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Research article

Boundedness on variable exponent Morrey-Herz space for fractional multilinear Hardy operators

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Abstract: In this treatise, the boundedness of the multilinear fractional Hardy operators is scrutinized within the context of variable exponent Morrey-Herz spaces, denoted as $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$. Analogous estimations are derived for their commutators, contingent upon the symbol functions residing in the space of bounded mean oscillation (BMO) with variable exponents.

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1. Introduction

In the year 1920, Hardy [1] introduced an operator for a locally integrable function g on $\mathbb{R}$, now referred to as the Hardy operator. This operator is defined as:

$$\mathcal{H}g(\zeta) = \zeta^{-1}\int_0^{\zeta} g(\mu)d\mu, \quad \zeta > 0, \tag{1.1}$$

and Hardy established the following inequality:

$$\|\mathcal{H}g\|_{L^p(\mathbb{R}^+)} \leq p'\|g\|_{L^p(\mathbb{R}^+)}, \quad \infty > p > 1, \tag{1.2}$$

where $p' = p/(p-1)$ is shown to be the optimal constant. Subsequently, Faris in [2] proposed an n-dimensional generalization of (1.1), which in an equivalent form is expressed as:

$$Hg(\zeta) = |B(0,|\zeta|)|^{-1}\int_{B(0,|\zeta|)} g(\mu)d\mu, \tag{1.3}$$

where $|B(0,|\zeta|)|$ denotes the Lebesgue measure of the ball $B(0,|\zeta|)$ in n-dimensional Euclidean space $\mathbb{R}^n$. Recently, it has been demonstrated in [3] that H satisfies:

$$\|Hg\|_{L^p(\mathbb{R}^n)} \le p'\|g\|_{L^p(\mathbb{R}^n)}, \qquad 1 < p \le \infty, \tag{1.4}$$

with p' being a precise constant. Inequalities (1.2) and (1.4) have been extended to power-weighted Lebesgue spaces in [4, 5], where the sharp constants are dependent on the weight indices. The inequalities referenced as (1.2) and (1.4) are denominated as strong-type (p, p) Hardy inequalities, owing to the fact that within these expressions, the Hardy operator effectively maps the space L^p onto itself, L^p. In contrast, the authors in [6] have derived weak-type (p, p) Hardy inequalities, wherein the Hardy operator instead maps L^p to the weak Lebesgue space $L^{p,\infty}$. Notably, it was demonstrated that the optimal constant for such weak-type Hardy inequalities is 1, a value that is inferior to $p/(p-1)$. Furthermore, the exact or "sharp" constants for weak-type Hardy inequalities applicable to Morrey-type spaces were subsequently determined in [7, 8]. Similarly, the precise constant for the high-dimensional fractional Hardy operator, as described in [9],

$$H_\beta g(\zeta) = |B(0,|\zeta|)|^{\frac{\beta}{n}-1}\int_{B(0,|\zeta|)} g(\mu)d\mu, \qquad 0 \le \beta < n, \tag{1.5}$$

on Lebesgue spaces remained undetermined until the year 2015. This conundrum was resolved by Zhao and Lu [10], who extended Bliss's results pertinent to the one-dimensional fractional Hardy operator. In their work, the boundedness of the Hardy operator H_β was established, along with the following inequality:

$$\|H_\beta g\|_{L^q(\mathbb{R}^n)} \le A\|g\|_{L^p(\mathbb{R}^n)}, \tag{1.6}$$

where

$$A = \left(\frac{p'}{q}\right)^{1/q}\left(\frac{n}{q\beta}\cdot B\left(\frac{n}{q\beta}, \frac{n}{q'\beta}\right)\right)^{-\beta/n}.$$

Moreover, for functions $g_1, g_2, \ldots, g_m \in L^1_{loc}(\mathbb{R}^n)$ and for $m \in \mathbb{N}$, the multilinear Hardy operator was introduced by Fu and Grafakos in [4], expressed as follows:

$$H(g_1, ..., g_m) = \frac{1}{|\vartheta|^{nm}}\int_{|(\zeta_1,...,\zeta_m)|<|\vartheta|}\prod_{i=1}^{m} g_i(\zeta_i)d\zeta_1, ..., d\zeta_m,$$

and they further developed the sharp bounds for the multilinear Hardy operator. The 2-linear operator, more commonly recognized as the bilinear operator, has been the subject of considerable exploration. In [11], the authors employed the commutator of the bilinear Hardy operator, expressed as:

$$[b_i, H^i](g_1, ..., g_m)(\zeta) = b_i(\zeta)H(g_1, ..., g_m)(\zeta) - H(g_1, ..., g_{i-1}, g_i b_i, g_{i+1}, ..., g_m)(\zeta),$$

and successfully established the boundedness of bilinear commutators engendered by the bilinear Hardy operator. For the inaugural occasion, the fractional multilinear Hardy operators were expounded in [12], delineated as follows:

$$H_\beta(g_1, ..., g_m) = \frac{1}{|\vartheta|^{nm-\beta}}\int_{|(\zeta_1,...,\zeta_m)|<|\vartheta|}\prod_{i=1}^{m} g_i(\zeta_i)d\zeta_1, ..., d\zeta_m,$$

$$H^*_\beta(g_1, ..., g_m) = \int_{|(\zeta_1,...,\zeta_m)|>|\vartheta|} \frac{1}{|\zeta|^{nm-\beta}} \prod_{i=1}^{m} g_i(\zeta_i) d\zeta_1, ..., d\zeta_m,$$

where $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_m)$. Additionally, in [12], a definition for the commutator of fractional multilinear Hardy operators is introduced as follows:

$$[b, H_\beta](g_1, ..., g_m)(\zeta) = \sum_{i=1}^{m} [b_i, H^i_\beta](g_1, ..., g_m)(\zeta),$$

$$[b, H^*_\beta](g_1, ..., g_m)(\zeta) = \sum_{i=1}^{m} [b_i, H^{*i}_\beta](g_1, ..., g_m)(\zeta),$$

$$[b_i, H^i_\beta](g_1, ..., g_m)(\zeta) = b_i(\zeta) H_\beta(g_1, ..., g_m)(\zeta) - H_\beta(g_1, ..., g_{i-1}, g_i b_i, g_{i+1}, ..., g_m)(\zeta),$$

$$[b_i, H^{*i}_\beta](g_1, ..., g_m)(\zeta) = b_i(\zeta) H^*_\beta(g_1, ..., g_m)(\zeta) - H^*_\beta(g_1, ..., g_{i-1}, g_i b_i, g_{i+1}, ..., g_m)(\zeta).$$

Hardy inequalities have been a focal point of interest in numerous scholarly treatises [13, 14]. The determination of optimal bounds for Hardy-type inequalities has been achieved in only a limited number of instances, and research in this domain remains a vibrant and evolving component of contemporary mathematical analysis. Noteworthy recent contributions to this field include [15, 16]. Furthermore, the sharp constants for Hardy-type inequalities over the product of certain function spaces have been thoroughly elucidated in the published work [17]. We discern several pivotal works concerning the analysis of Hardy operators across diverse function spaces, which encompass [4, 5, 18–20].

The foundational work presented in [21] catalyzed the notion of extending traditional function spaces. The concept of variable Lebesgue spaces $L^{p(\cdot)}$ was first introduced by Rákosník and Kováčik in [22]. This development heralded the inception of variable exponent Lebesgue spaces, alongside a burgeoning interest in examining the boundedness properties of various operators, with particular attention given to the maximal operator within the variable exponent Lebesgue space $L^{p(\cdot)}$ [23, 24]. In recent years, the theory of generalized function spaces has garnered substantial attention across multiple branches of mathematical analysis, including but not limited to image processing [25], the modeling of electrorheological fluids [26], and the study of partial differential equations [27].

Moreover, Izuki advanced the field by introducing variable exponent Herz spaces $\dot{K}^{\alpha}_{q,p(\cdot)}$ in [28]. Building upon this, Drihem and Almeida [29] proposed a revised formulation of Herz spaces that incorporated α as a variable exponent. In a further development, [30] presented Herz spaces wherein all exponents were treated as variables, marking a significant expansion in the theory. The notion of variable exponent Morrey-Herz spaces $M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}$ was first articulated in [31], where the generalized concept of Morrey-Herz spaces was proposed by replacing the exponent α with the function $\alpha(\cdot)$. This advancement is further explored in the seminal works [32, 33].

The recent introduction of weighted theories based on Muckenhoupt weights [34] represents a notable achievement in the study of variable exponent function spaces. Cruz-Uribe, in [35], established the boundedness of the Hardy-Littlewood maximal operator M, defined as

$$Mg(\zeta) = \sup_{E:\text{ball},\zeta\in \text{Q}} \frac{1}{|Q|} \int_Q |g(\mu)| d\mu,$$

on the variable exponent weighted Lebesgue space $L^{p(\cdot)}(w)$. Hästö and Diening, in their work [36], demonstrated the equivalence between the continuity criteria for M on $L^{p(\cdot)}(w)$ and the Muckenhoupt condition.

In this discourse, we shall explore the boundedness properties of the multilinear fractional Hardy operator within the framework of Herz-Morrey spaces with variable exponents. Furthermore, this article delves into the boundedness of commutators generated by the multilinear fractional Hardy operator, also within the context of Herz-Morrey spaces and variable exponents.

In doing so, we extend several results previously introduced in [18]. To effectively manage the continuity criteria of the multilinear fractional Hardy operator, we shall leverage the boundedness of the fractional integral, defined as

$$I_\beta(g)(\zeta) = \int_{\mathbb{R}^n} \frac{g(\mu)}{|\zeta-\mu|^{n-\beta}} d\mu.$$

The boundedness of the Riesz potential on variable exponent Lebesgue spaces is documented in [37], while the boundedness of the fractional integral operator on Herz spaces was established by Noi and Izuki [38]. The essential outcomes of our work are outlined as follows:

Theorem 1.1. *Consider $0 < q, q_i < \infty$, where $q(\cdot) \in P(\mathbb{R}^n) \cap \mathcal{G}^{log}(\mathbb{R}^n)$, and let $p(\cdot)$ be such that $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\beta}{n}$, with $\frac{1}{p(\cdot)} = \sum_{i=1}^m \frac{1}{p_i(\cdot)}$. Furthermore, take $\lambda = \sum_{i=1}^m \lambda_i$, and let $\alpha(\cdot) \in \mathcal{G}^{log}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be logarithmically Hölder continuous at the origin, satisfying $\alpha(0) = \sum_{i=1}^m \alpha_i(0)$ and $\alpha(\infty) = \sum_{i=1}^m \alpha_i(\infty)$, with $\alpha(0) \le \alpha(\infty) < n\delta_{ii} + \lambda$, where $\delta_{ii} \in (0,1)$ are constants arising from (3.3). Then, it holds that:*

$$\|H_\beta(f_1, f_2, ..., f_m)\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,q(\cdot)}(\mathbb{R}^n)} \le C \prod_{i=1}^m \|f_i\|_{M\dot{K}^{\alpha_i(\cdot),\lambda_i}_{q_i,p_i(\cdot)}(\mathbb{R}^n)}.$$

Theorem 1.2. *Let $q_i(\cdot), q, p_i(\cdot), p(\cdot), \alpha(\cdot)$, and β retain the same definitions as those established in Theorem 1.1. Moreover, if the condition $\alpha(\infty) \ge \alpha(0) > \lambda - n\delta$ holds, where $\delta \in (0,1)$ is the constant introduced in Lemma 3.3, then it follows that*

$$\|H^*_\beta(f_1, f_2, ..., f_m)\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,q(\cdot)}(\mathbb{R}^n)} \le C \prod_{i=1}^m \|f_i\|_{M\dot{K}^{\alpha_i(\cdot),\lambda_i}_{q_i,p_i(\cdot)}(\mathbb{R}^n)}.$$

Theorem 1.3. *Assume $0 < q, q_i < \infty$, for $i = 1, 2, ..., m$, where $q(\cdot) \in P(\mathbb{R}^n) \cap \mathcal{G}^{log}(\mathbb{R}^n)$ and $p(\cdot)$ satisfies $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\beta}{n}$ with $\frac{1}{p(\cdot)} = \sum_{i=1}^m \frac{1}{p_i(\cdot)}$. Let $\lambda = \sum_{i=1}^m \lambda_i$, and consider $\alpha(\cdot) \in \mathcal{G}^{log}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ as a logarithmic Hölder continuous function at the origin, fulfilling $\alpha(0) = \sum_{i=1}^m \alpha_i(0)$ and $\alpha(\infty) = \sum_{i=1}^m \alpha_i(\infty)$, with the condition $\alpha(0) \le \alpha(\infty) < \sum_{i=1}^m n\delta_{ii} + \lambda$, where $\delta_{ii} \in (0,1)$ are constants as described in (3.3). Under these assumptions, the commutator $[b, H_\beta]$ is bounded from the product space $\prod_i^m M\dot{K}^{\alpha_i(\cdot),\lambda_i}_{q_i,p_i(\cdot)}(\mathbb{R}^n)$ to $M\dot{K}^{\alpha(\cdot),\lambda}_{q,q(\cdot)}(\mathbb{R}^n)$, where $b = (b_1, b_2, ..., b_m)$ with $b_1, b_2, ..., b_m \in BMO$.*

Theorem 1.4. *Assume that $q_i, q, p_i(\cdot), p(\cdot), \alpha(\cdot)$, and β are defined in the same manner as in the aforementioned Theorem 1.1. Moreover, if $\alpha(\infty) \ge \alpha(0) > \lambda - n\delta$, where $\delta \in (0,1)$ is the constant that emerges in Lemma 3.3, then the commutator $[b, H^*_\beta]$ is bounded from $\prod_{i=1}^m M\dot{K}^{\alpha_i(\cdot),\lambda_i}_{q_i,p_i(\cdot)}(\mathbb{R}^n)$ to $M\dot{K}^{\alpha(\cdot),\lambda}_{q,q(\cdot)}(\mathbb{R}^n)$, where $b = (b_1, b_2, ..., b_m)$ and $b_1, b_2, ..., b_m \in BMO$.*

2. Terminology and formal definitions

The symbol C is employed throughout this manuscript to denote a constant, the value of which may vary from one instance to another. The spaces $L^1_{loc}(\mathbb{R}^n)$, $L^{p(\cdot)}_{loc}(\mathbb{R}^n)$, $L^\infty(\mathbb{R}^n)$ are defined by Grafakos in [39]. Let S be a nonempty, measurable subset of $\mathbb{R}^n$, and let χ_S represent the characteristic function of S, where $|S|$ designates the Lebesgue measure of the set. We shall commence by delineating variable exponent Lebesgue spaces, drawing upon foundational works and references such as [22, 24, 40, 41].

Definition 2.1. Consider a measurable function $q(\cdot) : \mathbb{R}^n \to [1, \infty]$. The Lebesgue space characterized by a variable exponent, symbolically represented as $L^{q(\cdot)}(\mathbb{R}^n)$ constitutes the assemblage of all measurable functions ξ for which the ensuing integral expression, identified as $F_q(\xi)$, is bounded by a finite quantity:

$$F_q(\xi) = \int_{\mathbb{R}^n} \Big(|\xi(\mu)|\Big)^{q(\mu)} d\mu < \infty.$$

The space $L^{q(\cdot)}(\mathbb{R}^n)$ is endowed with the following norm, making it a Banach space:

$$\|\xi\|_{L^{q(\cdot)}} = \inf\Big\{\sigma > 0 : F_q(\frac{\xi}{\sigma}) = \int_{\mathbb{R}^n} \Big(\frac{|\xi(\mu)|}{\sigma}\Big)^{q(\mu)} d\mu \leq 1\Big\}.$$

Definition 2.2. We denote by $P(\mathbb{R}^n)$ the collection of all measurable functions $q(\cdot) : \mathbb{R}^n \to (1, \infty)$ such that

$$1 < q_- \leq q(\mu) \leq q_+ < \infty,$$

where

$$q_- := \operatorname*{essinf}_{\mu\in\mathbb{R}^n} q(\mu), \; q_+ := \operatorname*{esssup}_{\mu\in\mathbb{R}^n} q(\mu).$$

Definition 2.3. Let $q(\cdot)$ be a real-valued function defined on $\mathbb{R}^n$. We define the following:
(i) $\mathcal{G}^{\log}_{\text{loc}}(\mathbb{R}^n)$ denotes the set of all locally logarithmically Hölder continuous functions $q(\cdot)$ satisfying

$$|q(\mu) - q(\zeta)| \lesssim \frac{-C}{\log(|\mu - \zeta|)}, \;\; |\zeta - \mu| < \frac{1}{2}, \;\; \mu, \zeta \in \mathbb{R}^n.$$

(ii) If $q(\cdot) \in \mathcal{G}^{\log}_0(\mathbb{R}^n)$, then it satisfies the following condition at the origin:

$$|q(\mu) - q(0)| \lesssim \frac{C}{\log(|\frac{1}{|\mu|} + e|)}, \qquad \mu \in \mathbb{R}^n.$$

(iii) If $q(\cdot) \in \mathcal{G}^{\log}_\infty(\mathbb{R}^n)$, then it fulfills the following inequality at infinity:

$$|q(\mu) - q_\infty| \leq \frac{C_\infty}{\log(|\mu| + e)}, \;\; \mu \in \mathbb{R}^n.$$

(iv) $\mathcal{G}^{\log} = \mathcal{G}^{log}_{\text{loc}} \bigcap \mathcal{G}^{log}_\infty$ represents the set of all globally logarithmically Hölder continuous functions $q(\cdot)$.

We denote by $\mathbb{D}(\mathbb{R}^n)$ the class of functions $q(\cdot)$ belonging to $P(\mathbb{R}^n) \cap \mathcal{G}^{\log}(\mathbb{R}^n)$, which satisfy the condition that the Hardy-Littlewood maximal operator is bounded on $L^{q(\cdot)}(\mathbb{R}^n)$. It was demonstrated in [42] that if $q(\cdot) \in P(\mathbb{R}^n) \cap \mathcal{G}^{\log}(\mathbb{R}^n)$, then M is bounded on $L^{q(\cdot)}(\mathbb{R}^n)$.

Definition 2.4. [39] Let $b \in L^1_{loc}(\mathbb{R}^n)$. The norm is defined as

$$\|b\|_{BMO(\mathbb{R}^n)} = \sup_S \frac{1}{|S|} \int_S |b(\mu) - b_S| d\mu,$$

where the supremum is taken over all balls $S \subset \mathbb{R}^n$ and $b_S = \frac{1}{|S|}\int_S f(y)dy$. The function b is said to have bounded mean oscillation if $\|b\|_{BMO(\mathbb{R}^n)} < \infty$, and $BMO(\mathbb{R}^n)$ is the space consisting of all functions $b \in L^1_{loc}(\mathbb{R}^n)$ for which $BMO(\mathbb{R}^n) < \infty$.

Variable exponent Morrey-Herz space $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ is defined now. Let $\chi_k = \chi_{A_k}$, $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$ and $A_k = B_k \setminus B_{k-1}$ for $k \in \mathbb{Z}$.

Definition 2.5. [31] Given $0 < q < \infty$, $0 \le \lambda < \infty$, a function $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ with $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$, $p(\cdot) \in P(\mathbb{R}^n)$, the space $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ is defined as

$$M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n) = \Big\{ f \in L^{p(\cdot)}_{loc}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} < \infty \Big\},$$

where

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \Big(\sum_{k=-\infty}^{k_0} 2^{k\alpha(\cdot)q} \|f\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \Big)^{1/q}.$$

In the special case when $\lambda = 0$, this space reduces to the Herz space with variable exponent, denoted by $\dot{K}^{\alpha(\cdot)}_{q,p(\cdot)}(\mathbb{R}^n)$.

3. Key theorems and findings

In this segment, we shall undertake the demonstration of the principal results enunciated in the inaugural section. Nonetheless, it is imperative to first present certain auxiliary lemmas, which will prove instrumental in this endeavor.

Lemma 3.1. *[43] If F is a Banach function space, then the following conditions hold:*
(1) The associated space F' must also be a Banach function space.
(2) The norms $\|\cdot\|_F$ and $\|\cdot\|_{(F')'}$ are equivalent.
(3) (Generalized Hölder inequality) If $\xi \in F'$ and $\eta \in F$, then the inequality

$$\int_{\mathbb{R}^n} |\eta(x)\xi(x)| \le \|\xi\|_{F'}\|\eta\|_F.$$

Lemma 3.2. *[44] If the function $q(\cdot)$ belongs to the class $P(\mathbb{R}^n)$, then for any ball S in $\mathbb{R}^n$, there exists a constant $C > 0$ such that the following inequality is satisfied:*

$$C^{-1} < |S|^{-1}\|\chi_S\|_{L^{q'(\cdot)}}\|\chi_S\|_{L^{q(\cdot)}} < C.$$

Lemma 3.3. *[38] If Y is a Banach function space and M is bounded on Y', then for any $E \subset \mathbb{R}^n$ and $S \subset E$, there exists a constant $\delta \in (0,1)$, such that*

$$\frac{\|\chi_S\|_Y}{\|\chi_E\|_Y} \lesssim \Big(\frac{|S|}{|E|}\Big)^\delta.$$

Proposition 3.4. *[35] Consider an open set E, and assume that $p(\cdot) \in P(E)$ fulfills the following conditions:*

$$|p(\zeta) - p(z\mu)| \leq \frac{-c}{log(|\zeta - \mu|)}, \frac{1}{2} \geq |\zeta - \mu| \tag{3.1}$$

$$|p(\zeta) - p(\mu)| \leq \frac{-c}{log(|\zeta| + e)}, |\zeta| \leq |\mu| \tag{3.2}$$

then it follows that $p(\cdot) \in \mathbb{D}(\mathbb{R}^n)$, where C denotes a positive constant that does not depend on ζ and μ.

Lemma 3.5. *[45] If $q(\cdot) \in \mathbb{D}(\mathbb{R}^n)$, then there exists a constant $0 < \delta < 1$ and a positive constant C such that for any ball S in $\mathbb{R}^n$ and any measurable subset $W \subset S$, the subsequent inequalities are satisfied:*

$$\frac{\|\chi_W\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|W|}{|S|}\right)^{\delta}.$$

Remark: *Assume $q(\cdot) \in P(\mathbb{R}^n)$ and fulfills the conditions (3.1) and (3.2) as articulated in Proposition 3.4. Consequently, $q'(\cdot)$ adheres to these conditions as well, indicating that both $q(\cdot)$ and $q'(\cdot)$ are members of $\mathbb{D}(\mathbb{R}^n)$. By invoking Lemma 3.5 and utilizing [12], we derive the existence of constants $\delta_{ii} \in (0, \frac{1}{(q_i)+})$ such that the inequalities*

$$\frac{\|\chi_W\|_{L^{q_i'(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{q'i(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|W|}{|S|}\right)^{\delta_{ii}} \tag{3.3}$$

are satisfied for all balls $S \subset \mathbb{R}^n$ and for all subsets $W \subset S$.

Proposition 3.6. *[37] Let $q(\cdot)$ be a member of the set $P(\mathbb{R}^n)$, where $0 < \beta < \frac{n}{(q_1)_+}$. Define $q'(\cdot)$ by the relation:*

$$\frac{1}{q'(\cdot)} = \frac{1}{q(\cdot)} - \frac{\beta}{n}.$$

Then, the following inequality holds:

$$\|I_\beta f\|_{L^{q'}(\cdot)(\mathbb{R}^n)} \leq C\|f\|_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

Proposition 3.6 plays a crucial role in deriving the subsequent lemma (see [46]).

Lemma 3.7. *Let β, $q(\cdot)$, and $q'(\cdot)$ be defined as outlined in Proposition 3.6 . For any ball $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$ with $k \in \mathbb{Z}$, the following inequality is satisfied:*

$$\|\chi_{B_k}\|_{L^{q'}(\cdot)(\mathbb{R}^n)} \leq C2^{-k\beta}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

Lemma 3.8. *[46] Assume that $q(\cdot)$ is an element of $P(\mathbb{R}^n)$. Then, for any $b \in BMO$ and for all integers $j, i \in \mathbb{Z}$ with $j > i$, the following inequalities hold:*

$$C^{-1}\|b\|_{BMO} \leq \sup_{B:Ball} \frac{1}{\|\chi_B\|_{L^{q(\cdot)}}}\|(b - b_B)\chi_B\|_{L^{q(\cdot)}} \leq C\|b\|_{BMO}$$

$$\|(b - b_{B_i})\chi_{B_j}\|_{L^{q(\cdot)}} \leq C(j - i)\|b\|_{BMO}\|\chi_{B_j}\|_{L^{q(\cdot)}}.$$

Here, C is a positive constant independent of the choices of j and i.

Lemma 3.9. *Given that* $q(\cdot) \in P(\mathbb{R}^n) \cap \mathcal{G}^{log}(\mathbb{R}^n)$ *and* $p(\cdot)$ *satisfies the condition* $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\beta}{n}$ *with* $\frac{1}{p(\cdot)} = \sum_{i=1}^{m} \frac{1}{p_i(\cdot)}$*, it follows that:*

$$\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le C2^{k(mn-\beta)} \prod_{i=1}^{m} \|\chi_{B_k}\|^{-1}_{L^{p_i'(\cdot)}(\mathbb{R}^n)}.$$

Proof. Let us assume $f = \chi_{B_k}$ and employ the definition of the operator I_β

$$I_\beta(\chi_{B_k})(x) \ge C2^{k\beta}\chi_{B_k}(x),$$

$$\chi_{B_k}(x) \le C2^{-k\beta} I_\beta(\chi_{B_k})(x).$$

Taking the norm on both sides and applying the results from Lemmas 3.2 and 3.7, respectively, we deduce:

$$\begin{aligned}
\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\le C2^{-k\beta}\|I_\beta\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\le C2^{-k\beta}\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\le C2^{-k\beta} \prod_{i=1}^{m} \|\chi_{B_k}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)} \\
&\le C2^{k(mn-\beta)} \prod_{i=1}^{m} \|\chi_{B_k}\|^{-1}_{L^{p_i'(\cdot)}(\mathbb{R}^n)}. \qquad (3.4)
\end{aligned}$$

Proposition 3.10. *[11] If* $\alpha(\cdot)$ *resides within the intersection of* $L^\infty(\mathbb{R}^n)$ *and* $\mathcal{G}^{log}(\mathbb{R}^n)$*, and let* $p(\cdot)$ *belong to the class* $P(\mathbb{R}^n)$*, with the parameters* $0 < q < \infty$ *and* $0 \le \lambda < \infty$*, then*

$$\begin{aligned}
&\|H_\beta(f_1, f_2, ..., f_m) \cdot \chi_k\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} \\
&= \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} 2^{k\alpha(\cdot)q} \|H_\beta(f_1, f_2, ..., f_m) \cdot \chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\approx \max\Bigg\{ \sup_{\substack{k_0 \in \mathbb{Z} \\ k_0 < 0}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q} \|H_\beta(f_1, f_2, ..., f_m) \cdot \chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}, \\
&\quad \sup_{\substack{k_0 \in \mathbb{Z} \\ k_0 \ge 0}} 2^{-k_0\lambda q} \Bigg(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q} \|H_\beta(f_1, f_2, ..., f_m) \cdot \chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&+ \sum_{k=0}^{k_0} 2^{k\alpha(\infty)q} \|H_\beta(f_1, f_2, ..., f_m) \cdot \chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \Bigg) \Bigg\}.
\end{aligned}$$

Proof of Theorem 1.1. For each $f_1, f_2, ..., f_m \in M\dot{K}^{\alpha(\cdot),\lambda}_{q_i,p_i(\cdot)}(\mathbb{R}^n)$, let us denote $f_{ij} = f_i \cdot \chi_j = f_i \cdot \chi_{A_j}$ for any $j \in \mathbb{Z}$ and $i \in \mathbb{Z}^+$. Then, we express

$$f_i(x) = \sum_{j=-\infty}^{\infty} f_i(x) \cdot \chi_j(x) = \sum_{j=-\infty}^{\infty} f_{ij}(x).$$

Utilizing the generalized Hölder inequality, we derive

$$\begin{aligned}|H_\beta(f_1, f_2, ..., f_m)(x)\cdot\chi_k(x)| &\le \frac{1}{|x|^{mn-\beta}}\int_{B_k}\int_{B_k}...\int_{B_k}(|f_1(t_1)||f_2(t_2)|...|f_m(t_m)|)dt_1dt_2...dt_m\cdot\chi_k(x)\\ &= \frac{1}{|x|^{mn-\beta}}\int_{B_k}|f_1(t_1)|dt_1\int_{B_k}|f_2(t_2)|dt_2...\int_{B_k}|f_m(t_m)|dt_m\cdot\chi_k(x)\\ &\le C2^{-kmn}\sum_{j=-\infty}^{k}\prod_{i=1}^{m}\|f_{ij}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)}2^{k\beta}\chi_k(x),\end{aligned}$$

$$\begin{aligned}&\|H_\beta(f_1, f_2, ..., f_m)\cdot\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\ &\le C2^{k\beta}\sum_{j=-\infty}^{k}\prod_{i=1}^{m}\|f_{ij}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)}2^{-kmn}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}.\end{aligned} \tag{3.5}$$

To advance further, we substitute Lemma 3.9 into inequality (3.5):

$$\begin{aligned}\|H_\beta(f_1, f_2, ..., f_m)\cdot\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\le C\sum_{j=-\infty}^{k}\prod_{i=1}^{m}\|f_{ij}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)'}\|\chi_k\|^{-1}_{L^{p_i'(\cdot)}(\mathbb{R}^n)}\\ &\le C\sum_{j=-\infty}^{k}\prod_{i=1}^{m}\|f_{ij}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_j\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)}}{\|\chi_k\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)}}\\ &\le C\sum_{j=-\infty}^{k}\prod_{i=1}^{m}2^{n\delta_{ii}(j-k)}\|f_{ij}\|_{L^{p_i(\cdot)}(\mathbb{R}^n).}\end{aligned}$$

In the remainder of the proof, to evaluate $\|f_{ij}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}$, we consider the following two distinct cases.
Case 1: For $j < 0$,

$$\begin{aligned}\|f_{ij}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)} &= \prod_{i=1}^{m}2^{-j\alpha_i(0)}\Big(2^{j\alpha_i(0)q_i}\|f_{ij}\|^{q_i}_{L^{p_i(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{q_i}}\\ &\le \prod_{i=1}^{m}2^{-j\alpha_i(0)}\Big(\sum_{l=-\infty}^{j}2^{l\alpha_i(0)q_i}\|f_{il}\|^{q_i}_{L^{p_i(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{q_i}}\\ &\le \prod_{i=1}^{m}2^{j(\lambda_i-\alpha_i(0))}2^{-j\lambda_i}\Big(\sum_{l=-\infty}^{j}2^{l\alpha_i(\cdot)q_i}\|f_{il}\|^{q_i}_{L^{p_i(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{q_i}}\\ &\le C\prod_{i=1}^{m}2^{j(\lambda_i-\alpha_i(0))}\|f_i\|_{M\dot{K}^{\alpha_i(\cdot),\lambda_i}_{q_i,p_i(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Here, we use $\alpha(0) = \sum_{i=1}^{m}\alpha_i(0)$ and $\lambda = \sum_{i=1}^{m}\lambda_i$,

$$\|f_{ij}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \le C2^{j(\lambda-\alpha(0))}\prod_{i=1}^{m}\|f_i\|_{M\dot{K}^{\alpha_i(\cdot),\lambda_i}_{q_i,p_i(\cdot)}(\mathbb{R}^n)}.$$

Case 2: For $j \geq 0$,

$$\begin{aligned}
\|f_{ij}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} &= \prod_{i=1}^{m} 2^{-j\alpha_i(\infty)}\Big(2^{j\alpha_i(\infty)q_i}\|f_{ij}\|^{q_i}_{L^{p_i(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{q_i}} \\
&\leq \prod_{i=1}^{m} 2^{-j\alpha_i(\infty)}\Big(\sum_{l=0}^{j} 2^{l\alpha_i(\infty)q_i}\|f_{il}\|^{q_i}_{L^{p_i(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{q_i}} \\
&\leq \prod_{i=1}^{m} 2^{j(\lambda_i-\alpha_i(\infty))}2^{-j\lambda_i}\Big(\sum_{l=-\infty}^{j} 2^{l\alpha_i(\cdot)q_i}\|f_{il}\|^{q_i}_{L^{p_i(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{q_i}} \\
&\leq C\prod_{i=1}^{m} 2^{j(\lambda_i-\alpha_i(\infty))}\|f_i\|_{M\dot{K}^{\alpha_i(\cdot),\lambda_i}_{q_i,p_i(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

We use $\alpha(\infty) = \sum_{i=1}^{m} \alpha_i(\infty)$ and $\lambda = \sum_{i=1}^{m} \lambda_i$ to obtain

$$\|f_{ij}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)} = C2^{j(\lambda-\alpha(\infty))}\prod_{i=1}^{m}\|f_i\|_{M\dot{K}^{\alpha_i(\cdot),\lambda_i}_{q_i,p_i(\cdot)}(\mathbb{R}^n)}.$$

Utilizing the definition of variable exponent Herz-Morrey spaces and Proposition 3.10, we derive the following inequality:

$$\begin{aligned}
\|H_\beta(f_1,f_2,...,f_m)\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,q(\cdot)}(\mathbb{R}^n)} &= \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(\cdot)q}\|H_\beta(f_1,f_2,...,f_m)\cdot\chi_k\|^q_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{q}} \\
&\leq \max\left\{\sup_{\substack{k_0\in\mathbb{Z}\\k_0<0}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q}\|H_\beta(f_1,f_2,...,f_m)\cdot\chi_k\|^q_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{q}},\right. \\
&\sup_{\substack{k_0\in\mathbb{Z}\\k_0\geq0}} 2^{-k_0\lambda}\left(\left(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q}\|H_\beta(f_1,f_2,...,f_m)\cdot\chi_k\|^q_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{q}}\right. \\
&\left.\left.+\left(\sum_{k=0}^{k_0} 2^{k\alpha(\infty)q}\|H_\beta(f_1,f_2,...,f_m)\cdot\chi_k\|^q_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{q}}\right)\right\} \\
&=: \max\{Y_1, Y_2+Y_3\}, \qquad (3.6)
\end{aligned}$$

where

$$Y_1 = \sup_{\substack{k_0\in\mathbb{Z}\\k_0<0}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q}\|H_\beta(f_1,f_2,...,f_m)\cdot\chi_k\|^q_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{q}},$$

$$Y_2 = \sup_{\substack{k_0\in\mathbb{Z}\\k_0\geq0}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q}\|H_\beta(f_1,f_2,...,f_m)\cdot\chi_k\|^q_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{q}},$$

$$Y_3 = \sup_{\substack{k_0\in\mathbb{Z}\\ k_0\geq 0}} 2^{-k_0\lambda}\left(\sum_{k=0}^{k_0} 2^{k\alpha(\infty)q}\|H_\beta(f_1, f_2, ..., f_m)\cdot\chi_k\|^q_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{q}}.$$

First, let us estimate Y_1. Given that $\alpha(0) \leq \alpha(\infty) < n\delta_{ii} + \lambda$,

$$\begin{aligned}
Y_1 &\leq C\sup_{\substack{k_0\in\mathbb{Z}\\ k_0<0}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q}\Big(\sum_{j=-\infty}^{k}\prod_{i=1}^{m} 2^{(-n\delta_{ii})(k-j)}\|f_i\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\Big)^q\right)^{\frac{1}{q}}\\
&\leq C\sup_{\substack{k_0\in\mathbb{Z}\\ k_0<0}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q}\Big(\sum_{j=-\infty}^{k}\prod_{i=1}^{m} 2^{(-n\delta_{ii})(k-j)}2^{j(\lambda-\alpha(0))}\|f_i\|_{M\dot{K}^{\alpha_i(\cdot),\lambda_i}_{q_i,p_i(\cdot)}(\mathbb{R}^n)}\Big)^q\right)^{\frac{1}{q}}\\
&\leq C\prod_{i=1}^{m}\|f_i\|_{M\dot{K}^{\alpha_i(\cdot),\lambda_i}_{q_i,p_i(\cdot)}(\mathbb{R}^n)}\sup_{\substack{k_0\in\mathbb{Z}\\ k_0<0}} 2^{-k_0\lambda}\left(\sum_{k=-;2\infty}^{k_0} 2^{k\lambda q}\Big(\sum_{j=-\infty}^{k} 2^{(n\delta_{ii}+\lambda-\alpha(0))(j-k)}\Big)^q\right)^{\frac{1}{q}}\\
&\leq C\prod_{i=1}^{m}\|f_i\|_{M\dot{K}^{\alpha_i(\cdot),\lambda_i}_{q_i,p_i(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

The estimate for Y_2 follows the same approach as for Y_1. Finally, let us approximate Y_3

$$\begin{aligned}
Y_3 &= C\sup_{\substack{k_0\in\mathbb{Z}\\ k_0\geq 0}} 2^{-k_0\lambda}\left(\sum_{k=0}^{k_0} 2^{k\alpha(\infty)q}\Big(\sum_{j=-\infty}^{k}\prod_{i=1}^{m} 2^{(-n\delta_{ii})(k-j)}\|f_i\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\Big)^q\right)^{\frac{1}{q}}\\
&= C\sup_{\substack{k_0\in\mathbb{Z}\\ k_0\geq 0}} 2^{-k_0\lambda}\left(\sum_{k=0}^{k_0} 2^{k\alpha(\infty)q}\Big(\sum_{j=-\infty}^{0}\prod_{i=1}^{m} 2^{(-n\delta_{ii})(k-j)}\|f_i\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\Big)^q\right)^{\frac{1}{q}}\\
&+ C\sup_{\substack{k_0\in\mathbb{Z}\\ k_0\geq 0}} 2^{-k_0\lambda}\left(\sum_{k=0}^{k_0} 2^{k\alpha(\infty)q}\Big(\sum_{j=1}^{k}\prod_{i=1}^{m} 2^{(-n\delta_{ii})(k-j)}\|f_i\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\Big)^q\right)^{\frac{1}{q}}\\
&= Y_3' + Y_3''
\end{aligned}$$

$$Y_3' = C\sup_{\substack{k_0\in\mathbb{Z}\\ k_0\geq 0}} 2^{-k_0\lambda}\left(\sum_{k=0}^{k_0} 2^{k\alpha(\infty)q}\Big(\sum_{j=-\infty}^{0}\prod_{i=1}^{m} 2^{(-n\delta_{ii})(k-j)}\|f_i\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\Big)^q\right)^{\frac{1}{q}}$$

$$Y_3'' = \sup_{\substack{k_0\in\mathbb{Z}\\ k_0\geq 0}} 2^{-k_0\lambda}\left(\sum_{k=0}^{k_0} 2^{k\alpha(\infty)q}\Big(\sum_{j=1}^{k}\prod_{i=1}^{m} 2^{(-n\delta_{ii})(k-j)}\|f_i\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\Big)^q\right)^{\frac{1}{q}}$$

$$Y_3' \leq C\sup_{\substack{k_0\in\mathbb{Z}\\k_0\geq 0}} 2^{-k_0\lambda}\left(\sum_{k=0}^{k_0}2^{k\alpha(\infty)q}\Big(\sum_{j=-\infty}^{0}\prod_{i=1}^{m}2^{(-n\delta_{ii})(k-j)}2^{j(\lambda-\alpha(\infty))}\|f_i\|_{M\dot{K}_{q_i,p_i(\cdot)}^{\alpha_i(\cdot),\lambda_i}(\mathbb{R}^n)}\Big)^q\right)^{\frac{1}{q}}$$

$$\leq C\prod_{i=1}^{m}\|f_i\|_{M\dot{K}_{q_i,p_i(\cdot)}^{\alpha_i(\cdot),\lambda_i}(\mathbb{R}^n)}\sup_{\substack{k_0\in\mathbb{Z}\\k_0\geq 0}} 2^{-k_0\lambda}\left(\sum_{k=0}^{k_0}2^{k\lambda q}\Big(\sum_{j=-\infty}^{0}2^{(n\delta_{ii}+\lambda-\alpha(\infty))(j-k)}\Big)^q\right)^{\frac{1}{q}}$$

$$\leq C\prod_{i=1}^{m}\|f_i\|_{M\dot{K}_{q_i,p_i(\cdot)}^{\alpha_i(\cdot),\lambda_i}(\mathbb{R}^n)}.$$

$$Y_3'' \leq C\sup_{\substack{k_0\in\mathbb{Z}\\k_0\geq 0}} 2^{-k_0\lambda}\left(\sum_{k=0}^{k_0}2^{k\alpha(\infty)q}\Big(\sum_{j=1}^{k}\prod_{i=1}^{m}2^{(-n\delta_{ii})(k-j)}2^{j(\lambda-\alpha(\infty))}\|f_i\|_{M\dot{K}_{q_i,p_i(\cdot)}^{\alpha_i(\cdot),\lambda_i}(\mathbb{R}^n)}\Big)^q\right)^{\frac{1}{q}}$$

$$\leq C\prod_{i=1}^{m}\|f_i\|_{M\dot{K}_{q_i,p_i(\cdot)}^{\alpha_i(\cdot),\lambda_i}(\mathbb{R}^n)}\sup_{\substack{k_0\in\mathbb{Z}\\k_0\geq 0}} 2^{-k_0\lambda}\left(\sum_{k=0}^{k_0}2^{k\lambda q}\Big(\sum_{j=1}^{k}2^{(n\delta_{ii}+\lambda-\alpha(\infty))(j-k)}\Big)^q\right)^{\frac{1}{q}}$$

$$\leq C\prod_{i=1}^{m}\|f_i\|_{M\dot{K}_{q_i,p_i(\cdot)}^{\alpha_i(\cdot),\lambda_i}(\mathbb{R}^n)}.$$

Hence, we obtain:

$$Y_3 \leq C\prod_{i=1}^{m}\|f_i\|_{M\dot{K}_{q_i,p_i(\cdot)}^{\alpha_i(\cdot),\lambda_i}(\mathbb{R}^n)}.$$

Substituting the estimates for Y_1, Y_2, and Y_3 into (3.6) results in the desired conclusion.

Proof of Theorem 1.2. By deploying Hölder's inequality, the following relation is ascertained:

$$|H_\beta^*(f_1,f_2,...,f_m)(x)\cdot\chi_k(x)| \leq \int_{\mathbb{R}^n\setminus B_k}\int_{\mathbb{R}^n\setminus B_k}\cdots\int_{\mathbb{R}^n\setminus B_k}\frac{1}{|t|^{mn-\beta}}|f_1(t_1)||f_2(t_2)|...|f_m(t_m)|dt_1dt_2...dt_m\cdot\chi_k(x)$$
$$\leq C\sum_{j=k+1}^{\infty}\prod_{i=1}^{m}2^{-j(mn-\beta)}\|f_{ij}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{p_i'(\cdot)}(\mathbb{R}^n))}\chi_k(x).$$

This further leads to:

$$\|H_\beta^*(f_1,f_2,...,f_m)(x)\cdot\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C\sum_{j=k+1}^{\infty}\prod_{i=1}^{m}2^{-j(mn-\beta)}\|f_{ij}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

Taking into account the inequality in (3.4), we derive:

$$\|H_\beta^*(f_1,f_2,...,f_m)(x)\cdot\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C\sum_{j=k+1}^{\infty}\prod_{i=1}^{m}\|f_{ij}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\|\chi_j\|^{-1}_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}$$
$$\leq C\sum_{j=k+1}^{\infty}\prod_{i=1}^{m}2^{n\delta(k-j)}\|f_{ij}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)},$$

where, in the final step, the result from Lemma 3.5 has been utilized.

Following a similar approach as in Theorem 1.1, we achieve:

$$\|H_\beta^*(f_1, f_2, ..., f_m)\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,q(\cdot)}(\mathbb{R}^n)} = \max\{Z_1, Z_2 + Z_3\},$$

where:

$$Z_1 = \sup_{\substack{k_0\in\mathbb{Z}\\ k_0<0}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q}\|H_\beta^*(f_1, f_2, ..., f_m)(x)\cdot\chi_k\|^q_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

$$Z_2 = \sup_{\substack{k_0\in\mathbb{Z}\\ k_0\ge 0}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q}\|H_\beta^*(f_1, f_2, ..., f_m)(x)\cdot\chi_k\|^q_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

$$Z_3 = \sup_{\substack{k_0\in\mathbb{Z}\\ k_0\ge 0}} 2^{-k_0\lambda q} \sum_{k=0}^{k_0} 2^{k\alpha(\infty)q}\|H_\beta^*(f_1, f_2, ..., f_m)(x)\cdot\chi_k\|^q_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

The boundedness properties of Z_l for l=1,2,3 mirror those of Y_l for l=1,2,3 as discussed in Theorem 1.1. We now draw close to the final result.

Proof of Theorem 1.3.

$$|[b_1, H_\beta](f_1, f_2, ..., f_m)(z)\cdot\chi_k(z)| \le \frac{1}{|z|^{mn-\beta}} \int_{B_k}\int_{B_k}...\int_{B_k} |f_1(t_1)f_2(t_2)...f_m(t_m)(b_1(z) - b(t_1))|dt_1dt_2...dt_m\cdot\chi_k(z)$$

$$\le C2^{-k(mn-\beta)} \sum_{j=-\infty}^{k} \int_{B_k}\int_{B_k}...\int_{B_k} |f_1(t_1)f_2(t_2)...f_m(t_m)(b_1(z) - (b_1)_{B_j} + (b_1)_{B_j} - b_1(t_1))|dt_1dt_2...dt_m\cdot\chi_k(z)$$

$$\le C2^{-k(mn-\beta)} \sum_{j=-\infty}^{k} \int_{B_k}\int_{B_k}...\int_{B_k} |f_1(t_1)f_2(t_2)...f_m(t_m)(b_1(z) - (b_1)_{B_j})|dt_1dt_2...dt_m\cdot\chi_k(z)$$

$$+ C2^{-k(mn-\beta)} \sum_{j=-\infty}^{k} \int_{B_k}\int_{B_k}...\int_{B_k} |f_1(t_1)f_2(t_2)...f_m(t_m)(b_1(t_1) - (b_1)_{B_j})|dt_1dt_2...dt_m\cdot\chi_k(z)$$

$$= I + II,$$

$$I = C2^{-k(mn-\beta)} \sum_{j=-\infty}^{k} \int_{B_k}\int_{B_k}...\int_{B_k} |f_1(t_1)f_2(t_2)...f_m(t_m)(b_1(z) - (b_1)_{B_j})|dt_1dt_2...dt_m\cdot\chi_k(z).$$

By invoking the Hölder inequality, we derive the following estimate:

$$I \le C2^{-k(mn-\beta)} \sum_{j=-\infty}^{k} \prod_{i=1}^{m} |(b_1(z) - (b_1)_{B_j})\cdot\chi_k(z)|\|f_{ij}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)}.$$

By applying Lemmas 3.5, 3.2, and 3.8 successively, we obtain

$$\begin{aligned}
&\|I\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\leq C2^{k\beta}\sum_{j=-\infty}^{k}\prod_{i=1}^{m}\|f_{ij}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)}2^{-mkn}\|(b_1(z)-(b_1)_{B_j})\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\leq C2^{-k(mn-\beta)}\sum_{j=-\infty}^{k}\prod_{i=1}^{m}\|f_{ij}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)}(k-j)\|b_1\|_{BMO}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \qquad (3.7)
\end{aligned}$$

$$\begin{aligned}
II &= 2^{-k(mn-\beta)}\sum_{j=-\infty}^{k}\int_{B_k}\int_{B_k}\dots\int_{B_k}|f_1(t_1)f_2(t_2)\dots f_m(t_m)(b_1(t_1)-(b_1)_{B_j})|dt_1dt_2\dots dt_m\cdot\chi_k(z)\\
&= 2^{-k(mn-\beta)}\sum_{j=-\infty}^{k}\int_{B_k}f_1(t_1)(b_1(t_1)-(b_1)_{B_j})dt_1\int_{B_k}f_2(t_2)dt_2\dots\int_{B_k}f_m(t_m)dt_m\cdot\chi_k(z)\\
&\leq C2^{-k(mn-\beta)}\sum_{j=-\infty}^{k}\prod_{i=1}^{m}\|f_{ij}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\|(b_1(t_1)-(b_1)_{B_j})\chi_j\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)}\cdot\chi_k(z)
\end{aligned}$$

$$\begin{aligned}
\|II\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\leq C2^{-k(mn-\beta)}\sum_{j=-\infty}^{k}\prod_{i=1}^{m}\|f_{ij}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\|(b_1(t_1)-(b_1)_{B_j})\chi_j\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)}\|\chi_k(x)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\leq C2^{-k(mn-\beta)}\sum_{j=-\infty}^{k}\prod_{i=1}^{m}\|f_{ij}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\|b_1\|_{BMO}\|\chi_j\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)}\|\chi_k(x)\|_{L^{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \qquad (3.8)$$

Deriving from inequalities (3.7) and (3.8), we arrive at the following expression:

$$\begin{aligned}
&\|[b_1,H_\beta](f_1,f_2,...,f_m)(z)\cdot\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\leq C2^{-k(mn-\beta)}\sum_{j=-\infty}^{k}\prod_{i=1}^{m}\|f_{ij}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)}(k-j)\|b_1\|_{BMO}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

In proceeding further, we follow the approach laid out in Theorem 1.1:

$$\begin{aligned}
&\|[b_1,H_\beta](f_1,f_2,...,f_m)(z)\cdot\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\leq C\sum_{j=-\infty}^{k}\prod_{i=1}^{m}2^{n\delta_{ii}(j-k)}(k-j)\|b_1\|_{BMO}\|f_{ij}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Utilizing the definition of the variable exponent Herz-Morrey space along with Proposition 3.10, we establish the following inequality:

$$
\begin{aligned}
\|[b_1,H_\beta](f_1,f_2,...,f_m)(z)\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,q(\cdot)}(\mathbb{R}^n)} &= \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(\cdot)q}\|[b_1,H_\beta](f_1,f_2,...,f_m)(z)\cdot\chi_k\|^q_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{q}}\\
&\le \max\Bigg\{\sup_{\substack{k_0\in\mathbb{Z}\\k_0<0}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q}\|[b_1,H_\beta](f_1,f_2,...,f_m)(z)\cdot\chi_k\|^q_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{q}},\\
&\quad \sup_{\substack{k_0\in\mathbb{Z}\\k_0\ge 0}} 2^{-k_0\lambda}\Bigg(\left(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q}\|[b_1,H_\beta](f_1,f_2,...,f_m)(z)\cdot\chi_k\|^q_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{q}}\\
&\quad + \left(\sum_{k=0}^{k_0} 2^{k\alpha(\infty)q}\|[b_1,H_\beta](f_1,f_2,...,f_m)(z)\cdot\chi_k\|^q_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{q}}\Bigg)\Bigg\}\\
&= \max\{A_1, A_2+A_3\},
\end{aligned}
$$

where

$$
A_1 = \sup_{\substack{k_0\in\mathbb{Z}\\k_0<0}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q}\|[b_1,H_\beta](f_1,f_2,...,f_m)(z)\cdot\chi_k\|^q_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{q}},
$$

$$
A_2 = \sup_{\substack{k_0\in\mathbb{Z}\\k_0\ge 0}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q}\|[b_1,H_\beta](f_1,f_2,...,f_m)(z)\cdot\chi_k\|^q_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{q}},
$$

$$
A_3 = \sup_{\substack{k_0\in\mathbb{Z}\\k_0\ge 0}} 2^{-k_0\lambda}\left(\sum_{k=0}^{k_0} 2^{k\alpha(\infty)q}\|[b_1,H_\beta](f_1,f_2,...,f_m)(z)\cdot\chi_k\|^q_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{q}}.
$$

Proceeding similarly to the computations carried out in Theorem 1.1, we arrive at the ensuing bound:

$$
\|[b_1,H_\beta](f_1,f_2,...,f_m)\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,q(\cdot)}(\mathbb{R}^n)} \le C\prod_{i=1}^{m}\|b_1\|_{BMO}\|f_i\|_{M\dot{K}^{\alpha_i(\cdot),\lambda_i}_{q_i,p_i(\cdot)}(\mathbb{R}^n)}.
$$

Analogously, one can readily establish the following estimate:

$$
\|[b_2,H_\beta](f_1,f_2,...,f_m)\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,q(\cdot)}(\mathbb{R}^n)} \le C\prod_{i=1}^{m}\|b_2\|_{BMO}\|f_i\|_{M\dot{K}^{\alpha_i(\cdot),\lambda_i}_{q_i,p_i(\cdot)}(\mathbb{R}^n)}.
$$

$$
\begin{array}{ccc}
, & , & ,\\
\vdots & \vdots & \vdots\\
, & , & ,
\end{array}
$$

$$
\|[b_m,H_\beta](f_1,f_2,...,f_m)\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,q(\cdot)}(\mathbb{R}^n)} \le C\prod_{i=1}^{m}\|b_m\|_{BMO}\|f_i\|_{M\dot{K}^{\alpha_i(\cdot),\lambda_i}_{q_i,p_i(\cdot)}(\mathbb{R}^n)}.
$$

Proof of Theorem 1.4.

$$
\begin{aligned}
&|[b_1, H_\beta^*](f_1, f_2, ..., f_m)(z)\cdot\chi_k(z)| \le \int_{\mathbb{R}^n\backslash B_k}\int_{\mathbb{R}^n\backslash B_k}...\int_{\mathbb{R}^n\backslash B_k}\frac{1}{|t|^{mn-\beta}}|f_1(t_1)f_1(t_2)...f_m(t_m)(b_1(z)-b(t_1))|dt_1dt_2...dt_m\cdot\chi_k(z)\\
&\le C\sum_{j=k+1}^{\infty}2^{-j(mn-\beta)}\int_{\mathbb{R}^n\backslash B_k}\int_{\mathbb{R}^n\backslash B_k}...\int_{\mathbb{R}^n\backslash B_k}|f_1(t_1)f_2(t_2)...f_m(t_m)(b_1(z)-(b_1)_{B_j}+(b_1)_{B_j}-b_1(t_1))|dt_1dt_2...dt_m\cdot\chi_k(z)\\
&\le C\sum_{j=k+1}^{\infty}2^{-j(mn-\beta)}\int_{\mathbb{R}^n\backslash B_k}\int_{\mathbb{R}^n\backslash B_k}...\int_{\mathbb{R}^n\backslash B_k}|f_1(t_1)f_2(t_2)...f_m(t_m)(b_1(z)-(b_1)_{B_j})|dt_1dt_2...dt_m\cdot\chi_k(z)\\
&+C\sum_{j=k+1}^{\infty}2^{-j(mn-\beta)}\int_{\mathbb{R}^n\backslash B_k}\int_{\mathbb{R}^n\backslash B_k}...\int_{\mathbb{R}^n\backslash B_k}|f_1(t_1)f_2(t_2)...f_m(t_m)(b_1(t_1)-(b_1)_{B_j})|dt_1dt_2...dt_m\cdot\chi_k(z)\\
&= III + IV,
\end{aligned}
$$

$$
III = C\sum_{j=k+1}^{\infty}2^{-j(mn-\beta)}\int_{\mathbb{R}^n\backslash B_k}\int_{\mathbb{R}^n\backslash B_k}...\int_{\mathbb{R}^n\backslash B_k}|f_1(t_1)f_2(t_2)...f_m(t_m)(b_1(z)-(b_1)_{B_j})|dt_1dt_2...dt_m\cdot\chi_k(z).
$$

By invoking Hölder inequality, the following estimate is derived:

$$
III \le C\sum_{j=k+1}^{\infty}2^{-j(mn-\beta)}\prod_{i=1}^{m}|(b_1(z)-(b_1)_{B_j})\cdot\chi_k(z)|\|f_{ij}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)}.
$$

Utilizing Lemmas 3.5, 3.2, and 3.8 sequentially, we obtain:

$$
\begin{aligned}
&\|III\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\le C\sum_{j=k+1}^{\infty}\prod_{i=1}^{m}2^{-j(mn-\beta)}\|f_{ij}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)}\|(b_1(z)-(b_1)_{B_j})\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\le C\sum_{j=k+1}^{\infty}2^{-j(mn-\beta)}\prod_{i=1}^{m}\|f_{ij}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)}(k-j)\|b_1\|_{BMO}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \qquad (3.9)
\end{aligned}
$$

$$
\begin{aligned}
IV &= \sum_{j=k+1}^{\infty}2^{-j(mn-\beta)}\int_{\mathbb{R}^n\backslash B_k}\int_{\mathbb{R}^n\backslash B_k}...\int_{\mathbb{R}^n\backslash B_k}|f_1(t_1)f_2(t_2)...f_m(t_m)(b_1(t_1)-(b_1)_{B_j})|dt_1dt_2...dt_m\cdot\chi_k(z)\\
&= \sum_{j=k+1}^{\infty}2^{-j(mn-\beta)}\int_{\mathbb{R}^n\backslash B_k}\int_{\mathbb{R}^n\backslash B_k}...\int_{\mathbb{R}^n\backslash B_k}|f_1(t_1)f_2(t_2)...f_m(t_m)(b_1(t_1)-(b_1)_{B_j})|dt_1dt_2...dt_m\cdot\chi_k(z)\\
&\le C\sum_{j=k+1}^{\infty}2^{-j(mn-\beta)}\prod_{i=1}^{m}\|f_{ij}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\|(b_1(t_1)-(b_1)_{B_j})\chi_j\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\cdot\chi_k(x)
\end{aligned}
$$

$$
\begin{aligned}
\|IV\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\le C\sum_{j=k+1}^{\infty}2^{-j(mn-\beta)}\prod_{i=1}^{m}\|f_{ij}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\|(b_1(t_1)-(b_1)_{B_j})\chi_j\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|\chi_k(x)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\le C\sum_{j=k+1}^{\infty}2^{-j(mn-\beta)}\prod_{i=1}^{m}\|f_{ij}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\|b_1\|_{BMO}\|\chi_j\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)}\|\chi_k(x)\|_{L^{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \qquad (3.10)
$$

From inequalities (3.9) and (3.10), we derive the following:

$$
\begin{aligned}
&\|[b_1, H^*_\beta](f_1, f_2, ..., f_m)(z)\cdot\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\leq C\sum_{j=k+1}^{\infty}\prod_{i=1}^{m}2^{-j(mn-\beta)}(k-j)\|b_1\|_{BMO}\|f_{ij}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{p_i'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

By invoking Proposition 3.10, we obtain the following inequality:

$$
\begin{aligned}
\|[b_1, H^*_\beta](f_1, f_2, ..., f_m)\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,q(\cdot)}(\mathbb{R}^n)} &= \sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0}2^{k\alpha(\cdot)q}\|[b_1, H^*_\beta](f_1, f_2, ..., f_m)(z)\cdot\chi_k\|^q_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{q}}\\
&\leq \max\left\{\sup_{\substack{k_0\in\mathbb{Z}\\ k_0<0}}2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0}2^{k\alpha(0)q}\|[b_1, H^*_\beta](f_1, f_2, ..., f_m)(z)\cdot\chi_k\|^q_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{q}},\right.\\
&\quad \sup_{\substack{k_0\in\mathbb{Z}\\ k_0\geq 0}}2^{-k_0\lambda}\Bigg(\left(\sum_{k=-\infty}^{-1}2^{k\alpha(0)q}\|[b_1, H^*_\beta](f_1, f_2, ..., f_m)(z)\cdot\chi_k\|^q_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{q}}\\
&\quad \left.+\left(\sum_{k=0}^{k_0}2^{k\alpha(\infty)q}\|[b_1, H^*_\beta](f_1, f_2, ..., f_m)(z)\cdot\chi_k\|^q_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{q}}\Bigg)\right\}\\
&= \max\{B_1, B_2 + B_3\}.
\end{aligned}
$$

The estimates for B_i (where $i = 1, 2, 3$) can be readily determined by employing methods analogous to those in Theorem 1.3.

4. Conclusions

In this note, we have scrutinized the boundedness of the multilinear Hardy operator and its commutators within the framework of variable exponent Herz-Morrey spaces, predicated on the assumption that the symbol functions are drawn from BMO spaces. Under specific conditions, our findings yielded affirmative results. These outcomes may incite further scholarly inquiry into establishing analogous bounds in other function spaces characterized by variable exponents.

Author contributions

The authors contributed equally to this work. All authors read and approved the final copy of this paper.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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