the degree of neutral membership, thereby using the latter as a reference point. For instance, an expert migis "movie A?" Suppose the expert assigns $< 0.55, 0.32 >$, indicating that the degree of positive membership is 0.55 times the degree of neutral membership, and the degree of negative membership is 0.32 times the degree of neutral membership. This model is more intuitive and relevant but faces a notable issue: When the expert is unable to provide the degree of neutral membership-such as when the degree of neutral membership is 0-both the degrees of positive and negative memberships will also be 0. Additionally, when an expert estimates the degree of positive membership as 0.55 times the degree of neutral membership and the degree of negative membership as [0.3, 0.4] times the degree of neutral membership, then the PPFS is unable to effectively handle this situation.

Based on this, the paper proposes a more rational model to express the membership degrees using proportions. We use three elements to represent the fuzzy number and develop the proportionht assign a proportional picture fuzzy number $< x; k_{\pi\mu}, k_{\pi\nu} >$ to express their opinion on "How good al grey picture fuzzy set (PGPFS) for its capacity to encapsulate more information." To avoid the issue where any element cannot be equal to 0, this model does not use the degree of neutral membership as a benchmark, but rather introduces the concept of a "unit". For example, in response to the proposition "How good is movie A?", if movie B is taken as the unit and assigned $< e, e, e >$, then movie A can be assigned $< 3e, 0.2e, 0.5e >$. To address the uncertainty of proportions, we incorporate the concept of grey numbers, as they can flexibly represent values where only the range is known, rather than the exact figure. Consequently, movie A can be assigned $< 3e, [0.3, 0.4]e, 0.5e >$ when the expert believes the degree of positive membership is 3 times the unit membership degree, the degree of negative membership is 0.5, and the degree of neutral membership is [0.3, 0.4] times the unit non-membership degree.

With research in FS extensions progressing rapidly, their application in various methods and techniques, particularly in multi-criteria decision making (MCDM), has become prominent [14, 15]. FS can address real-life decision-making problems by handling high levels of uncertainty and providing missing information, thereby resolving the complex issues in MCDM that involve fuzzy and contradictory data [16]. Two critical aspects of MCDM problems need to be addressed: The calculation of entropy, which is crucial for determining the weights of criteria or experts, and the aggregation operator, which can aggregate data from multiple decision makers in MCDM problems.

There are many entropy-based MCDM methods. For instance, Van Pham et al. [17] proposed an entropy measure based on picture fuzzy distances and developed an ideal solution similarity preference ranking technique to solve MCDM problems. On the other hand, Hezam et al. [18] and Thao [19] proposed weighting models based on picture fuzzy similarity metrics, which were respectively applied to the MCDM problems of siting a biofuel production plant and supplier selection. Kumar et al. [16] introduced a new picture fuzzy entropy (PFE) measure that satisfied the axiomatic definition of PFE and combined the advantages of portuguese acronym for interactive multicriteria decision-making (TODIM) and elimination et choice translating reality (ELECTRE) to evaluate the selection of sustainable partners in the PFS environment. Han [20] proposed a new class of PFE based on the cosine function and picture fuzzy weighted symmetric cross-entropy, thereby applying it to the problem of selecting innovative projects. Ma et al. [21] introduced entropy under Interval-Valued Picture Fuzzy Sets (IVPFS) and an improved MCDM method with a grey correlation projection (GRP), which was applied to determine the optimal design in yacht schemes.

MCDM methods based on aggregation operators have seen significant advancements. Punetha et al. [22] introduced two new operators named picture fuzzy power partitioned Hamymean (PFPPtHM$^{q}_{DST}$) and picture fuzzy weighted power partitioned Hamymean (PFW PPtHM$^{q}_{DST}$); based on these operators, they designed a new method for MCDM in a PFS environment. Jaikumar et al. [23] proposed an MCDM technique based on the Optimized Picture Fuzzy Weighted Geometric Operator (OPFWGO) to address the shortcomings of existing picture fuzzy-weighted geometric aggregation operators (PFWGO). Kahraman [13] proposed the proportional picture fuzzy-weighted average (PPFWA) operator and proportional picture fuzzy-weighted geometric (PPFWG) operator for the PPFS environment, along with a new AHP method to determine the optimal solution for MCDM problems. Additionally, there are MCDM methods based on both entropy measures and aggregation operators. For example, Sun et al. [24] used picture fuzzy numbers (PFN) and picture fuzzy weighted interactive average (PFWIA) operators and entropy measures of PFS to improve the ranking performance by combining TOPSIS and the weighted Martensian distance to enhance the GRP method. Ma et al. [31] introduced the interval-valued picture fuzzy number (IVPFN) and interval-valued picture fuzzy ordered weighted interactive averaging (IVPFOWIA) operators, and proposed an entropy weighting method based on IVPFS, which was combined with the extended TOPSIS method and applied to the selection of optimal design conceptualization schemes.

Since MCDM methods that use fuzzy set extensions provide more efficient decision-making results [2], PGPFS-based MCDM methods need to be proposed. We introduce a new MCDM method in the PGPFS environment that can simultaneously handle unknown criteria and expert weights while simplifying the operations of weighting and aggregation operators. Specifically, the method introduces the concept of a fuzzy tensor, which can manage high-dimensional data, thus allowing a single formula to be used in cases with multiple experts and multiple metrics.

The main contributions of this paper are as follows:

(1) The proposition of proportional grey picture fuzzy sets (PGPFS) and their arithmetic rules;

(2) The introduction of an entropy measure based on the fuzzy tensor of PGPFS, integrating

the fuzzy and grey degrees;

(3) The introduction of an aggregation operator using the proportional grey picture fuzzy tensor;

(4) The proposal of a suitable aggregation decision-making method for PGPFS.

The paper is organized as follows: Section 2 reviews the basic concepts of proportional picture fuzzy sets, grey numbers, and fuzzy tensors; Section 3 introduces proportional grey picture fuzzy sets, thereby detailing their arithmetic, entropy, and aggregation operators; Section 4 presents an integral tensor decision method for proportional grey picture fuzzy sets; Section 5 applies this decision-making method to a case study of a haze management program selection; and Section 6 provides a concluding discussion. The flow chart of this study is shown in Figure 1.

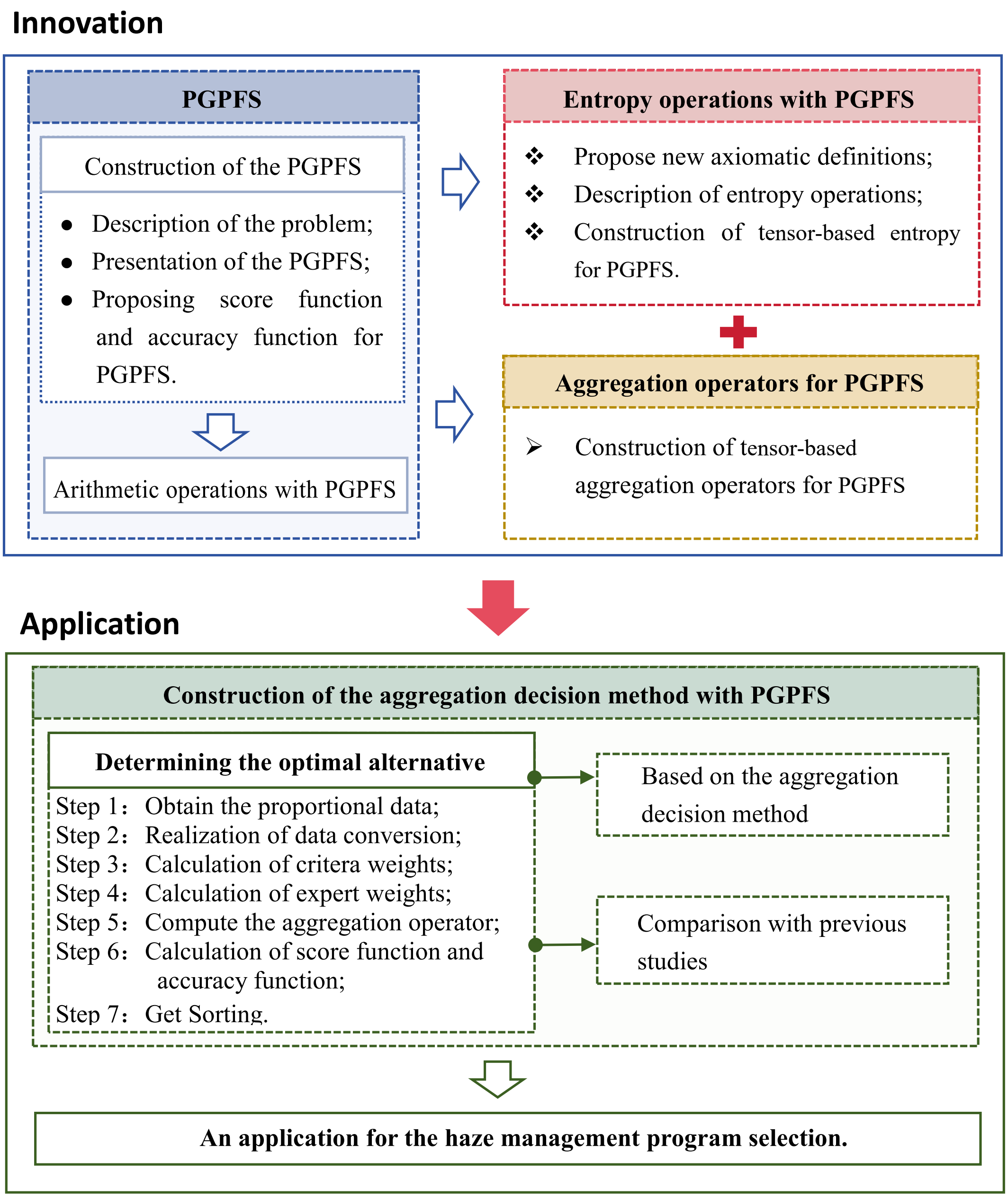

Figure 1. Flow chart of the study.

2. Preliminaries

2.1. Proportional picture fuzzy sets

Definition 1. *[13] Let $P = \{< x; \mu(x), \nu(x), \pi(x) > | x \in X\}$ be a picture fuzzy set, and the expert judges the proportions between $\mu(x)$, $\nu(x)$, and $\pi(x)$ as follows:*

$$\mu(x) = k_1 \pi(x), \tag{2.1}$$

$$\nu(x) = k_2 \pi(x), \tag{2.2}$$

where $\mu(x) \in [0, 1]$ is the degree of positive membership of x in P, $\nu(x) \in [0, 1]$ is the degree of neutral membership of x in P, and $\pi(x) \in [0, 1]$ is the degree of negative membership of x in P, which satisfies the following:

$$\pi(x) + k_1 \pi(x) + k_2 \pi(x) \leq 1. \tag{2.3}$$

Thus, the proportional picture fuzzy sets can be represented by the following:

$$P = \{< x; k_{\pi 1}, k_{\pi 2} > | x \in X\}. \tag{2.4}$$

Definition 2. *[13] Let $A =< k_{\pi A1}, \pi_A, k_{\pi A2} >$ and $B =< k_{\pi B1}, \pi_B, k_{\pi B2} >$ be two proportional picture fuzzy numbers (PPFNs) and $\lambda > 0$. Some mathematical operations are given in Eqs (2.5)–(2.9):*

$$A + B =< k_{\pi A1}\pi_A + k_{\pi B1}\pi_B - k_{\pi A1}\pi_A \times k_{\pi B1}\pi_B, \pi_A \times \pi_B, k_{\pi B2}\pi_B \times k_{\pi A2}\pi_A > . \tag{2.5}$$

$$A \times B =< k_{\pi A1}\pi_A \times k_{\pi B1}\pi_B \pi_A + \pi_B - \pi_A \times \pi_B, k_{\pi B2}\pi_B + k_{\pi A2}\pi_A - k_{\pi B2}\pi_B \times k_{\pi A2}\pi_A > . \tag{2.6}$$

$$\lambda A =< 1 - (1 - k_{\pi A1}\pi_A)^{\lambda}, {\pi_A}^{\lambda}, (k_{\pi A2}\pi_A)^{\lambda} > . \tag{2.7}$$

$$A^{\lambda} =< (k_{\pi A1}\pi_A)^{\lambda}, 1 - (1 - \pi_A)^{\lambda}, 1 - (1 - k_{\pi A2}\pi_A)^{\lambda} > . \tag{2.8}$$

$$A^C =< k_{\pi A2}, \pi_A, k_{\pi A1} > . \tag{2.9}$$

Definition 3. *[13] Let $a_i (i = 1, 2, ..., n)$ be a collection of PPFNs. The proportional picture fuzzy weighted averaging (PPFWA) operator is a mapping $PP^n \rightarrow PP$ such that*

$$PPFWA_w(a_1, a_2, ..., a_n) = \oplus_{i=1}^{n} (w_i a_i), \tag{2.10}$$

where $w = (w_1, w_2, ..., w_n)^T$ is the weight vector of $a_i (i = 1, 2, ..., n)$ and $w_i > 0$, $\sum_{i=1}^{n} w_i = 1$. Then,

$$PPFWA_w(a_1, a_2, ..., a_n) = (1 - \prod_{i=1}^{n} (1 - k_{\pi 1i}\pi_i)^{w_i}, \prod_{i=1}^{n} (\pi_i)^{w_i}, \prod_{i=1}^{n} (k_{\pi 2i}\pi_i)^{w_i}). \tag{2.11}$$

2.2. Grey number

Definition 4. *[26] The following is called a general grey number:*

$$g^{\pm} \in \bigcup_{i=1}^{n} [\underline{a_i}, \overline{a_i}] = \bigcup_{i=1}^{n} \otimes_i; \quad i = 1, 2, ..., n, \tag{2.12}$$

where $\otimes_i$ *is an interval grey number and*

$$\otimes_i \in [\underline{a_i}, \overline{a_i}] \subset \bigcup_{i=1}^{n} [\underline{a_i}, \overline{a_i}], \tag{2.13}$$

where $\underline{a_i}$ *and* $\overline{a_i}$ *are the lower and upper limits of the information separately in formula (2.13), and meet the following two conditions:*

$\underline{a_i}, \overline{a_i} \in R$ *and* $\overline{a_{i-1}} \leq \underline{a_i} \leq \overline{a_i} \leq \underline{a_{i+1}}$. $g^- = \inf_{\underline{a_i} \in g^{\pm}\underline{a_i}}$, $g^+ = \sup_{\overline{a_i} \in g^{\pm}\overline{a_i}}$ *are called lower bound and upper bound of* $g^{\pm}$ *respectively, where the interval grey number and the real number are special cases of the general grey number.*

Definition 5. *[26] (1) For a general grey number* $g^{\pm} \in \bigcup_{i=1}^{n} [\underline{a_i}, \overline{a_i}]$,

$$\hat{g} = 1/n \sum_{i=1}^{n} \hat{a}_i \tag{2.14}$$

is called the "kernel" of it.

(2) Suppose a general grey number $g^{\pm} \in \bigcup_{i=1}^{n} [\underline{a_i}, \overline{a_i}] (i = 1, 2, ..., n)$ *with a known probability distribution; in the case of a probability,* p_i *and* $p_i > 0$, $i = 1, 2, ..., n$, $\sum_{i=1}^{n} p_i = 1$,

$$\hat{g} = 1/n \sum_{i=1}^{n} p_i \hat{a}_i \tag{2.15}$$

is called the "kernel" of it.

Definition 6. *[27] Let* $\Omega \in R$ *be the domain of universe,* $g^{\pm} \in \bigcup_{i=1}^{n} [\underline{a_i}, \overline{a_i}] \subseteq \Omega$, *and* $d_{\min}, d_{\max} \in \Omega$, $d_{\min} = \min\{\Omega\}$, $d_{\max} = \max\{\Omega\}$; *then, the degree of greyness of a grey number is defined as follows:*

$$g^{\circ} = \frac{|g^+ - g^-|}{|d_{\max} - d_{\min}|}. \tag{2.16}$$

2.3. Fuzzy tensor

Definition 7. *[28] Suppose* $n_1 n_2 \cdots n_m$ *elements form the following multidimensional array:*

$$\mathrm{A} = (a_{i_1 i_2 \ldots i_m}), i_1 \in [n_1], i_2 \in [n_2], ..., i_m \in [n_m], \tag{2.17}$$

which is an m-order $n_1 n_2 \cdots n_m$*-dimensional tensor, denoted as follows:*

$$\mathrm{A} \in T_R(m, n_1 \times n_2 \times \cdots \times n_m), \tag{2.18}$$

where $a_{i_1 i_2 \ldots i_m}$ *is an entry of A, and* $a_{i_1 i_2 \ldots i_m} \in R$.

Note: According to the definition, vectors are considered first-order tensors, matrices represent second-order tensors, and collections of vectors across multiple dimensions constitute higher-order tensors. A visual representation of these basis tensors is provided in Figure 2.

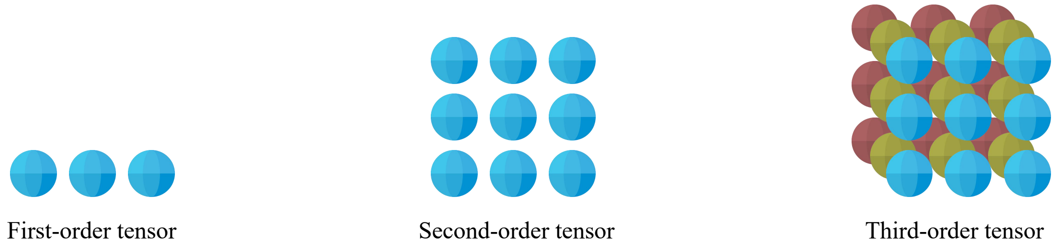

Figure 2. Schematic visualizations of basis tensors.

Definition 8. *[29] Suppose* $\tilde{A} \in T_R(m, n_1 \times n_2 \times \cdots \times n_m)$ *and its entries are* $a_{i_1 i_2 \ldots i_m} \in [0, 1]$, *where* $i_1 \in [n_1], i_2 \in [n_2], ..., i_m \in [n_m]$. *Then,* $\tilde{A}$ *is referred to as a fuzzy tensor of order m.*

3. Proportional grey picture fuzzy sets

Definition 9. *Let* $P = \{< x; \mu(x), \pi(x), \nu(x) > | x \in X\}$ *be a picture fuzzy set, e is the unit of assessment, and the expert judges the proportions between* $\mu(x)$, $\nu(x)$, $\pi(x)$, *and e as follows:*

$$\mu(x) = k_1 e, \tag{3.1}$$

$$\pi(x) = k_2 e, \tag{3.2}$$

$$\nu(x) = k_3 e. \tag{3.3}$$

The refusal degree can be given by the following:

$$r(x) = k_4 e. \tag{3.4}$$

At this point, the values of $\mu(x), \pi(x)$, *and* $\nu(x)$ *may exceed 1. To address this issue, set the following:*

$$\mu_P(x) = \frac{k_1}{k_1 + k_2 + k_3 + k_4}, \tag{3.5}$$

$$\pi_P(x) = \frac{k_2}{k_1 + k_2 + k_3 + k_4}, \tag{3.6}$$

$$\nu_P(x) = \frac{k_3}{k_1 + k_2 + k_3 + k_4}, \tag{3.7}$$

which satisfies the following:

$$0 \leq \mu_P(x) + \upsilon_P(x) + \pi_P(x) \leq 1. \tag{3.8}$$

The refusal degree can be given by the following:

$$r_P(x) = 1 - \frac{k_1 + k_2 + k_3}{k_1 + k_2 + k_3 + k_4}. \tag{3.9}$$

Thus, the improved proportional picture fuzzy sets can be represented by the following:

$$P = \{< x; \frac{k_1}{k_1 + k_2 + k_3 + k_4}, \frac{k_2}{k_1 + k_2 + k_3 + k_4}, \frac{k_3}{k_1 + k_2 + k_3 + k_4} > | x \in X\}. \tag{3.10}$$

Experts often use fuzzy terminology when predicting ratios, such as the membership degree being "about twice" or "between two and three times" the unit. Kahraman [13] advocated using triangular and trapezoidal ratios to predict these cases. However, in practice, experts use both fuzzy expressions, and triangular and trapezoidal fuzzy numbers cannot simultaneously address this situation. Therefore, grey numbers are introduced to solve this problem. Let e be the unit of assessment. The expert judges the proportions between $\mu(x), \pi(x), \nu(x)$, and e as follows:

$$\mu(x) = [\underline{k_1}, \overline{k_1}]e, \tag{3.11}$$

$$\pi(x) = [\underline{k_2}, \overline{k_2}]e, \tag{3.12}$$

$$\nu(x) = [\underline{k_3}, \overline{k_3}]e. \tag{3.13}$$

The refusal degree can be given by the following:

$$r(x) = [\underline{k_4}, \overline{k_4}]e. \tag{3.14}$$

Definition 10. *Let P_g be proportional picture grey fuzzy sets,*

$$P_g = \{< \otimes_i; \mu_g(\otimes_i), \pi_g(\otimes_i), \nu_g(\otimes_i) > | \otimes_i \in g^{\pm}\}, \tag{3.15}$$

where

$$\mu_g(\otimes_i) = \frac{\hat{g}_{k_1}(\otimes_i)}{\hat{g}_{k_1}(\otimes_i) + \hat{g}_{k_2}(\otimes_i) + \hat{g}_{k_3}(\otimes_i) + \hat{g}_{k_4}(\otimes_i)}, \tag{3.16}$$

$$\pi_g(\otimes_i) = \frac{\hat{g}_{k_2}(\otimes_i)}{\hat{g}_{k_1}(\otimes_i) + \hat{g}_{k_2}(\otimes_i) + \hat{g}_{k_3}(\otimes_i) + \hat{g}_{k_4}(\otimes_i)}, \tag{3.17}$$

$$\upsilon_g(\otimes_i) = \frac{\hat{g}_{k_3}(\otimes_i)}{\hat{g}_{k_1}(\otimes_i) + \hat{g}_{k_2}(\otimes_i) + \hat{g}_{k_3}(\otimes_i) + \hat{g}_{k_4}(\otimes_i)}, \tag{3.18}$$

$$r_g(\otimes_i) = 1 - \frac{\hat{g}_{k_1}(\otimes_i) + \hat{g}_{k_2}(\otimes_i) + \hat{g}_{k_3}(\otimes_i)}{\hat{g}_{k_1}(\otimes_i) + \hat{g}_{k_2}(\otimes_i) + \hat{g}_{k_3}(\otimes_i) + \hat{g}_{k_4}(\otimes_i)}, \tag{3.19}$$

$\hat{g}_{k_1}(\otimes_i)$ refers to the kernel of the interval grey number $[\underline{k_1}, \overline{k_1}]$, $\hat{g}_{k_2}(\otimes_i)$ refers to the kernel of the interval grey number $[\underline{k_2}, \overline{k_2}]$, $\hat{g}_{k_3}(\otimes_i)$ refers to the kernel of the interval grey number $[\underline{k_3}, \overline{k_3}]$, and $\hat{g}_{k_4}(\otimes_i)$ refers to the kernel of the interval grey number $[\underline{k_4}, \overline{k_4}]$. It is worth noting that for the case "approximately k times the temporary unit e", $\hat{g}_{k_j}(\overline{\otimes_i}) = k$, $j = 1, 2, 3, 4$, and is satisfied by the following:

$$0 \leq \mu_g(\otimes_i) + \upsilon_g(\otimes_i) + \pi_g(\otimes_i) \leq 1. \tag{3.20}$$

Then, the proportional grey picture fuzzy number(PGPFN) can be expressed as follows:

$$\begin{aligned} P_g = \{< \otimes_i; & \frac{\hat{g}_{k_1}(\otimes_i)}{\hat{g}_{k_1}(\otimes_i)+\hat{g}_{k_2}(\otimes_i)+\hat{g}_{k_3}(\otimes_i)+\hat{g}_{k_4}(\otimes_i)}, \frac{\hat{g}_{k_2}(\otimes_i)}{\hat{g}_{k_1}(\otimes_i)+\hat{g}_{k_2}(\otimes_i)+\hat{g}_{k_3}(\otimes_i)+\hat{g}_{k_4}(\otimes_i)}, \\ & \frac{\hat{g}_{k_3}(\otimes_i)}{\hat{g}_{k_1}(\otimes_i)+\hat{g}_{k_2}(\otimes_i)+\hat{g}_{k_3}(\otimes_i)+\hat{g}_{k_4}(\otimes_i)} > | \otimes_i \in g^{\pm}\}. \end{aligned} \tag{3.21}$$

Definition 11. *For any PGPFN, $P_g = (\mu_g(\otimes_i), \pi_g(\otimes_i), \nu_g(\otimes_i))$, its score function $S(P_g)$ and accuracy function $h(P_g)$ can be defined as follows:*

$$S(P_g) = \mu_g - \upsilon_g, S(P_g) \in [-1, 1]; \tag{3.22}$$

$$h(P_g) = \mu_g + \pi_g + \upsilon_g, h(P_g) \in [0, 1]. \tag{3.23}$$

Definition 12. *Let P_g and Q_g represent two PGPFNs. The rule for comparing these two numbers is defined as follows:*

(1) If $s(P_g) > s(Q_g)$, then $P_g > Q_g$;

(2) If $s(P_g) = s(Q_g)$, then

(i) If $h(P_g) > h(Q_g)$, then $P_g > Q_g$;

(ii) If $h(P_g) > h(Q_g)$, then $P_g \sim Q_g$.

3.1. Arithmetic operations with PGPFS

Let A_g and B_g be two proportional grey picture fuzzy sets (PGPFS), and some of their algorithms are as follows:

$$\begin{aligned} A_g + B_g =< & \frac{\hat{g}_{Ak_1}(\otimes_i)}{\hat{g}_{Ak_1}(\otimes_i)+\hat{g}_{Ak_2}(\otimes_i)+\hat{g}_{Ak_3}(\otimes_i)+\hat{g}_{Ak_4}(\otimes_i)} + \frac{\hat{g}_{Bk_1}(\otimes_i)}{\hat{g}_{Bk_1}(\otimes_i)+\hat{g}_{Bk_2}(\otimes_i)+\hat{g}_{Bk_3}(\otimes_i)+\hat{g}_{Bk_4}(\otimes_i)} \\ & - \frac{\hat{g}_{Ak_1}(\otimes_i)}{\hat{g}_{Ak_1}(\otimes_i)+\hat{g}_{Ak_2}(\otimes_i)+\hat{g}_{Ak_3}(\otimes_i)+\hat{g}_{Ak_4}(\otimes_i)} \times \frac{\hat{g}_{Bk_1}(\otimes_i)}{\hat{g}_{Bk_1}(\otimes_i)+\hat{g}_{Bk_2}(\otimes_i)+\hat{g}_{Bk_3}(\otimes_i)+\hat{g}_{Bk_4}(\otimes_i)}, \\ & \frac{\hat{g}_{Ak_2}(\otimes_i)}{\hat{g}_{Ak_1}(\otimes_i)+\hat{g}_{Ak_2}(\otimes_i)+\hat{g}_{Ak_3}(\otimes_i)+\hat{g}_{Ak_4}(\otimes_i)} \times \frac{\hat{g}_{Bk_2}(\otimes_i)}{\hat{g}_{Bk_1}(\otimes_i)+\hat{g}_{Bk_2}(\otimes_i)+\hat{g}_{Bk_3}(\otimes_i)+\hat{g}_{Bk_4}(\otimes_i)}, \\ & \frac{\hat{g}_{Ak_3}(\otimes_i)}{\hat{g}_{Ak_1}(\otimes_i)+\hat{g}_{Ak_2}(\otimes_i)+\hat{g}_{Ak_3}(\otimes_i)+\hat{g}_{Ak_4}(\otimes_i)} \times \frac{\hat{g}_{Bk_3}(\otimes_i)}{\hat{g}_{Bk_1}(\otimes_i)+\hat{g}_{Bk_2}(\otimes_i)+\hat{g}_{Bk_3}(\otimes_i)+\hat{g}_{Bk_4}(\otimes_i)} >; \end{aligned} \tag{3.24}$$

$$\begin{aligned} A_g \times B_g =< & \frac{\hat{g}_{Ak_1}(\otimes_i)}{\hat{g}_{Ak_1}(\otimes_i)+\hat{g}_{Ak_2}(\otimes_i)+\hat{g}_{Ak_3}(\otimes_i)+\hat{g}_{Ak_4}(\otimes_i)} \times \frac{\hat{g}_{Bk_1}(\otimes_i)}{\hat{g}_{Bk_1}(\otimes_i)+\hat{g}_{Bk_2}(\otimes_i)+\hat{g}_{Bk_3}(\otimes_i)+\hat{g}_{Bk_4}(\otimes_i)}, \\ & \frac{\hat{g}_{Ak_2}(\otimes_i)}{\hat{g}_{Ak_1}(\otimes_i)+\hat{g}_{Ak_2}(\otimes_i)+\hat{g}_{Ak_3}(\otimes_i)+\hat{g}_{Ak_4}(\otimes_i)} + \frac{\hat{g}_{Bk_2}(\otimes_i)}{\hat{g}_{Bk_1}(\otimes_i)+\hat{g}_{Bk_2}(\otimes_i)+\hat{g}_{Bk_3}(\otimes_i)+\hat{g}_{Bk_4}(\otimes_i)} - \\ & \frac{\hat{g}_{Ak_2}(\otimes_i)}{\hat{g}_{Ak_1}(\otimes_i)+\hat{g}_{Ak_2}(\otimes_i)+\hat{g}_{Ak_3}(\otimes_i)+\hat{g}_{Ak_4}(\otimes_i)} \times \frac{\hat{g}_{Bk_2}(\otimes_i)}{\hat{g}_{Bk_1}(\otimes_i)+\hat{g}_{Bk_2}(\otimes_i)+\hat{g}_{Bk_3}(\otimes_i)+\hat{g}_{Bk_4}(\otimes_i)}, \\ & \frac{\hat{g}_{Ak_3}(\otimes_i)}{\hat{g}_{Ak_1}(\otimes_i)+\hat{g}_{Ak_2}(\otimes_i)+\hat{g}_{Ak_3}(\otimes_i)+\hat{g}_{Ak_4}(\otimes_i)} + \frac{\hat{g}_{Bk_3}(\otimes_i)}{\hat{g}_{Bk_1}(\otimes_i)+\hat{g}_{Bk_2}(\otimes_i)+\hat{g}_{Bk_3}(\otimes_i)+\hat{g}_{Bk_4}(\otimes_i)} - \\ & \frac{\hat{g}_{Ak_3}(\otimes_i)}{\hat{g}_{Ak_1}(\otimes_i)+\hat{g}_{Ak_2}(\otimes_i)+\hat{g}_{Ak_3}(\otimes_i)+\hat{g}_{Ak_4}(\otimes_i)} \times \frac{\hat{g}_{Bk_3}(\otimes_i)}{\hat{g}_{Bk_1}(\otimes_i)+\hat{g}_{Bk_2}(\otimes_i)+\hat{g}_{Bk_3}(\otimes_i)+\hat{g}_{Bk_4}(\otimes_i)} >; \end{aligned} \tag{3.25}$$

$$\begin{aligned} \lambda A_g =< & 1-\Big(1-\frac{\hat{g}_{Ak_1}(\otimes_i)}{\hat{g}_{Ak_1}(\otimes_i)+\hat{g}_{Ak_2}(\otimes_i)+\hat{g}_{Ak_3}(\otimes_i)+\hat{g}_{Ak_4}(\otimes_i)}\Big)^{\lambda}, \Big(\frac{\hat{g}_{Ak_2}(\otimes_i)}{\hat{g}_{Ak_1}(\otimes_i)+\hat{g}_{Ak_2}(\otimes_i)+\hat{g}_{Ak_3}(\otimes_i)+\hat{g}_{Ak_4}(\otimes_i)}\Big)^{\lambda}, \\ & \Big(\frac{\hat{g}_{Ak_3}(\otimes_i)}{\hat{g}_{Ak_1}(\otimes_i)+\hat{g}_{Ak_2}(\otimes_i)+\hat{g}_{Ak_3}(\otimes_i)+\hat{g}_{Ak_4}(\otimes_i)}\Big)^{\lambda} >; \end{aligned} \tag{3.26}$$

$$\begin{aligned} {A_g}^{\lambda} =< & \Big(\frac{\hat{g}_{Ak_1}(\otimes_i)}{\hat{g}_{Ak_1}(\otimes_i)+\hat{g}_{Ak_2}(\otimes_i)+\hat{g}_{Ak_3}(\otimes_i)+\hat{g}_{Ak_4}(\otimes_i)}\Big)^{\lambda}, 1-\Big(1-\frac{\hat{g}_{Ak_2}(\otimes_i)}{\hat{g}_{Ak_1}(\otimes_i)+\hat{g}_{Ak_2}(\otimes_i)+\hat{g}_{Ak_3}(\otimes_i)+\hat{g}_{Ak_4}(\otimes_i)}\Big)^{\lambda}, \\ & 1-\Big(1-\frac{\hat{g}_{Ak_3}(\otimes_i)}{\hat{g}_{Ak_1}(\otimes_i)+\hat{g}_{Ak_2}(\otimes_i)+\hat{g}_{Ak_3}(\otimes_i)+\hat{g}_{Ak_4}(\otimes_i)}\Big)^{\lambda} > \end{aligned} ; \tag{3.27}$$

$$\begin{aligned} {A_g}^{C} =< & \frac{\hat{g}_{Ak_3}(\otimes_i)}{\hat{g}_{Ak_1}(\otimes_i)+\hat{g}_{Ak_2}(\otimes_i)+\hat{g}_{Ak_3}(\otimes_i)+\hat{g}_{Ak_4}(\otimes_i)}, \frac{\hat{g}_{Ak_2}(\otimes_i)}{\hat{g}_{Ak_1}(\otimes_i)+\hat{g}_{Ak_2}(\otimes_i)+\hat{g}_{Ak_3}(\otimes_i)+\hat{g}_{Ak_4}(\otimes_i)}, \\ & \frac{\hat{g}_{Ak_1}(\otimes_i)}{\hat{g}_{Ak_1}(\otimes_i)+\hat{g}_{Ak_2}(\otimes_i)+\hat{g}_{Ak_3}(\otimes_i)+\hat{g}_{Ak_4}(\otimes_i)} > \end{aligned} . \tag{3.28}$$

3.2. Tensor-based proportional grey picture fuzzy entropy

According to Deng & Wang [30] and Xin & Ying [31], fuzziness can be used to measure uncertainty. Fuzziness is related to the gap between the PGPFS and $\{0.25, 0.25, 0.25\}$. Additionally, some researchers believe that the grey scale similarly conveys uncertainty, thereby responding to the greyness of the information [32–34]. For these reasons, and inspired by Xin &

Ying [31], we introduce the fuzziness and greyness of PGPFSs. The fuzziness is borrowed from Thao [19]'s entropy measure, which is defined as follows:

$$F(P_g) = 1-\frac{1}{n}\sum_{i=1}^{n}\frac{2}{3}(|\mu_g(\otimes_i) - 0.25| + |\nu_g(\otimes_i) - 0.25| + |\pi_g(\otimes_i) - 0.25| + |r_g(\otimes_i) - 0.25|). \quad (3.29)$$

$$G(P_g) = \frac{(\overline{k_1} - \underline{k_1}) + (\overline{k_2} - \underline{k_2}) + (\overline{k_3} - \underline{k_3}) + (\overline{k_4} - \underline{k_4})}{4(d_{\max} - d_{\min})}. \quad (3.30)$$

In the scenario where the expression is "approximately k times the temporary unit e," it is specified that

$$G(P_g) = 0. \quad (3.31)$$

Drawing on the axiomatic definitions of entropy by Thao [19] and Xin & Ying [31], we provide an axiomatic definition of entropy with respect to PGPFS.

Definition 13. *Let $E(P_g)$ denote the proportional grey picture fuzzy entropy(PGPFE), which possesses the following characteristics:*

(1) $E(P_g) = 0 \Leftrightarrow P_g$ is a crisp set;
(2) $E(P_g) = 1 \Leftrightarrow \mu_g(\otimes_i) = \nu_g(\otimes_i) = \pi_g(\otimes_i) = 0.25$;
(3) If $F(P_{g1}) < F(P_{g2})$ and $G(P_{g1}) < G(P_{g2})$, then $E(P_{g1}) < E(P_{g2})$;
(4) $E(P_g^C) = E(P_g)$.

3.2.1. Entropy operations

Combining the fuzzy degree and grey degree, the mathematical expression for the PGPFE is presented in Definition 14.

Definition 14. *Let $E(P_g)$ denote the proportional grey picture fuzzy entropy, which can be expressed as follows:*

$$E(P_g) = \frac{\partial F(P_g) + (1-\partial)G(P_g)}{\partial + (1-\partial)G(P_g)}. \quad (3.32)$$

Proof.

Since $(\overline{k_1} - \underline{k_1})$, $(\overline{k_2} - \underline{k_2})$, $(\overline{k_3} - \underline{k_3})$, and $(\overline{k_4} - \underline{k_4})$ are all non-negative, $(d_{\max} - d_{\min})$ is also non-negative, thus leading to the conclusion that $G(P_g) \geq 0$. As $0 \leq \mu, \nu, \pi, r \leq 1$, $F(P_g)$ is also non-negative. From this reasoning, it follows straight forwardly that $0 \leq E(P_g) \leq 1$.

(1) Suppose $E(P_g) = 0$, since ∂ is non-zero; it follows that both $F(P_g)$ and $G(P_g)$ are equal to 0:

$$\begin{cases} 1 - \frac{2}{3}(|(\mu_g(\otimes_i) - 0.25| + |\upsilon_g(\otimes_i) - 0.25| + |\pi_g(\otimes_i) - 0.25| + |r_g(\otimes_i) - 0.25|) = 0 \\ \frac{(\overline{k_1}-\underline{k_1})+(\overline{k_2}-\underline{k_2})+(\overline{k_3}-\underline{k_3})+(\overline{k_4}-\underline{k_4})}{4(d_{\max}-d_{\min})} = 0 \end{cases}.$$

Consequently,

$$\begin{cases} \mu_g(\otimes_i) = 1 \\ \upsilon_g(\otimes_i) = 0 \\ \pi_g(\otimes_i) = 0 \end{cases} \text{or} \begin{cases} \mu_g(\otimes_i) = 0 \\ \upsilon_g(\otimes_i) = 1 \\ \pi_g(\otimes_i) = 0 \end{cases} \text{or} \begin{cases} \mu_g(\otimes_i) = 0 \\ \upsilon_g(\otimes_i) = 0 \\ \pi_g(\otimes_i) = 1 \end{cases}.$$

Conversely, if $\mu_g(\otimes_i) = 1$ and $\upsilon_g(\otimes_i) = \pi_g(\otimes_i) = 0$, then $r_g(\otimes_i) = 0$; we can deduce that

$F_g = 1 - \frac{2}{3} \times (|1 - 0.25| + |0 - 0.25| + |0 - 0.25| + |0 - 0.25|) = 0$.

At this point,

$\mu_g(\otimes_i) = \frac{\hat{g}_{k_1}(\otimes_i)}{\hat{g}_{k_1}(\otimes_i)+\hat{g}_{k_2}(\otimes_i)+\hat{g}_{k_3}(\otimes_i)+\hat{g}_{k_4}(\otimes_i)} = 1$;

$\pi_g(\otimes_i) = \frac{\hat{g}_{k_2}(\otimes_i)}{\hat{g}_{k_1}(\otimes_i)+\hat{g}_{k_2}(\otimes_i)+\hat{g}_{k_3}(\otimes_i)+\hat{g}_{k_4}(\otimes_i)} = 0$;

$\upsilon_g(\otimes_i) = \frac{\hat{g}_{k_3}(\otimes_i)}{\hat{g}_{k_1}(\otimes_i)+\hat{g}_{k_2}(\otimes_i)+\hat{g}_{k_3}(\otimes_i)+\hat{g}_{k_4}(\otimes_i)} = 0$;

$r_g(\otimes_i) = 1 - \frac{\hat{g}_{k_1}(\otimes_i)+\hat{g}_{k_2}(\otimes_i)+\hat{g}_{k_3}(\otimes_i)}{\hat{g}_{k_1}(\otimes_i)+\hat{g}_{k_2}(\otimes_i)+\hat{g}_{k_3}(\otimes_i)+\hat{g}_{k_4}(\otimes_i)} = 0$.

Then, we can obtain the following:

$$\begin{cases} \hat{g}_{k_1}(\otimes_i) = k \\ \hat{g}_{k_2}(\otimes_i) = 0 \\ \hat{g}_{k_3}(\otimes_i) = 0 \\ \hat{g}_{k_4}(\otimes_i) = 0 \end{cases} .$$

From the given conditions, it can be deduced that k_1 is a specific, nonzero value, while k_2–k_4 are both zero. Consequently, this leads to the conclusion that $G(P_g) = 0$.

If $\upsilon_g(\otimes_i) = 1$ and $\mu_g(\otimes_i) = \pi_g(\otimes_i) = 0$, or $\pi_g(\otimes_i) = 1$ and $\mu_g(\otimes_i) = \upsilon_g(\otimes_i) = 0$, then the reasoning applied earlier holds true under these conditions as well.

Therefore, Condition (1) holds.

(2) Suppose $E(P_g) = 1$:

$\frac{\partial F(P_g)+(1-\partial)G(P_g)}{\partial+(1-\partial)G(P_g)} = 1$, then, $F(P_g) = 1$; this implies that $1 - \frac{2}{3}(|\mu_g(\otimes_i) - 0.25| + |\nu_g(\otimes_i) - 0.25| + |\pi_g(\otimes_i) - 0.25| + |r_g(\otimes_i) - 0.25|) = 1$.

Furthermore, we obtain the following: $(|\mu_g(\otimes_i) - 0.25| + |\nu_g(\otimes_i) - 0.25| + |\pi_g(\otimes_i) - 0.25| + |r_g(\otimes_i) - 0.25|) = 0$, and subsequently $\mu_g(\otimes_i) = \pi_g(\otimes_i) = \upsilon_g(\otimes_i) = 0.25$.

Therefore, Condition (2) is satisfied.

(3) Let $F(P_g) = F$ and $G(P_g) = G$; Eq (3.22) can be expressed as follows:

$E(F, G) = \frac{\partial F+(1-\partial)G}{\partial+(1-\partial)G}$.

Calculate the partial derivatives of $E(F, G)$ with respect to F and G, respectively.

$\frac{\partial E}{\partial F} = \frac{\partial}{\partial+(1-\partial)G} \geq 0$, and thus $E(F, G)$ is monotonically increasing with respect to F.

$\frac{\partial E}{\partial G} = \frac{a(1-a)(1-F)}{[a+(1-a)G]^2} \geq 0$, and thus $E(F, G)$ is monotonically increasing with respect to G.

In summary, $E(F, G)$ is a monotonically increasing function with respect to both F and G. Thus, if $F(P1) < F(P2)$ and $G(P1) < G(P2)$, then it follows that $E(P1) \leq E(P2)$.

Therefore, Condition (3) holds.

(4) Since ${P_g}^C = (\nu_g(\otimes_i), \pi_g(\otimes_i), \mu_g(\otimes_i))$, it follows that

$$F_g^C = 1 - \frac{1}{n}\sum_{i=1}^{n}\frac{4}{3}(|\nu_g(\otimes_i) - 0.25| + |\mu_g(\otimes_i) - 0.25| + |\pi_g(\otimes_i) - 0.25| + |r_g(\otimes_i) - 0.25|)$$
$$= 1 - \frac{1}{n}\sum_{i=1}^{n}\frac{4}{3}(|\mu_g(\otimes_i) - 0.25| + |\nu_g(\otimes_i) - 0.25| + |\pi_g(\otimes_i) - 0.25| + |r_g(\otimes_i) - 0.25|)$$
$$= F_g$$

We attain the following: $E({P_g}^C) = E(P_g)$.

Therefore, Condition (4) holds.

3.2.2. Tensor-based entropy for PGPFS

When faced with multiple experts making decisions, the above entropy operation cannot directly derive the entropy of a certain index. Therefore, it is necessary to introduce the "expert" dimension. We borrow the concept of a tensor and propose the tensor-based PGPFE, which is applicable to the characteristics of multi-dimensional data.

Definition 15. *Let $\tilde{A}_{IF} = (a_{i_1 i_2 \cdots i_m})_{n_1\times n_2\times\cdots\times n_m}$ be defined with elements $a_{i_1 i_2\cdots i_m} = (\mu_{i_1 i_2\cdots i_m}, \pi_{i_1 i_2\cdots i_m}, \nu_{i_1 i_2\cdots i_m})$, where $\mu_{i_1 i_2\cdots i_m}, \nu_{i_1 i_2\cdots i_m}, \pi_{i_1 i_2\cdots i_m} \in [0,1]$. Additionally, they satisfy the following condition:*

$$\mu_{i_1 i_2\cdots i_m} + \nu_{i_1 i_2\cdots i_m} + \pi_{i_1 i_2\cdots i_m} \leq 1. \tag{3.33}$$

Then, $\tilde{A}_{IF}$ is defined as an m-order proportional grey picture fuzzy tensor(PGPFT).

Note that we denote the set of tensors of order m as $T_{IF} = (m, n_1 \times n_2 \times \cdots \times n_m)$.

Definition 16. *Let $\tilde{A}_{IF} = (a_{i_1 i_2\cdots i_m})_{n_1\times n_2\times\cdots\times n_m} \in T_{IF}(m, n_1 \times n_2 \times \cdots \times n_m)$; then, the expression for the i_1st component of PGPFE is as follows:*

$$E_{i_1}(a_{i_1 1\cdots 1}, \cdots, a_{i_1 i_2\cdots i_m}, \cdots, a_{i_1 n_2\cdots n_m}) = \frac{1}{n_2}\sum_{i_2=1}^{n_2}\cdots\frac{1}{n_m}\sum_{i_m=1}^{n_m}E(a_{i_1 i_2\cdots i_m}). \tag{3.34}$$

Then, $\tilde{A}_{IF}$ is defined as an m-order proportional grey picture fuzzy tensor.

Theorem 1. *When $m = 3$, PPGNFE is a 3-order tensor and Eq (3.34) can be simplified as follows:*

$$E(\tilde{P}_g) = \frac{1}{n_2}\sum_{i=1}^{n_2}\frac{1}{n_3}\sum_{j=1}^{n_3}E(P_g)_{ij}. \tag{3.35}$$

3.3. Tensor-based aggregation operators for PGPFS

Definition 17. *Let $\tilde{A}_{IF} = (a_{i_1 i_2\cdots i_m})_{n_1\times n_2\times\cdots\times n_m} \in T_{IF}(m, n_1 \times n_2 \times \cdots \times n_m)$; then, the expression for the i_1st component of the proportional grey picture fuzzy weighted aggregation (PGPFWA) operator, which is a mapping $PGPFWAIF^{n_2\times\cdots\times n_m} \to IF^{n_1}$ such that:*

$$\begin{aligned} &PGPFWA_{i_1}(a_{i_1 1\ldots 1}, \cdots, a_{i_1 i_2\ldots i_m}, \cdots, a_{i_1 n_2\ldots n_m}) \\ &= (A_{IF} \circ W_2 \circ W_3 \circ \cdots \circ W_m)_{i_1} \\ &= \sum_{i_2=1}^{n_2}\cdots\sum_{i_m=1}^{n_m} a_{i_1 i_2\ldots i_m} \times w_{i_2}^2 \times\cdots\times w_{i_m}^m \end{aligned}, \tag{3.36}$$

where $W_2 = (w_1^2, ..., w_{i_2}^2, ..., w_{n_2}^2)^T, \cdots, W_m = (w_1^m, ..., w_{i_m}^m, ..., w_{n_m}^m)^T$ *are the weight vectors for* $a_{:i_2:\cdots:}(i_2 = 1, 2, ..., n_2), \cdots, a_{:\cdots i_m}(i_m = 1, 2, ..., n_m)$*, respectively, which satisfy the following conditions:*

$$\sum_{i_2=1}^{n_2} w_{i_2}^2 = 1 w_{i_2}^2 \geq 0 \sum_{i_m=1}^{n_m} w_{i_m}^m = 1 w_{i_m}^m \geq 0. \tag{3.37}$$

Based on the provided definition, we can deduce the following theorem:

Theorem 2. *Let* $\tilde{A}_{IF} = (a_{i_1 i_2 \cdots i_m})_{n_1 \times n_2 \times \cdots \times n_m}$ *be defined such that its elements are* $a_{i_1 i_2 \cdots i_m} = (\mu_{i_1 i_2 \cdots i_m}, \pi_{i_1 i_2 \cdots i_m}, \nu_{i_1 i_2 \cdots i_m})$*, with* $\mu_{i_1 i_2 \cdots i_m}, \nu_{i_1 i_2 \cdots i_m}, \pi_{i_1 i_2 \cdots i_m} \in [0, 1]$*. Then, the integrated value expression for Eq (3.36) is as follows:*

$$\begin{aligned} & PGPFWA_{i_1}(a_{i_1 1 \ldots 1}, \cdots, a_{i_1 i_2 \ldots i_m}, \cdots, a_{i_1 n_2 \ldots n_m}) \\ &= \left(1 - \prod_{i_2=1}^{n_2} \cdots \prod_{i_m=1}^{n_m} (1 - \mu_{i_1 i_2 \cdots i_m}{}^2)^{w_{i_2}^2 \cdots w_{i_m}^m}, \prod_{i_2=1}^{n_2} \cdots \prod_{i_m=1}^{n_m} (\pi_{i_1 i_2 \cdots i_m})^{w_{i_2}^2 \cdots w_{i_m}^m}, \prod_{i_2=1}^{n_2} \cdots \prod_{i_m=1}^{n_m} (\nu_{i_1 i_2 \cdots i_m})^{w_{i_2}^2 \cdots w_{i_m}^m}\right), \end{aligned} \tag{3.38}$$

where $W_2 = (w_1^2, ..., w_{i_2}^2, ..., w_{n_2}^2)^T, \cdots, W_m = (w_1^m, ..., w_{i_m}^m, ..., w_{n_m}^m)^T$ *are the weight vectors for* $a_{:i_2:\cdots:}(i_2 = 1, 2, ..., n_2), \cdots, a_{:\cdots i_m}(i_m = 1, 2, ..., n_m)$*, respectively, which satisfy the following conditions:*

$$\sum_{i_2=1}^{n_2} w_{i_2}^2 = 1, w_{i_2}^2 \geq 0, \sum_{i_m=1}^{n_m} w_{i_m}^m = 1, w_{i_m}^m \geq 0. \tag{3.39}$$

Let $\tilde{A}_{IF} = (a_{i_1 i_2 \cdots i_m})_{n_1 \times n_2} \in T_{IF}(2, n_1 \times n_2)$ be a 2-order PGPFT; then, Eq (3.38) becomes the following:

$$PGPFWA_{i_1}(a_1, a_2, \cdots, a_n) = (1 - \prod_{i=1}^{n} (1 - \mu_g{}^2(\otimes_i))^{w_i}, \prod_{i=1}^{n} \pi_g(\otimes_i)^{w_i} \prod_{i=1}^{n} \nu_g(\otimes_i)^{w_i}), \tag{3.40}$$

where

$$\sum_{i=1}^{n} w_i = 1, w_i \in [0, 1], (i = 1, 2, ..., n). \tag{3.41}$$

Let $\tilde{A}_{IF} = (a_{i_1 i_2 \cdots i_m})_{n_1 \times n_2 \times n_3} \in T_{IF}(3, n_1 \times n_2 \times n_3)$ be a 3-order PGPFT; then, the following (3.38) becomes the following:

$$\begin{aligned} & PGPFWA_{i_1}(a_{i_1 i_2 i_3})_{n_1 \times n_2 \times n_3} \\ &= \left(1 - \prod_{i_2=1}^{n_2} \prod_{i_3=1}^{n_3} (1 - \mu_g{}^2(\otimes_i))^{w_{i2}^2 w_{i3}^3}, \prod_{i_2=1}^{n_2} \prod_{i_3=1}^{n_3} \pi_g(\otimes_i)^{w_{i2}^2 w_{i3}^3}, \prod_{i_2=1}^{n_2} \prod_{i_3=1}^{n_3} \nu_g(\otimes_i)^{w_{i2}^2 w_{i3}^3}\right), \end{aligned} \tag{3.42}$$

where

$$\sum_{i_2=1}^{n_2} w_{i_2}^2 = 1, w_{i_2}^2 \geq 0, \sum_{i_3=1}^{n_3} w_{i_3}^3 = 1, w_{i_3}^3 \geq 0. \tag{3.43}$$

4. Aggregation decision method with PGPFS

In this section, the algorithm will be designed using PGPFWA, thereby focusing on solving proportional grey picture fuzzy decision making problems that involve criteria, substitution schemes, and experts with 3-dimensional data features. The flow chart of the proportional grey picture fuzzy aggregation decision making method is shown in Figure 3. The specific steps of the algorithm are as follows:

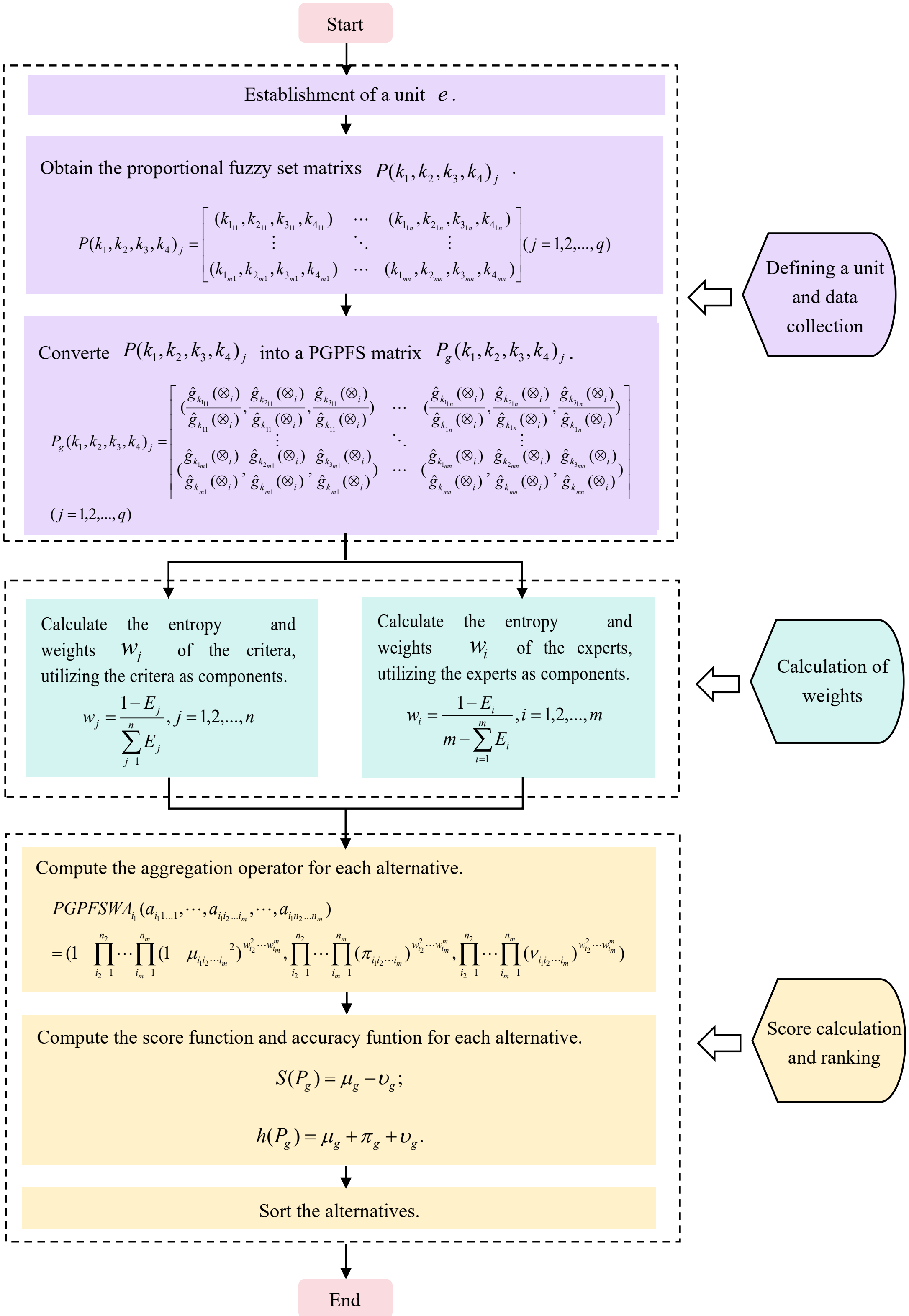

Figure 3. Flowchart of the aggregation decision making method with PGPFS.

Step 1. Based on the realistic problem context, a temporary unit is first established; then, the proportional decision matrices of q experts for m alternatives over n criteria are formulated. These matrices can be represented by (4.1):

$$P(k_1,k_2,k_3,k_4)_j = \begin{bmatrix} (k_{1_{11}},k_{2_{11}},k_{3_{11}},k_{4_{11}}) & \cdots & (k_{1_{1n}},k_{2_{1n}},k_{3_{1n}},k_{4_{1n}}) \\ \vdots & \ddots & \vdots \\ (k_{1_{m1}},k_{2_{m1}},k_{3_{m1}},k_{4_{m1}}) & \cdots & (k_{1_{mn}},k_{2_{mn}},k_{3_{mn}},k_{4_{mn}}) \end{bmatrix} (j=1,2,...,q). \quad (4.1)$$

Step 2. The above proportional decision matrix is transformed into PGPFS matrices through Eqs (3.16)–(3.18):

$$P_g(k_1,k_2,k_3,k_4)_j = \begin{bmatrix} (\frac{\hat{g}k_{1_{11}}(\otimes_i)}{\hat{g}k_{11}(\otimes_i)},\frac{\hat{g}k_{2_{11}}(\otimes_i)}{\hat{g}k_{11}(\otimes_i)},\frac{\hat{g}k_{3_{11}}(\otimes_i)}{\hat{g}k_{11}(\otimes_i)}) & \cdots & (\frac{\hat{g}k_{1_{1n}}(\otimes_i)}{\hat{g}k_{1n}(\otimes_i)},\frac{\hat{g}k_{2_{1n}}(\otimes_i)}{\hat{g}k_{1n}(\otimes_i)},\frac{\hat{g}k_{3_{1n}}(\otimes_i)}{\hat{g}k_{1n}(\otimes_i)}) \\ \vdots & \ddots & \vdots \\ (\frac{\hat{g}k_{1_{m1}}(\otimes_i)}{\hat{g}k_{m1}(\otimes_i)},\frac{\hat{g}k_{2_{m1}}(\otimes_i)}{\hat{g}k_{m1}(\otimes_i)},\frac{\hat{g}k_{3_{m1}}(\otimes_i)}{\hat{g}k_{m1}(\otimes_i)}) & \cdots & (\frac{\hat{g}k_{1_{mn}}(\otimes_i)}{\hat{g}k_{mn}(\otimes_i)},\frac{\hat{g}k_{2_{mn}}(\otimes_i)}{\hat{g}k_{mn}(\otimes_i)},\frac{\hat{g}k_{3_{mn}}(\otimes_i)}{\hat{g}k_{mn}(\otimes_i)}) \end{bmatrix}. \quad (4.2)$$

$(j=1,2,...,q)$

Step 3. Following Eq (3.35), the entropy is computed, with criteria as components; subsequently, the weight of each criterion is determined using Eq (4.3):

$$w_j = \frac{1-E_j}{\sum\limits_{j=1}^{n} E_j}, j=1,2,...,n. \quad (4.3)$$

Step 4. Following Eq (3.35), the entropy is computed, with experts as components; subsequently, the weight of each attribute is determined using Eq (4.4):

$$w_i = \frac{1-E_i}{m-\sum\limits_{i=1}^{m} E_i}, i=1,2,...,m. \quad (4.4)$$

Step 5. Aggregate all elements in accordance with Eq (3.42) to obtain the fuzzy values of PGPFNs to the alternatives.
Step 6. Based on Definitions 11, the score function and the accuracy for each fuzzy number are calculated.
Step 7. In accordance with Definition 12, the alternatives are ranked to identify the optimal solution.

5. Application

In this section, we apply the PPGNFS and aggregation decision-making method to a practical scenario that involves movie recommendations. We detail the decision-making steps of the proposed model, conduct a sensitivity analysis of critical parameters, and compare its performance with other models.

5.1. *Case analysis*

In recent years, China's rapid industrialization and urbanization have garnered global attention. However, the long-standing model of rough development has exacted a substantial environmental toll. In early 2024, extensive haze conditions emerged in North China, Huanghuai, Jianghuai, and other regions. Haze not only threatens public health, but also negatively impacts the economy and society, making it one of China's most significant environmental pollution issues since the 21st century. The frequent occurrence of haze has raised national concern. In 2023, the State Council issued the Action Plan for Continuous Improvement of Air Quality, thereby emphasizing the need for improved air quality, a reduction of heavily polluted weather, and addressing prominent atmospheric environmental problems through precise, scientific, and legal pollution control. Consequently, amidst frequent haze pollution incidents, there is an urgent need for effective haze management.

Using the methodology of this paper, three experts $E = \{E1, E2, E3\}$ from the relevant environmental protection departments formed a decision-making group to evaluate four haze control measures $N = \{N1, N2, N3, N4\}$ based on the following four indices [35]: $S1$ indicates the effect of the control measures on the formation of secondary particulate matter (aerosol); $S2$ indicates the effect of the control measures on the particle "hygroscopicity" of particulate matter, $S3$ represents the effect of the control measures on the emission level of PM2.5; and $S4$ represents the effect of the control measures on the regional transport of dust and particulate matter. $N1$ is designated as the assessment benchmark, termed the "unit", with its PGPFN in each criterion initially set to (1,1,1,1). The three experts assess other governance programs in relation to $N1$, and the assessment matrix of the decision-making experts is displayed in Tables 1–3.

Table 1. Initial proportional fuzzy number scoring matrix (E1).

	S1	S2	S3	S4
N1	(1,1,1,1)	(1,1,1,1)	(1,1,1,1)	(1,1,1,1)
N2	(0.5,0.75,2,0.2)	([2,3],[0,2],[0.2,0.4],[0.6,0.8])	([2,2.5],[1.5,2],[1.3,1.6],[0.5,1])	(1,[1.2,1.5],[1.7,2],0)
N3	([0.6,1],[0.1,1.2],0,0)	([0.5,2],[0.6,0.7],[3,5],0.75)	(1,[1,1.5],[1.6,2],[0.75,1.4])	([1,3],[0.3,0.4],[0.3,0.7].[0.5,1])
N4	([0.3,0.6],0.8,0.2,[2,3])	([1.3,1.5],[1,3],[0.3,1],[0.3,0.7])	([1,3],1.5,[0.2,0.3],0)	([0.75,0.8],[0.4,0.9],[0.3,1],[0.25,0.5])

Table 2. Initial proportional fuzzy number scoring matrix (E2).

	S1	S2	S3	S4
N1	(1,1,1,1)	(1,1,1,1)	(1,1,1,1)	(1,1,1,1)
N2	(5,[2.5,3],[2,3],[0.3,0.5])	([1,2.3],[1.2,1.5],[0.25,0.6],0)	([0.5,0.6],[0.5,2.5],[0.7,2],[0.3,0.6])	([1.2,2],[2,4],[0.05,1],[0.5,0.9])
N3	([3,4],[0.6,1.2],[1,3],[2,2.5])	(2,[0.8,1],[0.5,2],[0.2,0.4])	([0.9,1],[0.75,1.5],5,0)	([0.5,0.6],[3,4],[0.1,1],[0.3,0.6])
N4	([2,3],[0.6,1.6],[0.1,0.2],[0.4,0.7])	([0.2,3],[0.5,0.75],[0.3,0.6],[0.85,1])	([0.5,0.6],[0.75,2],[2,4],[0.4,1])	([0.5,1],[1,3],0,0)

Table 3. Initial proportional fuzzy number scoring matrix (E3).

	S1	S2	S3	S4
N1	(1,1,1,1)	(1,1,1,1)	(1,1,1,1)	(1,1,1,1)
N2	([0.7,1],[0.5,0.6],[0.5,1],[0.3,0.55])	([4,5],[1.5,2],[0.3,1],0)	(1,2,[0.3,0.4],[0.4,0.6])	([0.7,1],[1.5,2.4],[0.3,0.4],[3.5,4])
N3	([0.5,1],[0.5,0.7],[0.5,0.75],[0.4,0.5])	([2,4],[1.5,2],[0.3,1],[0.2,1])	([1.2,1.5],[1.25,1.6],[0.4,0.5],[2,3])	([1,1.2],[1.5,2],1,0)
N4	([0.6,1],[0.6,1],0.9,[0.25,0.4])	([1.5,2.5],0.3,[1,2],[0.25,0.3])	([1.2,2],[1.5,1.6],[0.25,1],[0.5,1])	([0.25,0.3],[0.5,1],[1,1.5],[0.4,1])

These matrices were converted to the corresponding PGPFNs, as shown in Tables 4–6.

Table 4. Proportional grey picture fuzzy set matrix (E1).

	S1			S2			S3			S4		
	μ	π	ν	μ	π	ν	μ	π	ν	μ	π	ν
N1	0.250000	0.250000	0.250000	0.250000	0.250000	0.250000	0.250000	0.250000	0.250000	0.250000	0.250000	0.250000
N2	0.144928	0.217391	0.579710	0.555556	0.222222	0.066667	0.362903	0.282258	0.233871	0.238095	0.321429	0.440476
N3	0.551724	0.448276	0.000000	0.187970	0.097744	0.601504	0.195122	0.243902	0.351220	0.555556	0.097222	0.138889
N4	0.113924	0.202532	0.050633	0.307692	0.439560	0.142857	0.533333	0.400000	0.066667	0.316327	0.265306	0.265306

Table 5. Proportional grey picture fuzzy set matrix (E2).

	S1			S2			S3			S4		
	μ	π	ν	μ	π	ν	μ	π	ν	μ	π	ν
N1	0.250000	0.250000	0.250000	0.250000	0.250000	0.250000	0.250000	0.250000	0.250000	0.250000	0.250000	0.250000
N2	0.469484	0.258216	0.234742	0.481752	0.394161	0.124088	0.142857	0.389610	0.350649	0.274678	0.515021	0.090129
N3	0.404624	0.104046	0.231214	0.449438	0.202247	0.280899	0.134276	0.159011	0.706714	0.108911	0.693069	0.108911
N4	0.588235	0.258824	0.035294	0.444444	0.173611	0.125000	0.097778	0.244444	0.533333	0.272727	0.727273	0.000000

Table 6. Proportional grey picture fuzzy set matrix (E3).

	S1			S2			S3			S4		
	μ	π	ν	μ	π	ν	μ	π	ν	μ	π	ν
N1	0.250000	0.250000	0.250000	0.250000	0.250000	0.250000	0.250000	0.250000	0.250000	0.250000	0.250000	0.250000
N2	0.439024	0.178862	0.243902	0.652174	0.253623	0.094203	0.259740	0.519481	0.090909	0.123188	0.282609	0.050725
N3	0.309278	0.247423	0.257732	0.500000	0.291667	0.108333	0.235808	0.248908	0.078603	0.285714	0.454545	0.259740
N4	0.283186	0.283186	0.318584	0.490798	0.073620	0.368098	0.353591	0.342541	0.138122	0.092437	0.252101	0.420168

Equation (3.35) to compute the indicator entropy and the Eq (4.3) utilized to compute the weights, the results of which are shown in Table 7. At this point, because it involves three vectors of governance measures-indicators-experts, the third-order tensor entropy is used, with $\partial = 0.5$ in this case. Similarly, the expert weights are calculated using the Eqs (3.35) and (4.4), as shown in Table 8.

Table 7. The criterion entropy and weight.

	Entropy	Weight
N1	0.636322	0.239155
N2	0.595048	0.266296
N3	0.646571	0.232415
N4	0.601378	0.262134

Table 8. The expert entropy and weight.

	Entropy	Weight
DM1	0.605616	0.345796
DM2	0.582513	0.366053
DM3	0.671360	0.288151

Equation (3.42) is used to calculate the aggregated PGPFN for each treatment measure, and the final results are shown in Table 9. According to Definitions 11 and 12, the final scores are calculated, as shown in Table 10. Therefore, the final ranking of haze management measures is $N3 > N4 > N2 > N1$.

Table 9. Computation of the final scores for alternatives.

	μ	π	ν	Score
N1	0.250000	0.257233	0.250000	-0.007233
N2	0.289051	0.165266	0.291738	0.123785
N3	0.357130	0.000000	0.269326	**0.357130**
N4	0.330255	0.000000	0.298762	0.330255

Table 10. Aggregated matrix.

	S1			S2			S3			S4		
N1	0.062500	0.250000	0.250000	0.062500	0.250000	0.250000	0.062500	0.250000	0.250000	0.062500	0.250000	0.250000
N2	0.149086	0.215732	0.321370	0.326799	0.281114	0.092024	0.074321	0.385137	0.195352	0.049435	0.360328	0.126280
N3	0.192736	0.225980	0.000000	0.167313	0.179317	0.263542	0.037358	0.212925	0.269207	0.143924	0.312877	0.157795
N4	0.159517	0.245767	0.082881	0.179937	0.177772	0.187323	0.147249	0.322343	0.169977	0.061777	0.365039	0.000000

5.2. Sensitivity analysis

In this section, a sensitivity analysis was performed on the balance parameter ∂, which was varied from 0.1 to 1 in steps of 0.1. Then, the weights of the indicators, the expert weights, and the scores of the haze management measures were calculated for different values of ∂. The calculation results are shown in Figures 4–6.

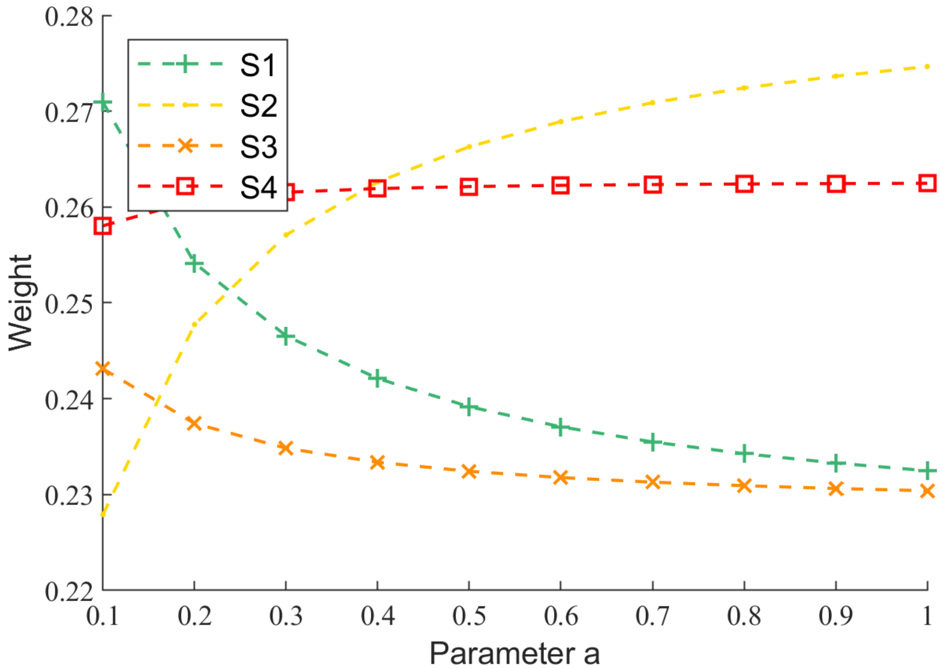

Figure 4. Curve of critera weight with ∂.

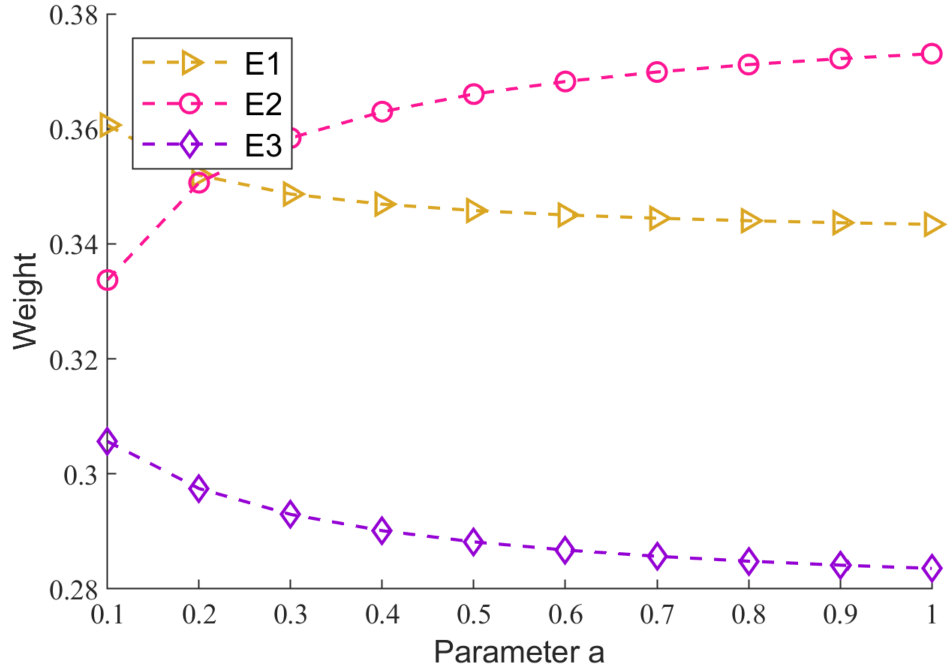

Figure 5. Curve of expert weight with ∂.

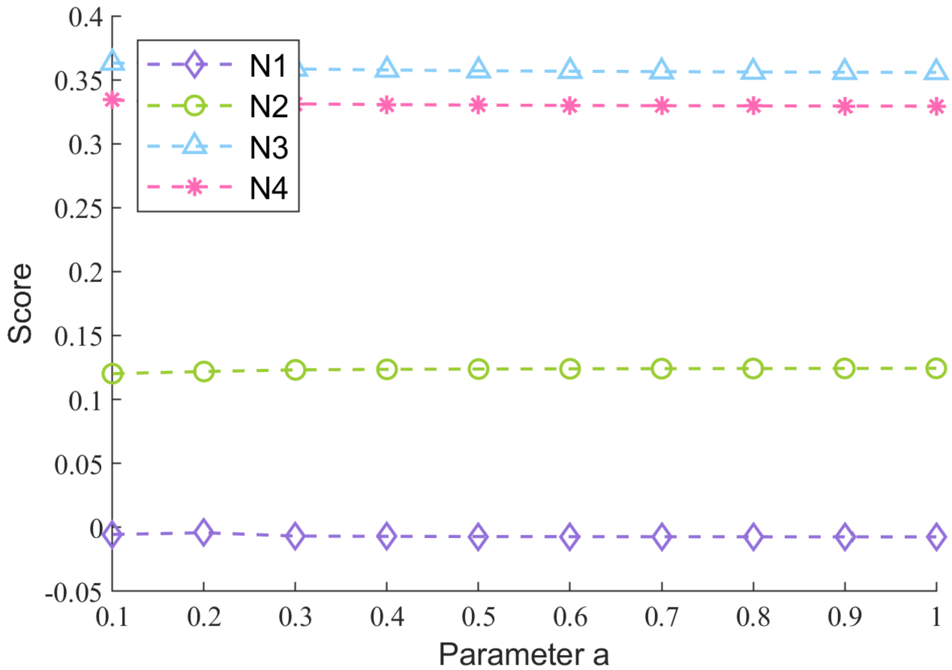

Figure 6. Curve of scores of the movies with ∂.

Figure 4 shows changes in the ordering of the criteria weights as the value of ∂ varies. When $\partial < 0.4$, the ordering of criteria weights fluctuates, while for $\partial > 0.4$, the ordering stabilizes at $S2 > S4 > S1 > S3$. This indicates that the entropy measure used in this study has a low sensitivity to the balancing parameter when calculating the weights of the indicators.

Specifically, the weights of $S2$ and $S4$ show an increasing trend, while the weights of $S1$ and $S3$ show a decreasing trend, thus suggesting that the parameters need to be set flexibly according to the actual situation.

Figure 5 demonstrates that the ranking of the expert weights changes with the value of ∂. When $\partial \leq 0.2$, the expert weights are ranked as $E1 > E2 > E3$. When $\partial > 0.2$, the expert weights are ranked as $E2 > E1 > E3$. This indicates that the entropy measure used in this study is less sensitive to the balance parameter when calculating the expert weights. However, as the value of ∂ increases, the weight values diverge more. Specifically, the value of $E2$ slowly increases, while the values of $E1$ and $E3$ slowly decrease, thus underscoring the importance of flexible parameterization.

Figure 6 shows that the final rankings of the four haze management scenarios remain unchanged with the increase in ∂, consistently maintaining the order $N3 > N4 > N2 > N1$. This indicates that the decision-making method used in this study is insensitive to the balance parameter.

5.3. Comparative analysis

To further verify the method's effectiveness, the outcomes of the proposed entropy measure and the multi-criteria decision-making method are compared with those of existing methods.

Since the PGPFS proposed in this paper is an extension of PFS, the entropy measure results are compared with those of the existing PFS [16, 19] (For ease of presentation, Nguyen's work [19] involves three entropy measures, referred to as W_{ET}, W_{EM}, and W_{ES}.). For a more authoritative validation, only the criteria weights are selected for the comparative analysis. As the existing methods cannot simultaneously handle multiple expert decision matrices simultaneously, we first aggregate the data from Tables 1–3 using Eq (3.40), given that this method does not require expert weights when determining multidimensional data. All experts are assigned equal weights during aggregation. The aggregated matrix is presented in Table 10.

Table 11 presents the results of the comparative analysis. First, despite differences in the criteria weight assignments, S2 consistently holds the highest weight. Second, Nguyen Xuan Thao's W_{ES} method fails to measure the criteria weights due to its reliance on logarithmic functions, making it incapable of handling cases with a parameter of 0 in the PFN. Third, S3 ranks second in weight across the three methods of Sunit Kumar, Nguyen Xuan Thao W_{ET}, and Nguyen Xuan Thao W_{EM}. In contrast, the entropy weighting method in this paper ranks S4 second. This difference may arise because the first three methods only account for fuzziness, while the method in this paper considers both fuzziness and the grey degree of the data. Therefore, the results in Table 11 demonstrate that the PGPFS entropy measurement method proposed in this paper better captures data uncertainty compared to existing methods, thus yielding more reasonable and effective criteria weights.

Table 11. Comparative analysis results of criteria weights.

	S1	S2	S3	S4	Rank
Sunit Kumar	0.243536	**0.261857**	0.256314	0.238293	Ws2>Ws3>Ws1>Ws4
Nguyen Xuan Thao(W_{ET})	0.248658	**0.273950**	0.256471	0.220921	Ws2>Ws3>Ws1>Ws4
Nguyen Xuan Thao(W_{EM})	0.238996	**0.260573**	0.259011	0.241420	Ws2>Ws3>Ws4>Ws1
Nguyen Xuan Thao(W_{ES})	Error	Error	Error	Error	-
Proposed method	0.239155	**0.266296**	0.232415	0.262134	Ws2>Ws4>Ws1>Ws3

To verify the rationality and effectiveness of the MCDM method in this paper, the method is compared with MCDM approaches based on the PFS environment. The PFS-MCDM method [20], which also employs an aggregation operator, along with the widely used PFS-TOPSIS [36], PFS-VIKOR methods [37], and the latest MCDM method, MEREC-MABAC [38], are chosen for comparison. To ensure more valid comparison results, Tables 4–6 are selected as the initial data matrices for these methods, as the transformed fuzzy gray numbers are equivalent to the Picture Fuzzy Sets. To avoid the influence of the indicator and the expert weights on the results, both indicator weights are set to the following:

$$\{w_{S1}, w_{S2}, w_{S3}, w_{S4}\} = \{0.239155, 0.266296, 0.232415, 0.262134\},$$

and the expert weights are set to the following:

$$\{w_{E1}, w_{E2}, w_{E3}\} = \{0.345796, 0.366053, 0.288151\}.$$

Additionally, other allocation parameters for the comparison methods are presented in Table 12.

Table 12. Main characteristics and parameterization of the methods.

Method	Number of DMs	Weight on DMs	Weight on criteria	Parameters
PFS-TOPSIS	≥ 1	known	known	-
PFS-VIKOR	≥ 1	unknown	unknown	The L_∂ norm $\partial = 1$; the level of uncertainty $t = 2$; the level of uncertainty $t = 2$; the weight of the strategy with the maximum overall utility $\tau = 0.5$.
PFS-MCDM	≥ 1	known	unknown	Attenuation factor of the losses $q = 2.5$.
MEREC-MABAC	≥ 1	known	unknown	Risk–benefit coefficient $\tau = 0.61$ and risk-loss coefficient $\delta = 0.69$;the degree of convexity of the gain and loss area value function $\varphi = \phi = 0.88$; loss aversion $\sigma = 2.25$.
Proposed method	≥ 1	unknown	unknown	∂=0.5

Table 13 presents the final comparison results. Despite the differences among various decision-making methods, the final outcomes of the best haze management programs exhibit a high level of consistency, with the N3 management program consistently ranking first across various assessment indicators. Second, the ranking results of the PFS-MCDM and PFS-TOPSIS methods completely align with those of the proposed method, thus demonstrating the reasonableness and effectiveness of this paper's approach. Third, the ranking of the PFSB-VIKOR method differs from that of the proposed method, likely due to its emphasis on image fuzzy similarity in determining the ranking of alternatives. Additionally, the MEREC-MABAC method yields results that differ from those of the other methods, as it accounts for the decision-makers' mental behavior regarding potential gains and losses. Different decision-making methods serve distinct functions and should be emphasized according to the specific context. The proposed method, which relies on the original data, consistently selects the same optimal solution for haze management, further proving its effectiveness in addressing MCDM problems.

Table 13. Comparison of ranking results.

	N1	N2	N3	N4	RANK
PFS-TOPSIS	0.396882	0.471066	**0.572785**	0.534007	N3>N4>N2>N1
PFSB-VIKOR	0.439700	0.223372	**0.000000**	1.000000	N3>N2>N1>N4
PFS-MCDM	0.255115	0.282408	**0.406632**	0.377557	N3>N4>N2>N1
MEREC-MABAC	-0.582886	-0.278269	**-0.075692**	-0.360145	N3>N2>N4>N1
Proposed method	-0.007233	0.123785	**0.357130**	0.330255	N3>N4>N2>N1

6. Conclusions

In this study, a simpler and more effective method to assign membership degree values was proposed, thereby drawing on the concept of PPFS introduced by Kahraman [13]. The PGPFS approach requires the proportional value between the membership parameters relative to a customized unit, rather than the relative proportional value between the membership parameters. This method addresses the defect in PPFS where the degree of hesitation cannot be zero. The data required is the proportion of each alternative in the expert's mind relative to the unit program, supporting predictions such as "about 2-3 times" or "about 2 times," thereby reducing the difficulty of the expert's assignment. Additionally, the concept of a fuzzy tensor is introduced, and tensor-based entropy and aggregation operators for PGPFS were proposed to simplify the computation of high-dimensional data. The tensor-based entropy for PGPFS considers both the fuzziness and the greyness of the data, thus making the entropy calculation more reasonable and accurate. Moreover, the tensor-based aggregation decision-making method in the PGPFS environment leverages the advantages of the proportional grey picture fuzzy tensor entropy and the proportional grey picture fuzzy tensor aggregation operator. This approach can solve MCDM problems where both the criteria weights and the expert weights are unknown. The application in the haze management scheme demonstrated the method's applicability. The proposed proportional grey picture fuzzy aggregation decision-making method successfully calculated the criteria weights and expert weights based on proportional data and selected the optimal haze management scheme.

Obviously, there are two shortcomings of the method in this paper that can be addressed. One is that the complex symbolic representation may affect the reader's reading experience. The other is that the method requires setting a maximum threshold for the proportion of each membership parameter relative to the unit, which may not be easy to achieve in real life, and may even generate some errors due to this limitation.

Therefore, in future research, we will continue to optimize PGPFS and explore improved methods for setting proportions. Additionally, by leveraging its advantages, we will investigate other extensions such as proportional grey spherical fuzzy sets, proportional spherical fuzzy sets, proportional complex fuzzy sets, and proportional decomposition fuzzy sets. Furthermore, other MCDM methods such as TOPSIS, VIKOR, ELECTRE, CRADIS, and MAIRCA can be extended using PGPFS.

Author contributions

Jingjie Zhao, Jiale Zhang: Conceptualization, methodology, writing-original draft, software, editing, preparing the figures and the table; Yu Lei, Baolin Yi: Conceptualization, formalAnalysis, writing-review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors state that they have no conflicts of interest to disclose.

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