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Research article

Geometric analysis of the pseudo-projective curvature tensor in doubly and twisted warped product manifolds

Ayman Elsharkawy[1,*], **Hoda Elsayied**[1], **Abdelrhman Tawfiq**[2] **and Fatimah Alghamdi**[3]

[1] Department of Mathematics, Faculty of Science, Tanta University, Tanta, Egypt; hkelsayied1998@yahoo.com

[2] Department of Mathematics, Faculty of Education, Ain-Shams University, Cairo, Egypt; abdelrhmanmagdi@edu.asu.edu.eg

[3] Department of Mathematics and Statistics, College of Science, University of Jeddah, Jeddah, Saudi Arabia; fmalghamdi@uj.edu.sa

* **Correspondence:** Email: ayman_ramadan@science.tanta.edu.eg.

Abstract: This study investigates the pseudo-projective curvature tensor within the framework of doubly and twisted warped product manifolds. It offers significant insights into the interaction between the pseudo-projective curvature tensor and both the base and fiber manifolds. The research highlights key geometric characteristics of the base and fiber manifolds as influenced by the pseudo-projective curvature tensor in these structures. Additionally, the paper extends its analysis to examine the behavior of the pseudo-projective curvature tensor in the context of generalized doubly and twisted generalized Robertson-Walker space-times.

Keywords: doubly and twisted warped product; pseudo-projective curvature tensor; generalized doubly and twisted Robertson-Walker space-times
Mathematics Subject Classification: 53C20, 53C21, 53C25, 53C30, 53C50

1. Introduction

Bishop and O'Neill [1] first introduced the concept of warped products within Riemannian manifolds to develop a broad class of complete manifolds characterized by negative curvature. This concept emerged from the study of surfaces of revolution. Subsequently, Nölker [2] extended this idea by formulating the notion of multiply warped products, which generalizes the original concept. Warped products hold significant relevance in differential geometry, particularly in mathematical physics and general relativity. Many exact solutions to Einstein's field equations and their modifications can be represented using warped products.

Doubly warped product manifolds (DWPMs) generalize the concept of warped products by introducing a warping function that depends on two distinct factors, typically associated with the base and fiber manifolds. This generalized structure has been instrumental in advancing the study of complex geometric configurations [3–5].

Moreover, DWPMs provide a robust framework for analyzing spaces characterized by varying curvature, enabling significant insights into their geometric properties [6, 7]. These manifolds also find extensive applications in mathematical physics and general relativity, where they serve as effective models for describing intricate theoretical phenomena [8, 9].

Further generalization is achieved with twisted warped product manifolds (TWPMs), which include an additional twist factor to modify the warping function. This twist enriches the geometric structure, allowing for the examination of more intricate relationships between the base and fiber manifolds. These manifolds are particularly advantageous for investigating the curvature properties of spaces that emerge in advanced theoretical physics, including various cosmological models [10].

The pseudo-projective curvature (PPC) tensor, originally introduced by Prasad [11], serves as an extension of the projective curvature tensor. This tensor has been extensively investigated by numerous researchers, reflecting its significance in mathematical and physical studies [12–14]. Further developments in this area include the work of Shenawy and Ünal [15], who specifically analyzed the W_2-curvature tensor within the context of warped product manifolds. Building upon these foundational studies, this paper focuses on the examination of the PPC in the settings of DWPMs, TWPMs, and space-times [16].

The structure of the paper is as follows: Section 2 presents the essential concepts and definitions related to DWPMs and TWPMs, which underpin the study. Sections 3 and 4 delve into the analysis of the PPC within DWPMs and TWPMs, respectively, offering a comprehensive description of the geometric properties of the base and fiber manifolds in relation to the pseudo-projective tensor. Section 5 applies these findings to analyze the behavior of the PPC in the context of generalized doubly and twisted Robertson-Walker space-times.

2. Preliminaries

This section introduces the key concepts and definitions for DWPMs and TWPMs and explores the PPC within a pseudo-Riemannian manifold (PRM).

2.1. DWPMs

Consider two Riemannian manifolds (M_1, g_1) and (M_2, g_2), with positive, smooth functions f_1 on M_1 and f_2 on M_2. Let π_1 and π_2 be the standard projection maps from $M_1 \times M_2$ onto M_1 and M_2, respectively. The DWPM ${}_{f_2}M_1 \times_{f_1} M_2$ is constructed as the product manifold $M_1 \times M_2$, endowed with a metric g defined by

$$g = (f_2 \circ \pi_2)^2 \pi_1^*(g_1) + (f_1 \circ \pi_1)^2 \pi_2^*(g_2),$$

where the expression $\pi_i^*(g_i)$ represents the pullback of the metric g_i by π_i, for $i = \{1, 2\}$ [3, 10]. The functions f_i are referred to as the warping functions of the DWPM $({}_{f_2}M_1 \times_{f_1} M_2, g)$. If one of the functions f_1 or f_2 is constant, the manifold reduces to a warped product manifold. If both f_1 and f_2 are constant, the result is a direct product manifold. A DWPM is considered non-trivial if neither f_1 nor f_2 is constant (see [12]).

Let $\mathcal{L}(M_i)$ represent the collection of lifted vector fields from M_i, with

$$k = \ln f_1 \ \ (\text{or} \ \ l = \ln f_2),$$

and the same notation is used for the function k (or l) and its pullback $k \circ \pi_1$ (or $l \circ \pi_2$). The symbols ${}^1R$ (or ${}^1Ric$) and ${}^2R$ (or ${}^2Ric$) represent the lifted Riemann (or Ricci) curvature tensors from (M_1, g_1) and (M_2, g_2), respectively, while R (or Ric) denotes the Riemann (or Ricci) curvature tensor of the DWPM.

Lemma 2.1. *If $A_j \in \mathcal{L}(M_1)$ and $B_j \in \mathcal{L}(M_2)$ for $j \in \{1, 2, 3\}$, then the Riemann curvature tensor is given by*

$$\begin{aligned}
R(A_1, A_2)A_3 &= {}^1R(A_1, A_2)A_3 + g(A_1, A_3)H^l(A_2) - g(A_2, A_3)H^l(A_1),\\
R(A_1, A_2)B_1 &= B_1(l)\Big(A_2(k)A_1 - A_1(k)A_2\Big),\\
R(B_1, B_2)A_1 &= A_1(k)\Big(B_2(l)B_1 - B_1(l)B_2\Big),\\
R(A_1, B_1)A_2 &= \Big(h_1^k(A_1, A_2) + A_1(k)A_2(k)\Big)B_1 + A_2(k)B_1(l)A_1 + g(A_1, A_2)\Big(H^l(B_1) + B_1(l)\nabla l\Big),\\
R(B_1, A_1)B_2 &= \Big(h_2^l(B_1, B_2) + B_1(l)B_2(l)\Big)A_1 + B_2(l)A_1(k)B_1 + g(B_1, B_2)\Big(H^k(A_1) + A_1(k)\nabla k\Big),\\
R(B_1, B_2)B_3 &= {}^2R(B_1, B_2)B_3 + g(B_1, W)H^k(B_2) - g(B_2, B_3)H^k(B_1).
\end{aligned} \tag{2.1}$$

Here, H^k, ∇ denote the Hessian tensor of k and the Levi-Civita connection on $({}_{f_2}M_1 \times_{f_1} M_2, g)$, which is defined by

$$H^k(X) = \nabla_X \nabla k$$

for any vector field X on the DWPM.

Let ${}^1Ric$ and ${}^2Ric$ denote the lifted Ricci curvature tensors of (M_1, g_1) and (M_2, g_2), respectively, while Ric represents the Ricci curvature tensor of the DWPM.

Lemma 2.2. *If $A_j \in \mathcal{L}(M_1)$ and $B_j \in \mathcal{L}(M_2)$ for $j \in \{1, 2\}$, then the Ricci curvature tensor is given by*

$$\begin{aligned}
Ric(A_1, A_2) &= {}^1Ric(A_1, A_2) - \frac{m_2}{f_1} h_1^{f_1}(A_1, A_2) - g(A_1, A_2)\Delta l,\\
Ric(A_1, B_1) &= (n_1 + n_2 - 2)A_1(k)B_1(l),\\
Ric(B_1, B_2) &= {}^2Ric(B_1, B_2) - \frac{m_1}{f_2} h_2^{f_2}(B_1, B_2) - g(B_1, B_2)\Delta k.
\end{aligned} \tag{2.2}$$

Here, Δ denotes the Laplacian operator on DWPM, and m_i represents the dimension of M_i, for $i \in \{1, 2\}$.

2.2. TWPMs

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds with corresponding Riemannian metrics, and let f be a smooth positive function on $M_1 \times M_2$. The canonical projections from $M_1 \times M_2$ onto M_1 and M_2 are denoted by π_1 and π_2, respectively. The TWPM $M_1 \times_f M_2$, as introduced in [10], is the product manifold $M_1 \times M_2$ equipped with the metric g, defined by

$$g = \pi_1^*(g_1) + f^2 \pi_2^*(g_2),$$

where $\pi_i^*(g_i)$ denotes the pullback of the metric g_i via π_i for $i = \{1, 2\}$. In this construction, f is termed the twisting function of the TWPM. When f depends only on points in M_1, the manifold forms a warped product, and if f is constant, the structure becomes a direct product manifold.

Let $\mathcal{L}(M_i)$ represent the set of lifted vector fields on M_i, and define

$$k = \ln f$$

with ∇k being the gradient of k. The Riemann curvature tensors of (M_1, g_1) and (M_2, g_2) are denoted by ${}^1R$ and ${}^2R$, respectively, while R is the Riemann curvature tensor of the TWPM.

Lemma 2.3. *If $A_j \in \mathcal{L}(M_1)$ and $B_j \in \mathcal{L}(M_2)$ for $j \in \{1, 2, 3\}$, then the Riemann curvature tensor is given by*

$$\begin{aligned}
R(A_1, A_2)A_3 &= {}^1R(A_1, A_2)A_3,\\
R(A_1, A_2)B_1 &= 0,\\
R(B_1, B_2)A_1 &= B_1A_1(k)B_2 - B_2A_1(k)B_1,\\
R(A_1, B_1)A_2 &= \Big(h_1^k(A_1, A_2) + A_1(k)A_2(k)\Big)B_1,\\
R(B_1, A_1)B_2 &= -A_1B_2(k)B_1 + \Big(A_1(k)\nabla k + H^k(A_1)\Big)g(B_1, B_2),\\
R(B_1, B_2)B_3 &= {}^2R(B_1, B_2)B_3 - \Big(h_2^k(B_2, B_3) - B_3(k)B_2(k)\Big)B_1 + \Big(h_2^k(B_1, B_3) - B_3(k)B_1(k)\Big)B_2\\
&\quad - \Big(H^k(B_1) + B_1(k)\nabla k\Big)g(B_2, B_3) + \Big(H^k(B_2) + V(B_2)\nabla k\Big)g(B_1, B_3).
\end{aligned} \tag{2.3}$$

Here, H^k denotes the Hessian tensor of k on TWPM, defined as

$$H^k(X) = \nabla_X \nabla k$$

for any vector field X on TWPM.

Let ${}^1Ric$ and ${}^2Ric$ represent the Ricci curvature tensors of (M_1, g_1) and (M_2, g_2), respectively, while Ric denotes the Ricci curvature tensor of the TWPM.

Lemma 2.4. *If $A_j \in \mathcal{L}(M_1)$ and $B_j \in \mathcal{L}(M_2)$ for $j \in \{1, 2\}$, then the Ricci curvature tensor is given by*

$$\begin{aligned}
Ric(A_1, A_2) &= {}^1Ric(A_1, A_2) - n_2\Big(h_1^k(A_1, A_2) + A_1(k)A_2(k)\Big),\\
Ric(A_1, B_1) &= -(m_2 - 1)A_1B_1(k),\\
Ric(B_1, B_2) &= {}^2Ric(B_1, B_2) - (m_2 - 2)h_2^k(B_1, B_2) + (m_2 - 2)B_1(k)B_2(k) - g(B_1, B_2)\Delta k.
\end{aligned} \tag{2.4}$$

Here, Δ represents the Laplacian operator on TWPM, and m_i denotes the dimension of M_i, for $i \in \{1, 2\}$.

2.3. PPC Tensor

The PPC $\bar{P}^*$ on a PRM M with

$$dim(M) = m$$

is given by

$$\begin{aligned}
\bar{P}^*(A, B, C, D) &= a_1\bar{R}(A, B, C, D) + a_2\Big(Ric(B, C)g(A, D) - Ric(A, C)g(B, D)\Big)\\
&\quad - \frac{\tau}{m}\Big(\frac{a_1}{m - 1} + a_2\Big)\Big[g(B, C)g(A, D) - g(A, C)g(B, D)\Big],
\end{aligned} \tag{2.5}$$

where a_1 and a_2 ($\neq 0$) are constants, Ric represents the Ricci tensor of $(0,2)$-type, and τ denotes the scalar curvature of the manifold. Additionally,

$$\bar{P}^*(A,B,C,D) = g(P^*(A,B)C,D)$$

and

$$\bar{R}(A,B,C,D) = g(R(A,B)C,D),$$

with R being the Riemannian curvature tensor and $A,B,C,D \in \mathcal{L}(M)$.

When

$$a_1 = 1$$

and

$$a_2 = -\frac{1}{m-1},$$

the expression in Eq (2.5) simplifies to the projective curvature tensor. Furthermore, if

$$P^* = 0$$

for $m > 3$, the PRM is referred to as pseudo-projectively flat (PPF).

It is evident from Eq (2.5) that the manifold is characterized by

$$\begin{aligned} \bar{P}^*(A,B)C = {} & a_1 R(A,B)C + a_2\Big(Ric(B,C)A - Ric(A,C)B\Big) \\ & - \frac{\tau}{m}\Big(\frac{a_1}{m-1} + a_2\Big)\Big[g(B,C)A - g(A,C)B\Big]. \end{aligned} \tag{2.6}$$

3. Properties of the PPC on the DWPM

This section explores the properties of the PPC on the DWPM manifold

$$M = {}_{f_2}M_1 \times_{f_1} M_2.$$

Several theorems are presented concerning the PPC for such manifolds, which shed light on the relationship between the warped geometry and its underlying base and fiber manifolds. We use the notation $\bar{P}^*$ and P^* for the PPC and the tensor P^* on M, and $\bar{P}^*_i$ and P^*_i for their counterparts on M_i.

Theorem 3.1. *If*

$$M, g = f_2^2 g_1 \oplus f_1^2 g_2$$

is a DWPM and the vector fields $A_j \in \mathcal{L}(M_1)$ and $B_j \in \mathcal{L}(M_2)$ for $j \in \{1,2,3\}$, then the non-zero components of PPC are given by

$$\begin{aligned} P^*(A_1,A_2)A_3 = {} & P^*_1(A_1,A_2)A_3 + a_1 f_2^2\Big(g_1(A_1,A_3)H^l(A_2) - g_1(A_2,A_3)H^l(A_1)\Big) \\ & - \frac{a_2 m_2}{f_1}\Big(h_1^{f_1}(A_2,A_3)A_1 - h_1^{f_1}(A_1,A_3)A_2\Big) + f_2^2\tau\Big[\Big(\frac{m_2(m+m_1-1)}{mm_1(m_1-1)(m-1)}\Big)a_1 \\ & + \Big(\frac{m_2}{mm_1} - \frac{\Delta l}{\tau}\Big)\Big][g_1(A_2,A_3)A_1 - g_1(A_1,A_3)A_2], \end{aligned} \tag{3.1}$$

$$P^*(A_1,A_2)B_1 = \Big(a_1+(m-2)a_2\Big)\Big[A_2(k)A_1 - A_1(k)A_2\Big]B_1(l) - \frac{\tau}{m}\Big(\frac{a_1}{m-1}+a_2\Big)[g(A_2,B_1)A_1 - g(A_1,B_1)A_2], \tag{3.2}$$

$$P^*(B_1,B_2)A_1 = \Big(a_1+(m-2)a_2\Big)\Big[B_2(l)B_1 - B_1(l)B_2\Big]A_1(k) - \frac{\tau}{m}\Big(\frac{a_1}{m-1}+a_2\Big)[g(B_2,A_1)B_1 - g(B_1,A_1)B_2], \tag{3.3}$$

$$\begin{aligned}P^*(A_1,B_1)A_2 = {}& a_1\Big[\Big(h_1^k(A_1,A_2)+A_1(k)A_2(k)\Big)B_1 + A_2(k)B_1(l)A_1 + g(A_1,A_2)\Big(H^l(B_1)\\ &+ B_1(l)\nabla l\Big)\Big] + a_2\Big[(m-2)A_2(k)B_1(l)A_1 - \Big({}^1Ric(A_1,A_2) - \frac{m_2}{f_1}h_1^{f_1}(A_1,A_2)\\ &- g(A_1,A_2)\Delta l\Big)B_1\Big] - \frac{\tau}{m}\Big(\frac{a_1}{m-1}+a_2\Big)[g(B_1,A_2)A_1 - g(A_1,A_2)B_1],\end{aligned} \tag{3.4}$$

$$\begin{aligned}P^*(A_1,B_1)B_2 = {}& a_1\Big[\Big(h_2^l(B_1,B_2)+B_1(l)B_2(l)\Big)A_1 + B_2(l)A_1(k)B_1 + g(B_1,B_2)\Big(H^k(A_1)\\ &+ A_1(k)\nabla k\Big)\Big] + a_2\Big[\Big({}^2Ric(B_1,B_2) - \frac{m_1}{f_2}h_2^{f_2}(B_1,B_2) - g(B_1,B_2)\Delta k\Big)A_1\\ &- (m-2)A_1(k)B_2(l)B_1\Big] - \frac{\tau}{m}\Big(\frac{a_1}{m-1}+a_2\Big)[g(B_1,B_2)A_1 - g(A_1,B_2)B_1],\end{aligned} \tag{3.5}$$

$$\begin{aligned}P^*(B_1,B_2)B_3 = {}& P_2^*(B_1,B_2)B_3 + a_1 f_1^2\Big(g_2(B_1,B_2)H^k(B_2) - g_2(B_1,B_3)H^k(B_1)\Big)\\ &- \frac{a_2 m_1}{f_2}\Big(h_2^{f_2}(B_2,B_3)B_1 - h_2^{f_2}(B_1,B_3)B_2\Big) + \tau f_1^2\Big[\frac{m_1(m+m_2-1)}{mm_2(m_2-1)(m-1)}a_1\\ &+ \Big(\frac{m_1}{mm_2} - \frac{\Delta k}{\tau}\Big)a_2\Big][g_2(B_2,B_3)B_1 - g_2(B_1,B_3)B_2].\end{aligned} \tag{3.6}$$

Proof. If

$$M, g = f_2^2 g_1 \oplus f_1^2 g_2$$

is a DWPM. Assume that

$$dim(M) = m$$

and

$$dim(M_i) = m_i$$

for $i \in 1,2$. For vector fields $A_j \in \mathcal{L}(M_1)$ and $B_j \in \mathcal{L}(M_2)$ for $j \in \{1,2,3\}$, applying Eq (2.6) yields

$$\begin{aligned}P^*(A_1,A_2)A_3 &= a_1 R(A_1,A_2)A_3 + a_2\Big(Ric(A_2,A_3)A_1 - Ric(A_1,A_3)A_2\Big)\\ &\quad - \frac{\tau}{m}\Big(\frac{a_1}{m-1}+a_2\Big)[g(A_2,A_3)A_1 - g(A_1,A_3)A_2]\\ &= a_1\Big[{}^1R(A_1,A_2)A_3 + f_2^2 g_1(A_1,A_3)H^l(A_2) - f_2^2 g_1(A_2,A_3)H^l(A_1)\Big]\\ &\quad + a_2\Big[({}^1Ric(A_2,A_3) - \frac{m_2}{f_1}h_1^{f_1}(A_2,A_3) - f_2^2 g_1(A_2,A_3)\Delta l)A_1\\ &\quad - ({}^1Ric(A_1,A_3) - \frac{m_2}{f_1}h_1^{f_1}(A_1,A_3) - f_2^2 g_1(A_1,A_3)\Delta l)A_2\Big]\\ &\quad - \frac{\tau f_2^2}{m}\Big(\frac{a_1}{m-1}+a_2\Big)[g(A_2,A_3)A_1 - g_1(A_1,A_3)A_2]\end{aligned}$$

$$
\begin{aligned}
&= a_1{}^1R(A_1,A_2)A_3 + a_2({}^1Ric(A_2,A_3)A_1 - {}^1Ric(A_1,A_3)A_2) - \frac{\tau f_2^2}{m_1}\Big(\frac{a_1}{m_1-1+a_2}\Big)\\
&\quad [g_1(A_2,A_3)A_1 - g_1(A_1,A_3)A_2] + a_1 f_2^2(h_1^{f_1}(A_2,A_3)A_1 - h_1^{f_1}(A_1,A_3)A_2)\\
&\quad - \frac{a_2m_2}{f_1}[g_1(A_2,A_3)A_1 - g_1(A_1,A_3)A_2]\Delta l + \Big[\frac{\tau f_2^2}{m_1}\Big(\frac{a_1}{m_1-1}+a_2\Big)\\
&\quad - \frac{\tau f_2^2}{m}\Big(\frac{a_1}{m-1}+a_2\Big)\Big][g_1(A_2,A_3)A_1 - g_1(A_1,A_3)A_2],\\
P^*(A_1,A_2)A_3 &= P_1^*(A_1,A_2)A_3 + a_1 f_2^2\Big(g_1(A_1,A_3)H^l(A_2) - g_1(A_2,A_3)H^l(A_1)\Big)\\
&\quad - \frac{a_2m_2}{f_1}\Big(h_1^{f_1}(A_2,A_3)A_1 - h_1^{f_1}(A_1,A_3)A_2\Big) + f_2^2\tau\Big[\Big(\frac{m_2(m+m_1-1)}{mm_1(m_1-1)(m-1)}\Big)a_1\\
&\quad + \Big(\frac{m_2}{mm_1} - \frac{\Delta l}{\tau}\Big)\Big][g_1(A_2,A_3)A_1 - g_1(A_1,A_3)A_2],\\
P^*(A_1,A_2)B_1 &= a_1R(A_1,A_2)B_1 + a_2\Big(Ric(A_2,B_1)A_1 - Ric(A_1,B_1)A_2\Big)\\
&\quad - \frac{\tau}{m}\Big(\frac{a_1}{m-1}+a_2\Big)[g(A_2,B_1)A_1 - g(A_1,B_1)A_2]\\
&= a_1B_1(l)\Big[A_2(k)A_1 - A_1(k)A_2\Big] + a_2(m-2)\Big[A_2(k)B_1(l)A_1 - A_1(k)B_1(l)A_2\Big]\\
&\quad - \frac{\tau}{m}\Big(\frac{a_1}{m-1}+a_2\Big)[g(A_2,B_1)A_1 - g(A_1,B_1)A_2]\\
&= \Big(a_1+(m-2)a_2\Big)\Big[A_2(k)A_1 - A_1(k)A_2\Big]B_1(l)\\
&\quad - \frac{\tau}{m}\Big(\frac{a_1}{m-1}+a_2\Big)[g(A_2,B_1)A_1 - g(A_1,B_1)A_2],\\
P^*(B_1,B_2)A_1 &= a_1R(B_1,B_2)A_1 + a_2\Big(Ric(B_2,A_1)B_1 - Ric(B_1,A_1)B_2\Big)\\
&\quad - \frac{\tau}{m}\Big(\frac{a_1}{m-1}+a_2\Big)[g(B_2,A_1)B_1 - g(B_1,A_1)B_2],\\
P^*(B_1,B_2)A_1 &= a_1A_1(k)\Big[B_2(l)B_1 - B_1(l)B_2\Big] + a_2(m-2)\Big[A_1(k)B_2(l)B_1 - A_1(k)B_1(l)B_2\Big]\\
&\quad - \frac{\tau}{m}\Big(\frac{a_1}{m-1}+a_2\Big)[g(B_2,A_1)B_1 - g(B_1,A_1)B_2]\\
&= \Big(a_1+(m-2)a_2\Big)\Big[B_2(l)B_1 - B_1(l)B_2\Big]A_1(k)\\
&\quad - \frac{\tau}{m}\Big(\frac{a_1}{m-1}+a_2\Big)[g(A_1,B_2)B_1 - g(B_1,A_1)B_2],\\
P^*(A_1,B_1)A_2 &= a_1R(A_1,B_1)A_2 + a_2\Big(Ric(B_1,A_2)A_1 - Ric(A_1,A_2)B_1\Big)\\
&\quad - \frac{\tau}{m}\Big(\frac{a_1}{m-1}+a_2\Big)[g(B_1,A_2)A_1 - g(A_1,A_2)B_1]\\
&= a_1\Big[\Big(h_1^k(A_1,A_2) + A_1(k)A_2(k)\Big)B_1 + A_2(k)B_1(l)A_1 + g(A_1,A_2)\Big(H^l(B_1) + B_1(l)\nabla l\Big)\Big]\\
&\quad + a_2\Big[(m-2)A_2(k)B_1(l)A_1 - \Big({}^1Ric(A_1,A_2) - \frac{m_2}{f_1}h_1^{f_1}(A_1,A_2) - g(A_1,A_2)\Delta l\Big)B_1\Big]\\
&\quad - \frac{\tau}{m}\Big(\frac{a_1}{m-1}+a_2\Big)[g(B_1,A_2)A_1 - g(A_1,A_2)B_1],\\
P^*(A_1,B_1)B_2 &= a_1R(A_1,B_1)B_2 + a_2\Big(Ric(B_1,B_2)A_1 - Ric(A_1,B_2)B_1\Big)\\
&\quad - \frac{\tau}{m}\Big(\frac{a_1}{m-1}+a_2\Big)[g(B_1,B_2)A_1 - g(A_1,B_2)B_1]
\end{aligned}
$$

$$
\begin{aligned}
&= a_1\Big[\Big(h_2^l(B_1,B_2)+B_1(l)B_2(l)\Big)A_1+B_2(l)A_1(k)B_1+g(B_1,B_2)\Big(H^k(A_1)+A_1(k)\nabla k\Big)\Big]\\
&\quad+a_2\Big[\Big({}^2Ric(B_1,B_2)-\frac{m_1}{f_2}h_2^{f_2}(B_1,B_2)-g(B_1,B_2)\Delta k\Big)A_1-(m-2)A_1(k)B_2(l)B_1\Big]\\
&\quad-\frac{\tau}{m}\Big(\frac{a_1}{m-1}+a_2\Big)[g(B_1,B_2)A_1-g(A_1,B_2)B_1],\\
P^*(B_1,B_2)B_3 &= a_1R(B_1,B_2)B_3+a_2\Big(Ric(B_2,B_3)B_1-Ric(B_1,B_3)B_2\Big)\\
&\quad-\frac{\tau}{m}\Big(\frac{a_1}{m-1}+a_2\Big)[g(B_2,B_3)B_1-g(B_1,B_3)B_2]\\
&= a_1\Big[R(B_1,B_2)B_3+g(B_1,B_2)H^k(B_2)-g(B_2,B_3)H^k(B_1)\Big]\\
&\quad+a_2\Big[\Big({}^2Ric(B_2,B_3)-\frac{m_1}{f_2}h_2^{f_2}(B_2,B_3)-g(B_2,B_3)\Delta k\Big)B_1\\
&\quad-\Big({}^2Ric(B_1,B_3)-\frac{m_1}{f_2}h_2^{f_2}(B_1,B_3)-g(B_1,B_3)\Delta k\Big)\Big]\\
&\quad-\frac{\tau}{m}\Big(\frac{a_1}{m-1}+a_2\Big)[g(B_2,B_3)B_1-g(B_1,B_3)B_2]\\
&= a_1R(B_1,B_2)B_3+a_2\Big({}^2Ric(B_2,B_3)B_1-{}^2Ric(B_1,B_3)B_2\Big)\\
&\quad-\frac{\tau f_1^2}{m_2}\Big(\frac{a_1}{m_2-1}+a_2\Big)[g_2(B_2,B_3)B_1-g_2(B_1,B_3)B_2]\\
&\quad+a_1f_1^2\Big(g_2(B_1,B_2)H^k(B_2)-g_2((B_2,B_3)H^kB_1)\Big)\\
&\quad-\frac{a_2m_1}{f_2}\Big[h_2^{f_2}(B_2,B_3)B_1-h_2^{f_2}(B_1,B_3)B_2\Big]-a_2f_1^2\Big(g_2(B_2,B_3)B_1-g(B_1,B_3)B_2\Big)\Delta k\\
&\quad+\Big[\frac{\tau f_1^2}{m_2}\Big(\frac{a_1}{m_2-1}+a_2\Big)-\frac{\tau f_1^2}{m}\Big(\frac{a_1}{m-1}+a_2\Big)\Big][g_2(B_2,B_3)B_1-g_2(B_1,B_3)B_2],\\
P^*(B_1,B_2)B_3 &= P_2^*(B_1,B_2)B_3+a_1f_1^2\Big(g_2(B_1,B_2)H^k(B_2)-g_2(B_1,B_3)H^k(B_1)\Big)\\
&\quad-\frac{a_2m_1}{f_2}\Big(h_2^{f_2}(B_2,B_3)B_1-h_2^{f_2}(B_1,B_3)B_2\Big)\\
&\quad+\tau f_1^2\Big[\frac{m_1(m+m_2-1)}{mm_2(m_2-1)(m-1)}a_1+\Big(\frac{m_1}{mn_2}-\frac{\Delta k}{\tau}\Big)a_2\Big][g_2(B_2,B_3)B_1-g_2(B_1,B_3)B_2].
\end{aligned}
$$

This concludes the proof. □

Theorem 3.2. *If*

$$M, g = f_2^2 g_1 \oplus f_1^2 g_2$$

is a PPF DWPM, then pseudo-curvature tensor is given by

$$
\begin{aligned}
\bar{P}_1^*(A_1,A_2,A_3,\zeta) &= a_1f_2^4\Big(g_1(A_2,A_3)g_1(H^l(A_1),\zeta)-g_1(A_1,A_3)g_1(H^l(A_2),\zeta)\Big)\\
&\quad+\frac{a_2m_2f_2}{f_1}\Big(h_1^{f_1}(A_2,A_3)g_1(A_1,\zeta)-h_1^{f_1}(A_1,A_3)g_1(A_2,\zeta)\Big)\\
&\quad+f_4^2\tau\Big[\Big(\frac{m_2(m+m_1-1)}{mm_1(m_1-1)(m-1)}\Big)a_1+\Big(\frac{m_2}{mm_1}-\frac{\Delta l}{\tau}\Big)\Big]\\
&\quad[g_1(A_1,A_3)g_1(A_2,\zeta)-g_1(A_2,A_3)g_1(A_1,\zeta)].
\end{aligned}
$$

Here, $A_j, \zeta \in \mathcal{L}(M_1)$ for $j \in \{1,2,3\}$.

Proof. If M is a PPF DWPM. Consequently, according to Theorem 3.1, we obtain

$$\begin{aligned} P_1^*(A_1, A_2)A_3 &= a_1 f_2^2\Big(g_1(A_2, A_3)H^l(A_1) - g_1(A_1, A_3)H^l(A_2)\Big) \\ &- \frac{a_2 m_2}{f_1}\Big(h_1^{f_1}(A_1, A_3)A_2 - h_1^{f_1}(A_2, A_3)A_1\Big) + f_2^2\tau\Big[\Big(\frac{m_2(m + m_1 - 1)}{mm_1(m_1 - 1)(m - 1)}\Big)a_1 \\ &+ \Big(\frac{m_2}{mm_1} - \frac{\Delta l}{\tau}\Big)\Big][g_1(A_1, A_3)A_2 - g_1(A_2, A_3)A_1]. \end{aligned}$$

As a result, we derive

$$\begin{aligned} \bar{P}_1^*(A_1, A_2, A_3, \zeta) &= g_1(P_1^*(A_1, A_2)A_3, \zeta) \\ &= a_1 f_2^4\Big(g_1(A_2, A_3)g_1(H^l(A_1), \zeta) - g_1(A_1, A_3)g_1(H^l(A_2), \zeta)\Big) \\ &+ \frac{a_2 m_2 f_2}{f_1}\Big(h_1^{f_1}(A_2, A_3)g_1(A_1, \zeta) - h_1^{f_1}(A_1, A_3)g_1(A_2, \zeta)\Big) \\ &+ f_4^2\tau\Big[\Big(\frac{m_2(m + m_1 - 1)}{mm_1(m_1 - 1)(m - 1)}\Big)a_1 + \Big(\frac{m_2}{mm_1} - \frac{\Delta l}{\tau}\Big)\Big] \\ &[g_1(A_1, A_3)g_1(A_2, \zeta) - g_1(A_2, A_3)g_1(A_1, \zeta)]. \end{aligned}$$

This concludes the proof. □

Theorem 3.3. *If*

$$M, g = f_2^2 g_1 \oplus f_1^2 g_2$$

is a PPF DWPM with the metric, then the base manifold M_1 is PPF if and only if

$$\begin{aligned} &a_1 f_2^4\Big(g_1(A_2, A_3)g_1(H^l(A_1), \zeta) - g_1(A_1, A_3)g_1(H^l(A_2), \zeta)\Big) \\ &+ \frac{a_2 m_2 f_2}{f_1}\Big(h_1^{f_1}(A_2, A_3)g_1(A_1, \zeta) - h_1^{f_1}(A_1, A_3)g_1(A_2, \zeta)\Big) \\ &+ f_4^2\tau\Big[\Big(\frac{m_2(m + m_1 - 1)}{mm_1(m_1 - 1)(m - 1)}\Big)a_1 + \Big(\frac{m_2}{mm_1} - \frac{\Delta l}{\tau}\Big)\Big] \\ &[g_1(A_1, A_3)g_1(A_2, \zeta) - g_1(A_2, A_3)g_1(A_1, \zeta)] = 0. \end{aligned}$$

Here $A_j, \zeta \in \mathcal{L}(M_1)$ for $j \in \{1, 2, 3\}$.

Proof. Let the base manifold M_1 be PPF. Then

$$\bar{P}_1^*(A_1, A_2, A_3, \zeta) = 0.$$

It is clear that the proof can be derived from Theorem 3.2. □

Theorem 3.4. *If*

$$Mg = f_2^2 g_1 \oplus f_1^2 g_2$$

is a PPF DWPM, then the PPC of M_2 is expressed as

$$\begin{aligned}\bar{P}_2^*(B_1,B_2,B_3,\eta) = {} & a_1 f_1^4\Big(g_2(B_1,B_3)g_2(H^k(B_1),\eta) - g_2(B_1,B_2)g_2(H^k(B_2),\eta)\Big)\\ & - \frac{a_2 m_1 f_1^2}{f_2}\Big(h_2^{f_2}(B_1,B_3)g_2(B_2,\eta) - h_2^{f_2}(B_2,B_3)g_2(B_1,\eta)\Big)\\ & + \tau f_1^4\Big[\frac{m_1(m+m_2-1)}{mm_2(m_2-1)(m-1)}a_1 + \Big(\frac{m_1}{mm_2} - \frac{\Delta k}{\tau}\Big)a_2\Big]\\ & [g_2(B_1,B_3)g_2(B_2,\eta) - g_2(B_2,B_3)g_2(B_1,\eta)],\end{aligned}$$

where $B_j, \eta \in \mathcal{L}(M_2)$ for $j \in \{1,2,3\}$.

Proof. If M is a PPF DWPM. Thus, according to Theorem 3.1, we obtain

$$\begin{aligned}0 = {} & P_2^*(B_1,B_2)B_3 + a_1 f_1^2\Big(g_2(B_1,B_2)H^k(B_2) - g_2(B_1,B_3)H^k(B_1)\Big) - \frac{a_2 m_1}{f_2}\Big(h_2^{f_2}(B_2,B_3)B_1 - h_2^{f_2}(B_1,B_3)B_2\Big)\\ & + \tau f_1^2\Big[\frac{m_1(m+m_2-1)}{mm_2(m_2-1)(m-1)}a_1 + \Big(\frac{m_1}{mm_2} - \frac{\Delta k}{\tau}\Big)a_2\Big][g_2(B_2,B_3)B_1 - g_2(B_1,B_3)B_2].\end{aligned}$$

Hence, we derive

$$\begin{aligned}\bar{P}_2^*(B_1,B_2,B_3,\eta) = {} & g_2(P_2^*(B_1,B_2)B_3,\eta)\\ = {} & a_1 f_1^4\Big(g_2(B_1,B_3)g_2(H^k(B_1),\eta) - g_2(B_1,B_2)g_2(H^k(B_2),\eta)\Big)\\ & - \frac{a_2 m_1 f_1^2}{f_2}\Big(h_2^{f_2}(B_1,B_3)g_2(B_2,\eta) - h_2^{f_2}(B_2,B_3)g_2(B_1,\eta)\Big)\\ & + \tau f_1^4\Big[\frac{m_1(m+m_2-1)}{mm_2(m_2-1)(m-1)}a_1 + \Big(\frac{m_1}{mm_2} - \frac{\Delta k}{\tau}\Big)a_2\Big]\\ & [g_2(B_1,B_3)g_2(B_2,\eta) - g_2(B_2,B_3)g_2(B_1,\eta)],\end{aligned}$$

and this concludes the proof. □

Theorem 3.5. *If*

$$M, g = f_2^2 g_1 \oplus f_1^2 g_2$$

is a PPF DWPM, then the fiber manifold M_2 is considered PPF if and only if

$$\begin{aligned}& a_1 f_1^4\Big(g_2(B_1,B_3)g_2(H^k(B_1),\eta) - g_2(B_1,B_2)g_2(H^k(B_2),\eta)\Big)\\ & - \frac{a_2 m_1 f_1^2}{f_2}\Big(h_2^{f_2}(B_1,B_3)g_2(B_2,\eta) - h_2^{f_2}(B_2,B_3)g_2(B_1,\eta)\Big)\\ & + \tau f_1^4\Big[\frac{m_1(m+m_2-1)}{mm_2(m_2-1)(m-1)}a_1 + \Big(\frac{m_1}{mm_2} - \frac{\Delta k}{\tau}\Big)a_2\Big]\\ & [g_2(B_1,B_3)g_2(B_2,\eta) - g_2(B_2,B_3)g_2(B_1,\eta)] = 0,\end{aligned}$$

where $B_j, \eta \in \mathcal{L}(M_2)$ for $j \in \{1,2,3\}$.

Proof. Assuming that the fiber manifold M_2 is PPF, it follows that

$$\bar{P}_1^*(B_1,B_2,B_3,\eta) = 0.$$

This result directly follows from Theorem 3.4. □

4. Properties of the PPC on the TWPM

In this section, we analyze the PPC for the TWPM

$$\tilde{M} = M_1 \times_f M_2.$$

We present the following theorems related to the PPC of TWPMs, which clarify the relationship between the warped geometry and its base and fiber manifolds. The PPC and the tensor P^* on $\tilde{M}$ and M_i are represented by $\tilde{P}^*$, P^*, and $\tilde{P}^*_i$, P^*_i, respectively.

Theorem 4.1. *If*

$$\tilde{M}, \tilde{g} = g_1 \oplus f^2 g_2$$

is a TWPM with $A_j \in \mathfrak{L}(M_1)$ and $B_j \in \mathfrak{L}(M_2)$ for $j \in \{1, 2, 3\}$, then the PPC is given by

$$\begin{aligned} P^*(A_1, A_2)A_3 = {} & P^*_1(A_1, A_2)A_3 - m_2 a_2\Big(h^k_1(A_2, A_3)A_1 - h^k_1(A_1, A_3)A_2\Big) \\ & - a_2\Big(A_2(k)A_3(k)A_1 + A_1(k)A_3(k)A_2\Big) \\ & + \tau\Big(\frac{m_2(m + m_1 - 1)}{mm_1(m-1)(m_1-1)}\Big)[g_1(A_2, A_3)A_1 - g_1(A_1, A_3)A_2], \end{aligned} \tag{4.1}$$

$$\begin{aligned} P^*(A_1, A_2)B_1 = {} & a_2(m_2 - 1)\Big(A_1 B_1(k)A_2 - A_2 B_1(k)A_1\Big) - \frac{\tau}{m}\Big(\frac{a_1}{m-1} + a_2\Big) \\ & [\tilde{g}(A_2, B_1)A_1 - \tilde{g}(A_1, B_1)A_2], \end{aligned} \tag{4.2}$$

$$\begin{aligned} P^*(B_1, B_2)A_1 = {} & a_1\Big(B_1 A_1(k)B_2 - B_2 A_1(k)B_1\Big) + a_2(m_2 - 1)\Big(A_1 B_1(k)B_2 - A_1 B_2(k)B_1\Big) \\ & - \frac{\tau}{m}\Big(\frac{a_1}{m-1} + a_2\Big)[\tilde{g}(B_2, A_1)B_1 - \tilde{g}(B_1, A_1)B_2], \end{aligned} \tag{4.3}$$

$$\begin{aligned} P^*(A_1, B_1)A_2 = {} & (a_1 + a_2 m_2)\Big(h^k_1(A_1, A_2) + A_1(k)A_2(k)\Big)B_1 - (m_2 - 1)a_2 A_2 B_1(k)A_1 \\ & - {}^1Ric(A_1, A_2)B_1 - \frac{\tau}{m}\Big(\frac{a_1}{m-1} + a_2\Big)[\tilde{g}(B_1, A_2)A_1 - g_1(A_1, A_2)B_1], \end{aligned} \tag{4.4}$$

$$\begin{aligned} P^*(A_1, B_1)B_2 = {} & (m_2 - 1 - a_1)A_1 B_2(k)B_1 + f^2\Big[a_1\Big(A_1(k)\nabla k + H^k(A_1)\Big) - a_2 A_1 \Delta k\Big]g_2(B_1, B_2) \\ & + a_2\Big[{}^2Ric(B_1, B_2)A_1 - (m_2 - 2)\Big(h^k_2(B_1, B_2) - B_1(k)B_2(k)\Big)A_1\Big] \\ & - \frac{\tau}{m}\Big(\frac{a_1}{m-1} + a_2\Big)[f^2 g_2(B_1, B_2)A_1 - \tilde{g}(A_1, B_2)B_1], \end{aligned} \tag{4.5}$$

$$\begin{aligned} P^*(B_1, B_2)B_3 = {} & P^*_2(B_1, B_2)B_3 + \Big(a_1 + a_2(m_2 - 2)\Big)\Big[\Big(h^k_2(B_1, B_3) - B_1(k)B_3(k)\Big)B_2 \\ & - \Big(h^k_2(B_2, B_3) - B_2(k)B_3(k)\Big)B_1\Big] + f^2\Big(H^k(B_2) + B_2(k)\nabla k\Big)g_2(B_1, B_3) \\ & + f^2\tau\Big(\frac{m_1(m + m_2 - 1)}{mm_2(m-1)(m_2-1)}a_1 + \Big(\frac{m_1}{mm_2} - \frac{\Delta k}{\tau}\Big)a_2\Big)[g_2(B_2, B_3)B_1 - g_2(B_1, B_3)B_2]. \end{aligned} \tag{4.6}$$

Theorem 4.2. *If*

$$\tilde{M}, \tilde{g} = g_1 \oplus f^2 g_2$$

is a PPF TWPM, then the PPC is given by

$$P_1^*(A_1,A_2)A_3 = m_2a_2\Big(h_1^k(A_2,A_3)A_1 - h_1^k(A_1,A_3)A_2\Big) + a_2\Big(A_2(k)A_3(k)A_1 + A_1(k)A_3(k)A_2\Big)$$
$$+ \tau\Big(\frac{m_2(m+m_1-1)}{mm_1(m-1)(m_1-1)}\Big)[g_1(A_1,A_3)A_2 - g_1(A_2,A_3)A_1],$$

where $A_j, \zeta \in \mathcal{L}(M_1)$ *for* $j \in \{1,2,3\}$.

Theorem 4.3. *If*

$$\tilde{M}, \tilde{g} = g_1 \oplus f^2 g_2$$

is a PPF TWPM, then the PPC of M_2 *is given by*

$$m_2a_2\Big(h_1^k(A_2,A_3)A_1 - h_1^k(A_1,A_3)A_2\Big) + a_2\Big(A_2(k)A_3(k)A_1 + A_1(k)A_3(k)A_2\Big)$$
$$+ \tau\Big(\frac{m_2(m+m_1-1)}{mm_1(m-1)(m_1-1)}\Big)[g_1(A_1,A_3)A_2 - g_1(A_2,A_3)A_1], = 0,$$

where $A_j, \zeta \in \mathcal{L}(M_1)$ *for* $j \in \{1,2,3\}$.

Theorem 4.4. *If*

$$\tilde{M}, \tilde{g} = g_1 \oplus f^2 g_2$$

is a PPF TWPM, then the fiber manifold M_2 *is PPF if and only if*

$$\bar{P}_2^*(B_1,B_2,B_3,\eta) = f^2\Big(a_1 + a_2(m_2-2)\Big)\Big[\Big(h_2^k(B_2,B_3) - B_2(k)B_3(k)\Big)g_2(B_1,\eta)$$
$$- \Big(h_2^k(B_1,B_3) - B_1(k)B_3(k)\Big)g_2(B_2,\eta)\Big] - f^4\Big(H^k(B_2) + B_2(k)\nabla k\Big)g_2(g_2(B_1,B_3),\eta)$$
$$+ f^4\tau\Big(\frac{m_1(m+m_2-1)}{mm_2(m-1)(m_2-1)}a_1 + \Big(\frac{m_1}{mm_2} - \frac{\Delta k}{\tau}\Big)a_2\Big)$$
$$[g_2(B_1,B_3)g_2(B_2,\eta) - g_2(B_2,B_3)g_2(B_1,\eta)],$$

where $B_j, \eta \in \mathcal{L}(M_2)$ *for* $j \in \{1,2,3\}$.

Theorem 4.5. *If*

$$\tilde{M}, \tilde{g} = g_1 \oplus f^2 g_2$$

is a PPF DWPM, then the fiber manifold M_2 *is PPF if and only if*

$$f^2\Big(a_1 + a_2(m_2-2)\Big)\Big[\Big(h_2^k(B_2,B_3) - B_2(k)B_3(k)\Big)g_2(B_1,\eta) - \Big(h_2^k(B_1,B_3) - B_1(k)B_3(k)\Big)g_2(B_2,\eta)\Big]$$
$$- f^4\Big(H^k(B_2) + B_2(k)\nabla k\Big)g_2(g_2(B_1,B_3),\eta) + f^4\tau\Big(\frac{m_1(m+m_2-1)}{mm_2(m-1)(m_2-1)}a_1 + \Big(\frac{m_1}{mm_2} - \frac{\Delta k}{\tau}\Big)a_2\Big)$$
$$[g_2(B_1,B_3)g_2(B_2,\eta) - g_2(B_2,B_3)g_2(B_1,\eta)] = 0,$$

where $B_j, \eta \in \mathcal{L}(M_2)$ *for* $j \in \{1,2,3\}$.

5. Applications

In this section, we utilize the findings presented in this paper to compute the PPCs for both doubly and twisted generalized Robertson-Walker space-times.

5.1. Doubly generalized Robertson-Walker space-times

Let (M, g) be an n-dimensional Riemannian manifold, and let f_1 and f_2 be smooth functions defined on I and M, respectively, where $I \subset \mathbb{R}$. The DWPM

$$\bar{M} = {}_{f_2}I \times_{f_1} M,$$

which has dimension $(m + 1)$ with metric

$$\bar{g} = -f_2^2 dt^2 \oplus f_1^2 g$$

is called a doubly generalized Robertson-Walker space-times. In this context, the term dt^2 represents the standard Euclidean metric defined on the interval I. This model generalizes the concept of doubly generalized Robertson-Walker space-times. For simplicity, we denote

$$\frac{\partial}{\partial t} \in \mathcal{L}(I)$$

by ∂_t in the following results.

Employing Lemmas 2.1 and 2.2, Theorem 3.1, we derive the following theorem:

Theorem 5.1. *If*

$$\bar{M}, \bar{g} = -f_2^2 dt^2 \oplus f_1^2 g$$

is a doubly generalized Robertson-Walker space-times, then the PPC $\bar{P}^*$ on $\bar{M}$ is expressed as

$$\begin{aligned}
\bar{P}^*(\partial_t, \partial_t)\partial_t &= \bar{P}^*(\partial_t, \partial_t)B_1 = 0,\\
\bar{P}^*(B_1, B_2)\partial_t &= \frac{\dot{f}_1}{f_1}\Big(a_1 + a_2(m-2)\Big)\Big(B_2(l)B_1 - B_1(l)B_2\Big)\\
&\quad - \frac{\tau}{m}\Big(\frac{a_1}{m-1} + a_2\Big)[\bar{g}(B_2, \partial_t)B_1 - \bar{g}(B_1, \partial_t)B_2],\\
\bar{P}^*(\partial_t, B_1)\partial_t &= a_1\Big[\Big(h_1^k(\partial_t, \partial_t) + \Big(\frac{\dot{f}_1}{f_1}\Big)^2\Big)B_1 + \frac{\dot{f}_1}{f_1}B_1(l)X - f_2^2(H^l(B_1) + B_1(l)\nabla l)dt^2\Big]\\
&\quad + a_2\Big[(m-2)\frac{\dot{f}_1}{f_1}B_1(l)\partial_t - \Big({}^1Ric(\partial_t, \partial_t) - \frac{m_2}{f_1}h_1^{f_1}(\partial_t, \partial_t) + f_2^2\Delta l dt^2\Big)B_1\Big]\\
&\quad - \frac{\tau}{m}\Big(\frac{a_1}{m-1} + a_2\Big)[\bar{g}(B_1, \partial_t)\partial_t + f_2^2 B_1 dt^2],\\
\bar{P}^*(\partial_t, B_1)B_2 &= a_1\Big[\Big(h_2^l(B_1, B_2) + B_1(l)B_2(l)\Big)\partial_t + \frac{\dot{f}_1}{f_1}B_2(l)B_1 + f_1^2 g_2(B_1, B_2)\Big(H^k(\partial_t) + \Big(\frac{\dot{f}_1}{f_1}\Big)^2\Big)\Big]\\
&\quad + a_2\Big[\Big({}^2Ric(B_1, B_2) - \frac{m_1}{f_2}h_2^{f_2}(B_1, B_2) - f_1\dot{f}_1 g_2(B_1, B_2)\Big)\partial_t - (m-2)\frac{\dot{f}_1}{f_1}B_2(l)B_1\Big]\\
&\quad - \frac{\tau}{m}\Big(\frac{a_1}{m-1} + a_2\Big)[f_1^2\bar{g}(B_1, B_2)\partial_t - \bar{g}(\partial_t, B_2)B_1],\\
P^*(B_1, B_2)B_3 &= P_2^*(B_1, B_2)B_3 + a_1 f_1^2\Big(g_2(B_1, B_2)H^k(V) - g_2(B_1, B_3)H^k(B_1)\Big)\\
&\quad - \frac{a_2 m_1}{f_2}\Big(h_2^{f_2}(V, B_3)B_1 - h_2^{f_2}(B_1, B_3)V\Big) + \tau f_1^2\Big[\frac{m_1(m + m_2 - 1)}{mm_2(m_2 - 1)(m-1)}a_1\\
&\quad + \Big(\frac{m_1}{mm_2} - \frac{\Delta k}{\tau}\Big)a_2\Big][g_2(B_2, B_3)B_1 - g_2(B_1, B_3)B_2],
\end{aligned}$$

for $B_j \in \mathcal{L}(M)$ for $j \in \{1, 2, 3\}$ and $\partial_t \in \mathcal{L}(I)$.

5.2. Twisted generalized Robertson-Walker space-times

Let (M, g) be an n-dimensional Riemannian manifold, and let

$$f : I \times M \to (0, 1)$$

be a smooth function, where $I \subset \mathbb{R}$. The manifold

$$\bar{M} = I \times_f M, \bar{g} = -dt^2 \oplus f^2 g$$

with dimension $(m+1)$ is called a twisted generalized Robertson-Walker space-times. In this context, the term dt^2 represents the standard Euclidean metric defined on the interval I. This construction generalizes the notion of twisted generalized Robertson-Walker space-times. For simplicity, in the following results, we will use ∂_t to represent

$$\frac{\partial}{\partial t} \in \mathcal{L}(I).$$

Employing Lemmas 2.1 and 2.2, Theorem 3.1, we derive the following theorem.

Theorem 5.2. *If*

$$\breve{M}, \breve{g} = -f_2^2 dt^2 \oplus f^2 g$$

is a twisted generalized generalized Robertson-Walker space-times, then, the PPC $\breve{P}^*$ on $\breve{M}$ is expressed as

$$\begin{aligned} P^*(\partial_t, \partial_t)\partial_t &= P^*(\partial_t, \partial_t)B_1 = 0, \\ P^*(B_1, B_2)\partial_t &= a_1 \frac{\dot{f}_1}{f_1}\Big(B_1 B_2 - B_2 B_1\Big) + a_2(m_2 - 1)\partial_t\Big(B_1(k)B_2 - B_2(k)B_1\Big) \\ &\quad - \frac{\tau}{m}\Big(\frac{a_1}{m-1} + a_2\Big)[\breve{g}(B_2, \partial_t)B_1 - \breve{g}(B_1, \partial_t)B_2], \end{aligned}$$

$$\begin{aligned} P^*(\partial_t, B_1)Y &= (a_1 + a_2 m_2)\Big(h_1^k(X, Y) + \Big(\frac{\dot{f}_1}{f_1}\Big)^2\Big)B_1 - (m_2 - 1)a_2 \partial_t^2 B_1(k) - {}^1Ric(\partial_t, \partial_t)B_1 \\ &\quad - \frac{\tau}{m}\Big(\frac{a_1}{m-1} + a_2\Big)[f^2 \breve{g}(B_1, Y)\partial_t + B_1 dt^2], \\ P^*(\partial_t, B_1)B_2 &= (m_2 - 1 - a_1)\partial_t(B_2(k)B_1) + f^2\Big[a_1\Big(\frac{\dot{f}_1}{f_1}\nabla k + H^k(\partial_t)\Big) - a_2 \partial_t \Delta k\Big] f^2 \breve{g}(B_1, B_2) \\ &\quad + a_2\Big[{}^2Ric(B_1, B_2)\partial_t - (m_2 - 2)\Big(h_2^k(B_1, B_2) - B_1(k)B_2(k)\Big)\partial_t\Big] \\ &\quad - \frac{\tau}{m}\Big(\frac{a_1}{m-1} + a_2\Big)[f^2 \breve{g}(B_1, B_2)\partial_t - \breve{g}(X, B_2)B_1], \\ P^*(B_1, B_2)B_3 &= P_2^*(B_1, B_2)B_3 + \Big(a_1 + a_2(m_2 - 2)\Big)\Big[\Big(h_2^k(B_1, B_3) - B_1(k)B_3(k)\Big)B_2 \\ &\quad - \Big(h_2^k(B_2, B_3) - B_2(k)B_3(k)\Big)B_1\Big] + f^2\Big(H^k(B_2) + B_2(k)\nabla k\Big)g_2(B_1, B_3) \\ &\quad + f^2\tau\Big(\frac{m_1(m + m_2 - 1)}{m m_2(m-1)(m_2 - 1)}a_1 + \Big(\frac{m_1}{m m_2} - \frac{\Delta k}{\tau}\Big)a_2\Big)[g(B_2, B_3)B_1 - g(B_1, B_3)B_2], \end{aligned}$$

for $B_j \in \mathcal{L}(M)$ for $j \in \{1, 2, 3\}$ and $\partial_t \in \mathcal{L}(I)$.

6. Conclusions

This study focused on the pseudo-projective curvature tensor in relation to doubly and twisted warped product manifolds. Key findings demonstrated how the pseudo-projective curvature tensor interacted with both the base and fiber manifolds. Additionally, the study emphasized the geometric characteristics of the base and fiber manifolds as shaped by the pseudo-projective tensor. The investigation was also expanded to include an analysis of the pseudo-projective curvature tensor in generalized doubly and twisted generalized Robertson-Walker space-times.

An important avenue for future research would be to explore further properties, including a detailed analysis of the relationship between the pseudo-projective curvature tensor and Killing vector fields, which encapsulate the manifold's symmetries, and to examine how these symmetries impact the curvature.

Author contributions

Ayman Elsharkawy: conceptualizations, supervision of the research, review, and editing, guided the theoretical framework; Hoda Elsayied: data collection, supervision the study, provided critical insights, and contributed to refining the manuscript; Abdelrhman Tawfiq: methodology, conducted the theoretical analysis, developed the main results, and prepared the manuscript draft; Fatimah Alghamdi: reviewing and editing the manuscript, provided critical insights to refine interpretations, ensured adherence to publication standards, and contributed to improving the overall clarity and coherence of the work. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no known financial conflicts of interest or personal relationships that could have influenced the work presented in this paper.

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