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Research article

Monotone iterative and quasilinearization method for a nonlinear integral impulsive differential equation

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Abstract: In this paper, we discuss the existence and approximation of solution sequences for a class of nonlinear ordinary differential equations with an impulsive integral condition. Our major methods were the monotone iterative and quasilinearization techniques. Interestingly, many new results could be obtained, which were different from the functional differential equation.

Keywords: impulsive integral condition; monotone iterative technique; quasilinearization method; quadratic convergence
Mathematics Subject Classification: 34B05, 34B37

1. Introduction

Impulsive differential equations arise in the real world, such as in biology, physics, population dynamics and economics [1–3]. It is a basic mathematical tool for studying evolution processes that suddenly change their states at certain moments. For the theory and application of the impulsive differential equation, readers can refer to references [4–8].

To the best of our knowledge, to obtain solution sequences that uniformly converge to the minimal and maximal solutions of the impulsive differential equation, the upper and lower solutions coupled with the monotone iterative technique are common methods [9–11]. The basic idea is that by using the upper and lower solutions as an initial iteration, one can construct monotone sequences from a corresponding linear system, and these monotone sequences can monotonically converge to the minimal and maximal solutions of the nonlinear system [9–11]. Moreover, in order to obtain a faster convergent solution sequence, such as quadratic convergence, many scholars use the quasilinearization (QLM) method [12]. The quasilinearization method is a very powerful approximation technique, whose iterations are constructed to yield monotonically and rapidly convergent solution sequences, which has given many excellent results [13–16]. For the application of the QLM method in ordinary differential equations, one can see [17]. For the application of the QLM method in functional differential equations, readers can see reference [18].

Note that, traditionally, the form of the impulsive condition is usually supposed as $\triangle y(t_k) = I_k(y(t_k))$, i.e., the state at impulse point t_k depends only on the left side of limit of $y(t_k)$ [1–3]. Many non-instantaneous impulse conditions are proposed [19–21]. For example, Tariboon develop an impulsive integral condition in the form of $\triangle y(t_k) = I_k\left(\int_{t_k-\tau_k}^{t_k} y(s)ds - \int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}} y(s)ds\right)$ and use the monotonic iteration technique to discuss the solutions of a class of delay differential equations under this condition [22]. In fact, the non-instantaneous integral impulse condition is not abrupt and is dependent on past states and evolution processes; the non-instantaneous integral impulse condition takes into account that the time of pulse action cannot be ignored relative to the development process, which is closer to the physical process and improves the accuracy and applicability of the model [19–22]. Therefore, non-instantaneous pulse differential equations are an extension of classical pulse differential equations, which can handle more complex pulse phenomena and has more practical applications in many fields [19–22]. However, we note that there is limited study on using the QLM method to consider the high-order convergence of solutions for differential equations under non-instantaneous integral impulse conditions.

Based on the above background, we propose an impulsive integral condition in the form of $\triangle y(t_k) = I_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} y(s)ds\right)$. Under this condition, we discuss the existence, uniformly convergence and quadratic convergence of solution sequences for a class of nonlinear first order ordinary differential equations with anti-periodic boundary values. Interestingly, we can effectively obtain a quadratic convergence solution sequence in this impulsive ordinary differential equation. More importantly, the new results are more abundant than those in other studies, which cannot be obtained in functional differential equations. The specific differential equations are described as follows:

$$\begin{cases} y'(t) = f(t, y(t)), & t \neq t_k, t \in J = [0, T], \\ \triangle y(t_k) = I_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} y(s)ds\right), & k = 1, 2, \cdots, m, \\ y(0) = -y(T), \end{cases} \tag{1}$$

where $f \in C(J \times R, R)$, $0 < t_1 < t_2 < \cdots < t_m < T, I_k \in C(R, R)$, $0 < q_{k-1} \leq (t_k - t_{k-1})/2$, $\quad 0 \leq p_k \leq (t_k - t_{k-1})/2$, $\triangle y(t_k) = y(t_k^+) - y(t_k^-), k = 1, 2, \cdots, m$. We denote $a = \max_{k=1,2,\cdots,m}\{t_k - t_{k-1}\}$.

In order to define the solution for (1), we introduce the following spaces [23–26]: Let $J^- = J^+ \backslash \{t_1, t_2, \cdots, t_m\}$; $PC(J^+, R) = \{y : J^+ \to R; y(t)$ is continuous everywhere except for some t_k, at which $y(t_k^+)$ and $y(t_k^-)$ exist and $y(t_k^-) = y(t_k), k = 1, \cdots, m\big\}$; $PC'(J^+, R) = \{y \in PC(J^+, R); y'$ is continuous on J^-, where $y'(0^+), y'(T^-), y'(t_k^+)$ and $y'(t_k^-)$ exist, $k = 1, 2, \cdots, m\}$; $E_0 = \{y \in PC(J^+, R) : y(t) = y(0), t \in [-r, 0]\}$, then E_0 is a Banach space with the norm $\|y\|_{E_0} = \sup_{t \in J^+} |y(t)|$; $E = PC(J^+, R) \bigcap PC'(J^+, R)$. Then a function $y \in E$ is called a solution of boundary value problem (BVP) (1) if it satisfies (1).

In Section 2, we discuss the existence and uniqueness of the solution for a linear problem, and a key comparison principle is established. In Section 3, we obtain the monotonic convergence and quadratic convergence of solution sequences for BVP(1) by using the QLM method and the monotone iterative technique. In Section 4, we give a series of corollaries for the major results.

2. Some key lemmas for a linear system

Now, we consider the following linear system:

$$\begin{cases} y'(t) + My(t) = \sigma(t), \qquad t \neq t_k, t \in J = [0, T], \\ \triangle y(t_k) = -L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} y(s)ds\right) + I_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \eta(s)ds\right) \\ +L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \eta(s)ds\right), \qquad k = 1, 2, \cdots, m, \\ y(0) = -y(T), \end{cases} \tag{2}$$

where $M > 0, L_k \leq 0$, and $\sigma(t) \in E_0, \eta(t) \in E$.

Below, we provide the expression and uniqueness proofs for the solution of system (2) in Lemmas 2.1 and 2.2, respectively. An important comparative principle about system (2) is given in Lemma 2.3. These three lemmas are the key conclusions that prove the main results in Section 3.

Lemma 2.1. $y \in E$ is a solution of (2) if and only if $y \in E_0$ satisfying:

$$\begin{aligned} y(t) &= \int_0^T G(t,s)\sigma(s)ds + \sum_{k=1}^{m} G(t,t_k)\left[-L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} y(s)ds\right)\right. \\ &+ \left. I_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \eta(s)ds\right) + L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \eta(s)ds\right)\right], \end{aligned} \tag{3}$$

where

$$G(t,s) = \frac{1}{e^{MT}+1}\begin{cases} e^{M(T-t+s)}, & 0 \leq s \leq t \leq T, \\ -e^{M(s-t)}, & 0 \leq t < s \leq T. \end{cases}$$

Proof. Setting $y(t)$ is a solution of (2). Letting $u(t) = e^{Mt}y(t)$, then

$$\Delta u(t_k) = e^{Mt_k}\Delta y(t_k),$$

and

$$u'(t) = e^{Mt}\sigma(t). \tag{4}$$

Integrating (4) from 0 to t_1, we obtain:

$$u(t_1) - u(0) = \int_0^{t_1} e^{Ms}\sigma(s)ds.$$

Once more integrating (4) from t_1 to t, where $t \in (t_1, t_2]$, then

$$\begin{aligned} u(t) &= u(t_1^+) + \int_{t_1}^t e^{Ms}\sigma(s)ds = u(0) + \int_0^t e^{Ms}\sigma(s)ds \\ &+ e^{Mt_1}\left[-L_1\left(\int_{t_0+q_0}^{t_1-p_1} y(s)ds\right) + I_1\left(\int_{t_0+q_0}^{t_1-p_1} \eta(s)ds\right) + L_1\left(\int_{t_0+q_0}^{t_1-p_1} \eta(s)ds\right)\right]. \end{aligned}$$

Repeating the above process, then for all $t \in J$, we have

$$u(t) = u(0) + \int_0^t e^{Ms}\sigma(s)ds + \sum_{0<t_k<t} e^{Mt_k}\left[-L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} y(s)ds\right)\right.$$

$$+ \; I_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}\eta(s)ds\right) + L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}\eta(s)ds\right)\Bigg].$$

Since $u(0) = y(0)$, so

$$\begin{aligned} e^{Mt}y(t) &= y(0) + \int_0^t e^{Ms}\sigma(s)ds + \sum_{0<t_k<t} e^{Mt_k}\left[-L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}y(s)ds\right)\right. \\ &+ \left. I_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}\eta(s)ds\right) + L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}\eta(s)ds\right)\right]. \end{aligned}$$

Therefore, by the boundary value condition $y(0) = -y(T)$, we have

$$\begin{aligned} y(0) &= -(e^{MT}+1)^{-1}\left\{\int_0^T e^{Ms}\sigma(s)ds + \sum_{0<t_k<T} e^{Mt_k}\left[-L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}y(s)ds\right)\right.\right. \\ &+ \left.\left. I_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}\eta(s)ds\right) + L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}\eta(s)ds\right)\right]\right\}. \end{aligned}$$

Then,

$$\begin{aligned} y(t) &= -(e^{MT}+1)^{-1}\left\{\int_0^T e^{M(s-t)}\sigma(s)ds + \sum_{0\le t_k<T} e^{M(t_k-t)}\left[-L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}y(s)ds\right)\right.\right. \\ &+ \left. I_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}\eta(s)ds\right) + L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}\eta(s)ds\right)\right] \\ &- (e^{MT}+1)\left[\int_0^t e^{M(s-t)}\sigma(s)ds - \sum_{0<t_k<t} e^{M(t_k-t)}\left[-L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}y(s)ds\right)\right.\right. \\ &+ \left.\left.\left. I_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}\eta(s)ds\right) + L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}\eta(s)ds\right)\right]\right]\right\} \\ &= -(e^{MT}+1)^{-1}\left\{\int_t^T e^{M(s-t)}\sigma(s)ds + \sum_{t\le t_k<T} e^{M(t_k-t)}\left[-L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}y(s)ds\right)\right.\right. \\ &+ \left. I_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}\eta(s)ds\right) + L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}\eta(s)ds\right)\right] \\ &- \int_0^t e^{M(T+s-t)}\sigma(s)ds - \sum_{0<t_k<t} e^{M(T+t_k-t)}\left[-L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}y(s)ds\right)\right. \\ &+ \left.\left. I_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}\eta(s)ds\right) + L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}\eta(s)ds\right)\right]\right\}. \end{aligned}$$

Since,

$$\begin{aligned} &\sum_{k=1}^{m} G(t,t_k)\left[-L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}y(s)ds\right) + I_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}\eta(s)ds\right)\right. \\ &+ \left. L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}\eta(s)ds\right)\right] = \sum_{0<t_k<T} G(t,t_k)\left[-L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}y(s)ds\right)\right. \end{aligned}$$

$$
\begin{aligned}
&+\ I_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}\eta(s)ds\right)+L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}\eta(s)ds\right)\Bigg]\\
&=\ \sum_{t\le t_k<T}G(t,t_k)\left[-L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}y(s)ds\right)+I_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}\eta(s)ds\right)\right.\\
&+\ L_k\left.\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}\eta(s)ds\right)\right]\\
&+\ \sum_{0<t_k<t}G(t,t_k)\left[-L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}y(s)ds\right)+I_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}\eta(s)ds\right)\right.\\
&+\ L_k\left.\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}\eta(s)ds\right)\right].
\end{aligned}
$$

Finally, we obtain that

$$
\begin{aligned}
y(t)\ =\ &\int_0^T G(t,s)\sigma(s)ds+\sum_{k=1}^m G(t,t_k)\left[-L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}y(s)ds\right)\right.\\
&+\ I_k\left.\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}\eta(s)ds\right)+L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}\eta(s)ds\right)\right],\qquad t\in J.
\end{aligned}
$$

I.e., $y(t)$ is also a solution of (3).

On the other hand, we suppose that $y(t)$ is a solution of (3), then obviously $y(t)\in E$. By direct computation, we have

$$
\begin{cases}
y'(t)+My(t)=\sigma(t),\qquad t\neq t_k,\quad t\in J,\\
\triangle y(t_k)=-L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}y(s)ds\right)+I_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}\eta(s)ds\right)\\
+L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}\eta(s)ds\right),\qquad k=1,2,\cdots,m.
\end{cases}
$$

Because of $G(0,s)=-G(T,s)$, $s\in J$, thus $y(0)=-y(T)$. Therefore, $y(t)$ is also a solution of (2). The proof is complete. □

Lemma 2.2. Suppose that there exist constants $M>0, L_k\le 0, 0<q_{k-1}\le(t_k-t_{k-1})/2,\quad 0\le p_k\le(t_k-t_{k-1})/2$, $k=1,2,\cdots,m$, such that:

$$
\frac{e^{MT}}{e^{MT}+1}\sum_{k=1}^m|L_k|(a-(p_k+q_{k-1}))<1. \tag{5}
$$

Then (2) has a unique solution.

Proof. For any $y\in E_0$, we define an operator F:

$$
\begin{aligned}
(Fy)(t)\ =\ &\int_0^T G(t,s)\sigma(s)ds+\sum_{k=1}^m G(t,t_k)\left[-L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}y(s)ds\right)\right.\\
&+\ I_k\left.\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}\eta(s)ds\right)+L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}\eta(s)ds\right)\right].
\end{aligned}
$$

Clearly, $(Fy)\in E_0$. Since,

$$
\max_{t\in[0,T],s\in[0,T]}|G(t,s)|=\frac{e^{MT}}{(e^{MT}+1)},
$$

then for any $x, y \in E_0$, we have

$$\begin{aligned}
\|Fx - Fy\|_{E_0} &= \sup_{t\in J}\left|\sum_{k=1}^{m} G(t,t_k)\left[-L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} x(s)ds\right)\right]\right.\\
&\left. - \sum_{k=1}^{m} G(t,t_k)\left[-L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} y(s)ds\right)\right]\right|\\
&\le \left(\sup_{t\in J}|G(t,t_k)|\sum_{k=1}^{m}|L_k|(a-(p_k+q_{k-1})\right)\|x-y\|_{E_0}\\
&= \left(\frac{e^{MT}}{e^{MT}+1}\sum_{k=1}^{m}|L_k|(a-(p_k+q_{k-1}))\right)\|x-y\|_{E_0}.
\end{aligned}$$

So by the condition (5) and Banach fixed point theorem, we know that F has an unique fixed point $y^* \in E_0$. Then, by Lemma 2.1, y^* is also a unique solution of (2). The proof is complete. □

Lemma 2.3. (Comparison principle) Suppose that there exist constants $M > 0, L_k \le 0, 0 < q_{k-1} \le (t_k - t_{k-1})/2, \quad 0 \le p_k \le (t_k - t_{k-1})/2, k = 1, 2, \cdots, m$, such that $y \in E$ satisfying:

$$\begin{cases}
y'(t) + My(t) \le 0, & t \ne t_k, t \in J = [0,T],\\
\triangle y(t_k) \le -L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} y(s)ds\right), & k = 1,2,\cdots,m,\\
y(0) \le 0.
\end{cases}$$

Then $y(t) \le 0$ for all $t \in J$.

Proof. Set $u(t) = e^{Mt}y(t)$, then

$$\begin{cases}
u'(t) \le 0, & t \ne t_k, t \in J = [0,T],\\
\triangle u(t_k) \le -L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} e^{-M(s-t_k)}u(s)ds\right), & k = 1,2,\cdots,m,\\
u(0) \le 0.
\end{cases}$$

Clearly, u(t) is a non-increasing function, so $u(t) \le 0$. Since u(t) and y(t) have the same sign, thus $y(t) \le 0$. The proof is complete. □

3. Major results

First, we give the definition of upper and lower solutions. Then, we prove our major results by using the upper and lower solutions coupled with the monotone iterative technique and the method of quasilinearization.

Definition 3.1. A function $\alpha_0 \in E \bigcap E_0$ is called a lower solution of BVP(1) if

$$\begin{cases}
\alpha_0{}'(t) \le f(t,\alpha_0(t)), & t \ne t_k, t \in J = [0,T],\\
\triangle \alpha_0(t_k) \le I_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \alpha_0(s)ds\right), & k = 1,2,\cdots,m,\\
\alpha_0(0) \le -\beta_0(T).
\end{cases}$$

Definition 3.2. A function $\beta_0 \in E \bigcap E_0$ is called an upper solution of BVP (1) if

$$\begin{cases}
\beta_0{}'(t) \ge f(t,\beta_0(t)), & t \ne t_k, t \in J = [0,T],\\
\triangle \beta_0(t_k) \ge I_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \beta_0(s)ds\right), & k = 1,2,\cdots,m,\\
\beta_0(0) \ge -\alpha_0(T).
\end{cases}$$

Theorem 3.1. Suppose that the following assumptions hold:

(A_1): The functions $\alpha_0(t), \beta_0(t)$ are lower and upper solutions of the BVP (1) respectively, such that $\alpha_0(t) \leq \beta_0(t)$ on J^+;

(A_2): The function f satisfies $f_y(t, y(t)) < 0$ and the quadratic form is given by

$$K(f(t,y)) = (y-u)^2 f_{yy}(t, y_1) \leq 0,$$

where $\alpha_0 \leq u \leq y_1 \leq y \leq \beta_0, \quad t \neq t_k, t \in J$;

(A_3): For $k = 1, 2, \cdots, m$, all functions I_k satisfy $I_k'(.) \geq 0$ and $I_k''(.) \geq 0$.

Then, there are two monotone sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ of lower and upper solutions respectively, which uniformly and quadratically converge to the extreme solutions of the BVP (1) in $[\alpha_0, \beta_0]$.

Proof. Using the Taylor′s theorem and (A_2), we have

$$f(t, y(t)) \leq Q(t, y(t), U(t)),$$

where $Q(t, y(t), U(t)) = f(t, u(t)) + f_y(t, u(t))(y(t) - u(t))$. Similarly, using the Taylor′s theorem together with (A_3), we get that

$$\begin{aligned} I_k & \left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} x(s)ds \right) - I_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} y(s)ds\right) \\ & \geq I_k'\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} y(s)ds\right)\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} (x(s)-y(s))ds\right), \end{aligned}$$

where $\alpha_0(t_k) \leq y(t_k) \leq x(t_k) \leq \beta_0(t_k)$.

Now, we give two sequences $\alpha_i(t)$ and $\beta_i(t)$ satisfying:

$$\begin{cases} \alpha_i'(t) - f_y(t, \alpha_{i-1}(t))\alpha_i(t) = f(t, \alpha_{i-1}(t)) - f_y(t, \alpha_{i-1}(t))\alpha_{i-1}(t), \qquad t \neq t_k, t \in J = [0, T], \\ \triangle\alpha_i(t_k) = I_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \alpha_{i-1}(s)ds\right) + I_k'\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \alpha_{i-1}(s)ds\right)\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} (\alpha_i(s) - \alpha_{i-1}(s))ds\right), \\ \qquad k = 1, 2, \cdots, m, \\ \alpha_i(0) = -\beta_{i-1}(T). \end{cases} \tag{6}$$

$$\begin{cases} \beta_i'(t) - f_y(t, \alpha_{i-1}(t))\beta_i(t) = f(t, \beta_{i-1}(t)) - f_y(t, \alpha_{i-1}(t))\beta_{i-1}(t), \qquad t \neq t_k, t \in J = [0, T], \\ \triangle\beta_i(t_k) = I_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \beta_{i-1}(s)ds\right) + I_k'\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \alpha_{i-1}(s)ds\right)\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} (\beta_i(s) - \beta_{i-1}(s))ds\right), \\ \qquad k = 1, 2, \cdots, m, \\ \beta_i(0) = -\alpha_{i-1}(T). \end{cases} \tag{7}$$

From Lemmas 2.1 and 2.2, we can see that both (6) and (7) have a unique solution. We complete our proof in five steps.

Step 1. We proof that $\alpha_i \leq \alpha_{i+1}$ and $\beta_i \leq \beta_{i-1}, i = 0, 1, 2, \cdots$.

Let $i = 1$ in (6), then α_1 satisfies:

$$\begin{cases} \alpha_1'(t) - f_y(t, \alpha_0(t))\alpha_1(t) = f(t, \alpha_0(t)) - f_y(t, \alpha_0(t))\alpha_0(t), \qquad t \neq t_k, t \in J = [0, T], \\ \triangle\alpha_1(t_k) = I_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \alpha_0(s)ds\right) + I_k'\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \alpha_0(s)ds\right)\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} (\alpha_1(s) - \alpha_0(s))ds\right), \\ \qquad k = 1, 2, \cdots, m, \\ \alpha_1(0) = -\beta_0(T). \end{cases}$$

To prove, we set $p(t) = \alpha_0(t) - \alpha_1(t)$, then

$$\begin{aligned}
p'(t) &- f_y(t,\alpha_0(t))p(t) = \alpha_0'(t) - \alpha_1'(t) - f_y(t,\alpha_0(t))\alpha_0(t) + f_y(t,\alpha_0(t))\alpha_1(t) \\
&\leq f(t,\alpha_0(t)) - f_y(t,\alpha_0(t))\alpha_1(t) - f(t,\alpha_0(t)) + f_y(t,\alpha_0(t))\alpha_0(t) - f_y(t,\alpha_0(t))\alpha_0(t) + f_y(t,\alpha_0(t))\alpha_1(t) \\
&= 0, \\
\Delta p(t_k) &= \Delta\alpha_0(t_k) - \Delta\alpha_1(t_k) \\
&\leq I_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \alpha_0(s)ds\right) - I_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \alpha_0(s)ds\right) \\
&- I_k'\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \alpha_0(s)ds\right)\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} (\alpha_1(s)-\alpha_0(s))ds\right) \\
&= I_k'\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \alpha_0(s)ds\right)\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} p(s)ds\right), \\
p(0) &\leq 0.
\end{aligned}$$

Thus, by Lemma 2.3, we know that $p(t) \leq 0$, i.e., $\alpha_0 \leq \alpha_1$. By the same way, we can show that $\beta_1 \leq \beta_0$. Then by the mathematic induction, we get that $\alpha_i \leq \alpha_{i+1}$ and $\beta_i \leq \beta_{i-1}, i = 0, 1, 2, \cdots$.

Step 2. We show that for all $t \in J, \alpha_1 \leq \beta_1$.

Letting $p(t) = \alpha_1(t) - \beta_1(t)$, then by $(A_1) - (A_3)$, we obtain

$$\begin{aligned}
p'(t) &- f_y(t,\alpha_0(t))p(t) = \alpha_1'(t) - \beta_1'(t) - f_y(t,\alpha_0(t))\alpha_1(t) + f_y(t,\alpha_0(t))\beta_1(t) \\
&= f(t,\alpha_0(t)) - f_y(t,\alpha_0(t))\alpha_0(t) - f(t,\beta_0(t)) + f_y(t,\alpha_0(t))\alpha_1(t) + f_y(t,\alpha_0(t))\beta_0(t) \\
&- f_y(t,\alpha_0(t))\beta_1(t) - f_y(t,\alpha_0(t))\alpha_1(t) + f_y(t,\alpha_0(t))\beta_1(t) \\
&= f(t,\alpha_0(t)) - f(t,\beta_0(t)) + f_y(t,\alpha_0(t))(\beta_0(t) - \alpha_0(t)) \leq 0, \\
\Delta p(t_k) &= \Delta\alpha_1(t_k) - \Delta\beta_1(t_k) \\
&= I_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \alpha_0(s)ds\right) - I_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \beta_0(s)ds\right) \\
&+ I_k'\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \alpha_0(s)ds\right)\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} (\alpha_1(s)-\alpha_0(s))ds\right) \\
&- I_k'\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \alpha_0(s)ds\right)\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} (\beta_1(s)-\beta_0(s))ds\right) \\
&\leq I_k'\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \alpha_0(s)ds\right)\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} p(s)ds\right), \\
p(0) &\leq 0.
\end{aligned}$$

Thus, we have $p(t) \leq 0$ by Lemma 2.3, i.e., $\alpha_1 \leq \beta_1$.

Step 3. From the above two steps, we get two monotone sequences $\alpha_i(t)$ and $\beta_i(t)$, such that

$$\alpha_0(t) \leq \alpha_1(t) \leq \cdots \leq \alpha_n(t) \leq \cdots \leq \beta_n(t) \leq \cdots \leq \beta_1(t) \leq \beta_0(t), \quad t \in J,$$

where $\alpha_i(t), \beta_i(t) \in E \bigcap E_0$ and satisfies (6) and (7), respectively.

Because it is easy to prove that $\alpha_n(t)$, $\beta_n(t)$ are uniformly bounded and equi-continuous, so by the Ascoli-Arzela criterion [27], we know that there exist two functions r(t),$\rho(t)$ such that the following expression holds for all $t \in J$:

$$lim_{n\to\infty}\alpha_n(t) = r(t), \quad lim_{n\to\infty}\beta_n(t) = \rho(t), \qquad \text{uniformly on J.}$$

Clearly, by letting $i \to \infty$ in (6) and (7), we know that r(t) and $\rho(t)$ are two solutions of (1).
Step 4. We prove that r(t) and $\rho(t)$ are the minimal solution and maximal solution of (1), respectively.

Set x(t) is an any solution of (1), and $\alpha_0(t) \le x(t) \le \beta_0(t)$. We suppose that $\alpha_n(t) \le x(t) \le \beta_n(t)$ holds for a positive integer n, then we prove that $\alpha_{n+1}(t) \le x(t) \le \beta_{n+1}(t)$.

Let $p(t) = \alpha_{n+1}(t) - x(t)$, then

$$\begin{aligned}
p'(t) &- f_y(t,\alpha_n(t))p(t) = \alpha'_{n+1}(t) - x'(t) - f_y(t,\alpha_n(t))\alpha_{n+1}(t) + f_y(t,\alpha_n(t))x(t)\\
&= f(t,\alpha_n(t)) - f_y(t,\alpha_n(t))\alpha_n(t) - f(t,x(t)) - f_y(t,\alpha_n(t))\alpha_{n+1}(t) + f_y(t,\alpha_n(t))\alpha_{n+1}(t)\\
&+ f_y(t,\alpha_n(t))x(t) \le 0,\\
\Delta p(t_k) &= \Delta\alpha_{n+1}(t_k) - \Delta x(t_k)\\
&= I_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}\alpha_n(s)ds\right) - I_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}x(s)ds\right)\\
&+ I'_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}\alpha_n(s)ds\right)\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}(\alpha_{n+1}(s)-\alpha_n(s))ds\right)\\
&= I_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}\alpha_n(s)ds\right) - I_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}x(s)ds\right)\\
&+ I'_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}\alpha_n(s)ds\right)\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}(x(s)-\alpha_n(s))ds\right)\\
&+ I'_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}\alpha_n(s)ds\right)\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}p(s)ds\right)\\
&\le I'_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}\alpha_n(s)ds\right)\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k}p(s)ds\right),\\
p(0) &\le 0.
\end{aligned}$$

Thus, by Lemma 2.3, we get that $p(t) \le 0$, i.e., $\alpha_{n+1} \le x$. Similarly, we know that $x \le \beta_{n+1}$. Therefore, $\alpha_{n+1}(t) \le x(t) \le \beta_{n+1}(t)$ holds. Finally, by letting $n \to \infty$, we can see that $r(t) \le x(t) \le \rho(t)$.
Step 5. We show that the two above monotone sequences satisfy quadratic convergence.

No lose generally, we show only that α_n satisfies quadratic convergence.

Letting $p_n(t) = r(t) - \alpha_n(t) \ge 0$, we consider the following problem:

$$\begin{aligned}
p'_n(t) &- f_y(t,\alpha_{n-1}(t))p_n(t) = r'(t) - \alpha'_n(t) - f_y(t,\alpha_{n-1}(t))r(t) + f_y(t,\alpha_{n-1}(t))\alpha_n(t)\\
&= f(t,r(t)) - f_y(t,\alpha_{n-1}(t))\alpha_n(t) - f(t,\alpha_{n-1}(t)) + f_y(t,\alpha_{n-1}(t))\alpha_{n-1}(t)\\
&- f_y(t,\alpha_{n-1}(t))r(t) + f_y(t,\alpha_{n-1}(t))\alpha_n(t)\\
&= \frac{1}{2}p^2_{n-1}(t)f_{yy}(t,y_1),
\end{aligned}$$

where $\alpha_{n-1}(t) \le y_1 \le r(t)$,

$$\Delta p_n(t_k) = \Delta r(t_k) - \Delta\alpha_n(t_k)$$

$$
\begin{aligned}
&= I_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} r(s)ds\right) - I_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \alpha_{n-1}(s)ds\right) \\
&- I_k'\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \alpha_{n-1}(s)ds\right)\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} (r(s)-\alpha_{n-1}(s))ds\right) \\
&+ I_k'\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \alpha_{n-1}(s)ds\right)\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} p_n(s)ds\right) \\
&= I_k'\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \alpha_{n-1}(s)ds\right)\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} p_n(s)ds\right) \\
&+ \frac{1}{2}I_k''(\xi)\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} p_{n-1}(s)ds\right)^2, \\
p_n(0) &= -p_n(T) + \eta, \qquad \eta = p_{n-1}(T) - \alpha_n(T),
\end{aligned}
$$

where $\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \alpha_{n-1}(s)ds < \xi < \int_{t_{k-1}+q_{k-1}}^{t_k-p_k} r(s)ds$.

Then, by Lemma 2.1, the solution of the above problem is:

$$
\begin{aligned}
p_n(t) &= \int_0^T G(t,s)[\frac{1}{2}p_{n-1}^2(s)f_{yy}(s,y_1)]ds + \frac{e^{M(T)-M(t)}}{1+e^{M(T)}}\eta \\
&+ \sum_{k=1}^m G(t,t_k)\Bigg[I_k'\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \alpha_{n-1}(s)ds\right)\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} (p_n(s))ds\right) \\
&+ \frac{1}{2}I_k''(\xi)\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} p_{n-1}(s)ds\right)^2\Bigg],
\end{aligned}
$$

where $M(t) = -\int_0^t f_y(u,\alpha_{n-1}(u))du$. Letting $|f_{yy}| \le \delta_1$ and we take the norm of p_{n-1} on J by $\|p_{n-1}\|_{E_0} = \max_{t\in J}\{p_{n-1}(t)\}$. Since $(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} p_{n-1}(s)ds)^2 \le (a-(p_k+q_{k-1}))^2\|p_{n-1}\|_{E_0}^2$, then by the expression of $p_n(t)$, we know that there is a constant λ such that

$$
\|p_n\|_{E_0} \le \lambda\|p_{n-1}\|_{E_0}^2.
$$

Thus, p_i is quadratic convergence. This completes the proof. □

4. Several corollaries

In this section, we provide a series of corollaries about the existence and convergence of solutions for system (1). It is interesting to note that it is difficult to obtain similar corollaries for the solution sequence of functional differential equations [18]. Therefore, the major results we obtained regarding system (1) may have broader applicability. Since the proof approaches are similar to that in Sections 2 and 3, we provide only relevant results and omit proof processes.

4.1. Corollary 1

We consider the following linear problem:

$$\begin{cases} y'(t) + My(t) = \sigma(t), & t \neq t_k, t \in J = [0, T], \\ \triangle y(t_k) = L_k \left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} y(s)ds\right) + I_k \left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \eta(s)ds\right) \\ -L_k \left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \eta(s)ds\right), & k = 1, 2, \cdots, m, \\ y(0) = -y(T), \end{cases} \tag{8}$$

where $M > 0, L_k \geq 0$, and $\sigma(t) \in E_0, \eta(t) \in E$.

Lemma 4.1. $y \in E$ is a solution of (8) if and only if $y \in E_0$ satisfying:

$$\begin{aligned} y(t) &= \int_0^T G(t,s)\sigma(s)ds + \sum_{k=1}^m G(t,t_k)\left[L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} y(s)ds\right)\right. \\ &+ \left. I_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \eta(s)ds\right) - L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \eta(s)ds\right)\right], \end{aligned}$$

where

$$G(t,s) = \frac{1}{e^{MT}+1}\begin{cases} e^{M(T-t+s)}, & 0 \leq s \leq t \leq T, \\ -e^{M(s-t)}, & 0 \leq t < s \leq T. \end{cases}$$

Lemma 4.2. Suppose that there exist constants $M > 0, L_k \geq 0, 0 < q_{k-1} \leq (t_k - t_{k-1})/2, \quad 0 \leq p_k \leq (t_k - t_{k-1})/2, k = 1, 2, \cdots, m$, such that:

$$\frac{e^{MT}}{e^{MT}+1}\sum_{k=1}^m L_k(a - (p_k + q_{k-1})) < 1.$$

Then (8) has a unique solution.

Lemma 4.3. Suppose that there exist constants $M > 0, L_k \geq 0, 0 < q_{k-1} \leq (t_k - t_{k-1})/2, \quad 0 \leq p_k \leq (t_k - t_{k-1})/2, k = 1, 2, \cdots, m$, such that $y \in E$ satisfying:

$$\begin{cases} y'(t) + My(t) \leq 0, & t \neq t_k, t \in J = [0, T], \\ \triangle y(t_k) \leq L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} y(s)ds\right), & k = 1, 2, \cdots, m, \\ y(0) \leq 0. \end{cases}$$

Then $y(t) \leq 0$ for all $t \in J$.

Theorem 4.1. Suppose that the following assumptions hold:

(A_1): The functions $\alpha_0(t)$,$\beta_0(t)$ are lower and upper solutions of the BVP (1) respectively, such that $\alpha_0(t) \leq \beta_0(t)$ on J^+;

(A_2): The function f satisfies $f_y(t, y(t)) < 0$ and the quadratic form is given by

$$K(f(t,y)) = (y-u)^2 f_{yy}(t, y_1) \leq 0,$$

where $\alpha_0 \leq u \leq y_1 \leq y \leq \beta_0, \quad t \neq t_k, t \in J$.

(A_3): For $k = 1, 2, \cdots, m$, all functions I_k satisfies $I_k^{'}(.) \leq 0$ and $I_k^{''}(.) \geq 0$.

Then, there exist two monotone sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ of lower and upper solutions respectively, which uniformly and quadratically converge to the extreme solutions of the BVP (1) in $[\alpha_0, \beta_0]$.

4.2. Corollary 2

We consider the following linear problem:

$$\begin{cases} y'(t) - My(t) = \sigma(t), \qquad t \neq t_k, t \in J = [0,T], \\ \triangle y(t_k) = -L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} y(s)ds\right) + I_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \eta(s)ds\right) \\ +L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \eta(s)ds\right), \qquad k = 1,2,\cdots,m, \\ y(0) = -y(T), \end{cases} \tag{9}$$

where $M > 0, L_k \leq 0$, and $\sigma(t) \in E_0, \eta(t) \in E$.

Lemma 4.4. $y \in E$ is a solution of (9) if and only if $y \in E_0$ satisfies:

$$\begin{aligned} y(t) &= \int_0^T G(t,s)\sigma(s)ds + \sum_{k=1}^{m} G(t,t_k)\left[-L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} y(s)ds\right)\right. \\ &+ \left. I_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \eta(s)ds\right) + L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \eta(s)ds\right)\right], \end{aligned}$$

where

$$G(t,s) = \frac{1}{e^{-MT}+1}\begin{cases} e^{-M(T-t+s)}, & 0 \leq s \leq t \leq T, \\ -e^{-M(s-t)}, & 0 \leq t < s \leq T. \end{cases}$$

Lemma 4.5. Suppose that there exist constants $M > 0, L_k \leq 0, 0 < q_{k-1} \leq (t_k - t_{k-1})/2, \quad 0 \leq p_k \leq (t_k - t_{k-1})/2, k = 1,2,\cdots,m$, such that:

$$\frac{e^{-MT}}{e^{-MT}+1}\sum_{k=1}^{m}|L_k|(a - (p_k + q_{k-1})) < 1.$$

Then, (9) has a unique solution.

Lemma 4.6. Suppose that there exist constants $M > 0, L_k \leq 0, 0 < q_{k-1} \leq (t_k - t_{k-1})/2, \quad 0 \leq p_k \leq (t_k - t_{k-1})/2, k = 1,2,\cdots,m$, such that $y \in E$ satisfying:

$$\begin{cases} y'(t) - My(t) \leq 0, \qquad t \neq t_k, t \in J = [0,T], \\ \triangle y(t_k) \leq -L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} y(s)ds\right), \qquad k = 1,2,\cdots,m, \\ y(0) \leq 0. \end{cases}$$

Then $y(t) \leq 0$ for all $t \in J$.

Theorem 4.2. Suppose that the following assumptions hold:

(A_1): The functions $\alpha_0(t)$,$\beta_0(t)$ are lower and upper solutions of the BVP(1) respectively, such that $\alpha_0(t) \leq \beta_0(t)$ on J^+;

(A_2): The function f satisfies $f_y(t, y(t)) > 0$ and the quadratic form is given by

$$K(f(t,y)) = (y-u)^2 f_{yy}(t,y_1) \leq 0,$$

where $\alpha_0 \leq u \leq y_1 \leq y \leq \beta_0, \quad t \neq t_k, t \in J$.

(A_3): For $k = 1,2,\cdots,m$, all functions I_k satisfies $I_k'(.) \geq 0$ and $I_k''(.) \geq 0$.

Then, there exist two monotone sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ of lower and upper solutions respectively, which uniformly and quadratically converge to the extreme solutions of the BVP (1) in $[\alpha_0,\beta_0]$.

4.3. Corollary 3

We consider the following linear problem:

$$\begin{cases} y'(t) - My(t) = \sigma(t), \qquad t \neq t_k, t \in J = [0, T], \\ \triangle y(t_k) = L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} y(s)ds\right) + I_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \eta(s)ds\right) \\ -L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \eta(s)ds\right), \qquad k = 1, 2, \cdots, m, \\ y(0) = -y(T), \end{cases} \tag{10}$$

where $M > 0, L_k \geq 0$, and $\sigma(t) \in E_0, \eta(t) \in E$.

Lemma 4.7. $y \in E$ is a solution of (10) if and only if $y \in E_0$ satisfying:

$$\begin{aligned} y(t) \ = \ & \int_0^T G(t, s)\sigma(s)ds + \sum_{k=1}^{m} G(t, t_k)\left[L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} y(s)ds\right)\right. \\ + \ & \left. I_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \eta(s)ds\right) - L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} \eta(s)ds\right)\right], \end{aligned}$$

where

$$G(t, s) = \frac{1}{e^{-MT} + 1}\begin{cases} e^{-M(T-t+s)}, & 0 \leq s \leq t \leq T, \\ -e^{-M(s-t)}, & 0 \leq t < s \leq T. \end{cases}$$

Lemma 4.8. Suppose that there exist constants $M > 0, L_k \geq 0, 0 < q_{k-1} \leq (t_k - t_{k-1})/2, \quad 0 \leq p_k \leq (t_k - t_{k-1})/2, k = 1, 2, \cdots, m$, such that:

$$\frac{e^{-MT}}{e^{-MT} + 1}\sum_{k=1}^{m} L_k(a - (p_k + q_{k-1})) < 1.$$

Then, (10) has a unique solution.

Lemma 4.9. Suppose that there exist constants $M > 0, L_k \geq 0, 0 < q_{k-1} \leq (t_k - t_{k-1})/2, \quad 0 \leq p_k \leq (t_k - t_{k-1})/2, k = 1, 2, \cdots, m$, such that $y \in E$ satisfying:

$$\begin{cases} y'(t) - My(t) \leq 0, \qquad t \neq t_k, t \in J = [0, T], \\ \triangle y(t_k) \leq L_k\left(\int_{t_{k-1}+q_{k-1}}^{t_k-p_k} y(s)ds\right), \qquad k = 1, 2, \cdots, m, \\ y(0) \leq 0, \end{cases}$$

then $y(t) \leq 0$ for all $t \in J$.

Theorem 4.3. Suppose that the following assumptions hold:

(A_1): The functions $\alpha_0(t)$,$\beta_0(t)$ are lower and upper solutions of the BVP (1) respectively, such that $\alpha_0(t) \leq \beta_0(t)$ on J^+;

(A_2): The function f satisfies $f_y(t, y(t)) > 0$ and the quadratic form is given by

$$K(f(t, y)) = (y - u)^2 f_{yy}(t, y_1) \leq 0,$$

where $\alpha_0 \leq u \leq y_1 \leq y \leq \beta_0, \quad t \neq t_k, t \in J$.

(A_3): For $k = 1, 2, \cdots, m$, all functions I_k satisfies $I_k^{'}(.) \leq 0$ and $I_k^{''}(.) \geq 0$.

Then, there exist two monotone sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ of lower and upper solutions respectively, which uniformly and quadratically converge to the extreme solutions of the BVP (1) in $[\alpha_0, \beta_0]$.

5. Conclusions

In this paper, we systematically explore the existence conditions of extreme solutions for an impulsive ordinary differential equation with an anti-periodic boundary value. Unlike traditional discrete impulsive conditions, the impulsive condition in this equation is in integral form. On the one hand, the impulsive integration condition depends on past states, which is more reasonable in describing the impulsive effects of many natural phenomena. On the other hand, we note that the impulsive integration condition proposed in previous studies (such as [22]) depends on the states near the impulsive point t_{k-1} and t_k. The impulsive integration condition used in this paper is dependent on the states within the interval (t_{k-1}, t_k). Therefore, the impulsive integration condition in this paper might have a stronger dependence on past states than previous studies.

In this paper, we mostly use the monotonic iteration technique and quasilinearization method to study the existence and convergence of solutions for the impulsive equation. For impulsive differential equation systems, monotonic iteration techniques generally can ensure only the uniform convergence of the solution sequence (i.e., first-order convergence). In order to obtain higher-order convergence of the solution sequence, we need to use quasilinearization methods. Therefore, in general, in impulsive differential equations, the sequences constructed by the quasilinearization technique have higher order convergence than the sequences constructed by the monotonic iteration technique. In this paper, by combining these two methods, we can obtain both first-order and second-order (quadratic) convergence conditions for the solution sequence of the integral impulsive differential system. Through rigorous argumentation, we find that the impulsive differential equation cannot obtain the condition for second-order convergence of the solution sequence under the previous integral impulsive condition ([22]). Therefore, from the perspective of convergence of the solution sequence, the impulsive integration condition used in this paper might be an improvement of previous research. Interestingly, we can also derive a series of corollaries regarding the impulsive ordinary differential equation proposed in this paper (Section 4), which provide richer conclusions than functional differential equations (such as [18]).

Of course, there are many types of impulsive differential equations. For example, delayed impulsive differential equations, impulsive integral-differential equations, stochastic, or fractional impulsive differential equations have wide applications in practice. Our methods and results may also be applicable to research in these fields in the future.

Author contributions

Yan Li: Writing-original draft, Writing-review & editing, Methodology, Validation; Zihan Rui: Writing-review & editing, Writing-original draft, Supervision, Validation; Bing Hu: Methodology, Conceptualization, Validation, Investigation, Writing-review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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