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Research article

Compactness and axioms of countability in soft hyperspaces

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Abstract: In this paper, we studied the compactness relationships, the local compactness relationships, the separability, and the axiom of countability relationships in a soft topological space and its soft hyperspaces. In particular, the compactness relationships, the local compactness relationships, the separability, and the axiom of countability relationships in a classical topological space and its hyperspace were treated as corollaries.

Keywords: soft topological space; soft T_i-space ($i = 0, 1, 2, 3, 4$); soft compactness; soft local compactness; soft separability; soft axioms of countability; soft hyperspace
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1. Introduction

The starting point of research on hyperspaces was by Riemann [1] in 1868. Then, Vietories [2, 3] studied it extensively (see [4–12] for further research). Recently, Macías and Nadler [13] obtained continua where the one-fold hyperspace is a cone.

The notion of soft sets was proposed by Molodtsov [14] in 1999. Shabir and Naz [15] applied topology and investigated separation axioms in a soft topological space. Nazmul and Samanta [16] defined soft metrics and discussed their basic properties. Bayramov and Aras [17] introduced the new concepts of separation axioms in a soft topological space and dealt with their properties (see [18–20] for further research). Baek et al. [21] defined separation axioms in an interval-valued soft topological space, studied their properties, and obtained some relationships among them (see [22, 23]). Zorlutuna et al. [24] investigated compactness in a soft topological space as well as basic properties related to soft topology (see [25–27] for further research). Bayramov and Gunduz [28] discussed soft local compactness in a soft topological space.

In 2015, Akdağ and Erol [29] initially proposed the notion of soft Vietoris topologies of soft topological spaces and gave the relationship between Vietoris continuity of soft multifunction and continuity of soft mapping (see [30]). Shakir [31], unlike Akdağ and Erol, defined Vietoies soft hyperspace and studied some basic properties. Demir [32] dealt with the axiom of countabilty in Vietoies soft hyperspace in the sense of Akdağ and Erol. Özkan [33] obtained some properties of the continuity of soft multifunction proposed by Akdağ and Erol. Recently, Baek et al. [34] obtained some basic properties for soft hyperspaces and discussed relationships between separation axioms in a soft topological space and its soft hyperspace.

The relationships between compactness, local compactness, and separability between a classical topological space and its hyperspace have been studied by many researchers. So, we believe that research into soft settings is necessary. The purpose of our research is to discuss more diverse relationships in soft topological space and soft hyperspace. To accomplish this, our research is conducted as follows. First, we study the compactness relationships in soft topological space and its soft hyperspaces. Next, we discuss the local compactness relationships. Finally, we deal with the separability and the axiom of countability relationships.

2. Preliminaries

In this section, we recall the basic concepts and results needed for the next sections. Throughout this paper, let X denote a non-empty universe set, E a set of parameters, and 2^X the power set of X.

Definition 2.1 ([14,35]). Let $E' \in 2^E$. Then an $F_{E'}$ is called a *soft set* over X, if $F_{E'} : E \to 2^X$ is a mapping such that $F_{E'}(e) = \emptyset$ for each $e \notin E'$. In this case, E' is called the *support* of $F_{E'}$.

For each $e \in E'$, $F_{E'}(e)$ may be considered as the set of e-approximate elements of the soft set $F_{E'}$.

Definition 2.2 ([36,37]). A soft set $F_{E'}$ over X is called:

(i) a *null soft set* or a *relative null soft set* (with respect to E'), denoted by $\emptyset_{E'}$, if $F_{E'}(e) = \emptyset$ for each $e \in E'$,

(ii) an *absolute soft set* or a *relative whole soft set* (with respect to E'), denoted by $X_{E'}$, if $F_{E'}(e) = X$ for each $e \in E'$.

The *empty* [resp. *whole*] *soft set* over X with respect to E, denoted by $\emptyset_E$ [resp. X_E], is a soft set over X defined by, for each $e \in E$,

$$\emptyset_E(e) = \emptyset \text{ [resp. } X_E(e) = X].$$

We will denote the set of all soft sets over X having all the subsets of E as the supports by $SS(X)$, while the set of all soft sets over X having E itself as the support by $SS_E(X)$ (see [36, 37] for the definitions of the inclusion, the equality, the complement, the intersection, and the union on $SS(X)$). Also, operations on $SS_E(X)$ can be defined similarly to those on $SS(X)$.

Definition 2.3 ([15]). Let $A,\ B \in SS_E(X)$. Then the *difference* of A and B, denoted by $A \setminus B$, is a soft set over X defined by:

$$(A \setminus B)(e) = A(e) \setminus B(e) \text{ for each } e \in E.$$

Definition 2.4 ([15]). Let $A \in SS_E(X)$ and $x \in X$. Then we say that x *belongs to* A, denoted by $x \in A$, if $x \in A(e)$ for each $e \in E$.

Definition 2.5 ([15]). Let $x \in X$, Y be a nonempty subset of X and $A \in SS_E(X)$.

(i) A soft set x_E over X is defined by $x_E(e) = \{x\}$ for each $e \in E$. In this case, x_E is called a *singleton soft set* of X.

(ii) A soft set Y_E over X is defined by $Y_E(e) = Y$ for each $e \in E$.

Definition 2.6 ([16,38,39]). Let $A \in SS_E(X)$. Then

(i) A is called a *soft point in* X *with the value* $x \in X$ *and the support* $e \in E$ or a *soft element*, denoted by e_x, if for each $f \in E$,

$$e_x(f) = \begin{cases} \{x\} & \text{if } f = e \\ \\ \emptyset & \text{otherwise.} \end{cases}$$

(ii) we say that e_x *belongs to* A, denoted by $e_x \in A$, if $e_x(e) = \{x\} \subset A(e)$.

We will denote the set of all soft points over X with respect to E by $SP_E(X)$.

Definition 2.7 ([16]). Let $e_x,\ f_y \in SP_E(X)$. Then we say that e_x and f_y are equal, denoted by $e_x = f_y$, if $e = f$ and $e_x(e) = f_y(f)$, i.e., $x = y$.

It is obvious that $e_x(e) \neq f_y(f)$ if and only if $x \neq y$ or $e \neq f$.

Result 2.8 (Proposition 3.5, [16]). *For each* $A \in SS_E(X)$, $A = \bigcup_{e_x \in A} e_x$.

Result 2.9 (Proposition 3.6, [16]). *Let* $A,\ B \in SS_E(X)$. *Then* $A \subset B$ *if and only if* $e_x \in B$ *for each* $e_x \in A$ *and thus* $A = B$ *if and only if* $e_x \in A \Leftrightarrow e_x \in B$.

Result 2.10 (Proposition 3.7, [16]). *Let* $A,\ B \in SS_E(X)$ *and* $e_x \in SP_E(X)$. *Then*

(1) $e_x \in A$ *if and only if* $e_x \notin A^c$,

(2) $e_x \in A \cup B$ *if and only if* $e_x \in A$ *or* $e_x \in B$,

(3) $e_x \in A \cap B$ *if and only if* $e_x \in A$ *and* $e_x \in B$.

Definition 2.11 ([15]). Let $\tau \subset SS_E(X)$. Then τ is called a *soft topology* on X, if it satisfies the following conditions:

(i) $\emptyset_E,\ X_E \in \tau$,

(ii) $A \cap B \in \tau$ for any $A,\ B \in \tau$,

(iii) $\bigcup_{j \in J} A_j \in \tau$ for each $(A_j)_{j \in J} \subset \tau$, where J denotes an index set.

The triple (X, τ, E) is called a *soft topological space* over X. Each member of τ is called a *soft open set* in X and a soft set A over X is called a *closed soft set* in X, if $A^c \in \tau$, where A^c is a soft set over X defined by: $A^c(e) = X - A(e)$ for each $e \in E$.

It is obvious that $\{\emptyset_E, X_E\}$ [resp. $SS_E(X)$] is a soft topology on X. In this case, $\{\emptyset_E, X_E\}$ [resp. $SS_E(X)$] is called the *soft indiscrete* [resp. *discrete*] *topology* on X and the triple $(X, \{\emptyset_E, X_E\}, E)$ [resp. $(X, SS_E(X), E)$] is called a *soft indiscrete* [resp. *discrete*] *space*.

Result 2.12 (Proposition 5, [15]). *Let* (X, τ, E) *be a soft topological space. Then the collection of subsets of* X,

$$\tau_e = \{A(e) \in 2^X : A \in \tau\} \text{ for each } e \in E,$$

is a topology on X.

In this case, τ_e is called the *topology on* X *induced by* E.

Definition 2.13 ([35]). Let (X, τ, E) be a soft topological space and $A \in SS_E(X)$. Then the *soft interior* and the *soft closure* of A, denoted by $S\,int(A)$ or A° and $S\,cl(A)$ or $\overline{A}$, are soft sets over X, respectively, defined as follows:

$$S\,int(A) = \bigcup\{U \in \tau : U \subset A\},\ S\,cl(A) = \bigcap\{F \in \tau^c : A \subset F\}.$$

Result 2.14 (Theorems 8 and 11, [35]). *Let (X, τ, E) be a soft topological space and $A,\ B \in SS_E(X)$. Then the following hold:*

(1) $S\,cl(A)^c = S\,int(A^c)$,

(3) $S\,int(S\,int(A)) = S\,int(A)$, $S\,cl(S\,cl(A)) = S\,cl(A)$,

(3) *if $A \subset B$, then $S\,int(A) \subset S\,int(B)$, $S\,cl(A) \subset S\,cl(B)$,*

(4) $S\,int(A) \cap S\,int(B) = S\,int(A \cap B)$, $S\,cl(A) \cap S\,cl(B) \subset S\,cl(A \cap B)$,

(5) $S\,int(A) \cup S\,int(B) \subset S\,int(A \cup B)$, $S\,cl(A) \cup S\,cl(B) = S\,cl(A \cup B)$.

Definition 2.15 ([39]). Let (X, τ, E) be a soft topological space, $e_x \in SP_E(X)$, and $A \in SS_E(X)$. Then A is called a *soft neighborhood* (briefly, soft nbd) of e_x, if there is $U \in \tau$ such that $e_x \in U \subset A$, i.e., e_x is a soft interior point of A. The set of all soft nbds of e_x will be denoted by $\widetilde{\mathcal{N}}(e_x)$, i.e.,

$$\mathcal{N}(e_x) = \{A \in SS_E(X) : \textit{there is } U \in \tau \textit{ such that } e_x \in U \subset A\}.$$

In particular, the family of all soft open nbds of e_x, denoted by $\widetilde{\mathcal{SN}}(e_x)$,

$$\mathcal{SN}(e_x) = \{U \in \tau : e_x \in U\},$$

will be called the *system of soft open neighborhoods* of e_x.

Result 2.16 ([39]). *Let (X, τ, E) be a soft topological space, $e_x \in SP_E(X)$, and $A,\ B \in SS_E(X)$. Then the following hold:*

(1) *if $A \in \mathcal{SN}(e_x)$, then $e_x \in A$,*

(2) *if $A,\ B \in \mathcal{SN}(e_x)$, then $A \cap B \in \mathcal{SN}(e_x)$,*

(3) *if $A \in \mathcal{SN}(e_x)$ and $A \subset B$, then $B \in \mathcal{SN}(e_x)$,*

(4) *if $A \in \mathcal{SN}(e_x)$, then there is $U \in \mathcal{SN}(f_y)$ such that $A \in \mathcal{SN}(e_x)$ for each $f_y \in SP_E(X)$ such that $f_y \in U$,*

(5) *$A \in \tau$ if and only if A contains a soft nbd of each of its points.*

Definition 2.17 (Proposition 3, [40]). Let (X, τ, E) be a soft topological space, $e_x \in SP_E(X)$, and A, $B \in SS_E(X)$. Then e_x is called a *soft limit point* of A, if $U \cap (A \setminus \{e_x\}) \neq \emptyset_E$. The set of all soft limit points of A is called the *derived soft set* over X and will be denoted by $S\,d(A)$.

Result 2.18 (Theorems 13 and 15, [35]). *Let (X, τ, E) be a soft topological space and $A,\ B \in SS_E(X)$. Then the following hold:*

(1) $A \cup S\,d(A) = S\,cl(A)$,

(2) $S\,d(A) \subset S\,cl(A)$,

(3) *if $A \subset B$, then $S\,d(A) \subset S\,d(B)$,*

(4) $S\,d(A \cap B) \subset S\,d(A) \cap S\,d(B)$,

(5) $S\,d(A \cup B) = S\,d(A) \cup S\,d(B)$,

(6) *$A \in \tau^c$ if and only if $S\,d(A) \subset A$.*

Definition 2.19 ([17]). A soft topological space (X, τ, E) is called a:

(i) *soft T_0-space*, if for any $e_x,\ f_y \in SP_E(X)$ with $e_x \neq f_y$, there are $U \in \mathcal{SN}(e_x),\ V \in \mathcal{SN}(f_y)$ such that either $e_x \in U,\ f_y \notin U$ or $f_y \in V,\ e_x \notin V$,

(ii) *soft T_1-space*, if for any $e_x,\ f_y \in SP_E(X)$ with $e_x \neq f_y$, there are $U \in \mathcal{SN}(e_x),\ V \in \mathcal{SN}(f_y)$ such that $e_x \in U,\ f_y \notin U$ and $f_y \in V,\ e_x \notin V$,

(iii) *soft T_2-space*, if for any $e_x,\ f_y \in SP_E(X)$ with $e_x \neq f_y$, there are $U \in \mathcal{SN}(e_x),\ V \in \mathcal{SN}(f_y)$ such that $e_x \in U, f_y \in V$ and $U \cap V = \emptyset_E$,

(iv) *soft regular space*, if for each $A \in \tau^c$ with $e_x \notin A$, there are $U,\ V \in \tau$ such that $e_x \in U,\ A \subset V$ and $U \cap V = \emptyset_E$,

(v) *soft T_3-space*, if it is both a soft regular and a soft T_1-space.

From Remarks 4.1–4.3 in [17], the following implication holds:

$$soft\,T_4 \Rightarrow soft\,T_3 \Rightarrow soft\,T_2 \Rightarrow soft\,T_1 \Rightarrow soft\,T_0. \tag{2.1}$$

Result 2.20 (Theorems 4.1 and 4.4, [17]). *Let (X, τ, E) be a soft topological space. Then*

(1) *X is a soft T_1-space if and only if $e_x \in \tau^c$ for each $e_x \in SP_E(X)$,*

(2) *X is a soft T_3-space if and only if for each $e_x \in U \in \tau$, there is $V \in \tau$ such that $e_x \in V \subset Scl(V) \subset U$.*

Result 2.21 (Proposition 4.1 and Theorem 4.5, [17]). *Let (X, τ, E) be a soft topological space. If X is a soft T_i-space, then (X, τ_e) is a T_i-space for each $e \in E$, where $i \in \{0, 1, 2, 3\}$.*

Definition 2.22 ([15]). A soft topological space (X, τ, E) is called a:

(i) *soft normal space*, if for any $A,\ B \in \tau^c$ with $A \cap B = \emptyset_E$, there is $U,\ V \in \tau$ such that $A \subset U$, $B \subset V$, and $U \cap V = \emptyset_E$,

(ii) *soft T_4-space*, if it is both a soft normal space and a soft T_1-space.

Result 2.23 (Theorem 4.6, [17]). *Let (X, τ, E) be a soft topological space. Then X is a soft T_4-space if and only if for each $A \in \tau^c$ and each $U \in \tau$ with $A \subset U$, there is $V \in \tau$ such that $A \subset V \subset Scl(V) \subset U$.*

Result 2.24 (Proposition 4.1, Theorem 4.5, [17]). *Let (X, τ, E) be a soft topological space. If X is a soft T_i-space, then (X, τ_e) is a T_i-space for each $i \in \{0, 1, 2, 3\}$ and each $e \in E$. However, we can see that (X, τ, E) is a soft T_4-space but (X, τ_e) is not a T_4-space for each $e \in E$ (see Remark 5(2) and Example 10 in [15]).*

Definition 2.25 ([38]). Let (X, τ, E) be a soft topological space and $\beta \subset \tau$. Then β is called a *soft base* for τ, if every member of τ can be expressed as the union of some members of β.

Notation 2.26 ([34]). Let (X, τ, E) be a soft topological space and $A \in SS_E(X)$. Then

(i) $2^X_E = \{F \in \tau^c : F \neq \emptyset_E\}$,

(ii) $2^A_E = \{F \in 2^X_E : F \subset A\}$,

(iii) $2^X_e = \{F(e) \in \tau^c_e : F \neq \emptyset\}$ for each $e \in E$,

(iv) $2^A_e = \{F(e) \in 2^X_e : F(e) \subset A(e)\}$ for each $e \in E$.

Result 2.27 (Proposition 3.10, [34]). *Let (X, τ, E) be a soft topological space and $\mathcal{B}_{SV}$ a family of the form $\langle U_1, U_2, \cdots, U_n\rangle$ such that $U_i \in \tau$ for each $i = 1,\ 2,\ \cdots, n$, where*

$$\langle U_1, U_2, \cdots, U_n\rangle = \{F \in 2^X_E : F \subset \bigcup_{i=1}^{n} U_i,\ F \cap U_i \neq \emptyset_E \ \textit{for each}\ i \in \{1, 2, \cdots, n\}\}.$$

Then $\mathcal{B}_{S\nu}$ is a soft base for some topology $\mathcal{T}_{S\nu}$ on 2^X_E. In fact,

$$\mathcal{T}_{S\nu} = \{\emptyset_E\} \cup \{U \in SS_E(X) : U = \bigcup \mathcal{B}\ for\ some\ \mathcal{B} \subset \mathcal{B}_{S\nu}\}.$$

In this case, $\mathcal{T}_{S\nu}$ is called the *soft Vietories* $(finite)$ *topology* on 2^X_E. The pair $(2^X_E, \mathcal{T}_{S\nu})$ is called a *soft hyperspace with soft Vietories topology* (briefly, soft hyperspace).

Result 2.28 (See Corollary 3.11, [34])**.** *Let (X, τ, E) be a soft topological space and $\mathcal{B}_{\nu,e}$ a family of the form $\langle U_1(e), U_2, \cdots, U_n(e)\rangle$ such that $U_i(e) \in \tau_e$ for each $i = 1,\ 2,\ \cdots, n$ and each $e \in E$. Then $\mathcal{B}_{\nu,e}$ is a base for some topology $\mathcal{T}_{\nu,e}$ on 2^X_e. In fact,*

$$\mathcal{T}_{\nu,e} = \{\emptyset\} \cup \{U \in 2^X : U = \bigcup \mathcal{B}\ for\ some\ \mathcal{B} \subset \mathcal{B}_{\nu,e}\}.$$

In this case, $\mathcal{T}_{\nu,e}$ is called the *Vietories* $(finite)$ *topology* on 2^X_e for $e \in E$. The pair $(2^X_e, \mathcal{T}_{\nu,e})$ is called a *hyperspace with Vietories topology* (briefly, hyperspace) for $e \in E$.

Notation 2.29 ([34])**.** Let (X, τ, E) be a soft topological space and $e \in E$. Then
(i) $\mathcal{SF}_n(X) = \{F \in 2^X_E : F\ has\ at\ most\ n\ soft\ points\}$,
(ii) $\mathcal{SF}(X) = \{F \in 2^X_E : F\ is\ finite\}$, where F is finite if and only if $F(e)$ is finite for each $e \in X$,
(iii) $\mathcal{SK}(X) = \{F \in 2^X_E : F\ is\ soft\ compact\}$, where the concept of soft compacteness is given in Definition 3.4,
(iv) $SC(X) = \{F \in 2^X_E : F\ is\ soft\ connected\}$, where the notion of soft connectedness is given in [41],
(v) $SC_K(X) = \mathcal{SK}(X) \cap SC(X)$,
(vi) $\mathcal{F}_{n,e}(X) = \{F(e) \in 2^X_e : F(e)\ has\ at\ most\ n\ elements\}$,
(vii) $\mathcal{F}_e(X) = \{F(e) \in 2^X_e : F(e)\ is\ finite\}$,
(viii) $\mathcal{K}_e(X) = \{F(e) \in 2^X_e : F(e)\ is\ compact\}$,
(ix) $C_e(X) = \{F(e) \in 2^X_e : F(e)\ is\ connected\}$,
(x) $C_{K_e}(X) = \mathcal{K}_e(X) \cap C_e(X)$.

The topology on $\mathcal{SK}(X)$ [resp. $\mathcal{SF}(X)$, $\mathcal{SF}_n(X)$, $SC(X)$, and $SC_K(X)$] is the subspace topology induced by $\mathcal{T}_{S\nu}$. Also, the topology on $\mathcal{K}_e(X)$ [resp. $\mathcal{F}_e(X)$, $\mathcal{F}_{n,e}(X)$, $C_e(X)$, and $C_{K_e}(X)$] is the subspace topology induced by $\mathcal{T}_{\nu,e}$. Moreover, $\mathcal{SF}(X)$ [resp. $\mathcal{SF}_n(X)$ and $SC_K(X)$] is a subspace of $\mathcal{SK}(X)$ and $\mathcal{F}_e(X)$ [resp. $\mathcal{F}_{n,e}(X)$ and $C_{K_e}(X)$] is a subspace of $\mathcal{K}_e(X)$.

3. Compactness and local compactness in soft hyperspaces

First of all, we recall the concept of soft compact sets, some of their results and an example. Next, we study some relationships between compactness in a soft topological space and its soft hyperspace. Finally, we discuss various relationships between local compactness in a soft topological space and its soft hyperspace.

Result 3.1 (Proposition 3.13, [38])**.** *Let (X, τ, E) be a soft topological space and $\beta \subset \tau$. Then β is a soft base for τ if and only if for each $A \in \tau$ and each $e_x \in A$, there is $B \in \beta$ such that $e_x \in B \subset A$.*

Result 3.2 (Proposition 3.14, [38])**.** *Let $\beta \subset SS_E(X)$. Then β is a soft base for a soft topology on X if and only if it satisfies the following conditions:*

(1) $\emptyset_E \in \beta$,

(2) $X_E = \bigcup \beta$,

(3) *if* $B_1,\ B_2 \in \beta$, *then there is* $\beta' \subset \beta$ *such that* $B_1 \cap B_2 = \bigcup \beta'$, *i.e., if* $B_1,\ B_2 \in \beta$ *and* $e_x \in B_1 \cap B_2$, *then* $B \in \beta$ *such that* $e_x \in B \subset B_1 \cap B_2$.

Definition 3.3 ([24,42])**.** Let (X, τ, E) be a soft topological space, Ψ a family of soft sets over X, and $A \in SS_E(X)$. Then Ψ is called a:

(i) *soft cover* of A, if $A \subset \bigcup \Psi$,

(ii) *soft open cover* of A, if it is a soft cover of A and $\Psi \subset \tau$.

If Ψ is a soft cover of A and $\Omega \subset \Psi$ is a soft cover of A, then Ω is called a *soft subcover* of Ψ.

Definition 3.4 ([24,42])**.** Let (X, τ, E) be a soft topological space, Ψ a family of soft sets over X, and $A \in SS_E(X)$. Then

(i) we say that Ψ has the *finite intersection property*, if $\bigcap \Omega \neq \emptyset_E$ for each finite $\Omega \subset \Psi$,

(ii) X is said to be *soft compact*, if each soft open cover of X_E has a finite soft subcover of X_E,

(iii) A is said to be a *soft compact set* in X, if each soft open cover of A has a finite soft subcover of A.

Result 3.5 (Theorem 2.23, [43])**.** *Every soft closed set in a soft compact space* (X, τ, E) *is a soft compact set in X.*

Result 3.6 (Theorem 4.9, [44])**.** *Let* (X, τ, E) *be a soft* T_2*-space and* $A \in SS_E(X)$*. If A is a soft compact set in X, then* $A \in \tau^c$*.*

Result 3.7 (Theorem 4.10, [44])**.** *Every soft compact* T_2*-space is soft regular.*

Result 3.8 (Theorem 4.12, [44])**.** *Every soft compact* T_2*-space is soft normal.*

Example 3.9 (See Examples 2.13 and 2.20, [43])**.** Let $X = [0, 1)$ and $E = \{e_1, \cdots, k\}$ be the set of parameters. For each $n \in \mathbb{N} \setminus \{0, 1\}$, consider the soft set A_n over X defined by, for each $i \in \{1, \cdots, k\}$,

$$A_n(e_i) = [0, 1 - \frac{1}{n}).$$

Let $\tau = \{\emptyset_E, [0, 1)_E\} \bigcup_{n \in \mathbb{N}\setminus\{0,1\}} A_n$.

Then (X, τ, E) is a soft topological space but is not soft compact.

Definition 3.10. A soft topological space (X, τ, E) is said to be *soft countably compact*, if every countable soft open cover of X has a finite subcover.

It is obvious that every soft compact space is soft countably compact.

Theorem 3.11. *A soft topological space* (X, τ, E) *is soft countably compact if and only if every countable family of soft closed sets in X with the finite intersection property has a nonempty intersection.*

Proof. Suppose X is soft countably compact and let $\Omega = \{A_\alpha \in \tau^c : \alpha \in \Lambda\}$ be a countable family of soft closed sets in X with the finite intersection property. Assume that $\bigcap_{\alpha \in \Lambda} A_\alpha = \emptyset_E$ and let $\Omega^c = \{A_\alpha^c \in \tau : \alpha \in \Lambda\}$. Then we have

$$\bigcup_{\alpha \in \Lambda} A_\alpha^c = \left(\bigcap_{\alpha \in \Lambda} A_\alpha \right)^c = \emptyset_E^c = X_E.$$

Thus Ω^c is a soft open cover of X. Since X is soft countably compact, there are $\{\alpha_1, \cdots, \alpha_n\} \subset \Lambda$ such that $\{A^c_{\alpha_i} : i = 1, \cdots, n\} \subset \Omega^c$ is a cover of X. So $X_E = \bigcup_{i=1}^n A^c_{\alpha_i} = (\bigcap_{i=1}^n A_{\alpha_i})^c$. Hence $\bigcap_{i=1}^n A_{\alpha_i} = \emptyset_E$. This is a contradiction. Therefore $\bigcap_{i=1}^n A_{\alpha_i} \neq \emptyset_E$.

Conversely, suppose the necessary condition holds and assume that X is not soft countably compact. Then there is a countable soft open cover $\Omega = \{A_\alpha \in \tau : \alpha \in \Lambda\}$ of X such that Ω does not have a finite subcover. Note that $\Omega^c = \{A^c_\alpha \in \tau^c : \alpha \in \Lambda\}$ is a family of soft closed sets in X. Let Γ be a finite subset of Λ. Since Ω does not have a finite subcover, $\bigcap_{\alpha\in\Gamma} A^c_\alpha = (\bigcup_{\alpha\in\Gamma} A_\alpha)^c \neq \emptyset_E$. Thus Ω^c has the finite intersection property. So $\bigcap_{\alpha\in\Lambda} A^c_\alpha \neq \emptyset_E$. Since Ω is a soft open cover of X,

$$\bigcap_{\alpha\in\Lambda} A^c_\alpha = \left(\bigcup_{\alpha\in\Lambda} A_\alpha\right)^c = X^c_E = \emptyset_E.$$

This is a contradiction. Hence X is soft countably compact. □

Definition 3.12. A soft topological space (X, τ, E) has the *Bolzano-Weierstrass property*, if every infinite soft set over X has a soft limit point.

It is well-known (Theorem F.6 in [45]) that every infinite set contains a countably infinite set. Then we have the following.

Lemma 3.13. *Every soft infinite set over X contains a soft countably infinite set over X.*

Proposition 3.14. *Every soft countably compact T_1-space has the Bolzano-Weierstrass property.*

Proof. Let (X, τ, E) be a soft countably compact T_1-space and A a soft infinite set over X. Then by Lemma 3.13, there is a soft countably infinite set $B = \bigcup_{n\in\mathbb{N}} e_{n_{x_n}}$ over X such that $B \subset A$. We may assume that if $m \neq n \in \mathbb{N}$, then $e_{m_{x_m}} \neq e_{n_{x_n}}$. Assume that B has no soft limit point and let $C_n = \bigcup\{e_{i_{x_i}} \in B : i \geq n\}$ for each $n \in \mathbb{N}$. Since X is T_1, $C_n \in \tau^c$ for each $n \in \mathbb{N}$ and $\{C_n \in \tau^c : n \in \mathbb{N}\}$ has the finite intersection property. Then by Theorem 3.11, $\bigcap_{n=1}^\infty C_n \neq \emptyset_E$. On the other hand, if $e_{k_{x_k}} \in B$, then $e_{k_{x_k}} \notin C_{k+1}$. Thus $e_{k_{x_k}} \notin \bigcap_{n=1}^\infty C_n$. So $\bigcap_{n=1}^\infty C_n = \emptyset_E$. This is a contradiction. Hence B has a soft limit point. Since $B \subset A$, A has a soft limit point. Therefore X has the Bolzano-Weierstrass property. □

Lemma 3.15. *Let (X, τ, E) be a soft T_1-space, $A \in SS_E(X)$, and $e_x \in Sd(A)$. Then N contains an infinite number of distinct soft points in A for each $N \in \mathcal{N}(e_x)$.*

Proof. Assume that the necessary condition does not hold. Then there is $U \in \mathcal{SN}(e_x)$ such that U contains only a finite number of soft points $e_{1_{x_1}}, \cdots, e_{n_{x_n}}$ of A distinct from e_x. Since X is T_1, there is $U_i \in \mathcal{SN}(e_x)$ such that $e_{i_{x_i}} \notin U_i$ for each $i \in \{1, \cdots, n\}$. Thus $U \cap (\bigcap_{i=1}^n U_i) \in \mathcal{SN}(e_x)$ and $[U \cap (\bigcap_{i=1}^n U_i)] \cap A \setminus \{e_x\} = \emptyset_E$. So $e_x \notin Sd(A)$. This is a contradiction. Hence the necessary condition holds. □

Theorem 3.16. *Let (X, τ, E) be a soft T_1-space. Then X is soft countably compact if and only if it has the Bolzano-Weierstrass property.*

Proof. Suppose X is soft countably compact. Then by Proposition 3.14, X has the Bolzano-Weierstrass property.

Conversely, suppose X has the Bolzano-Weierstrass property and assume that X is not soft countably compact. Then there is a countable soft open cover $\Omega = \{U_n \in \tau : n \in \mathbb{N}\}$ of X having no finite subcover.

Let $C_n = \bigcap_{i=1}^n U_i^c$ for each $n \in \mathbb{N}$. Then clearly, $\emptyset_E \neq C_n \in \tau^c$ for each $n \in \mathbb{N}$. Let us take $e_{n_{x_n}} \in C_n$ for each $n \in \mathbb{N}$ and let $A = \bigcup_{n\in\mathbb{N}} U_n$.

Case 1: Suppose A is finite. Then there is $e_x \in A$ such that $e_x = e_{n_{x_n}}$ for an infinite number of $n \in \mathbb{N}$. Thus $e_x \in C_n$ for each $n \in \mathbb{N}$. Since Ω covers X_E, this is a contradiction.

Case 2: Suppose A is infinite. Then by the hypothesis, there is $e_x \in SP_E(X)$ such that $e_x \in Sd(A)$. Since X is T_1, by Lemma 3.15, N contains an infinite number of soft points of A for each $N \in \mathcal{SN}(e_x)$. Thus $e_x \in Sd(A_n)$ for each $n \in \mathbb{N}$, where $A_n = \bigcup\{e_{i_{x_i}} \in A : i > n\}$. Moreover, $A_n \subset C_n$ and $C_n \in \tau^c$ for each $n \in \mathbb{N}$. So $e_x \in C_n$ for each $n \in \mathbb{N}$. Since Ω covers X_E, this is a contradiction. Hence X is soft countably compact. □

Theorem 3.17. *Let (X, τ, E) be a soft T_1-space. Then X is soft compact if and only if 2^X_E is compact.*

Proof. Suppose X is soft compact. Then it is proved for 2^X_E to be compact by using either Theorem 5.6 in [46] or Theorem 4.25 in [45].

Conversely, suppose 2^X_E is compact and let $\Omega = (U_\alpha)_{\alpha\in\Gamma}$ be a soft open cover, i.e., $\bigcup_{\alpha\in\Gamma} U_\alpha = X_E$. Then clearly, $(\langle X_E, U_\alpha\rangle)_{\alpha\in\Gamma}$ is an open cover of 2^X_E. Thus there is a finite subcover $\{\langle X_E, U_1\rangle, \cdots, \langle X_E, U_n\rangle\}$, i.e., $2^X_E = \bigcup_{i=1}^n \langle X_E, U_i\rangle$. Let $e_x \in X_E$. Since X is soft T_1, $e_x \in \tau^c$. Then $e_x \in 2^X_E$. Thus there is $i \in \{1, \cdots, n\}$ such that $e_x \in \langle X_E, U_i\rangle$, i.e., $e_x \in U_i$. So $e_x \in \bigcup_{i=1}^n U_i$, i.e., $X_E \subset \bigcup_{i=1}^n U_i$. Hence $\{U_1, \cdots, U_n\}$ is a finite soft subcover of X_E. Therefore X is soft compact. □

Corollary 3.18 (Theorem 4.2, [47])**.** *Let (X, τ, E) be a soft T_1-space and $e \in E$. Then (X, τ_e) is compact if and only if 2^X_e is compact.*

Theorem 3.19. *Let (X, τ, E) be a soft topological space. Then X is soft compact T_2 if and only if 2^X_E is compact T_2.*

Proof. The proof follows from Theorem 3.17, Result 3.7, and Proposition 4.10 (1) in [34]. □

Corollary 3.20 (Theorem 4.9.6, [47])**.** *Let (X, τ, E) be a soft T_1-space and $e \in E$. Then (X, τ_e) is compact T_2 if and only if 2^X_e is compact T_2.*

Theorem 3.21. *Let (X, τ, E) be a soft topological space. Then X is soft compact T_2 if and only if $\mathcal{SK}(X)$ is compact T_2.*

Proof. Suppose X is soft compact T_2. Since X is soft T_2, by Theorem 4.6 in [34], $\mathcal{SK}(X)$ is T_2. By the hypothesis an Theorem 3.19, 2^X_E is compact. Since $\mathcal{SK}(X)$ is a subspace of 2^X_E, $\mathcal{SK}(X)$ is compact. Thus $\mathcal{SK}(X)$ is compact T_2.

Conversely, suppose $\mathcal{SK}(X)$ is compact T_2. Then clearly, $\mathcal{SF}_1(X)$ is compact T_2. Thus X is soft compact T_2. □

Corollary 3.22 (Theorem 4.9.12, [47])**.** *Let (X, τ, E) be a soft topological space and $e \in E$. Then (X, τ_e) is compact T_2 if and only if $\mathcal{K}_e(X)$ is compact T_2.*

The following is an immediate consequence of Theorem 3.21.

Corollary 3.23. *Let (X, τ, E) be a soft topological space. Then the following are equivalent:*

(1) *X is soft compact T_2,*
(2) *$\mathcal{SK}(X)$ is compact T_2,*
(3) *$SC_K(X)$ is compact T_2,*
(4) *$\mathcal{F}_n(X)$ is compact T_2 for each $n \in \mathbb{N}$.*

From Corollary 3.22, we have the following.

Corollary 3.24. *Let (X, τ, E) be a soft topological space and $e \in E$. Then the following statements are equivalent:*
(1) (X, τ_e) *is compact* T_2,
(2) $\mathcal{K}_e(X)$ *is compact* T_2,
(3) $C_{K_e}(X)$ *is compact* T_2,
(4) $\mathcal{F}_{n,e}(X)$ *is compact* T_2 *for each* $n \in \mathbb{N}$.

Theorem 3.25. *Let (X, τ, E) be a soft topological space. Then X is soft compact T_2 if and only if $SC(X)$ is compact T_2.*

Proof. Suppose X is soft compact T_2. By Result 3.8, X is soft normal. Then by Theorem 4.23 in [34], $SC(X)$ is closed in 2^X_E. By Theorem 3.19, 2^X_E is compact T_2. Thus $SC(X)$ is compact T_2.

Conversely, suppose $SC(X)$ is compact T_2. Then clearly, $\mathcal{SF}_1(X)$ is a closed subspace of $SC(X)$. Thus $\mathcal{SF}_1(X)$ is compact T_2. Thus X is soft compact T_2. By Lemma 4.5 in [34], X is homeomorphic to $\mathcal{SF}_1(X)$. So X is soft compact T_2. □

Corollary 3.26 (See Proposition 3.1, [47])**.** *Let (X, τ, E) be a soft topological space and $e \in E$. Then (X, τ_e) is compact T_2 if and only if $C_e(X)$ is compact T_2.*

Theorem 3.27. *Let (X, τ, E) be a soft topological space. Then the following are equivalent:*
(1) *X is soft compact* T_2,
(2) 2^X_E *is compact* T_2,
(3) $SC(X)$ *is compact* T_2.

Proof. The proof follows from Theorems 3.19 and 3.25. □

Corollary 3.28 (See Corollary 3.1.1, [47])**.** *Let (X, τ, E) be a soft topological space and $e \in E$. Then the following statements are equivalent:*
(1) (X, τ_e) *is compact* T_2,
(2) 2^X_e *is compact* T_2,
(3) $C_e(X)$ *is compact* T_2.

Remark 3.29. It is obvious that if X is soft compact T_2, then $2^X_E = \mathcal{SK}(X)$ and $SC(X) = SC_K(X)$.

Definition 3.30. A soft topological space (X, τ, E) is said to be:
(i) *soft locally compact at* $e_x \in SP_E(X)$, if there is a $U \in \tau$ and a soft compact set K in X such that $e_x \in U \subset K$,
(ii) *soft locally compact*, if it is soft locally compact at e_x for each $e_x \in SP_E(X)$.

Remark 3.31. See Example 3.5 in [28].

Theorem 3.32. *Let (X, τ, E) be a soft T_2-space and $e_x \in SP_E(X)$. Then X is soft locally compact at e_x if and only if for each $V \in \mathcal{N}(e_x)$, there is a $U \in \tau$ and a soft compact set K in X such that $e_x \in U \subset V$ and $U \subset K \subset Scl(U)$.*

Proof. Suppose X is soft locally compact at e_x and let $V \in \mathcal{N}(e_x)$. Then there is a $W \in \tau$ and a soft compact set C in X such that $e_x \in W \subset C$. Let $U = V \cap W$ and $K = Scl(U)$. Then clearly, $e_x \in U \in \tau$ and $U \subset K \subset Scl(U)$. Since $U \subset W$, $K = Scl(U) \subset Scl(W)$. Since X is soft T_2 and C is a soft compact set in X, by Result 3.6, $C \in \tau^c$. Since $W \subset C$, $Scl(W) \subset C$. Thus $K \subset Scl(W) \subset C$. So by Result 3.5, K is soft compact in X.

The proof of the sufficient condition is straightforward. □

Theorem 3.33 (See Definition 3.4 in [28])**.** *Let (X, τ, E) be a soft T_2-space and $e_x \in SP_E(X)$. Then X is soft locally compact at e_x if and only if there is $U \in \mathcal{N}(e_x)$ such that $Scl(U)$ is soft compact in X.*

Proof. X is soft locally compact at e_x. Then there is a $U \in \tau$ and a soft compact set K in X such that $e_x \in U \subset K$. Since X is soft T_2 and K is a soft compact set in X, by Result 3.6, $K \in \tau^c$. Thus $Scl(U) \subset K$. So by Result 3.5, $Scl(U)$ is soft compact in X. Hence the necessary condition holds.

The proof of the sufficient condition is straightforward. □

The following is an immediate consequence Definition 3.30 and Theorem 3.33.

Corollary 3.34. *A soft T_2-space (X, τ, E) is soft locally compact if and only if for each $e_x \in SP_E(X)$ and each $V \in \mathcal{N}(e_x)$, there is a $U \in \mathcal{N}(e_x)$ such that $Scl(U) \subset V$ and $Scl(U)$ is soft compact in X.*

Lemma 3.35. *If (X, τ, E) is a soft locally compact T_2-space and $\langle U_1, \cdots, U_n\rangle$ is a basic open set in 2^X_E, then $cl(\langle U_1, \cdots, U_n\rangle)$ is compact in 2^X_E if and only if $Scl(U) = \bigcup_{i=1}^n Scl(U_i)$ is soft compact in X.*

Proof. Suppose $Scl(U)$ is soft compact in X. Then by Theorem 3.17, $2^{Scl(U)}_E = \langle Scl(U)\rangle$ is compact in 2^X_E. On the other hand, by Proposition 3.22 (2) in [34], we have

$$cl(\langle U_1, \cdots, U_n\rangle) = \langle Scl(U_1), \cdots, Scl(U_n)\rangle \subset \langle Scl(U)\rangle .$$

Thus $cl(\langle U_1, \cdots, U_n\rangle)$ is compact in 2^X_E.

The proof of the necessary condition is similar to one of Xie's theorems in [48]. □

Theorem 3.36. *Let (X, τ, E) be a soft locally compact T_2-space. Then $A \in 2^X_E$ has a compact neighborhood in 2^X_E if and only if A is soft compact in X.*

Proof. Suppose A is soft compact in X. Then by the hypothesis and Theorem 3.12 in [28], there is $U \in \tau$ such that $A \subset U$ and $Scl(U)$ is soft compact in X. Thus by Theorem 3.17, $2^{Scl(U)}_E = \langle Scl(U)\rangle$ is compact in 2^X_E. On the other hand, by Proposition 3.22 (2) in [34], $2^{Scl(U)}_E = \langle Scl(U)\rangle = cl(\langle U\rangle)$. Since X is soft T_2 and A is soft compact in X, by Result 3.6, $A \in \tau^c$. Since $A \subset U$, $A \in 2^{Scl(U)}_E$. So $2^{Scl(U)}_E$ is a compact neighborhood of A.

The proof of the converse follows from Lemma 3.35. □

Corollary 3.37 (See Theorem 4.3.2, [47])**.** *Let (X, τ, E) be a soft locally compact T_2-space and $e \in E$. Then $A(e) \in 2^X_e$ has a compact neighborhood in 2^X_e if and only if $A(e)$ is soft compact in (X, τ_e).*

Theorem 3.38. *Let (X, τ, E) be a soft T_1-space. Then 2^X_E is locally compact at each $F \in \mathcal{SK}(X)$ if and only if X is a soft locally compact space.*

Proof. Suppose 2^X_E is locally compact at each $F \in \mathcal{SK}(X)$ and let $e_x \in SP_E(X)$. Since X is soft T_1, $e_x \in 2^X_E$. Then clearly, $\{e_x\} \in \mathcal{SK}(X)$. Thus by the hypothesis and Theorem 3.33, there is $\langle U\rangle \in \mathcal{N}(e_x)$ such that $cl(\langle U\rangle) = \langle Scl(U)\rangle = 2^{Scl(U)}_E$ is compact in 2^X_E. So by Theorem 3.17, $Scl(U)$ is soft compact at e_x. Hence X is a soft locally compact space.

The proof of the converse follows from Theorem 3.36. □

Corollary 3.39 (See Theorem 4.4.1, [47])**.** *Let (X, τ, E) be a soft T_1-space and $e \in E$. Then 2^X_e is locally compact at each $F \in \mathcal{K}_e(X)$ if and only if (X, τ_e) is locally compact.*

Proposition 3.40. *If (X, τ, E) is a soft locally compact space, then $\mathcal{SK}(X)$ is open in 2^X_E.*

Proof. Suppose X is soft locally compact and let $F \in \mathcal{SK}(X)$. Then by Theorem 3.33, there is $U \in \tau$ such that $F \subset U$ and $Scl(U)$ is soft compact in X. Thus $K \subset Scl(U)$ for each $K \in \langle U\rangle$. Since $Scl(U)$ is soft compact, K is soft compact. So $\langle U\rangle \subset \mathcal{SK}(X)$. Hence $\mathcal{SK}(X)$ is open in 2^X_E. □

Corollary 3.41 (See Theorem 4.4.2, [47])**.** *Let (X, τ, E) be a soft topological space and $e \in E$. If (X, τ_e) is locally compact, then $\mathcal{K}_e(X)$ is open in 2^X_e.*

Theorem 3.42. *(X, τ, E) is a soft locally compact T_2-space if and only if $\mathcal{SK}(X)$ is locally compact T_2.*

Proof. Suppose X is locally compact T_2. Then by Theorem 3.10 in [28], X is soft regular. Thus by Proposition 4.10 (1) in [34], 2^X_E is T_2. Since X is soft locally compact, by Proposition 3.40, $\mathcal{SK}(X)$ is open in 2^X_E. So by Theorem 3.38, each $F \in \mathcal{SK}(X)$ has a compact neighborhood contained in $\mathcal{SK}(X)$. Hence $\mathcal{SK}(X)$ is locally compact T_2.

Conversely, suppose $\mathcal{SK}(X)$ is locally compact T_2 and let $e_x \in SP_E(X)$. Then clearly, $\{e_x\} \in \mathcal{SK}(X)$. Thus by the hypothesis, there is a neighborhood $\mathcal{U}$ of $\{e_x\}$ in $\mathcal{SK}(X)$ such that $cl(\mathcal{U})$ is compact in $\mathcal{SK}(X)$. By Proposition 3.25 (1) in [34], $U = \bigcup \mathcal{U} \in \tau$. Since $\mathcal{U} \in \mathcal{SK}(\mathcal{SK}(X))$, by Proposition 3.32 (2) in [34], $\bigcup cl(\mathcal{U}) \in \mathcal{SK}(X)$, i.e., $\bigcup cl(\mathcal{U})$ is soft compact in X. It is obvious that $e_x \in U \subset Scl(U) \subset \bigcup cl(\mathcal{U})$. So X is soft locally compact at e_x, i.e., X is soft locally compact. Since $\mathcal{SF}_1 \subset \mathcal{SK}(X)$ and $\mathcal{SK}(X)$ is T_2, $\mathcal{SF}_1$ is T_2. By Lemma 4.5 in [34], $\mathcal{SF}_1$ is homeomorphic to X. Hence X is T_2. Therefore X is locally compact T_2. □

Corollary 3.43. *Let (X, τ, E) be a soft T_2-space and $e \in E$. Then (X, τ_e) is locally compact T_2 if and only if 2^X_e is locally compact T_2.*

Proposition 3.44. *If (X, τ, E) is soft locally compact, then $SC_K(X)$ is open in $SC(X)$.*

Proof. Suppose X is soft locally compact and let $F \in SC_K(X)$. Let $\{U_1, \cdots, U_n\} \subset \tau$ such that $Scl(U_i)$ is soft compact in X for each $i \in \{1, \cdots, n\}$ and $F \in \langle U_1, \cdots, U_n\rangle \cap SC(X)$. Let $A \in \langle U_1, \cdots, U_n\rangle \cap SC(X)$. Then clearly, $A \subset \bigcup_{i=1}^n U_i \subset \bigcup_{i=1}^n Scl(U_i)$. Since $Scl(U_i)$ is soft compact in X for each $i \in \{1, \cdots, n\}$, $\bigcup_{i=1}^n Scl(U_i)$ is soft compact in X. Thus A is soft compact in X, i.e., $A \in SC_K(X)$. So $\langle U_1, \cdots, U_n\rangle \cap SC(X) \subset SC_K(X)$. Hence $SC_K(X)$ is open in $SC(X)$. □

Corollary 3.45 (See Proposition 1.5, [49])**.** *Let (X, τ, E) be a soft topological space and $e \in E$. If (X, τ_e) is locally compact, then $C_{K_e}(X)$ is open in $C_e(X)$.*

Theorem 3.46. *Let (X, τ, E) be a soft T_2-space and $e_x \in SP_E(X)$. Then the following are equivalent:*

(1) *X is soft locally compact at e_x,*

(2) *2^X_E is locally compact at $\{e_x\}$,*

(3) $\mathcal{SK}(X)$ *is locally compact at* $\{e_x\}$,
(4) $SC_K(X)$ *is locally compact at* $\{e_x\}$,
(5) $SC(X)$ *is locally compact at* $\{e_x\}$.

Proof. (1)⇒(2) Suppose X is soft locally compact at e_x and let $\langle U\rangle$ be a basic open set in 2^X_E such that $\{e_x\} \in \langle U\rangle$. Then clearly, $e_x \in U \in \tau$. By the hypothesis and Corollary 3.34, there is $V \in \tau$ such that $Scl(V)$ is soft compact in X and $e_x \in V \subset Scl(V) \subset U$. Thus $\{e_x\} \in \langle V\rangle \subset \langle Scl(V)\rangle = cl(\langle V\rangle) \subset \langle U\rangle$. Since $\langle Scl(V)\rangle = 2^{Scl(V)}_E$ and $Scl(V)$ is soft compact in X, $2^{Scl(V)}_E$ is compact in 2^X_E. So $\langle Scl(V)\rangle$ is compact in 2^X_E. Hence 2^X_E is locally compact at $\{e_x\}$.

(2)⇒(3) Suppose the condition (2) holds and let $\langle U\rangle \cap \mathcal{SK}(X)$ be a basic open set in $\mathcal{SK}(X)$ such that $\{e_x\} \in \langle U\rangle \cap \mathcal{SK}(X)$. Then clearly, $\{e_x\} \in \langle U\rangle$. Since 2^X_E is locally compact at $\{e_x\}$, there is an open set $\mathcal{U}$ such that $cl(\mathcal{U})$ is compact in 2^X_E and $\{e_x\} \in \mathcal{U} \subset cl(\mathcal{U}) \subset \langle U\rangle$. Now let $V \in \tau$ such that $\{e_x\} \in \langle V\rangle \subset \mathcal{U}$. Then

$$\{e_x\} \in \langle V\rangle \subset \langle Scl(V)\rangle \subset cl(\mathcal{U}) \subset \langle U\rangle \text{ and } \langle Scl(V)\rangle \text{ is compact in } 2^X_E.$$

Thus $\langle Scl(V)\rangle = 2^{Scl(V)}_E = \mathcal{SK}(Scl(V))$. So $\langle Scl(V)\rangle \cap \mathcal{SK}(X) = \langle Scl(V)\rangle$. Hence

$$\{e_x\} \in \langle V\rangle \cap \mathcal{SK}(X) \subset \langle Scl(V)\rangle \cap \mathcal{SK}(X) = \langle Scl(V)\rangle \subset \langle U\rangle \cap \mathcal{SK}(X).$$

Therefore $\mathcal{SK}(X)$ is locally compact at $\{e_x\}$.

(3)⇒(1) Suppose the condition (3) holds and let $U \in \tau$ such that $e_x \in U$. Then clearly, $\{e_x\} \in \langle U\rangle \cap \mathcal{SK}(X)$. Thus by the hypothesis, there is a basic open set $\mathcal{U}$ in $\mathcal{SK}(X)$ such that $cl(\mathcal{U}) \cap \mathcal{SK}(X)$ is compact in $\mathcal{SK}(X)$ and $\{e_x\} \in \mathcal{U} \subset cl(\mathcal{U}) \cap \mathcal{SK}(X) \subset \langle U\rangle \cap \mathcal{SK}(X)$. Now let $V \in \tau$ such that $\{e_x\} \in \langle V\rangle \cap \mathcal{SK}(X) \subset \mathcal{U} \subset cl(\mathcal{U}) \cap \mathcal{SK}(X) \subset \langle U\rangle \cap \mathcal{SK}(X)$. Then $e_x \in V \subset \bigcup\{F \in \tau^c : F \in cl(\mathcal{U})\mathcal{SK}(X)\} \subset U$. Since $cl(\mathcal{U}) \cap \mathcal{SK}(X)$ is compact in $\mathcal{SK}(X)$, by Proposition 3.32 (2) in [34], $\bigcup\{F \in \tau^c : F \in cl(\mathcal{U})\mathcal{SK}(X)\}$ is soft compact in X. Thus $Scl(V)$ is soft compact in X and $e_x \in V \subset Scl(V) \subset U$. So X is soft locally compact at e_x.

(1)⇒(4) Suppose X is soft locally compact at e_x and let $\langle U\rangle \cap SC_K(X)$ be a basic open set in $SC_K(X)$ such that $\{e_x\} \in \langle U\rangle \cap SC_K(X)$. Then clearly, $e_x \in U \in \tau$. By the hypothesis and Corollary 3.34, there is $V \in \tau$ such that $Scl(V)$ is soft compact in X and $e_x \in V \subset Scl(V) \subset U$. Thus we have

$$\{e_x\} \in \langle V\rangle \cap SC_K(X) \subset \langle Scl(V)\rangle \cap SC_K(X) \subset \langle U\rangle \cap SC_K(X).$$

Since $Scl(V)$ is soft compact in X, by Theorem 3.17, $2^{Scl(V)}_E$ is compact in 2^X_E. Since $SC_K(Scl(V)) \subset 2^{Scl(V)}_E$, $SC_K(Scl(V))$ is compact in 2^X_E. Since $Scl(V)$ is a soft compact T_2-subspace of X, by Proposition 4.23 in [34], $SC_K(Scl(V))$ is closed in 2^X_E. Since $\langle V\rangle \cap SC_K(X) \subset SC_K(Scl(V)) \subset SC_K(X)$, we have

$$\begin{aligned} cl_{SC_K(X)}(\langle V\rangle \cap SC_K(X)) &= cl(\langle V\rangle \cap SC_K(X)) \cap SC_K(X) \\ &= cl(\langle V\rangle \cap SC_K(X)) \\ &\subset SC_K(Scl(V)), \end{aligned}$$

where $cl_{SC_K(X)}$ denotes the closure in the subspace $SC_K(X)$. So $cl_{SC_K(X)}(\langle V\rangle \cap SC_K(X))$ is compact in $SC_K(X)$. Hence $SC_K(X)$ is locally compact at $\{e_x\}$.

(4)⇒(1) Suppose $SC_K(X)$ is locally compact at $\{e_x\}$ and let $U \in \tau$ such that $e_x \in U$. Then $\{e_x\} \in \langle U\rangle \cap SC_K(X)$. By the hypothesis, there is a basic open set $\mathcal{U}$ in $SC_K(X)$ such that $cl(\mathcal{U}) \cap SC_K(X)$ is compact in $SC_K(X)$ and $\{e_x\} \in \mathcal{U} \subset cl(\mathcal{U}) \cap SC_K(X) \subset \langle U\rangle \cap SC_K(X)$. Let $V \in \tau$ such that

$$\{e_x\} \in \langle V\rangle \cap SC_K(X) \subset \mathcal{U} \subset cl(\mathcal{U}) \cap SC_K(X) \subset \langle U\rangle \cap SC_K(X).$$

Then $e_x \in V \subset \bigcup\{F \in \tau^c : F \in cl(\mathcal{U}) \cap SC_K(X)\} \subset U$. Since $cl(\mathcal{U}) \cap SC_K(X)$ is compact in $SC_K(X)$, by Proposition 3.32 (2) in [34], $\bigcup\{F \in \tau^c : F \in cl(\mathcal{U}) \cap SC_K(X)\}$ is soft compact in X. Thus $Scl(V)$ is soft compact in X and $e_x \in V \subset Scl(V) \subset U$. So X is soft locally compact at e_x.

(1)⇒(5) Suppose X is soft locally compact at e_x and let $\langle U\rangle \cap SC(X)$ be a basic open set in $SC(X)$ such that $\{e_x\} \in \langle U\rangle \cap SC(X)$. Then clearly, $e_x \in U \in \tau$. By the hypothesis and Corollary 3.34, there is $V \in \tau$ such that $Scl(V)$ is soft compact in X and $e_x \in V \subset Scl(V) \subset U$. Thus we have

$$\{e_x\} \in \langle V\rangle \cap SC(X) \subset \langle Scl(V)\rangle \cap SC(X) \subset \langle U\rangle \cap SC(X).$$

Since $Scl(V)$ is soft compact in X, by Theorem 3.17, $2_E^{Scl(V)}$ is compact in 2_E^X. Since $SC(Scl(V)) \subset 2_E^{Scl(V)}$, $SC(Scl(V))$ is compact in 2_E^X. Since $Scl(V)$ is a soft compact T_2-subspace of X, by Proposition 4.23 in [34], $SC(Scl(V))$ is closed in 2_E^X. Since $\langle V\rangle \cap SC(X) \subset SC(Scl(V)) \subset SC(X)$, we have

$$\begin{aligned} cl_{SC(X)}(\langle V\rangle \cap SC(X)) &= cl(\langle V\rangle \cap SC(X)) \cap SC(X) \\ &= cl(\langle V\rangle \cap SC(X)) \\ &\subset SC(Scl(V)), \end{aligned}$$

where $cl_{SCX)}$ denotes the closure in the subspace $SC(X)$. So $cl_{SC(X)}(\langle V\rangle \cap SC(X))$ is compact in $SC_K(X)$. Hence $SC(X)$ is locally compact at $\{e_x\}$.

(5)⇒(1) Suppose $SC(X)$ is locally compact at $\{e_x\}$. Then there is a basic open set $\mathcal{U}$ in $SC(X)$ such that $\{e_x\} \in \mathcal{U}$ and $cl_{SC(X)}(\mathcal{U}) = cl(\mathcal{U}) \cap SC(X)$ is compact in $SC(X)$. Let $V \in \tau$ such that $e_x \in V$ and $\{e_x\} \in \langle V\rangle \cap SC(X) \subset \mathcal{U}$. It is clear that $\mathcal{SF}_1(X)$ is closed in 2_E^X and $\mathcal{SF}_1(X) \subset SC(X)$. Then we get

$$cl_{SC(X)}\mathcal{SF}_1(V) = cl(\mathcal{SF}_1(V)) \cap SC(X) = cl(\mathcal{SF}_1(V)).$$

Since $\mathcal{SF}_1(V) \subset \langle V\rangle \cap SC(X)$, we have

$$cl(\mathcal{SF}_1(V)) = cl_{SC(X)}\mathcal{SF}_1(V) \subset cl_{SC(X)}\mathcal{U} = cl(\mathcal{U}) \cap SC(X).$$

Since $cl_{SC(X)}(\mathcal{U})$ is compact in $SC(X)$, $cl(\mathcal{SF}_1(V))$ is compact in $SC(X)$. On the other hand, $cl(\mathcal{SF}_1(V)) = \mathcal{SF}_1(Scl(V))$ and $\mathcal{SF}_1(Scl(V))$ is homeomorphic to $Scl(V)$. Thus $Scl(V)$ is soft compact in X. So X is soft locally compact at e_x. □

Corollary 3.47 (See Propoisition 2.6, [49]). *Let (X, τ, E) be a soft T_2-space, $e \in E$, and $x \in X$. Then the following are equivalent:*

(1) *(X, τ_e) is soft locally compact at x,*
(2) *2_e^X is locally compact at $\{x\}$,*
(3) *$\mathcal{K}_e(X)$ is locally compact at $\{x\}$,*
(4) *$C_{K_e}(X)$ is locally compact at $\{x\}$,*
(5) *$C_e(X)$ is locally compact at $\{x\}$.*

Theorem 3.48. *Let (X, τ, E) be a soft T_2-space and $e_x \in SP_E(X)$. Then X is soft locally compact at e_x if and only if $\mathcal{SF}_n(X)$ is locally compact at $\{e_x\}$ for each $n \in \mathbb{N}$.*

Proof. Suppose X is soft locally compact at e_x and let $\mathcal{U}$ be a basic open set in $\mathcal{SF}_n(X)$ such that $\{e_x\} \in \mathcal{U}$. Then there is $U \in \tau$ such that $\{e_x\} \in \langle U\rangle \cap \mathcal{SF}_n(X) \subset \mathcal{U}$. Thus by the hypothesis, there is $V \in \tau$ such that $e_x \in V$, $Scl(V)$ is soft compact in X and $Scl(V) \subset U$. So $\langle Scl(V)\rangle \cap \mathcal{SF}_n(X)$ is compact in $\mathcal{SF}_n(X)$ and $\langle Scl(V)\rangle \cap \mathcal{SF}_n(X) \subset \langle U\rangle \cap \mathcal{SF}_n(X)$. Hence $\mathcal{SF}_n(X)$ is locally compact at $\{e_x\}$.

Conversely, suppose $\mathcal{SF}_n(X)$ is locally compact at $\{e_x\}$ for each $n \in \mathbb{N}$ and let $U \in \tau$ such that $e_x \in U$. Then clearly, $\{e_x\} \in \langle U \rangle \cap \mathcal{SF}_n(X)$. Thus by the hypothesis, there is a basic open set $\mathcal{U}$ in $\mathcal{SF}_n(X)$ such that $cl(\mathcal{U}) \cap \mathcal{SF}_n(X)$ is compact in $\mathcal{SF}_n(X)$ and $\{e_x\} \in \mathcal{U} \subset cl(\mathcal{U}) \cap \mathcal{SF}_n(X) \subset \langle U \rangle \cap \mathcal{SF}_n(X)$. Let $V \in \tau$ such that $\{e_x\} \in \langle V \rangle \cap \mathcal{SF}_n(X) \subset \mathcal{U} \subset cl(\mathcal{U}) \cap \mathcal{SF}_n(X) \subset \langle U \rangle \cap \mathcal{SF}_n(X)$. Then clearly, $V \subset \bigcup\{F \in \tau^c : F \in cl(\mathcal{U}) \cap \mathcal{SF}_n(X)\}$. Since $cl(\mathcal{U}) \cap \mathcal{SF}_n(X)$ is compact in $\mathcal{SF}_n(X)$, by Proposition 3.32 (2) in [34], $\bigcup\{F \in \tau^c : F \in cl(\mathcal{U}) \cap \mathcal{SF}_n(X)\}$ is soft compact in X. Thus $Scl(V)$ is soft compact in X and $e_x \in V \subset Scl(V) \subset U$. So X is soft locally compact at e_x. □

Corollary 3.49. *Let (X, τ, E) be a soft T_2-space, $e \in E$, and $x \in X$. Then (X, τ_e) is locally compact at x if and only if $\mathcal{F}_{n,e}(X)$ is locally compact at $\{x\}$ for each $n \in \mathbb{N}$.*

Lemma 3.50. *Let (X, τ, E) be a soft locally compact T_2-space and $F \in SC_K(X)$. If $\langle U_1, \cdots, U_n \rangle$ is a basic open set in 2^X_E containing F, then there is a soft compact set M in X such that $F \subset Sint(M) \subset M \subset \bigcup_{i=1}^n U_i$. Furthermore, there is a basic open set $\langle W_1, \cdots, W_m \rangle$ in 2^X_E such that $Scl(W_i)$ is soft compact in X for each $i \in \{1, \cdots, m\}$ and $F \in \langle W_1, \cdots, W_m \rangle \subset cl(\langle W_1, \cdots, W_m \rangle) \subset \langle U_1, \cdots, U_n \rangle$.*

Proof. Suppose $\langle U_1, \cdots, U_n \rangle$ is a basic open set in 2^X_E containing F. Then clearly, $F \cap U_i \neq \emptyset_E$, so that $e_{i_{x_i}} \in U_i$ for each $i \in \{1, \cdots, n\}$. Since X is soft locally compact, X is soft locally compact at $e_{i_{x_i}}$ for each $i \in \{1, \cdots, n\}$. Thus there is $V_{e_{i_{x_i}}} \in \tau$ such that $e_{i_{x_i}} \in V_{e_{i_{x_i}}}$, $Scl(V_{e_{i_{x_i}}})$ is soft compact in X, and $V_{e_{i_{x_i}}} \subset U_i$ for each $i \in \{1, \cdots, n\}$. Let Ω be the collection of all such $V_{e_{i_{x_i}}}$. Since $F \in SC_K(X)$, F is a soft compact set in X. Moreover, Ω covers F. So there is a finite subcollection $\{V_{e_{1_{x_1}}}, \cdots, V_{e_{k_{x_k}}}\}$ of Ω such that $F \subset \bigcup_{i=1}^k V_{e_{i_{x_i}}}$. Now let $f_{i_{y_i}} \in F \cap U_i$ for each $i \in \{1, \cdots, n\}$ and $M = \left(\bigcup_{j=1}^k V_{e_{j_{x_j}}}\right) \cap \left(\bigcup_{i=1}^n V_{f_{i_{y_i}}}\right)$. Then M is soft compact in X and $F \subset Sint(M) \subset M \subset \bigcup_{i=1}^n U_i$. Furthermore, by Proposition 3.22 (2) in [34],

$$\begin{aligned} F &\in \left\langle V_{e_{1_{x_1}}}, \cdots, V_{e_{k_{x_k}}}, V_{f_{1_{y_1}}}, \cdots, V_{f_{n_{y_n}}} \right\rangle \\ &\subset cl(\left\langle V_{e_{1_{x_1}}}, \cdots, V_{e_{k_{x_k}}}, V_{f_{1_{y_1}}}, \cdots, V_{f_{n_{y_n}}} \right\rangle) \\ &= \left\langle Scl(V_{e_{1_{x_1}}}), \cdots, Scl(V_{e_{k_{x_k}}}), Scl(V_{f_{1_{y_1}}}), \cdots, Scl(V_{f_{n_{y_n}}}) \right\rangle \\ &\subset \langle U_1, \cdots, U_n \rangle . \end{aligned}$$

□

Corollary 3.51 (See Lemma 1.7 (*b*), [49])**.** *Let (X, τ, E) be a soft T_2-space and $e \in E$. If (X, τ_e) is locally compact, $F \in C_{K_e}(X)$, and $\langle U_1, \cdots, U_n \rangle$ is a basic open set in 2^X_e containing F, then there is a compact set M in (X, τ_e) such that $F \subset int(M) \subset M \subset \bigcup_{i=1}^n U_i$. Furthermore, there is a basic open set $\langle W_1, \cdots, W_m \rangle$ in 2^X_e such that $cl(W_i)$ is compact in (X, τ_e) for each $i \in \{1, \cdots, m\}$ and $F \in \langle W_1, \cdots, W_m \rangle \subset cl(\langle W_1, \cdots, W_m \rangle) \subset \langle U_1, \cdots, U_n \rangle$.*

Theorem 3.52. *Let (X, τ, E) be a soft T_2-space. Then the following are equivalent:*

(1) *X is soft locally compact,*

(2) *$\mathcal{SK}(X)$ is locally compact,*

(3) *$SC_K(X)$ is locally compact.*

Proof. (1)⇒(2) Suppose X is soft locally compact and let $F \in \mathcal{SK}(X)$. Then clearly, F is soft compact in X. By the hypothesis, there is $U \in \tau$ such that $F \subset U$ and $scl(U)$ is soft compact in X. Thus $F \in \langle U \rangle \cap \mathcal{SK}(X)$. Note that A is soft compact in X for each $A \in \langle U \rangle$. So $\langle U \rangle \subset \mathcal{SK}(X)$. On the other hand, $cl(\langle U \rangle) = \langle Scl(U) \rangle = 2^{Scl(U)}_E$. Since $Scl(U)$ is soft compact in X, by Theorem 3.17, $\langle Scl(U) \rangle$ is compact in $\mathcal{SK}(X)$. So $\mathcal{SK}(X)$ is locally compact at F. Hence $\mathcal{SK}(X)$ is locally compact.

(2)⇒(1) Suppose $\mathcal{SK}(X)$ is locally compact and let $e_x \in SP_E(X)$. By the hypothesis, $\mathcal{SK}(X)$ is locally compact at $\{e_x\}$. Then there is a basic open set $\mathcal{U}$ in $\mathcal{SK}(X)$ such that $\{e_x\} \in \mathcal{U}$ and $cl(\mathcal{U})$ is compact in $\mathcal{SK}(X)$. Since $\mathcal{U}$ is open in $\mathcal{SK}(X)$, by Proposition 3.25 (1) in [34], $U = \bigcup \mathcal{U} \in \tau$. Since $cl(\mathcal{U})$ is compact in $\mathcal{SK}(X)$, $cl(\mathcal{U}) \in \mathcal{SK}(\mathcal{SK}(X))$. Thus by Proposition 3.32 (2) in [34], $\bigcup cl(\mathcal{U}) \in \mathcal{SK}(X)$. So $U \subset Scl(U) \subset \bigcup cl(\mathcal{U})$. Hence X is soft locally compact at e_x. Therefore X is soft locally compact.

(1)⇒(3) Suppose X is soft locally compact and let $F \in SC_K(X)$. Let $\langle U_1, \cdots, U_n\rangle \cap SC_K(X)$ be a basic open set in $SC_K(X)$ containing F. Then by Lemma 3.50, there is a basic open set $\langle W_1, \cdots, W_m\rangle$ in 2^X_E such that $Scl(W_i)$ is soft compact in X for each $i \in \{1, \cdots, m\}$ and $F \in \langle W_1, \cdots, W_m\rangle \subset cl(\langle W_1, \cdots, W_m\rangle) \subset \langle U_1, \cdots, U_n\rangle$. Thus we have

$$\begin{aligned} F \in \langle W_1, \cdots, W_m\rangle \cap SC_K(X) &\subset cl(\langle W_1, \cdots, W_m\rangle) \cap SC_K(X) \\ &\subset \langle U_1, \cdots, U_n\rangle \cap SC_K(X). \end{aligned}$$

Let $M = \bigcup_{i=1}^m Scl(W_i)$. Since $Scl(W_i)$ is soft compact in X for each $i \in \{1, \cdots, m\}$, M is soft compact in X. Then by Theorem 3.25, $SC(M)$ is compact in 2^X_E. Since $\langle W_1, \cdots, W_m\rangle \cap SC_K(X) \subset SC(M)$ and $SC(M)$ is closed in 2^X_E, we have

$$cl(\langle W_1, \cdots, W_m\rangle \cap SC_K(X)) \subset SC(M) \subset SC_K(X)).$$

Thus we get

$$\begin{aligned} &cl_{SC_K(X)}(\langle W_1, \cdots, W_m\rangle \cap SC_K(X)) \\ &= cl(\langle W_1, \cdots, W_m\rangle \cap SC_K(X)) \cap SC_K(X) \\ &= cl(\langle W_1, \cdots, W_m\rangle \cap SC_K(X)). \end{aligned}$$

So $cl_{SC_K(X)}(\langle W_1, \cdots, W_m\rangle \cap SC_K(X))$ is compact in $SC_K(X)$. Hence $SC_K(X)$ is locally compact at F. Therefore $SC_K(X)$ is locally compact.

(3)⇒(1) Suppose $SC_K(X)$ is locally compact and let $e_x \in SP_E(X)$. Let $U \in \tau$ such that $e_x \in U$. Then clearly, $\{e_x\} \in 2^X_E$ and $\{e_x\} \in \langle U\rangle \cap SC_K(X)$. By the hypothesis, $SC_K(X)$ is locally compact at $\{e_x\}$. Thus there is a basic open set $\mathcal{U}$ in $SC_K(X)$ such that $cl(\mathcal{U})$ is compact in $SC_K(X)$ and $\{e_x\} \in \mathcal{U} \subset cl(\mathcal{U}) \cap SC_K(X) \subset \langle U\rangle \cap SC_K(X)$. Let $V \in \tau$ such that $\{e_x\} \in \langle V\rangle \cap SC_K(X) \subset \mathcal{U}$. Then we have

$$e_x \in V \subset \bigcup \mathcal{U} \subset \bigcup \{F \in \tau^c : F \in cl(\mathcal{U}) \cap SC_K(X)\}.$$

By Proposition 3.32 (2) in [34], $\bigcup\{F \in \tau^c : F \in cl(\mathcal{U}) \cap SC_K(X)\}$ is soft compact in X. Thus $Scl(V)$ is soft compact in X. So X is soft locally compact at e_x. Hence X is soft locally compact. □

Corollary 3.53 (See Proposition 3.3, [49])**.** *Let (X, τ, E) be a soft T_2-space and $e \in E$. Then the following are equivalent:*

(1) (X, τ_e) *is locally compact,*

(2) $\mathcal{K}_e(X)$ *is locally compact,*

(3) $C_{K_e}(X)$ *is locally compact.*

Theorem 3.54. *Let (X, τ, E) be a soft T_2-space. Then X is soft locally compact if and only if $\mathcal{SF}_n(X)$ is locally compact for each $n \in \mathbb{N}$.*

Proof. Suppose X is soft locally compact and let $F \in \mathcal{SF}_n(X)$. Then clearly, F is soft compact in X. By the hypothesis, there is $U \in \tau$ such that $F \subset U$ and $scl(U)$ is soft compact in X. Thus $F \in \langle U\rangle \cap \mathcal{SF}_n(X)$. On the other hand, $cl(\langle U\rangle) = \langle Scl(U)\rangle = 2^{Scl(U)}_E$. Since $scl(U)$ is soft compact in X, by Theorem 3.17,

$\langle Scl(U)\rangle$ is compact in 2^X_E. Moreover, $cl_{\mathcal{SF}_n(X)}(\langle U\rangle \cap \mathcal{SF}_n(X) \subset \langle Scl(U)\rangle$. Thus $cl_{\mathcal{SF}_n(X(}(\langle U\rangle \cap \mathcal{SF}_n(X)$ is compact in $F \in \mathcal{SF}_n(X)$. So $\mathcal{SF}_n(X)$ is locally compact at F. Hence $\mathcal{SF}_n(X)$ is locally compact.

Conversely, suppose $\mathcal{SF}_n(X)$ is locally compact. It is obvious that $\mathcal{SF}_1(X)$ is closed in $\mathcal{SF}_n(X)$. Then $\mathcal{SF}_1(X)$ is locally compact. On the other hand, X is homeomorphic to $\mathcal{SF}_1(X)$. Thus X is soft locally compact. □

Corollary 3.55. *Let (X, τ, E) be a soft T_2-space and $e \in E$. Then (X, τ_e) is locally compact if and only if $\mathcal{F}_{n,e}(X)$ is locally compact for each $n \in \mathbb{N}$.*

4. Axioms of countability in soft hyperspaces

First, we discuss some of the axioms of countability and separability in soft topological spaces. Next, we study some relationships for axioms of countability and separability in a soft topological space and its soft hyperspace.

Definition 4.1 ([50])**.** Let $\mathcal{B}(\mathbb{R})$ be the collection of all non-empty bounded subsets of $\mathbb{R}$. Then a mapping $A : E \to \mathcal{B}(\mathbb{R})$ is called a *soft real set*. For each $r \in \mathbb{R}$, r_E is called a *soft real number* of $\mathbb{R}$, denoted by $\overline{r}$, and e_r is called a *soft real point* of $\mathbb{R}$, denoted by $\widetilde{r}$.

Definition 4.2 (See [51])**.** Let $r,\ s \in \mathbb{R}$. Then the order $\leq$ between soft real numbers $\overline{r}$ and $\overline{s}$ is defined by:

(i) $\overline{r} \leq \overline{s}$ if and only if $r \leq_{\mathbb{R}} s$, i.e., $\overline{r}(e) \leq_{\mathbb{R}} \overline{s}(e)$ for each $e \in E$,

(ii) $\overline{r} < \overline{s}$ if and only if $r <_{\mathbb{R}} s$, i.e., $\overline{r}(e) <_{\mathbb{R}} \overline{s}(e)$ for each $e \in E$,

where $\leq_{\mathbb{R}}$ and $<_{\mathbb{R}}$ denote the usual order on $\mathbb{R}$.

By using soft real numbers, we define soft real intervals of $\mathbb{R}$.

Definition 4.3. Let $\overline{a},\ \overline{b} \in \mathbb{R}_E$ such that $\overline{a} \leq \overline{b}$. Then

(i) (*Soft open interval*) $(\overline{a}, \overline{b}) = \{\overline{x} \in \mathbb{R}_E : \overline{a} < \overline{x} < \overline{b}\}$,

(ii) (*Soft closed interval*) $[\overline{a}, \overline{b}] = \{\overline{x} \in \mathbb{R}_E : \overline{a} \leq \overline{x} \leq \overline{b}\}$,

(ii) (*Soft half-open interval*) $(\overline{a}, \overline{b}] = \{\overline{x} \in \mathbb{R}_E : \overline{a} < \overline{x} \leq \overline{b}\}$,

$[\overline{a}, \overline{b}) = \{\overline{x} \in \mathbb{R}_E : \overline{a} \leq \overline{x} < \overline{b}\}$,

(iv) (*Soft half-real line*) $(\overline{a}, \infty) = \{\overline{x} \in \mathbb{R}_E : \overline{x} < \overline{a}\}$,

$[\overline{a}, \infty) = \{\overline{x} \in \mathbb{R}_E : \overline{x} \leq \overline{a}\}$,

$(-\infty, \overline{b}) = \{\overline{x} \in \mathbb{R}_E : \overline{x} > \overline{b}\}$,

$(-\infty, \overline{b}] = \{\overline{x} \in \mathbb{R}_E : \overline{x} \geq \overline{b}\}$.

Lemma 4.4. *Let σ* [resp. *β*] *be the collection of all soft-real lines* [resp. soft half-open intervals] *of the forms $(\overline{a}, \infty)$ and $(-\infty, \overline{b})$* [resp. $[\overline{a}, \overline{b})$]. *Then σ* [resp. *β*] *is a soft subbase* [resp. *base*] *for the soft topology τ* [resp. τ_l] *on $\mathbb{R}$.*

In this case, τ is called the *soft usual topology* [resp. *soft lower-limit topology*] on $\mathbb{R}$. The triple $(\mathbb{R}, \tau, E)$ [resp. $(\mathbb{R}, \tau_l, E)$] is called a *soft real space* [resp. *soft lower-limit real space*].

Proof. The proof follows from the definition of a soft subbase and Proposition 3.14 in [38]. □

Definition 4.5 ([16])**.** $A \in SS_E(X)$ is said to be *countable* [resp. *finite*], if $A(e)$ is countable [resp. finite] for each $e \in E$.

Definition 4.6. Let (X, τ, E) be a soft topological space and $e_x \in SP_E(X)$. Then $\beta_{e_x} \subset \tau$ is called a *soft local base at* e_x, provided that the following conditions hold:

(i) $B \in \beta_{e_x}$ implies $e_x \in B$,

(ii) $e_x \in U \in \tau$ implies there is $B \in \beta_{e_x}$ such that $B \subset U$.

It is clear that $\beta_{e_x} = \mathcal{SN}(e_x)$.

Definition 4.7 (See [52,53])**.** Let (X, τ, E) be a soft topological space. Then we say that

(i) X is *first countable* or *satisfies the first axiom of countability*, if there is a countable soft local base at e_x for each $e_x \in SP_E(X)$,

(ii) X is *second countable* or *satisfies the second axiom of countability*, if there is a countable soft base for τ.

Example 4.8. (1) Let $(\mathbb{R}, \tau, E)$ be a soft real space and $\beta = \{-(\overline{\frac{1}{n}}, \overline{\frac{1}{n}}) : n \in \mathbb{N}\}$. Then we can easily check that $\beta = \mathcal{SN}(\overline{0})$. Thus $(\mathbb{R}, \tau, E)$ is first countable.

(2) (See Proposition 4.1, [52]) Let $(\mathbb{R}, \tau, E)$ be a soft real space and $\beta = \{(\overline{a}, \overline{b}) : a,\ b \in \mathbb{R}\ \textit{are rational}\}$. Then clearly, β is a countable base for τ. Thus $(\mathbb{R}, \tau, E)$ is second countable.

(3) Consider the soft topological space $(\mathbb{R}, \tau_f, E)$ and assume that it is first countable, i.e., there is a soft local base $\beta = \{B_n \in \tau_f : n \in \mathbb{N}\}$ at $\overline{x} \in \mathbb{R}_E$. Let $y \in \mathbb{R}$ such that $x \neq y$. Then clearly, $\overline{x} \neq \overline{y}$ and $\mathbb{R} \setminus \{\overline{y}\} \in \mathcal{N}(\overline{x})$. Thus there is $n \in \mathbb{N}$ such that $\overline{y} \notin B_n$. So $\bigcap \beta = \{\overline{x}\}$. On the other hand, we have

$$\mathbb{R} \setminus \{\overline{x}\} = \mathbb{R} \setminus \bigcap_{n \in \mathbb{N}} B_n = \bigcup_{n \in \mathbb{N}} (\mathbb{R} \setminus B_n).$$

Since $\mathbb{R} \setminus B_n$ is finite for each $n \in \mathbb{N}$, $\bigcup_{n \in \mathbb{N}} (\mathbb{R} \setminus B_n)$ is countable. Hence $\mathbb{R} \setminus \{\overline{x}\}$ is countable. This is a contradiction. Therefore $(\mathbb{R}, \tau_f, E)$ is not first countable.

Proposition 4.9 (See Proposition 4.2, [52]; Theorem 4, [53])**.** *Every second countable soft topological space is first countable.*

Proof. Let (X, τ, E) be a soft topological space and suppose X is second countable. Then there is a countable soft base β for τ. Let $e_x \in SP_E(X)$. Since $\bigcup \beta = X_E$, $e_x \in \bigcup \beta$. Let $\beta_{e_x} = \{B \in SS_E(X) : e_x \in B \in \beta\}$. Since $\beta \subset \tau$, $\beta_{e_x} = \mathcal{SN}(e_x)$. Then β_{e_x} is a countable soft neighborhood base at e_x. Thus X is first countable. □

Remark 4.10. The converse of Proposition 4.9 is not true in general (see Example 4.11).

Example 4.11. Let $(\mathbb{R}, \tau_l, E)$ be a soft lower-limit real space. Then it is first countable but not second countable.

Definition 4.12 (See [53])**.** Let (X, τ, E) be a soft topological space and $A \in SS_E(X)$.

(i) A is called a *soft dense set* in X, if $Scl(A) = X_E$.

(ii) X is said to be *soft separable*, if there is a countable soft dense set in X.

Theorem 4.13. *Let (X, τ, E) be a soft T_1-space. Then X is first countable if and only if $\mathcal{SF}(X)$ is first countable.*

Proof. Suppose X is first countable and let $e_{i_{x_i}} \in SP_E(X)$ for each $i \in \{1, \cdots, n\}$, $F = \bigcup_{i=1}^{n} e_{i_{x_i}}$. Since X is T_1, $e_{i_{x_i}} \in \tau^c$. Then $F \in \mathcal{SF}(X)$. Since X is first countable, there is a countable soft local base $\beta_i = \mathcal{SN}(e_{i_{x_i}})$ for each $i \in \{1, \cdots, n\}$. Let $\mathcal{B}$ be the collection of all open sets in $\mathcal{SF}(X)$ of the form $\langle B_1, \cdots, B_n\rangle \cap \mathcal{SF}(X)$, where $B_i \in \beta_i$ for each $i \in \{1, \cdots, n\}$. Then clearly, $\mathcal{B}$ is countable. We will prove that $\mathcal{B}$ is a base at F. Let $\langle U_1, \cdots, U_m\rangle \cap \mathcal{SF}(X)$ be an open neighborhood of F. Then clearly, $F \in \langle U_1, \cdots, U_m\rangle$. Thus $F \subset \bigcup_{j=1}^{m} U_j$. Since $e_{i_{x_i}} \in F$ for each $i \in \{1, \cdots, n\}$, $e_{i_{x_i}} \in \bigcup_{j=1}^{m} U_j$ for each $i \in \{1, \cdots, n\}$, i.e., there is $j \in \{1, \cdots, m\}$ such that $e_{i_{x_i}} \in U_j$ for each $i \in \{1, \cdots, n\}$. So we choose $B_i \in \beta_i$ such that $B_i \subset \bigcap_{j=1}^{n}\{U_j : e_{i_{x_j}} \in U_j\}$. Hence we have

$$F \in \langle B_1, \cdots, B_n\rangle \cap \mathcal{SF}(X) \subset \langle U_1, \cdots, U_m\rangle \cap \mathcal{SF}(X).$$

It follows that $\mathcal{B}$ is a countable local base at F. Therefore $\mathcal{SF}(X)$ is first countable.

Conversely, suppose $\mathcal{SF}(X)$ is first countable. By Lemma 4.5 in [34], X is soft homeomorphic to $\mathcal{SF}_1(X) \subset \mathcal{SF}(X)$. Then X is first countable. □

We obtain the following consequences from Theorem 4.13.

Corollary 4.14. *Let (X, τ, E) be a soft T_1-space. Then X is first countable if and only if $\mathcal{SF}_n(X)$ is first countable for each $n \in \mathbb{N}$.*

Corollary 4.15. *Let (X, τ, E) be a soft T_1-space and $e \in E$. Then (X, τ_e) is first countable if and only if $\mathcal{F}_e(X)$ is first countable.*

Corollary 4.16. *Let (X, τ, E) be a soft T_1-space and $e \in E$. Then (X, τ_e) is first countable if and only if $\mathcal{F}_{n,e}(X)$ is first countable for each $n \in \mathbb{N}$.*

It is well-known that first countability and second countability are topological properties. Then we have:

Proposition 4.17. *Let (X, τ, E) be a soft topological space. If 2^X_E is first* [resp. *second*] *countable, then each one of the subspaces of 2^X_E is first* [resp. *second*] *countable.*

Corollary 4.18. *Let (X, τ, E) be a soft T_1-space and $e \in E$. If 2^X_e is first* [resp. *second*] *countable, then each one of the subspaces of 2^X_e is first* [resp. *second*] *countable.*

Theorem 4.19. *Let (X, τ, E) be a soft T_1-space. Then X is second countable if and only if $\mathcal{SK}(X)$ is second countable.*

Proof. Suppose X is second countable. Then there is a countable soft base $\beta = \{U_n \in \tau : n \in \mathbb{N}\}$ for $\mathcal{T}_v$. Let $\mathcal{U} = \{\langle U_{\alpha_1}, \cdots, U_{\alpha_n}\rangle \cap \mathcal{SK}(X) : U_{\alpha_i} \in \beta\}$. Then $\mathcal{U}$ is a countable base for $\mathcal{SK}(X)$. Thus $\mathcal{SK}(X)$ is second countable.

Suppose $\mathcal{SK}(X)$ is second countable. Then clearly, $\mathcal{SF}_1(X)$ is second countable. By Lemma 4.5 in [34], X is soft homeomorphic to $\mathcal{SF}_1(X) \subset \mathcal{SK}(X)$. Thus X is second countable. □

Corollary 4.20 (See Proposition 4.5.2, [47])**.** *Let (X, τ, E) be a soft T_1-space and $e \in E$. Then (X, τ_e) is second countable if and only if $\mathcal{K}_e(X)$ is second countable.*

Corollary 4.21. *Let (X, τ, E) be a soft T_1-space. Then the following are equivalent:*

(1) *X is second countable,*

(2) $\mathcal{SF}_n(X)$ *is second countable for each* $n \in \mathbb{N}$,
(3) $\mathcal{SF}(X)$ *is second countable,*
(4) $C_K(X)$ *is second countable,*
(5) $\mathcal{SK}(X)$ *is second countable.*

Proof. It is clear that $\mathcal{SF}_n(X) \subset \mathcal{SF}(X) \subset \mathcal{SK}(X)$ and $C_K(X) \subset \mathcal{SK}(X)$. By Lemma 4.5 in [34], X is soft homeomorphic to $\mathcal{SF}_1(X)$. Then X is second countable if and only if each of $\mathcal{SF}_n(X)$, $\mathcal{SF}(X)$, and $C_K(X)$ as a subspace is second countable. □

Corollary 4.22. *Let* (X, τ, E) *be a soft* T_1*-space and* $e \in E$*. Then the following are equivalent:*
(1) (X, τ_e) *is second countable,*
(2) $\mathcal{F}_{n,e}(X)$ *is second countable for each* $n \in \mathbb{N}$,
(3) $\mathcal{F}_e(X)$ *is second countable,*
(4) $C_{K_e}(X)$ *is second countable,*
(5) $\mathcal{K}_e(X)$ *is second countable.*

Lemma 4.23. *Let* (X, τ, E) *be a soft topological space,* β *a soft base for* τ *such that* $\emptyset_E \notin \beta$*, and* $D \in SS_E(X)$*. Then* D *is soft dense in* X *if and only if* $D \cap B \neq \emptyset_E$ *for each* $V \in \beta$*.*

Proof. Suppose D is soft dense in X and assume that there is $B \in \beta$ such that $D \cap B = \emptyset_E$. Then clearly, $D \subset B^c$ and $B^c \in \tau^c$. Thus $Scl(D) \subset B^c$. So $Scl(D) \neq X_E$. This is a contradiction. Hence the necessary condition holds.

Conversely, suppose the necessary condition holds, and let $e_x \in SP_E(X)$ and $N \in \mathcal{SN}(e_x)$. Then there is $B \in \beta$ such that $e_x \in B \subset N$. Since $B \cap D \neq \emptyset_E$, $N \cap D \neq \emptyset_E$. Thus by Theorem 3.3 in [17], $e_x \in Scl(D)$. So $X_E \subset Scl(D)$, i.e., $Scl(D) = X_E$. Hence D is soft dense in X. □

Theorem 4.24. *Let* (X, τ, E) *be a soft* T_1*-space. Then* X *is soft separable if and only if* 2^X_E *is separable.*

Proof. Suppose X is soft separable. Then there is a countable soft dense set D in X, i.e., $Scl(D) = X_E$ and $D(e)$ is countable for each $e \in E$. Let $\mathcal{D} = \{A \in SS_E(X) : A \subset D \text{ is finite}\}$. Then clearly, $\mathcal{D}$ is countable. Moreover, $\mathcal{D} \subset 2^X_E$ since X is T_1. Let $\langle U_1, \cdots, U_n\rangle$ be a basic open set in 2^X_E. Since $Scl(D) = X_E$ and $U_i \in \tau$ for each $i \in \{1, \cdots, n\}$, there is $e_{i_{x_i}} \in SP_E(X)$ such that $e_{i_{x_i}} \in D \cap U_i$ for each $i \in \{1, \cdots, n\}$. Since X is soft T_1, $F = \bigcup_{i=1}^n e_{i_{x_i}} \in \tau^c$. Thus $F \in 2^X_E$. Furthermore, $F \subset D$, $F \subset \bigcup_{i=1}^n U_i$ and $F \cap U_i \neq \emptyset_E$ for each $i \in \{1, \cdots, n\}$. So $F \in \mathcal{D} \cap \langle U_1, \cdots, U_n\rangle$, i.e., $\mathcal{D} \cap \langle U_1, \cdots, U_n\rangle \neq \emptyset$. Hence 2^X_E is separable.

Conversely, suppose 2^X_E is separable. Then a countable dense subset $\mathcal{D} = \{A_n : n \in \mathbb{N}\}$ of 2^X_E. For each $n \in \mathbb{N}$, let us take $e_{n_{x_n}} \in A_n$ and let $D = \bigcup_{n \in \mathbb{N}} e_{n_{x_n}}$. Let $U \in \tau$. Then clearly, $\langle U\rangle$ is a basic open set in 2^X_E. Since $\mathcal{D}$ is dense in 2^X_E, $\mathcal{D} \cap \langle U\rangle \neq \emptyset$. Thus there is $n \in \mathbb{N}$ such that $A_n \in \mathcal{D} \cap \langle U\rangle$. So $e_{n_{x_n}} \in D \cap U$, i.e., $D \cap U \neq \emptyset_E$. So by Lemma 4.23, D is a countable soft dense set in X. Hence X is soft separable. □

Corollary 4.25 (Proposition 4.5.1, [47])**.** *Let* (X, τ, E) *be a soft* T_1*-space and* $e \in E$*. Then* (X, τ_e) *is separable if and only if* 2^X_e *is separable.*

Theorem 4.26. *Let* (X, τ, E) *be a soft* T_1*-space. Then* X *is soft separable if and only if* $\mathcal{SK}(X)$ *is separable.*

Proof. Suppose X is soft separable. Then there is a countable soft dense set D in X. Let $\mathcal{D} = \{A \in SS_E(X) : A \subset D \; is \; finite\}$. Then $\mathcal{D}$ is countable. Let $\mathcal{U} = \langle U_1, \cdots, U_n\rangle \cap \mathcal{SK}(X)$, where $\langle U_1, \cdots, U_n\rangle$ is a basic open set in 2^X_E. Since D is soft dense in X and $U_i \in \tau$ for each $i \in \{1, \cdots, n\}$, by Lemma 4.23, $D \cap U_i \neq \emptyset_E$ for each $i \in \{1, \cdots, n\}$. Let us choose $e_{i_{x_i}} \in D \cap U_i$ for each $i \in \{1, \cdots, n\}$ and $F = \bigcup_{u=1}^{n} e_{i_{x_i}}$. Since X is soft T_1, $F \subset D$. Moreover, $F \subset \bigcup_{i=1}^{n} U_i$. Then $F \in \mathcal{D} \cap \mathcal{U}$, i.e., $\mathcal{D} \cap \mathcal{U} \neq \emptyset$. Thus $\mathcal{D}$ is dense in $\mathcal{SK}(X)$. So $\mathcal{SK}(X)$ is separable.

Conversely, suppose $\mathcal{SK}(X)$ is separable. Then there is a countable dense subset $\mathcal{D} = \{A_n : n \in \mathbb{N}\}$ of $\mathcal{SK}(X)$. Let us take $e_{n_{x_n}} \in A_n$ for each $n \in \mathbb{N}$ and let $D = \bigcup_{n\in\mathbb{N}} e_{n_{x_n}}$. Then clearly, D is a soft countable set in X. Let U be a nonempty basic open set in X. Since $\mathcal{D}$ is dense in $\mathcal{SK}(X)$, $\mathcal{D} \cap (\langle U\rangle \cap \mathcal{SK}(X)) \neq \emptyset$. Thus there is $n \in \mathbb{N}$ such that $A_n \in \mathcal{D} \cap (\langle U\rangle \cap \mathcal{SK}(X))$, i.e., $A_n \in \mathcal{D} \cap \langle U\rangle$. So $e_{n_{x_n}} \in A_n \cap U$, i.e., $D \cap U \neq \emptyset_E$. Hence D is a soft dense set in X. Therefore X is soft separable. □

Corollary 4.27. *Let (X, τ, E) be a soft T_1-space and $e \in E$. Then (X, τ_e) is separable if and only if $\mathcal{K}_e(X)$ is separable.*

Proposition 4.28. *Let (X, τ, E) be a soft T_1-space. If $SC(X)$* [resp. $SC_K(X)$] *is separable, then X is soft separable.*

Proof. Suppose $SC(X)$ is separable. Then there is a countable dense subset $\mathcal{D} = \{A_n : n \in \mathbb{N}\}$ of $SC(X)$. Let us choose $e_{n_{x_n}} \in A_n$ for each $n \in \mathbb{N}$ and let $D = \bigcup_{n\in\mathbb{N}} e_{n_{x_n}}$. Since X is soft T_1, $D \in \tau^c$ and D is countable. Let U be a nonempty soft basic open set X. Then clearly, $\langle U\rangle$ is a basic open set in 2^X_E. Thus $\langle U\rangle \cap SC(X)$ is a basic open set in $SC(X)$. Since $\mathcal{D}$ is dense in $SC(X)$, $\mathcal{D} \cap (\langle U\rangle \cap SC(X)) \neq \emptyset$. So there is $n \in \mathbb{N}$ such that $A_n \in \mathcal{D} \cap (\langle U\rangle \cap SC(X))$, i.e., $A_n \in \mathcal{D} \cap \langle U\rangle$. Hence $e_{n_{x_n}} \in A_n \subset U$, i.e., D is soft dense in X. Therefore X is soft separable.

The proof of the second part is similar. □

Corollary 4.29. *Let (X, τ, E) be a soft T_1-space and $e \in E$. If $C_e(X)$* [resp. $C_{K_e}(X)$] *is separable, then X is soft separable.*

5. Discussion

In this paper, we discussed various compactness, local compactness, separability, and countability relationships in a soft topological space and its soft hyperspace.

6. Conclusions

We obtained some compactness relationships in a soft topological space and its soft hyperspace. Also, we discussed various local compactness relationships in a soft topological space and its soft hyperspace. Also, we studied some separability and axiom of countability relationships in a soft topological space and its soft hyperspace. Furthermore, it was found that the properties obtained from Sections 3 and 4 were almost similar to the classical case.

In the future, we would like to find connectedness relationships, local connectedness relationships and local connected Klein relationships in a soft topological space and its soft hyperspace. Also, we will study metrization relationships in a soft topological space and its soft hyperspace. Moreover, according to the reviewers' suggestions, we expect to be able to conduct more enriched research by referring to references [54–58], etc.

Author contributions

Conceptualization, K. Hur; Methodology, J. I. Baek and G. Şenel; Validation, S. H. Han and M. Cheong; Formal analysis research, J. I. Baek, G. Şenel, and K. Hur; Writing—original draft, J.I.Baek and M.Cheong; Writing—review and editing, G. Şenel, K. Hur. and S. H. Han; Project administration, G. Şenel and K. Hur; Funding acquisition, G. Şenel. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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