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Research article

Some new estimations on the spectral radius of the Schur product of matrices

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Abstract: The present study investigates the Schur product of multiple nonnegative matrices $\boldsymbol{G}_1$, $\boldsymbol{G}_2$, $\cdots$, $\boldsymbol{G}_n$. By utilizing the Perron root estimation for nonnegative matrices and applying the Hölder inequality, we establish some upper bounds on $\rho\left(\boldsymbol{G}_1 \circ \boldsymbol{G}_2 \circ \cdots \circ \boldsymbol{G}_n\right)$. These novel findings encompass and extend certain earlier results. Some comparative analysis between our new results and existing results is conducted through numerical experiments. Theoretical analysis and data calculations demonstrate that our results outperform those reported in other studies.

Keywords: nonnegative matrix; Schur product; spectral radius; upper bound; irreducible
Mathematics Subject Classification: 15A15, 15A42

1. Introduction

We represent $C^{m\times n}$ as the collection of matrices composed of elements from the complex field, whereas $R^{m\times n}$ represents the collection of matrices formed by elements from the real field.

Let $\boldsymbol{G}=\left(g_{ij}\right)\in R^{m\times n}$, if $g_{ij}\geq 0$, then $\boldsymbol{G}$ is called a nonnegative matrix. If the strict inequality is true, we say that $\boldsymbol{G}$ is positive. The spectral radius $\rho(\boldsymbol{G})$ is

$$\rho(\boldsymbol{G}) \equiv \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \boldsymbol{G}\}.$$

If λ is any eigenvalue of $\boldsymbol{G}$, then $|\lambda| \le \rho(\boldsymbol{G})$; moreover, there is at least one eigenvalue λ for which $|\lambda| = \rho(\boldsymbol{G})$. Let $\boldsymbol{G} \in R^{n\times n}$, for $n \ge 2$, if no permutation matrix $\boldsymbol{A}$ exists such that

$$\boldsymbol{A}^T\boldsymbol{G}\boldsymbol{A} = \begin{pmatrix} W_{11} & W_{12} \\ 0 & W_{22} \end{pmatrix},$$

then $\boldsymbol{G}$ is irreducible, where $\boldsymbol{W}_{11} \in R^{s\times s}$ and $\boldsymbol{W}_{22} \in R^{(n-s)\times(n-s)}$. Otherwise, $\boldsymbol{G}$ is reducible.

The Schur product is an operation that is much simpler than matrix multiplication. For two matrices $\boldsymbol{G} = (g_{ij})$ and $\boldsymbol{H} = (h_{ij})$ with the same dimensions, the Schur product is represented as $\boldsymbol{G} \circ \boldsymbol{H} \equiv (g_{ij}h_{ij})$[1]. If $\boldsymbol{G} \ge 0$ and $\boldsymbol{H} \ge 0$, then so is $\boldsymbol{G} \circ \boldsymbol{H}$. For $\alpha > 0$, $\boldsymbol{G}^{(\alpha)} \equiv (g_{ij}^{\alpha})$ is called the α-th Schur power of $\boldsymbol{G} \ge 0$. The Schur product has extensive applications in various fields, such as trigonometric moments, integral equations, partial differential equations, probability theory, and combinatorial theory.

Let $\boldsymbol{G}_1 \ge 0$ and $\boldsymbol{G}_2 \ge 0$. The investigation of the Schur product, particularly its spectral radius, has emerged as a prominent research area among scholars in recent years. Some conclusions concerning the upper bounds of the spectral radius have been given. In [1], the following classical result was given $\rho(\boldsymbol{G}_1 \circ \boldsymbol{G}_2) \le \rho(\boldsymbol{G}_1)\rho(\boldsymbol{G}_2)$. Fang [2] and Huang [3] provided improved results respectively. Subsequently, literatures [4,5] improved the results of literatures [2,3]. Next, the result [4] was improved in [6,7], and the result [5] was improved in [8]. Audenaert [9] gave a result on $\rho(\boldsymbol{G}_1 \circ \boldsymbol{G}_2)$, that is, $\rho(\boldsymbol{G}_1 \circ \boldsymbol{G}_2) \le \rho(\boldsymbol{G}_1\boldsymbol{G}_2)$. However, the authors above only gave some estimations of the upper bound for the spectral radius of the Schur product of two nonnegative matrices by using the Gersgorin theorem and Brauer theorem, and these results were not accurate enough.

Huang [10] generalized the above result and gave the classic result $\rho(\boldsymbol{G}_1 \circ \boldsymbol{G}_2 \circ \cdots \circ \boldsymbol{G}_n) \le \rho(\boldsymbol{G}_1\boldsymbol{G}_2 \cdots \boldsymbol{G}_n)$. Inspired by Reference [10], the authors of Reference [11] gave several better results on the upper bounds for the spectral radius of the Schur product of several nonnegative matrices, but the accuracy was not high. Therefore, it is particularly important to seek higher precision computational methods.

Inspired by the existing research in [11], we continue our research in this specific field. We present four novel inequalities concerning the upper limit of $\rho(\boldsymbol{G}_1 \circ \boldsymbol{G}_2 \circ \cdots \circ \boldsymbol{G}_n)$. The obtained results generalize some previous conclusions, and certain findings can be regarded as specific instances of the given results. The numerical results confirm the superior accuracy of the obtained results compared to some existing findings.

2. Main conclusions

First, some lemmas that we use in the proof of our conclusions are given.

Lemma 2.1. [12] Let $\boldsymbol{G}=(g_{ij})\geq 0$, and for any principal submatrix $\overline{\boldsymbol{G}}$ of $\boldsymbol{G}$, it holds that $\rho(\overline{\boldsymbol{G}})\leq\rho(\boldsymbol{G})$. In particular, $\max\limits_{i=1,2,\cdots,n} g_{ii}\leq\rho(\boldsymbol{G})$.

Lemma 2.2. [2] Let $\boldsymbol{G}\geq 0$ and y be a nonnegative nonzero vector. If $\boldsymbol{G}y\leq\lambda y$ for some $\lambda\in R$, then $\rho(\boldsymbol{G})\leq\lambda$.

Lemma 2.3. [12] Let $\boldsymbol{G}\geq 0$ and suppose that $\boldsymbol{G}$ is irreducible. Then

(1) $\rho(\boldsymbol{G})>0$;

(2) The eigenvalue $\rho(\boldsymbol{G})$ is associated with the matrix $\boldsymbol{G}$;

(3) There is a vector u that is positive and satisfies $\boldsymbol{G}u=\rho(\boldsymbol{G})u$.

Lemma 2.4. [13] Let $\boldsymbol{G}=(g_{ij})\geq 0$. Then

$$\rho(\boldsymbol{G})\leq\max_{i\neq j}\frac{1}{2}\left\{g_{ii}+g_{jj}+\left[\left(g_{ii}-g_{jj}\right)^2+4\sum\nolimits_{l\neq i}g_{il}\sum\nolimits_{m\neq j}g_{jm}\right]^{1/2}\right\}.$$

Lemma 2.5. [14] If $a_q,b_q,\cdots,t_q>0$ and $\kappa_l>0$ satisfy $\sum\limits_{l=1}^{p}\kappa_l=1$, then

$$\sum_{q=1}^{s}a_qb_q\cdots t_q\leq\left(\sum_{q=1}^{s}a_q^{\kappa_1}\right)^{1/\kappa_1}\left(\sum_{q=1}^{s}b_q^{\kappa_2}\right)^{1/\kappa_2}\cdots\left(\sum_{q=1}^{s}t_q^{\kappa_p}\right)^{1/\kappa_p}.$$

Lemma 2.6. [11] If $\boldsymbol{E}_1,\boldsymbol{E}_2,\cdots,\boldsymbol{E}_n\in R^{n\times n}$ and $\boldsymbol{F}_1,\boldsymbol{F}_2,\cdots,\boldsymbol{F}_n$ are diagonal, then

$$\left(\boldsymbol{F}_n\cdots\boldsymbol{F}_2\boldsymbol{F}_1\right)^{-1}\left(\boldsymbol{E}_1\circ\boldsymbol{E}_2\circ\cdots\circ\boldsymbol{E}_n\right)\left(\boldsymbol{F}_n\cdots\boldsymbol{F}_2\boldsymbol{F}_1\right)=\left(\boldsymbol{F}_1^{-1}\boldsymbol{E}_1\boldsymbol{F}_1\right)\circ\left(\boldsymbol{F}_2^{-1}\boldsymbol{E}_2\boldsymbol{F}_2\right)\circ\cdots\circ\left(\boldsymbol{F}_n^{-1}\boldsymbol{E}_n\boldsymbol{F}_n\right).$$

The main conclusions of the upper bounds on $\rho(\boldsymbol{G}_1\circ\boldsymbol{G}_2\circ\cdots\circ\boldsymbol{G}_n)$ of this study are given below. For brevity, let $\boldsymbol{B}_1=\boldsymbol{G}_1^{(t)}$, $\boldsymbol{B}_2=\boldsymbol{G}_2^{(t)}$, $\cdots$, $\boldsymbol{B}_n=\boldsymbol{G}_n^{(t)}$ in Theorems 2.1 and 2.2.

Theorem 2.1. Let $\boldsymbol{G}_1=(a_{ij})\geq 0, \boldsymbol{G}_2=(b_{ij})\geq 0, \cdots, \boldsymbol{G}_n=(\tau_{ij})\geq 0$. Then

$$\rho(\boldsymbol{G}_1\circ\boldsymbol{G}_2\circ\cdots\circ\boldsymbol{G}_n)\leq\max_{1\leq i\leq n}\left\{a_{ii}b_{ii}\cdots\tau_{ii}+\left[\left(\rho(\boldsymbol{B}_1)-a_{ii}^t\right)\left(\rho(\boldsymbol{B}_2)-b_{ii}^t\right)\cdots\left(\rho(\boldsymbol{B}_n)-\tau_{ii}^t\right)\right]^{1/t}\right\}. \tag{2.1}$$

Proof. If $\boldsymbol{G}=\boldsymbol{G}_1\circ\boldsymbol{G}_2\circ\cdots\circ\boldsymbol{G}_n$ is irreducible. We can see that $\boldsymbol{G}_1,\boldsymbol{G}_2,\cdots,\boldsymbol{G}_n$ are irreducible; thus, $\boldsymbol{B}_1$, $\boldsymbol{B}_2,\cdots,\boldsymbol{B}_n$ are irreducible and nonnegative. According to Lemma 2.1, we have

$$\rho(\boldsymbol{B}_1)-a_{ii}^t>0,\quad \rho(\boldsymbol{B}_2)-b_{ii}^t>0,\quad \cdots,\quad \rho(\boldsymbol{B}_n)-\tau_{ii}^t>0,\qquad i=1,2,\cdots,n.$$

By Lemma 2.3, positive column vectors can be found for $\boldsymbol{\delta}=(\delta_i)$, $\boldsymbol{\varphi}=(\varphi_i)$, $\cdots$, $\varsigma=(\varsigma_i)$, such that

$$\boldsymbol{B}_1\boldsymbol{\delta}^{(t)}=\rho(\boldsymbol{B}_1)\boldsymbol{\delta}^{(t)},\quad \boldsymbol{B}_2\boldsymbol{\varphi}^{(t)}=\rho(\boldsymbol{B}_2)\boldsymbol{\varphi}^{(t)},\quad \cdots,\quad \boldsymbol{B}_n\varsigma^{(t)}=\rho(\boldsymbol{B}_n)\varsigma^{(t)}, \tag{2.2}$$

where $\boldsymbol{\delta}^{(t)}=(\delta_i^t)$, $\boldsymbol{\varphi}^{(t)}=(\varphi_i^t)$, $\cdots$, $\varsigma^{(t)}=(\varsigma_i^t)$.

Hence, Eq (2.2) can be expressed as follows:

$$a_{ii}^t\delta_i^t+\sum_{l\neq i}a_{il}^t\delta_l^t=\rho(\boldsymbol{B}_1)\delta_i^t, \tag{2.3}$$

$$b_{ii}^t\varphi_i^t+\sum_{l\neq i}b_{il}^t\varphi_l^t=\rho(\boldsymbol{B}_2)\varphi_i^t, \tag{2.4}$$

$$\cdots\cdots$$

$$\tau_{ii}^t\varsigma_i^t+\sum_{l\neq i}\tau_{il}^t\varsigma_l^t=\rho(\boldsymbol{B}_n)\varsigma_i^t. \tag{2.5}$$

From (2.3)–(2.5), we have

$$\sum_{l\neq i}a_{il}^t\delta_l^t=\left[\rho(\boldsymbol{B}_1)-a_{ii}^t\right]\delta_i^t,$$

$$\sum_{l\neq i}b_{il}^t\varphi_l^t=\left[\rho(\boldsymbol{B}_2)-b_{ii}^t\right]\varphi_i^t,$$

$$\cdots\cdots$$

$$\sum_{l\neq i}\tau_{il}^t\varsigma_l^t=\left[\rho(\boldsymbol{B}_n)-\tau_{ii}^t\right]\varsigma_i^t.$$

Let $\boldsymbol{u}=(u_i)=(\delta_i\varphi_i\cdots\varsigma_i)$ be a positive vector, For any i, by Lemma 2.5, we have

$$\begin{aligned}(\boldsymbol{Gu})_i&=a_{ii}b_{ii}\cdots\tau_{ii}u_i+\sum_{l\neq i}a_{il}b_{il}\cdots\tau_{il}u_l\\&=a_{ii}b_{ii}\cdots\tau_{ii}u_i+\sum_{l\neq i}(a_{il}\delta_l)(b_{il}\varphi_l)\cdots(\tau_{il}\varsigma_l)\end{aligned}$$

$$\le a_{ii}b_{ii}\cdots\tau_{ii}u_i+\left(\sum_{l\ne i}a_{il}^t\delta_l^t\right)^{1/t}\left(\sum_{l\ne i}b_{il}^t\varphi_l^t\right)^{1/t}\cdots\left(\sum_{l\ne i}\tau_{il}^t\varsigma_l^t\right)^{1/t}$$

$$=a_{ii}b_{ii}\cdots\tau_{ii}u_i+\left[\left(\rho\left(\boldsymbol{B}_1\right)-a_{ii}^t\right)\left(\rho\left(\boldsymbol{B}_2\right)-b_{ii}^t\right)\cdots\left(\rho\left(\boldsymbol{B}_n\right)-\tau_{ii}^t\right)\right]^{1/t}\delta_i\varphi_i\varsigma_i$$

$$=a_{ii}b_{ii}\cdots\tau_{ii}u_i+\left[\left(\rho\left(\boldsymbol{B}_1\right)-a_{ii}^t\right)\left(\rho\left(\boldsymbol{B}_2\right)-b_{ii}^t\right)\cdots\left(\rho\left(\boldsymbol{B}_n\right)-\tau_{ii}^t\right)\right]^{1/t}u_i$$

$$=\left\{a_{ii}b_{ii}\cdots\tau_{ii}u_i+\left[\left(\rho\left(\boldsymbol{B}_1\right)-a_{ii}^t\right)\left(\rho\left(\boldsymbol{B}_2\right)-b_{ii}^t\right)\cdots\left(\rho\left(\boldsymbol{B}_n\right)-\tau_{ii}^t\right)\right]^{1/t}\right\}u_i.$$

According to Lemma 2.2, this shows that

$$\rho\left(\boldsymbol{G}_1\circ\boldsymbol{G}_2\circ\cdots\circ\boldsymbol{G}_n\right)=\left\{a_{ii}b_{ii}\cdots\tau_{ii}u_i+\left[\left(\rho\left(\boldsymbol{B}_1\right)-a_{ii}^t\right)\left(\rho\left(\boldsymbol{B}_2\right)-b_{ii}^t\right)\cdots\left(\rho\left(\boldsymbol{B}_n\right)-\tau_{ii}^t\right)\right]^{1/t}\right\}u_i.$$

If $\boldsymbol{G}_1\circ\boldsymbol{G}_2\circ\cdots\circ\boldsymbol{G}_n$ is reducible. We define a permutation matrix $\boldsymbol{L}$ with

$$l_{12}=l_{23}=\cdots=l_{n1}=1,$$

the remaining $l_{ij}=0$. Let $\varepsilon>0$ be any chosen real number. Then $\boldsymbol{G}_1+\varepsilon\boldsymbol{L}$, $\boldsymbol{G}_2+\varepsilon\boldsymbol{L}$, $\cdots$, $\boldsymbol{G}_n+\varepsilon\boldsymbol{L}$ are nonnegative irreducible. Now, we replace $\boldsymbol{G}_1,\boldsymbol{G}_2,\cdots,\boldsymbol{G}_n$ with $\boldsymbol{G}_1+\varepsilon\boldsymbol{L}$, $\boldsymbol{G}_2+\varepsilon\boldsymbol{L}$, $\cdots$, $\boldsymbol{G}_n+\varepsilon\boldsymbol{L}$, and then, letting $\varepsilon\to 0$, the conclusion still holds by continuity.

If we suppose that $t=1$ in (2.1), the following Corollary 2.1 is presented below.

Corollary 2.1. [11] Let $\boldsymbol{G}_1=\left(a_{ij}\right)\ge 0,\boldsymbol{G}_2=\left(b_{ij}\right)\ge 0,\cdots,\boldsymbol{G}_n=\left(\tau_{ij}\right)\ge 0$. Then

$$\rho\left(\boldsymbol{G}_1\circ\boldsymbol{G}_2\circ\cdots\circ\boldsymbol{G}_n\right)$$
$$\le\max_{1\le i\le n}\left\{a_{ii}b_{ii}\cdots\tau_{ii}+\left(\rho\left(\boldsymbol{G}_1\right)-a_{ii}\right)\left(\rho\left(\boldsymbol{G}_2\right)-b_{ii}\right)\cdots\left(\rho\left(\boldsymbol{G}_n\right)-\tau_{ii}\right)\right\}. \tag{2.6}$$

Obviously, this conclusion is Theorem 2.1 in [11]. Let $t=1$, $n=2$; then, the following Corollary 2.2 is obtained, which is Theorem 4 in [2].

Corollary 2.2. [2] Let $\boldsymbol{G}_1=\left(a_{ij}\right)\ge 0,\boldsymbol{G}_2=\left(b_{ij}\right)\ge 0$. Then

$$\rho\left(\boldsymbol{G}_1\circ\boldsymbol{G}_2\right)\le\max_{1\le i\le n}\left\{a_{ii}b_{ii}+\left(\rho\left(\boldsymbol{G}_1\right)-a_{ii}\right)\left(\rho\left(\boldsymbol{G}_2\right)-b_{ii}\right)\right\}.$$

We obtain Corollary 2.3 by setting $t=n=2$.

Corollary 2.3. [7] Let $\boldsymbol{G}_1=\left(a_{ij}\right)\ge 0,\boldsymbol{G}_2=\left(b_{ij}\right)\ge 0$. Then

$$\rho\left(\boldsymbol{G}_1\circ\boldsymbol{G}_2\right)\le\max_{1\le i\le n}\left\{a_{ii}b_{ii}+\left[\left(\rho\left(\boldsymbol{G}_1^{(2)}\right)-a_{ii}^2\right)\left(\rho\left(\boldsymbol{G}_2^{(2)}\right)-b_{ii}^2\right)\right]^{1/2}\right\}.$$

Therefore, the results of the literature [2,7,11] are included in Theorem 2.1 of this paper.

Theorem 2.2. Let $G_1=(a_{ij})\geq 0, G_2=(b_{ij})\geq 0,\cdots, G_n=(\tau_{ij})\geq 0$. Then

$$\begin{aligned}&\rho(G_1\circ G_2\circ\cdots\circ G_n)\\&\leq\max_{1\leq i\leq n}\frac{1}{2}\Big\{a_{ii}b_{ii}\cdots\tau_{ii}+a_{jj}b_{jj}\cdots\tau_{jj}+\Big[\left(a_{ii}b_{ii}\cdots\tau_{ii}-a_{jj}b_{jj}\cdots\tau_{jj}\right)^2\\&\quad+4\Big(\left(\rho(B_1)-a_{ii}^t\right)\left(\rho(B_2)-b_{ii}^t\right)\cdots\left(\rho(B_n)-\tau_{ii}^t\right)\\&\quad\times\left(\rho(B_1)-a_{jj}^t\right)\left(\rho(B_2)-b_{jj}^t\right)\cdots\left(\rho(B_n)-\tau_{jj}^t\right)\Big)^{1/t}\Big]^{1/2}\Big\}.\end{aligned}\tag{2.7}$$

Proof. If $G=G_1\circ G_2\circ\cdots\circ G_n$ is irreducible. Obviously, $G_1, G_2,\cdots, G_n$ and $B_1, B_2,\cdots, B_n$ are irreducible. According to Lemma 2.1, we have

$$\rho(B_1)-a_{ii}^t>0,\quad \rho(B_2)-b_{ii}^t>0,\quad \cdots,\quad \rho(B_n)-\tau_{ii}^t>0,\qquad i=1,2,\cdots,n.$$

By Lemma 2.3, positive column vectors can be found for $\delta=(\delta_i)$, $\varphi=(\varphi_i)$, $\cdots$, $\varsigma=(\varsigma_i)$ such that

$$B_1\delta^{(t)}=\rho(B_1)\delta^{(t)},\quad B_2\varphi^{(t)}=\rho(B_2)\varphi^{(t)},\quad\cdots,\quad B_n\varsigma^{(t)}=\rho(B_n)\varsigma^{(t)},\tag{2.8}$$

where $\delta^{(t)}=(\delta_i^t)$, $\varphi^{(t)}=(\varphi_i^t)$, $\cdots$, $\varsigma^{(t)}=(\varsigma_i^t)$.

Hence, Eq (2.8) can be expressed as follows:

$$a_{ii}^t\delta_i^t+\sum_{l\neq i}a_{il}^t\delta_l^t=\rho(B_1)\delta_i^t,\tag{2.9}$$

$$b_{ii}^t\varphi_i^t+\sum_{l\neq i}b_{il}^t\varphi_l^t=\rho(B_2)\varphi_i^t,\tag{2.10}$$

……

$$\tau_{ii}^t\varsigma_i^t+\sum_{l\neq i}\tau_{il}^t\varsigma_l^t=\rho(B_n)\varsigma_i^t.\tag{2.11}$$

From (2.9)–(2.11), we have

$$\sum_{l\neq i}\frac{a_{il}^t\delta_l^t}{\delta_i^t}=\rho(B_1)-a_{ii}^t,$$

$$\sum_{l\neq i}\frac{b_{il}^t\varphi_l^t}{\varphi_i^t}=\rho(B_2)-b_{ii}^t,$$

……

$$\sum_{l\neq i}\frac{\tau_{il}^{t}\varsigma_{l}^{t}}{\varsigma_{i}^{t}}=\rho\left(\boldsymbol{B}_{n}\right)-\tau_{ii}^{t},\ \ i=1,2,\cdots,n\ .$$

Define the following nonsingular diagonal matrices:

$$\boldsymbol{C}_1=\operatorname{diag}\left(\delta_1,\delta_2,\cdots\delta_n\right),\quad \boldsymbol{C}_2=\operatorname{diag}\left(\varphi_1,\varphi_2,\cdots\varphi_n\right),\quad \cdots,\quad \boldsymbol{C}_n=\operatorname{diag}\left(\varsigma_1,\varsigma_2,\cdots\varsigma_n\right).$$

Let

$$\boldsymbol{P}_1=\left(a_{ij}'\right)=\boldsymbol{C}_1^{-1}\boldsymbol{G}_1\boldsymbol{C}_1=\left(\frac{a_{ij}\delta_j}{\delta_i}\right),\ \ \boldsymbol{P}_2=\left(b_{ij}'\right)=\boldsymbol{C}_2^{-1}\boldsymbol{G}_2\boldsymbol{C}_2=\left(\frac{b_{ij}\varphi_j}{\varphi_i}\right),\ \ \cdots,$$

$$\boldsymbol{P}_n=\left(\tau_{ij}'\right)=\boldsymbol{C}_n^{-1}\boldsymbol{G}_n\boldsymbol{C}_n=\left(\frac{\tau_{ij}\varsigma_j}{\varsigma_i}\right).$$

Then

$$\boldsymbol{P}=\boldsymbol{P}_1\circ\boldsymbol{P}_2\circ\cdots\circ\boldsymbol{P}_n=\left(p_{ij}\right),$$

where

$$p_{ij}=a_{ij}'b_{ij}'\cdots\tau_{ij}'=\begin{cases}a_{ii}b_{ii}\cdots\tau_{ii}, & i=j,\\ \dfrac{a_{ij}\delta_j}{\delta_i}\dfrac{b_{ij}\varphi_j}{\varphi_i}\cdots\dfrac{\tau_{ij}\varsigma_j}{\varsigma_i}, & i\neq j.\end{cases}$$

Obviously, $\boldsymbol{P}_1,\boldsymbol{P}_2,\cdots,\boldsymbol{P}_n$ are irreducible nonnegative matrices. From Lemma 2.6, we have

$$\begin{aligned}\rho(\boldsymbol{G})&=\rho\left[\left(\boldsymbol{C}_n\cdots\boldsymbol{C}_2\boldsymbol{C}_1\right)^{-1}\left(\boldsymbol{G}_1\circ\boldsymbol{G}_2\circ\cdots\circ\boldsymbol{G}_n\right)\left(\boldsymbol{C}_n\cdots\boldsymbol{C}_2\boldsymbol{C}_1\right)\right]\\&=\rho\left[\left(\boldsymbol{C}_1^{-1}\boldsymbol{G}_1\boldsymbol{C}_1\right)\circ\left(\boldsymbol{C}_2^{-1}\boldsymbol{G}_2\boldsymbol{C}_2\right)\circ\cdots\circ\left(\boldsymbol{C}_n^{-1}\boldsymbol{G}_n\boldsymbol{C}_n\right)\right]=\rho(\boldsymbol{P}).\end{aligned}$$

In addition, from Lemma 2.5, we obtain

$$\begin{aligned}\sum_{l\neq i}p_{il}&=\sum_{l\neq i}a_{il}'b_{il}'\cdots\tau_{il}'=\sum_{l\neq i}\frac{a_{il}\delta_l}{\delta_i}\frac{b_{il}\varphi_l}{\varphi_i}\cdots\frac{\tau_{il}\varsigma_l}{\varsigma_i}\\&\leq\left(\sum_{l\neq i}\frac{a_{il}^t\delta_l^t}{\delta_i^t}\right)^{1/t}\left(\sum_{l\neq i}\frac{b_{il}^t\varphi_l^t}{\varphi_i^t}\right)^{1/t}\cdots\left(\sum_{l\neq i}\frac{\tau_{il}^t\varsigma_l^t}{\varsigma_i^t}\right)^{1/t}\\&\leq\left[\left(\rho\left(\boldsymbol{B}_1\right)-a_{ii}^t\right)\left(\rho\left(\boldsymbol{B}_2\right)-b_{ii}^t\right)\cdots\left(\rho\left(\boldsymbol{B}_n\right)-\tau_{ii}^k\right)\right]^{1/t}.\end{aligned}\tag{2.12}$$

Similarly, we obtain

$$\sum_{m\neq j} p_{jm} \leq \left[\left(\rho(\boldsymbol{B}_1)-a_{jj}^t\right)\left(\rho(\boldsymbol{B}_2)-b_{jj}^t\right)\cdots\left(\rho(\boldsymbol{B}_n)-\tau_{jj}^t\right)\right]^{1/t}. \tag{2.13}$$

According to Lemma 2.4 and inequalities (2.12)–(2.13), we obtain

$$\begin{aligned}
&\rho(\boldsymbol{G}_1 \circ \boldsymbol{G}_2 \circ \cdots \circ \boldsymbol{G}_n) \\
&\leq \max_{i\neq j} \frac{1}{2}\left\{p_{ii}+p_{jj}+\left[\left(p_{ii}-p_{jj}\right)^2+4\sum_{l\neq i} p_{il}\sum_{m\neq j} p_{jm}\right]^{1/2}\right\} \\
&\leq \max_{1\leq i\leq n} \frac{1}{2}\left\{a_{ii}b_{ii}\cdots\tau_{ii}+a_{jj}b_{jj}\cdots\tau_{jj}+\left[\left(a_{ii}b_{ii}\cdots\tau_{ii}-a_{jj}b_{jj}\cdots\tau_{jj}\right)^2\right.\right. \\
&\quad +4\left(\left(\rho(\boldsymbol{B}_1)-a_{ii}^t\right)\left(\rho(\boldsymbol{B}_2)-b_{ii}^t\right)\cdots\left(\rho(\boldsymbol{B}_n)-\tau_{ii}^t\right)\right. \\
&\quad \left.\left.\left.\times\left(\rho(\boldsymbol{B}_1)-a_{jj}^t\right)\left(\rho(\boldsymbol{B}_2)-b_{jj}^t\right)\cdots\left(\rho(\boldsymbol{B}_n)-\tau_{jj}^t\right)\right)^{1/t}\right]^{1/2}\right\}.
\end{aligned}$$

If $\boldsymbol{G}_1 \circ \boldsymbol{G}_2 \circ \cdots \circ \boldsymbol{G}_n$ is reducible. The method of proof utilized is the same as that used in the previous Theorem 2.1.

Remark 2.1. By employing the proof methodology introduced in (2.7), we give a novel demonstration of (2.1). By [12, Theorem 8.1.22], we obtain

$$\begin{aligned}
\rho(\boldsymbol{G}_1 \circ \boldsymbol{G}_2 \circ \cdots \circ \boldsymbol{G}_n) &= \rho(\boldsymbol{G}) = \rho(\boldsymbol{P}) \\
&\leq \max_{1\leq i\leq n} \sum_{l=1}^n p_{il} = \max_{1\leq i\leq n} \sum_{l=1}^n a'_{il}b'_{il}\cdots\tau'_{il} \\
&= \max_{1\leq i\leq n}\left(a_{ii}b_{ii}\cdots\tau_{ii}+\sum_{l\neq i} a'_{il}b'_{il}\cdots\tau'_{il}\right).
\end{aligned}$$

Thus, from (2.12), we have

$$\begin{aligned}
&\rho(\boldsymbol{G}_1 \circ \boldsymbol{G}_2 \circ \cdots \circ \boldsymbol{G}_n) \\
&\leq \max_{1\leq i\leq n}\left(a_{ii}b_{ii}\cdots\tau_{ii}+\sum_{l\neq i} a'_{il}b'_{il}\cdots\tau'_{il}\right) \\
&\leq \max_{1\leq i\leq n}\left\{a_{ii}b_{ii}\cdots\tau_{ii}+\left[\left(\rho(\boldsymbol{B}_1)-a_{ii}^t\right)\left(\rho(\boldsymbol{B}_2)-b_{ii}^t\right)\cdots\left(\rho(\boldsymbol{B}_n)-\tau_{ii}^t\right)\right]^{1/t}\right\}.
\end{aligned}$$

Setting $t=1$ in (2.7), Corollary 2.4 is presented below:

Corollary 2.4. [11] Let $\boldsymbol{G}_1=(a_{ij})\geq 0, \boldsymbol{G}_2=(b_{ij})\geq 0, \cdots, \boldsymbol{G}_n=(\tau_{ij})\geq 0$. Then

$$\rho(\boldsymbol{G}_1 \circ \boldsymbol{G}_2 \circ \cdots \circ \boldsymbol{G}_n)$$

$$\le \max_{1\le i\le n}\frac{1}{2}\Big\{a_{ii}b_{ii}\cdots\tau_{ii}+a_{jj}b_{jj}\cdots\tau_{jj}+\Big[\left(a_{ii}b_{ii}\cdots\tau_{ii}-a_{jj}b_{jj}\cdots\tau_{jj}\right)^2$$

$$+4\left(\rho(\boldsymbol{G}_1)-a_{ii}\right)\left(\rho(\boldsymbol{G}_2)-b_{ii}\right)\cdots\left(\rho(\boldsymbol{G}_n)-\tau_{ii}\right)$$

$$\times\left(\rho(\boldsymbol{G}_1)-a_{jj}\right)\left(\rho(\boldsymbol{G}_2)-b_{jj}\right)\cdots\left(\rho(\boldsymbol{G}_n)-\tau_{jj}\right)\Big]^{1/2}\Big\}. \tag{2.14}$$

This conclusion is Theorem 2.2 in [11]. Let $t=1$, $n=2$; then, another Corollary 2.5 is obtained.

Corollary 2.5. [4] Let $\boldsymbol{G}_1=\left(a_{ij}\right)\ge 0, \boldsymbol{G}_2=\left(b_{ij}\right)\ge 0$. Then

$$\rho\left(\boldsymbol{G}_1\circ\boldsymbol{G}_2\right)\le \max_{1\le i\le n}\frac{1}{2}\Big\{a_{ii}b_{ii}+a_{jj}b_{jj}+\Big[\left(a_{ii}b_{ii}-a_{jj}b_{jj}\right)^2$$

$$+4\left(\rho(\boldsymbol{G}_1)-a_{ii}\right)\left(\rho(\boldsymbol{G}_2)-b_{ii}\right)\left(\rho(\boldsymbol{G}_1)-a_{jj}\right)\left(\rho(\boldsymbol{G}_2)-b_{jj}\right)\Big]^{1/2}\Big\}.$$

This conclusion is Theorem 4 in [4]. We obtain Corollary 2.6 by setting $t=n=2$.

Corollary 2.6. [7] Let $\boldsymbol{G}_1=\left(a_{ij}\right)\ge 0, \boldsymbol{G}_2=\left(b_{ij}\right)\ge 0$. Then

$$\rho\left(\boldsymbol{G}_1\circ\boldsymbol{G}_2\right)\le \max_{1\le i\le n}\frac{1}{2}\Big\{a_{ii}b_{ii}+a_{jj}b_{jj}+\Big[\left(a_{ii}b_{ii}-a_{jj}b_{jj}\right)^2$$

$$+4\left(\left(\rho\left(\boldsymbol{G}_1^{(2)}\right)-a_{ii}^2\right)\left(\rho\left(\boldsymbol{G}_2^{(2)}\right)-b_{ii}^2\right)\left(\rho\left(\boldsymbol{G}_1^{(2)}\right)-a_{jj}^2\right)\left(\rho\left(\boldsymbol{G}_2^{(2)}\right)-b_{jj}^2\right)\right)^{1/2}\Big]^{1/2}\Big\}.$$

Therefore, the results of [4,7,11] are included in Theorem 2.2.

Remark 2.2. We compare the results obtained from Theorem 2.1 with the findings derived through Theorem 2.2. For $i\ne j$, we assume that

$$a_{ii}b_{ii}\cdots\tau_{ii}+\left[\left(\rho(\boldsymbol{B}_1)-a_{ii}^t\right)\left(\rho(\boldsymbol{B}_2)-b_{ii}^t\right)\cdots\left(\rho(\boldsymbol{B}_n)-\tau_{ii}^t\right)\right]^{1/t}$$

$$\ge a_{jj}b_{jj}\cdots\tau_{jj}+\left[\left(\rho(\boldsymbol{B}_1)-a_{jj}^t\right)\left(\rho(\boldsymbol{B}_2)-b_{jj}^t\right)\cdots\left(\rho(\boldsymbol{B}_n)-\tau_{jj}^t\right)\right]^{1/t}. \tag{2.15}$$

From (2.15), we have

$$a_{ii}b_{ii}\cdots\tau_{ii}-a_{jj}b_{jj}\cdots\tau_{jj}+\left[\left(\rho(\boldsymbol{B}_1)-a_{ii}^t\right)\left(\rho(\boldsymbol{B}_2)-b_{ii}^t\right)\cdots\left(\rho(\boldsymbol{B}_n)-\tau_{ii}^t\right)\right]^{1/t}$$

$$\ge\left[\left(\rho(\boldsymbol{B}_1)-a_{jj}^t\right)\left(\rho(\boldsymbol{B}_2)-b_{jj}^t\right)\cdots\left(\rho(\boldsymbol{B}_n)-\tau_{jj}^t\right)\right]^{1/t}.$$

Thus, we have

$$a_{ii}b_{ii}\cdots\tau_{ii}+a_{jj}b_{jj}\cdots\tau_{jj}+\Big[\left(a_{ii}b_{ii}\cdots\tau_{ii}-a_{jj}b_{jj}\cdots\tau_{jj}\right)^2$$

$$+4\Big(\big(\rho(\boldsymbol{B}_1)-a_{ii}^t\big)\big(\rho(\boldsymbol{B}_2)-b_{ii}^t\big)\cdots\big(\rho(\boldsymbol{B}_n)-\tau_{ii}^t\big)$$

$$\times\big(\rho(\boldsymbol{B}_1)-a_{jj}^t\big)\big(\rho(\boldsymbol{B}_2)-b_{jj}^t\big)\cdots\big(\rho(\boldsymbol{B}_n)-\tau_{jj}^t\big)\Big)^{1/t}\Big]^{1/2}$$

$$\le a_{ii}b_{ii}\cdots\tau_{ii}+a_{jj}b_{jj}\cdots\tau_{jj}+\Big[\big(a_{ii}b_{ii}\cdots\tau_{ii}-a_{jj}b_{jj}\cdots\tau_{jj}\big)^2$$

$$+4\Big(\big(\rho(\boldsymbol{B}_1)-a_{ii}^t\big)\big(\rho(\boldsymbol{B}_2)-b_{ii}^t\big)\cdots\big(\rho(\boldsymbol{B}_n)-\tau_{ii}^t\big)\Big)^{1/t}$$

$$\times\Big(a_{ii}b_{ii}\cdots\tau_{ii}-a_{jj}b_{jj}\cdots\tau_{jj}+\Big(\big(\rho(\boldsymbol{B}_1)-a_{ii}^t\big)\big(\rho(\boldsymbol{B}_2)$$

$$-b_{ii}^t\big)^{\frac{1}{t}}\cdots\big(\rho(\boldsymbol{B}_n)-\tau_{ii}^t\big)\Big)^{1/t}\Big)\Big]^{1/2}$$

$$=a_{ii}b_{ii}\cdots\tau_{ii}+a_{jj}b_{jj}\cdots\tau_{jj}+\Big[\Big(a_{ii}b_{ii}\cdots\tau_{ii}-a_{jj}b_{jj}\cdots\tau_{jj}$$

$$+2\Big(\big(\rho(\boldsymbol{B}_1)-a_{ii}^t\big)\big(\rho(\boldsymbol{B}_2)-b_{ii}^t\big)\cdots\big(\rho(\boldsymbol{B}_n)-\tau_{ii}^t\big)\Big)^{1/t}\Big)^2\Big]^{1/2}$$

$$=2a_{ii}b_{ii}\cdots\tau_{ii}+2\Big[\big(\rho(\boldsymbol{B}_1)-a_{ii}^t\big)\big(\rho(\boldsymbol{B}_2)-b_{ii}^t\big)\cdots\big(\rho(\boldsymbol{B}_n)-\tau_{ii}^t\big)\Big]^{1/t}.$$

That is,

$$\rho(\boldsymbol{G}_1\circ\boldsymbol{G}_2\circ\cdots\circ\boldsymbol{G}_n)$$

$$\le\max_{1\le i\le n}\frac{1}{2}\Big\{a_{ii}b_{ii}\cdots\tau_{ii}+a_{jj}b_{jj}\cdots\tau_{jj}+\Big[\big(a_{ii}b_{ii}\cdots\tau_{ii}-a_{jj}b_{jj}\cdots\tau_{jj}\big)^2$$

$$+4\Big(\big(\rho(\boldsymbol{B}_1)-a_{ii}^t\big)\big(\rho(\boldsymbol{B}_2)-b_{ii}^t\big)\cdots\big(\rho(\boldsymbol{B}_n)-\tau_{ii}^t\big)$$

$$\times\big(\rho(\boldsymbol{B}_1)-a_{jj}^t\big)\big(\rho(\boldsymbol{B}_1)-b_{jj}^t\big)\cdots\big(\rho(\boldsymbol{B}_1)-\tau_{jj}^t\big)\Big)^{1/t}\Big]^{1/2}\Big\}$$

$$\le\max_{1\le i\le n}\frac{1}{2}\Big\{2a_{ii}b_{ii}\cdots\tau_{ii}+2\Big[\big(\rho(\boldsymbol{B}_1)-a_{ii}^t\big)\big(\rho(\boldsymbol{B}_2)-b_{ii}^t\big)\cdots\big(\rho(\boldsymbol{B}_n)-\tau_{ii}^t\big)\Big]^{1/t}\Big\}$$

$$\le\max_{1\le i\le n}\Big\{a_{ii}b_{ii}\cdots\tau_{ii}+\Big[\big(\rho(\boldsymbol{B}_1)-a_{ii}^t\big)\big(\rho(\boldsymbol{B}_2)-b_{ii}^t\big)\cdots\big(\rho(\boldsymbol{B}_n)-\tau_{ii}^t\big)\Big]^{1/t}\Big\}.$$

Hence, the conclusion drawn from Theorem 2.2 exceeds that of Theorem 2.1.

We provide an illustration to further confirm the excellence of our findings. Consider four nonnegative matrices.

$$G_1=\begin{pmatrix}4&1&0&2\\0&0.05&1&1\\0&0&4&0.5\\1&0.5&0&4\end{pmatrix},\qquad G_2=\begin{pmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix},$$

$$G_3=\begin{pmatrix}2&0&1&1\\1&4&0.5&0.5\\1&0&3&0.5\\0.5&1&1&2\end{pmatrix},\qquad G_4=\begin{pmatrix}2&0.5&0.5&0.5\\1&1&1&1\\0.5&0&2&0.5\\0&1&1&2\end{pmatrix}.$$

By direct calculation, we have $\rho\left(G_1^{(2)}\right)=18.0233$, $\rho\left(G_2^{(2)}\right)=4$, $\rho\left(G_3^{(2)}\right)=16.0292$, $\rho\left(G_4^{(2)}\right)=4.9132$. Let $A=G_1\circ G_2\circ G_3$, $B=G_1\circ G_2\circ G_3\circ G_4$. Then

$$\rho(A)=12.0014 \text{ and } \rho(B)=24.0001.$$

(1) According to inequalities (2.6) and (2.14), we get

$$\rho(A)\le 20.8846 \text{ and } \rho(A)\le 17.8268.$$

Setting $n=3, t=2$. From Theorems 2.1 and 2.2, we have

$$\rho(A)\le 18.5320 \text{ and } \rho(A)\le 17.7340.$$

(2) By inequalities (2.6) and (2.14), we obtain

$$\rho(B)\le 36.6608 \text{ and } \rho(B)\le 32.4451.$$

Setting $n=4, t=2$. From Theorems 2.1 and 2.2, we have

$$\rho(B)\le 30.2420 \text{ and } \rho(B)\le 28.1835.$$

Next, we present several other upper bounds of the spectral radius. For $G_1=(a_{ij})\ge 0$, $G_2=(b_{ij})\ge 0$, $\cdots$, $G_n=(\tau_{ij})\ge 0$, we write

$$M_1=G_1-D_1,\quad M_2=G_2-D_2,\quad \cdots,\quad M_n=G_n-D_n,$$

where $D_1=\operatorname{diag}(a_{ii})$, $D_2=\operatorname{diag}(b_{ii})$, $\cdots$, $D_n=\operatorname{diag}(\tau_{ii})$. Let $J_{G_1}=H^{-1}M_1$, $J_{G_2}=S^{-1}M_2$, $\cdots$, $J_{G_n}=Z^{-1}M_n$ with $H=\operatorname{diag}(h_{ii})$, $S=\operatorname{diag}(s_{ii})$, $\cdots$, $Z=\operatorname{diag}(z_{ii})$, where

$$h_{ii}=\begin{cases}a_{ii}, & a_{ii}\neq 0\\ 1, & a_{ii}=0\end{cases},\quad s_{ii}=\begin{cases}b_{ii}, & b_{ii}\neq 0\\ 1, & b_{ii}=0\end{cases},\quad \cdots,\quad z_{ii}=\begin{cases}\tau_{ii}, & \tau_{ii}\neq 0\\ 1, & \tau_{ii}=0\end{cases}.$$

Obviously, $\boldsymbol{J}_{\boldsymbol{G}_1}$, $\boldsymbol{J}_{\boldsymbol{G}_2}$, $\cdots$, $\boldsymbol{J}_{\boldsymbol{G}_n}$ are nonnegative matrices. For simplicity, we set $\boldsymbol{V}_1=\boldsymbol{J}_{\boldsymbol{G}_1}^{(t)}$, $\boldsymbol{V}_2=\boldsymbol{J}_{\boldsymbol{G}_2}^{(t)}$, $\cdots$, $\boldsymbol{V}_n=\boldsymbol{J}_{\boldsymbol{G}_n}^{(t)}$.

Theorem 2.3. Let $\boldsymbol{G}_1=(a_{ij})\geq 0, \boldsymbol{G}_2=(b_{ij})\geq 0, \cdots, \boldsymbol{G}_n=(\tau_{ij})\geq 0$. Then

$$\rho(\boldsymbol{G}_1\circ\boldsymbol{G}_2\circ\cdots\circ\boldsymbol{G}_n)\leq\max_{1\leq i\leq n}\left\{a_{ii}b_{ii}\cdots\tau_{ii}+h_{ii}s_{ii}\cdots z_{ii}\left[\rho(\boldsymbol{V}_1)\rho(\boldsymbol{V}_2)\cdots\rho(\boldsymbol{V}_n)\right]^{1/t}\right\}. \tag{2.16}$$

Proof. If $\boldsymbol{G}=\boldsymbol{G}_1\circ\boldsymbol{G}_2\circ\cdots\circ\boldsymbol{G}_n$ is irreducible. Obviously $\boldsymbol{G}_1,\boldsymbol{G}_2,\cdots,\boldsymbol{G}_n$ are irreducible, then $\boldsymbol{V}_1$, $\boldsymbol{V}_2$, $\cdots$, $\boldsymbol{V}_n$ are nonnegative and irreducible. According to Lemma 2.3, positive column vectors can be found for $\boldsymbol{\xi}=(\xi_i)$, $\boldsymbol{\mu}=(\mu_i)$, $\cdots$, $\boldsymbol{\omega}=(\omega_i)$, $i=1,2,\cdots,n$ such that

$$\boldsymbol{V}_1\boldsymbol{\xi}^{(t)}=\rho(\boldsymbol{V}_1)\boldsymbol{\xi}^{(t)},\quad \boldsymbol{V}_2\boldsymbol{\mu}^{(t)}=\rho(\boldsymbol{V}_2)\boldsymbol{\mu}^{(t)},\quad \cdots,\quad \boldsymbol{V}_n\boldsymbol{\omega}^{(t)}=\rho(\boldsymbol{V}_n)\boldsymbol{\omega}^{(t)},$$

where $\boldsymbol{\xi}^{(t)}=(\xi_i^t)$, $\boldsymbol{\mu}^{(t)}=(\mu_i^t)$, $\cdots$, $\boldsymbol{\omega}^{(t)}=(\omega_i^t)$.

Therefore, we have

$$\sum_{l\neq i}a_{il}^t\xi_l^t=\rho(\boldsymbol{V}_1)h_{ii}^t\xi_i^t,$$
$$\sum_{l\neq i}b_{il}^t\mu_l^t=\rho(\boldsymbol{V}_2)s_{ii}^t\mu_i^t,$$
$$\cdots\cdots$$
$$\sum_{l\neq i}\tau_{il}^t\omega_l^t=\rho(\boldsymbol{V}_n)z_{ii}^t\omega_i^t.$$

Let $\boldsymbol{x}=(x_i)=\boldsymbol{\xi}\circ\boldsymbol{\mu}\circ\cdots\circ\boldsymbol{\omega}=(\xi_i\mu_i\cdots\omega_i)$ be a positive vector. By Lemma 2.5, we have

$$\begin{aligned}(\boldsymbol{Gx})_i&=a_{ii}b_{ii}\cdots\tau_{ii}x_i+\sum_{l\neq i}a_{il}b_{il}\cdots\tau_{il}x_l\\ &=a_{ii}b_{ii}\cdots\tau_{ii}x_i+\sum_{l\neq i}(a_{il}\xi_l)(b_{il}\mu_l)\cdots(\tau_{il}\omega_l)\\ &\leq a_{ii}b_{ii}\cdots\tau_{ii}x_i+\left(\sum_{l\neq i}a_{il}^t\xi_l^t\right)^{1/t}\left(\sum_{l\neq i}b_{il}^t\mu_l^t\right)^{1/t}\cdots\left(\sum_{l\neq i}\tau_{il}^t\omega_l^t\right)^{1/t}\\ &=a_{ii}b_{ii}\cdots\tau_{ii}x_i+\left[\rho(\boldsymbol{V}_1)\xi_i^th_{ii}^t\right]^{1/t}\left[\rho(\boldsymbol{V}_2)\mu_i^ts_{ii}^t\right]^{1/t}\cdots\left[\rho(\boldsymbol{V}_n)\omega_i^tz_{ii}^t\right]^{1/t}\end{aligned}$$

$$=\left\{a_{ii}b_{ii}\cdots\tau_{ii}+h_{ii}s_{ii}\cdots z_{ii}\left[\rho(\boldsymbol{V}_1)\rho(\boldsymbol{V}_2)\cdots\rho(\boldsymbol{V}_n)\right]^{1/t}\right\}x_i.$$

Thus, according to Lemma 2.2, this shows that

$$\rho(\boldsymbol{G}_1\circ\boldsymbol{G}_2\circ\cdots\circ\boldsymbol{G}_n)\le\max_{1\le i\le n}\left\{a_{ii}b_{ii}\cdots\tau_{ii}+h_{ii}s_{ii}\cdots z_{ii}\left[\rho(\boldsymbol{V}_1)\rho(\boldsymbol{V}_2)\cdots\rho(\boldsymbol{V}_n)\right]^{1/t}\right\}.$$

Now, we consider that $\boldsymbol{G}_1\circ\boldsymbol{G}_2\circ\cdots\circ\boldsymbol{G}_n$ is reducible. The method of proof utilized is the same as that used in the previous Theorem 2.1.

Setting $t=1$ in [2.16], the following Corollary 2.7 is presented below:

Corollary 2.7. [11] Let $\boldsymbol{G}_1=(a_{ij})\ge 0,\boldsymbol{G}_2=(b_{ij})\ge 0,\cdots,\boldsymbol{G}_n=(\tau_{ij})\ge 0$. Then

$$\rho(\boldsymbol{G}_1\circ\boldsymbol{G}_2\circ\cdots\circ\boldsymbol{G}_n)\le\max_{1\le i\le n}\left\{a_{ii}b_{ii}\cdots\tau_{ii}+h_{ii}s_{ii}\cdots z_{ii}\rho\left(\boldsymbol{J}_{\boldsymbol{G}_1}\right)\rho\left(\boldsymbol{J}_{\boldsymbol{G}_2}\right)\cdots\rho\left(\boldsymbol{J}_{\boldsymbol{G}_n}\right)\right\}. \tag{2.17}$$

This conclusion is Theorem 2.3 in [11]. Let $t=1$, $n=2$; then, another Corollary 2.8 is obtained.

Corollary 2.8. [3] Let $\boldsymbol{G}_1=(a_{ij})\ge 0,\boldsymbol{G}_2=(b_{ij})\ge 0$. Then

$$\rho(\boldsymbol{G}_1\circ\boldsymbol{G}_2)\le\max_{1\le i\le n}\left\{a_{ii}b_{ii}+h_{ii}s_{ii}\rho\left(\boldsymbol{J}_{\boldsymbol{G}_1}\right)\rho\left(\boldsymbol{J}_{\boldsymbol{G}_2}\right)\right\}. \tag{2.18}$$

This conclusion is given in Theorem 6 in [3]. We obtain Theorem 2.4 by setting $t=n=2$.

Theorem 2.4. Let $\boldsymbol{G}_1=(a_{ij})\ge 0,\boldsymbol{G}_2=(b_{ij})\ge 0$. Then

$$\rho(\boldsymbol{G}_1\circ\boldsymbol{G}_2)\le\max_{1\le i\le n}\left\{a_{ii}b_{ii}+h_{ii}s_{ii}\left[\rho\left(\boldsymbol{J}_{\boldsymbol{G}_1}^{(2)}\right)\rho\left(\boldsymbol{J}_{\boldsymbol{G}_2}^{(2)}\right)\right]^{1/2}\right\}. \tag{2.19}$$

Remark 2.3. From [1, Lemma 5.7.8], we know that $\rho\left(\boldsymbol{G}^{(\alpha)}\right)\le\left[\rho(\boldsymbol{G})\right]^{\alpha}$. Thus, we have

$$\left[\rho\left(\boldsymbol{J}_{\boldsymbol{G}_1}^{(2)}\right)\rho\left(\boldsymbol{J}_{\boldsymbol{G}_2}^{(2)}\right)\right]^{1/2}\le\rho\left(\boldsymbol{J}_{\boldsymbol{G}_1}\right)\rho\left(\boldsymbol{J}_{\boldsymbol{G}_2}\right).$$

This shows that

$$\max_{1\le i\le n}\left\{a_{ii}b_{ii}+h_{ii}s_{ii}\left[\rho\left(\boldsymbol{J}_{\boldsymbol{G}_1}^{(2)}\right)\rho\left(\boldsymbol{J}_{\boldsymbol{G}_2}^{(2)}\right)\right]^{1/2}\right\}\le\max_{1\le i\le n}\left\{a_{ii}b_{ii}+h_{ii}s_{ii}\rho\left(\boldsymbol{J}_{\boldsymbol{G}_1}\right)\rho\left(\boldsymbol{J}_{\boldsymbol{G}_2}\right)\right\}.$$

Therefore, the result in (2.19) is superior to the result in (2.18).

Theorem 2.5. Let $\boldsymbol{G}_1=(a_{ij})\ge 0,\boldsymbol{G}_2=(b_{ij})\ge 0,\cdots,\boldsymbol{G}_n=(\tau_{ij})\ge 0$. Then

$$\rho\left(\boldsymbol{G}_1\circ\boldsymbol{G}_2\circ\cdots\circ\boldsymbol{G}_n\right)\le\max_{1\le i\le n}\frac{1}{2}\Big\{a_{ii}b_{ii}\cdots\tau_{ii}+a_{jj}b_{jj}\cdots\tau_{jj}+\Big[\left(a_{ii}b_{ii}\cdots\tau_{ii}-a_{jj}b_{jj}\cdots\tau_{jj}\right)^2$$
$$+4\left(h_{ii}s_{ii}\cdots z_{ii}\right)\left(h_{jj}s_{jj}\cdots z_{jj}\right)\left(\rho\left(\boldsymbol{V}_1\right)\rho\left(\boldsymbol{V}_2\right)\cdots\rho\left(\boldsymbol{V}_n\right)\right)^{2/t}\Big]^{1/2}\Big\}. \tag{2.20}$$

Proof. If $\boldsymbol{G}=\boldsymbol{G}_1\circ\boldsymbol{G}_2\circ\cdots\circ\boldsymbol{G}_n$ is irreducible. Obviously $\boldsymbol{G}_1,\boldsymbol{G}_2,\cdots,\boldsymbol{G}_n$ are irreducible, then $\boldsymbol{V}_1$, $\boldsymbol{V}_2$, $\cdots$, $\boldsymbol{V}_n$ are nonnegative and irreducible. According to Lemma 2.3, positive column vectors can be found for $\boldsymbol{\xi}=\left(\xi_i\right)$, $\boldsymbol{\mu}=\left(\mu_i\right)$, $\cdots$, $\boldsymbol{\omega}=\left(\omega_i\right)$, $i=1,2,\cdots,n$ such that

$$\boldsymbol{V}_1\boldsymbol{\xi}^{(t)}=\rho\left(\boldsymbol{V}_1\right)\boldsymbol{\xi}^{(t)},\quad \boldsymbol{V}_2\boldsymbol{\mu}^{(t)}=\rho\left(\boldsymbol{V}_2\right)\boldsymbol{\mu}^{(t)},\quad\cdots,\quad \boldsymbol{V}_n\boldsymbol{\omega}^{(t)}=\rho\left(\boldsymbol{V}_n\right)\boldsymbol{\omega}^{(t)}, \tag{2.21}$$

where $\boldsymbol{\xi}^{(t)}=\left(\xi_i^t\right)$, $\boldsymbol{\mu}^{(t)}=\left(\mu_i^t\right)$, $\cdots$, $\boldsymbol{\omega}^{(t)}=\left(\omega_i^t\right)$.

Therefore, from (2.21), we have

$$\sum_{l\ne i}\frac{a_{il}^t\xi_l^t}{\xi_i^t}=\rho\left(\boldsymbol{V}_1\right)h_{ii}^t,$$

$$\sum_{l\ne i}\frac{b_{il}^t\mu_l^t}{\mu_i^t}=\rho\left(\boldsymbol{V}_2\right)s_{ii}^t,$$

……

$$\sum_{l\ne i}\frac{\tau_{il}^t\omega_l^t}{\omega_i^t}=\rho\left(\boldsymbol{V}_n\right)z_{ii}^t,\quad i=1,2,\cdots,n.$$

Now, the following nonsingular diagonal matrices are defined:

$$\boldsymbol{K}_1=\operatorname{diag}\left(\xi_1,\xi_2,\cdots\xi_n\right),\quad \boldsymbol{K}_2=\operatorname{diag}\left(\mu_1,\mu_2,\cdots\mu_n\right),\quad\cdots,\quad \boldsymbol{K}_n=\operatorname{diag}\left(\omega_1,\omega_2,\cdots\omega_n\right).$$

Let

$$\boldsymbol{Q}_1=\left(a_{ij}''\right)=\boldsymbol{K}_1^{-1}\boldsymbol{G}_1\boldsymbol{K}_1=\left(\frac{a_{ij}\xi_j}{\xi_i}\right),\ \boldsymbol{Q}_2=\left(b_{ij}''\right)=\boldsymbol{K}_2^{-1}\boldsymbol{G}_2\boldsymbol{K}_2=\left(\frac{b_{ij}\mu_j}{\mu_i}\right),\ \cdots,$$

$$\boldsymbol{Q}_n=\left(\tau_{ij}''\right)=\boldsymbol{K}_n^{-1}\boldsymbol{G}_n\boldsymbol{K}_n=\left(\frac{\tau_{ij}\omega_j}{\omega_i}\right).$$

Then

$$\boldsymbol{Q}=\boldsymbol{Q}_1\circ\boldsymbol{Q}_2\circ\cdots\circ\boldsymbol{Q}_n=\left(q_{ij}\right),$$

where

$$q_{ij} = a''_{ij} b''_{ij} \cdots \tau''_{ij} = \begin{cases} a_{ii} b_{ii} \cdots \tau_{ii}, & j = i, \\ \dfrac{a_{ij}\xi_j}{\xi_i} \dfrac{b_{ij}\mu_j}{\mu_i} \cdots \dfrac{\tau_{ij}\omega_j}{\omega_i}, & j \neq i. \end{cases}$$

Clearly, $\boldsymbol{Q}_1, \boldsymbol{Q}_2, \cdots, \boldsymbol{Q}_n$ are irreducible and nonnegative matrices. From Lemma 2.6, we have

$$\begin{aligned} \rho(\boldsymbol{G}) &= \rho\left[\left(\boldsymbol{K}_n \cdots \boldsymbol{K}_2 \boldsymbol{K}_1 \right)^{-1} \left(\boldsymbol{G}_1 \circ \boldsymbol{G}_2 \circ \cdots \circ \boldsymbol{G}_n \right) \left(\boldsymbol{K}_n \cdots \boldsymbol{K}_2 \boldsymbol{K}_1 \right) \right] \\ &= \rho\left[\left(\boldsymbol{K}_1^{-1} \boldsymbol{G}_1 \boldsymbol{K}_1 \right) \circ \left(\boldsymbol{K}_2^{-1} \boldsymbol{G}_2 \boldsymbol{K}_2 \right) \circ \cdots \circ \left(\boldsymbol{K}_n^{-1} \boldsymbol{G}_n \boldsymbol{K}_n \right) \right] = \rho(\boldsymbol{Q}). \end{aligned}$$

In addition, from Lemma 2.5, we have

$$\begin{aligned} \sum_{l \neq i} q_{il} &= \sum_{l \neq i} a''_{il} b''_{il} \cdots \tau''_{il} = \sum_{l \neq i} \frac{a_{il}\xi_l}{\xi_i} \frac{b_{il}\mu_l}{\mu_i} \cdots \frac{\tau_{il}\omega_l}{\omega_i} \\ &\leq \left(\sum_{l \neq i} \frac{a_{il}^t \xi_l^t}{\xi_i^t} \right)^{1/t} \left(\sum_{l \neq i} \frac{b_{il}^t \mu_l^t}{\mu_i^t} \right)^{1/t} \cdots \left(\sum_{l \neq i} \frac{\tau_{il}^t \omega_l^t}{\omega_i^t} \right)^{1/t} \\ &\leq h_{ii} s_{ii} \cdots z_{ii} \left[\rho(\boldsymbol{V}_1) \rho(\boldsymbol{V}_2) \cdots \rho(\boldsymbol{V}_n) \right]^{1/t}. \end{aligned} \tag{2.22}$$

Similarly, we obtain

$$\sum_{m \neq j} q_{jm} = \sum_{m \neq j} a''_{jm} b''_{jm} \cdots \tau''_{jm} \leq h_{jj} s_{jj} \cdots z_{jj} \left[\rho(\boldsymbol{V}_1) \rho(\boldsymbol{V}_2) \cdots \rho(\boldsymbol{V}_n) \right]^{1/t}. \tag{2.23}$$

According to Lemma 2.4 and inequalities (2.22)–(2.23), we have

$$\begin{aligned} &\rho\left(\boldsymbol{G}_1 \circ \boldsymbol{G}_2 \circ \cdots \circ \boldsymbol{G}_n \right) \\ &\leq \max_{i \neq j} \frac{1}{2} \left\{ q_{ii} + q_{jj} + \left[\left(q_{ii} - q_{jj} \right)^2 + 4 \sum_{l \neq i} q_{il} \sum_{m \neq j} q_{jm} \right]^{1/2} \right\} \\ &\leq \max_{1 \leq i \leq n} \frac{1}{2} \left\{ a_{ii} b_{ii} \cdots \tau_{ii} + a_{jj} b_{jj} \cdots \tau_{jj} + \left[\left(a_{ii} b_{ii} \cdots \tau_{ii} - a_{jj} b_{jj} \cdots \tau_{jj} \right)^2 \right. \right. \\ &\quad \left. \left. + 4 \left(h_{ii} s_{ii} \cdots z_{ii} \right) \left(h_{jj} s_{jj} \cdots z_{jj} \right) \left(\rho(\boldsymbol{V}_1) \rho(\boldsymbol{V}_2) \cdots \rho(\boldsymbol{V}_n) \right)^{2/t} \right]^{1/2} \right\}. \end{aligned}$$

If $\boldsymbol{G}_1 \circ \boldsymbol{G}_2 \circ \cdots \circ \boldsymbol{G}_n$ is reducible. The method of proof utilized is the same as that used in the previous Theorem 2.1.

Remark 2.4. By employing the proof methodology introduced in [2.20], we present a novel demonstration of [2.16]. By [12, Theorem 8.1.22], we obtain

$$\rho(\boldsymbol{G}_1 \circ \boldsymbol{G}_2 \circ \cdots \circ \boldsymbol{G}_n) = \rho(\boldsymbol{G}) = \rho(\boldsymbol{Q})$$
$$\le \max_{1\le i\le n} \sum_{l=1}^{n} q_{il} = \max_{1\le i\le n} \sum_{l=1}^{n} a''_{il} b''_{il} \cdots \tau''_{il}$$
$$= \max_{1\le i\le n} \left(a_{ii} b_{ii} \cdots \tau_{ii} + \sum_{l\ne i} a''_{il} b''_{il} \cdots \tau''_{il} \right).$$

Thus, from (2.22), we have

$$\rho(\boldsymbol{G}_1 \circ \boldsymbol{G}_2 \circ \cdots \circ \boldsymbol{G}_n)$$
$$\le \max_{1\le i\le n} \left(a_{ii} b_{ii} \cdots \tau_{ii} + \sum_{l\ne i} a''_{il} b''_{il} \cdots \tau''_{il} \right)$$
$$\le \max_{1\le i\le n} \left\{ a_{ii} b_{ii} \cdots \tau_{ii} + h_{ii} s_{ii} \cdots z_{ii} \left[\rho(\boldsymbol{V}_1) \rho(\boldsymbol{V}_2) \cdots \rho(\boldsymbol{V}_n) \right]^{1/t} \right\}.$$

Setting $t = 1$ in [2.20], Corollary 2.9 is obtained as follows:

Corollary 2.9. [11] Let $\boldsymbol{G}_1 = (a_{ij}) \ge 0, \boldsymbol{G}_2 = (b_{ij}) \ge 0, \cdots, \boldsymbol{G}_n = (\tau_{ij}) \ge 0$. Then

$$\rho(\boldsymbol{G}_1 \circ \boldsymbol{G}_2 \circ \cdots \circ \boldsymbol{G}_n) \le \max_{1\le i\le n} \frac{1}{2} \Big\{ a_{ii} b_{ii} \cdots \tau_{ii} + a_{jj} b_{jj} \cdots \tau_{jj} + \Big[\left(a_{ii} b_{ii} \cdots \tau_{ii} - a_{jj} b_{jj} \cdots \tau_{jj} \right)^2$$
$$+4 \left(h_{ii} s_{ii} \cdots z_{ii} \right) \left(h_{jj} s_{jj} \cdots z_{jj} \right) \left(\rho(\boldsymbol{J}_{\boldsymbol{G}_1}) \rho(\boldsymbol{J}_{\boldsymbol{G}_2}) \cdots \rho(\boldsymbol{J}_{\boldsymbol{G}_n}) \right)^2 \Big]^{1/2} \Big\}. \quad (2.24)$$

This conclusion is Theorem 2.4 in [11]. Let $t = 1$, $n = 2$; then, we obtain Corollary 2.10, which is Theorem 3 in [5]:

Corollary 2.10. [5] Let $\boldsymbol{G}_1 = (a_{ij}) \ge 0, \boldsymbol{G}_2 = (b_{ij}) \ge 0$. Then

$$\rho(\boldsymbol{G}_1 \circ \boldsymbol{G}_2) \le \max_{1\le i\le n} \frac{1}{2} \left\{ a_{ii} b_{ii} + a_{jj} b_{jj} + \left[\left(a_{ii} b_{ii} - a_{jj} b_{jj} \right)^2 + 4 h_{ii} s_{ii} h_{jj} s_{jj} \rho^2(\boldsymbol{J}_{\boldsymbol{G}_1}) \rho^2(\boldsymbol{J}_{\boldsymbol{G}_2}) \right]^{1/2} \right\}. \quad (2.25)$$

We obtain Theorem 2.6 by setting $t = n = 2$.

Theorem 2.6. Let $\boldsymbol{G}_1 = (a_{ij}) \ge 0, \boldsymbol{G}_2 = (b_{ij}) \ge 0$. Then

$$\rho(\boldsymbol{G}_1 \circ \boldsymbol{G}_2) \le \max_{1\le i\le n} \frac{1}{2} \left\{ a_{ii} b_{ii} + a_{jj} b_{jj} + \left[\left(a_{ii} b_{ii} - a_{jj} b_{jj} \right)^2 + 4 h_{ii} s_{ii} h_{jj} s_{jj} \rho\left(\boldsymbol{J}_{\boldsymbol{G}_1}^{(2)}\right) \rho\left(\boldsymbol{J}_{\boldsymbol{G}_2}^{(2)}\right) \right]^{1/2} \right\}. \quad (2.26)$$

Remark 2.5. From [1, Lemma 5.7.8], we know that $\rho\left(\boldsymbol{G}^{(\alpha)}\right) \le \left[\rho(\boldsymbol{G})\right]^{\alpha}$. Thus, we have

$$\rho\left(\boldsymbol{J}_{\boldsymbol{G}_1}^{(2)}\right) \rho\left(\boldsymbol{J}_{\boldsymbol{G}_2}^{(2)}\right) \le \rho^2(\boldsymbol{J}_{\boldsymbol{G}_1}) \rho^2(\boldsymbol{J}_{\boldsymbol{G}_2}).$$

Therefore, the result in (2.26) is superior to the result in (2.25).

Remark 2.6. We compare the results obtained from Theorem 2.3 with the findings derived through Theorem 2.5. For $i \neq j$, we assume that

$$a_{ii}b_{ii}\cdots\tau_{ii}+h_{ii}s_{ii}\cdots z_{ii}\left[\rho(\boldsymbol{V}_1)\rho(\boldsymbol{V}_2)\cdots\rho(\boldsymbol{V}_n)\right]^{1/t}$$
$$\geq a_{jj}b_{jj}\cdots\tau_{jj}+h_{jj}s_{jj}\cdots z_{jj}\left[\rho(\boldsymbol{V}_1)\rho(\boldsymbol{V}_2)\cdots\rho(\boldsymbol{V}_n)\right]^{1/t}. \tag{2.27}$$

The above inequality (2.27) is equivalent to

$$a_{ii}b_{ii}\cdots\tau_{ii}-a_{jj}b_{jj}\cdots\tau_{jj}+h_{ii}s_{ii}\cdots z_{ii}\left[\rho(\boldsymbol{V}_1)\rho(\boldsymbol{V}_2)\cdots\rho(\boldsymbol{V}_n)\right]^{1/t}$$
$$\geq h_{jj}s_{jj}\cdots z_{jj}\left[\rho(\boldsymbol{V}_1)\rho(\boldsymbol{V}_2)\cdots\rho(\boldsymbol{V}_n)\right]^{1/t}.$$

Thus, we have

$$\begin{aligned}
&\left(a_{ii}b_{ii}\cdots n_{ii}-a_{jj}b_{jj}\cdots n_{jj}\right)^2\\
&+4\left(h_{ii}s_{ii}\cdots z_{ii}\right)\left(h_{jj}s_{jj}\cdots z_{jj}\right)\left[\rho(\boldsymbol{V}_1)\rho(\boldsymbol{V}_2)\cdots\rho(\boldsymbol{V}_n)\right]^{2/t}\\
\leq&\left(a_{ii}b_{ii}\cdots\tau_{ii}-a_{jj}b_{jj}\cdots\tau_{jj}\right)^2+4\left(h_{ii}s_{ii}\cdots z_{ii}\right)^2\left[\rho(\boldsymbol{V}_1)\rho(\boldsymbol{V}_2)\cdots\rho(\boldsymbol{V}_n)\right]^{2/t}\\
&+4\left(h_{ii}s_{ii}\cdots z_{ii}\right)\left[\rho(\boldsymbol{V}_1)\rho(\boldsymbol{V}_2)\cdots\rho(\boldsymbol{V}_n)\right]^{1/t}\left(a_{ii}b_{ii}\cdots\tau_{ii}-a_{jj}b_{jj}\cdots\tau_{jj}\right)\\
=&\left[a_{ii}b_{ii}\cdots\tau_{ii}-a_{jj}b_{jj}\cdots\tau_{jj}+2\left(h_{ii}s_{ii}\cdots z_{ii}\right)\left(\rho(\boldsymbol{V}_1)\rho(\boldsymbol{V}_2)\cdots\rho(\boldsymbol{V}_n)\right)^{1/t}\right]^2.
\end{aligned} \tag{2.28}$$

From (2.20) and (2.28), we obtain

$$\begin{aligned}
&\rho(\boldsymbol{G}_1\circ\boldsymbol{G}_2\circ\cdots\circ\boldsymbol{G}_n)\\
\leq&\max_{1\leq i\leq n}\frac{1}{2}\Big\{a_{ii}b_{ii}\cdots\tau_{ii}+a_{jj}b_{jj}\cdots\tau_{jj}+\Big[\left(a_{ii}b_{ii}\cdots\tau_{ii}-a_{jj}b_{jj}\cdots\tau_{jj}\right)^2\\
&+4\left(h_{ii}s_{ii}\cdots z_{ii}\right)\left(h_{jj}s_{jj}\cdots z_{jj}\right)\left(\rho(\boldsymbol{V}_1)\rho(\boldsymbol{V}_2)\cdots\rho(\boldsymbol{V}_n)\right)^{2/t}\Big]^{1/2}\Big\}\\
\leq&\max_{1\leq i\leq n}\frac{1}{2}\Big\{a_{ii}b_{ii}\cdots\tau_{ii}+a_{jj}b_{jj}\cdots\tau_{jj}+\Big[\Big(a_{ii}b_{ii}\cdots\tau_{ii}-a_{jj}b_{jj}\cdots\tau_{jj}\\
&+2\left(h_{ii}s_{ii}\cdots z_{ii}\right)\left(\rho(\boldsymbol{V}_1)\rho(\boldsymbol{V}_2)\cdots\rho(\boldsymbol{V}_n)\right)^{1/t}\Big)^2\Big]^{1/2}\Big\}\\
=&\max_{1\leq i\leq n}\left\{a_{ii}b_{ii}\cdots\tau_{ii}+h_{ii}s_{ii}\cdots z_{ii}\left[\rho(\boldsymbol{V}_1)\rho(\boldsymbol{V}_2)\cdots\rho(\boldsymbol{V}_n)\right]^{1/t}\right\}.
\end{aligned}$$

Hence, the conclusion drawn from Theorem 2.5 exceeds that of Theorem 2.3.

We provide an illustration to further confirm the excellence of our findings. Consider the following nonnegative matrices:

$$\boldsymbol{G}_1=\begin{pmatrix}4&1&1&1\\2&5&1&1\\0&2&4&1\\1&1&1&4\end{pmatrix},\quad \boldsymbol{G}_2=\begin{pmatrix}1&1&0&0\\1&3&2&0\\0&1&4&3\\0&0&1&5\end{pmatrix},\quad \boldsymbol{G}_3=\begin{pmatrix}1&0.5&2&0.5\\0.5&1&0.5&0\\0&0.5&1&0.5\\0&1&0.5&1\end{pmatrix},$$

$$\boldsymbol{G}_4=\begin{pmatrix}2&0&1&1\\1&4&0.5&0.5\\1&0&3&0.5\\0.5&1&1&2\end{pmatrix},\quad \boldsymbol{G}_5=\begin{pmatrix}2&0.5&0.5&0.5\\1&1&1&1\\0.5&0&2&0.5\\0&1&1&2\end{pmatrix},\quad \boldsymbol{G}_6=\begin{pmatrix}4&1&1&1\\2&2&1&1\\0&2&2&1\\1&1&1&1\end{pmatrix}.$$

By direct calculation, we have

$$\rho\left(\boldsymbol{J}_{\boldsymbol{G}_1}\right)=0.7652,\quad \rho\left(\boldsymbol{J}_{\boldsymbol{G}_2}\right)=0.7489,\quad \rho\left(\boldsymbol{J}_{\boldsymbol{G}_3}\right)=1.3343,$$

$$\rho\left(\boldsymbol{J}_{\boldsymbol{G}_4}\right)=0.8182,\quad \rho\left(\boldsymbol{J}_{\boldsymbol{G}_5}\right)=1.1218,\quad \rho\left(\boldsymbol{J}_{\boldsymbol{G}_6}\right)=1.7247.$$

$$\rho\left(\boldsymbol{J}^{(2)}_{\boldsymbol{G}_1}\right)=0.2287,\quad \rho\left(\boldsymbol{J}^{(2)}_{\boldsymbol{G}_2}\right)=0.3795,\quad \rho\left(\boldsymbol{J}^{(2)}_{\boldsymbol{G}_3}\right)=0.9351,$$

$$\rho\left(\boldsymbol{J}^{(2)}_{\boldsymbol{G}_4}\right)=0.3047,\quad \rho\left(\boldsymbol{J}^{(2)}_{\boldsymbol{G}_5}\right)=0.6263,\quad \rho\left(\boldsymbol{J}^{(2)}_{\boldsymbol{G}_6}\right)=1.1538.$$

$$\rho\left(\boldsymbol{G}_1\circ\boldsymbol{G}_2\right)=20.7439,\quad \rho\left(\boldsymbol{G}_1\circ\boldsymbol{G}_2\circ\boldsymbol{G}_3\right)=20.1878,\quad \rho\left(\boldsymbol{G}_1\circ\boldsymbol{G}_2\circ\cdots\circ\boldsymbol{G}_6\right)=192.0010.$$

(1) From inequalities (2.18) and (2.25), we otain

$$\rho\left(\boldsymbol{G}_1\circ\boldsymbol{G}_2\right)\le 31.4611 \text{ and } \rho\left(\boldsymbol{G}_1\circ\boldsymbol{G}_2\right)\le 28.4460.$$

According to Theorems 2.4 and 2.6, we have

$$\rho\left(\boldsymbol{G}_1\circ\boldsymbol{G}_2\right)\le 25.8921 \text{ and } \rho\left(\boldsymbol{G}_1\circ\boldsymbol{G}_2\right)\le 23.6368.$$

(2) By inequalities (2.17) and (2.24), we obtain

$$\rho\left(\boldsymbol{G}_1\circ\boldsymbol{G}_2\circ\boldsymbol{G}_3\right)\le 35.2926 \text{ and } \rho\left(\boldsymbol{G}_1\circ\boldsymbol{G}_2\circ\boldsymbol{G}_3\right)\le 31.8240.$$

Setting $n=3, t=2$. From Theorems 2.3 and 2.5, we have

$$\rho\left(\boldsymbol{G}_1\circ\boldsymbol{G}_2\circ\boldsymbol{G}_3\right)\le 25.6977 \text{ and } \rho\left(\boldsymbol{G}_1\circ\boldsymbol{G}_2\circ\boldsymbol{G}_3\right)\le 23.4746.$$

(3) From inequalities (2.17) and (2.24), we obtain

$$\rho\left(\boldsymbol{G}_1\circ\boldsymbol{G}_2\circ\cdots\circ\boldsymbol{G}_6\right)\le 425.2143 \text{ and } \rho\left(\boldsymbol{G}_1\circ\boldsymbol{G}_2\circ\cdots\circ\boldsymbol{G}_6\right)\le 343.8788.$$

Setting $n=6$, $t=2$. From Theorems 2.3 and 2.5, we have

$$\rho\left(\boldsymbol{G}_1 \circ \boldsymbol{G}_2 \circ \cdots \circ \boldsymbol{G}_6\right) \leq 217.6663 \quad \text{and} \quad \rho\left(\boldsymbol{G}_1 \circ \boldsymbol{G}_2 \circ \cdots \circ \boldsymbol{G}_6\right) \leq 197.3330 .$$

3. Conclusions

In this article, we have introduced several new inequalities regarding the maximum values of $\rho\left(\boldsymbol{G}_1 \circ \boldsymbol{G}_2 \circ \cdots \circ \boldsymbol{G}_n\right)$. These novel findings not only encompass and extend existing results but also offer improved accuracy. The results can be used as a useful supplement in the field of nonnegative matrix theory.

Use of Generative-AI tools declaration

The author declares that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that there is no conflict of interest.

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