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Research article

Modified likelihood approach for Wald-typed interval of the shape parameter in Weibull distribution

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Abstract: The Weibull distribution, widely used in lifetime analysis, is characterized by its shape parameter. We analytically derived Wald-type confidence intervals using standard and modified profile likelihood methods. Performance was assessed through a simulation study examining coverage probability (CP) and average length (AL) across twelve scenarios, varying the shape parameter from 0.5 to 10, the scale parameter from 0.5 to 5, and a range of sample sizes from 5 to 200. The proposed intervals were compared with traditional Wald, profile likelihood, and modified profile likelihood intervals. Our results indicated that the proposed intervals, especially those based on modified profile likelihood, consistently outperformed traditional methods, particularly with small sample sizes. Reductions in either the shape or scale parameter led to shorter AL, as the shape parameter was inversely related to CP. For larger sample sizes (over 30), all interval methods performed similarly, confirming the robustness of the derived intervals across sample sizes. Additionally, the methods were applied to real data on hospital-acquired urinary tract infections, demonstrating their practical utility in healthcare settings.

Keywords: confidence interval; lifetime distribution; profile likelihood; Wald-typed interval
Mathematics Subject Classification: 62F10, 62E10, 60E05

1. Introduction

The Weibull distribution, originally introduced by Waloddi Weibull [1], has become an essential tool in statistical analysis across a wide range of scientific and engineering fields. This model is frequently applied to characterize failures in various components and phenomena, particularly within reliability and survival analysis. Over the years, researchers have developed multiple Weibull-related distributions that extend beyond the conventional two- and three-parameter forms often discussed in reliability and statistics [2].

Applications of the Weibull model span multiple disciplines. For instance, Vallée et al. [3] used the Weibull model to estimate the strength of adhesively bonded joints, while Lewis and Withers [4] applied it to investigate particle cracking in metal matrix composites. Beyond engineering, the Weibull distribution is valuable in environmental and medical sciences. Albassam et al. [5] demonstrated its utility in scenarios with indeterminate factors, such as the assessment of unpredictable wind speed data. In medical research, the Weibull model has significant applications in survival analysis and reliability of medical treatments, as seen in studies by Ghazal and Radwan [6].

This flexibility has driven the development of many new Weibull-based models. Examples include the generalized Weibull-modified model by Emam and Alomani [7], which improves parameter estimation, and the weighted Weibull distribution by Xavier and Nadarajah [8], which broadens its applications to complex datasets. Lai et al. [9] added a modified Weibull distribution that effectively models bathtub-shaped hazard rates, and Silva et al. [10] developed the beta-modified Weibull distribution, which is especially useful in survival data analysis for its ability to capture varying hazard functions. Finally, Cousineau [11] provided a comprehensive evaluation of approaches to the three-parameter Weibull model, further extending its potential applications.

While standard Wald-type and profile likelihood intervals are widely used for estimating the Weibull distribution, their performance can be limited, especially with small sample sizes or high parameter interdependence. In this study, we derive formulas for Wald-type confidence intervals for the shape parameter, incorporating both standard and modified profile likelihood methods to address these challenges. The primary contribution is the application of a modified profile likelihood approach, designed to improve interval estimation accuracy under restrictive conditions. The effectiveness of the proposed intervals is assessed through Monte Carlo simulations, comparing them with traditional Wald-type and profile likelihood intervals based on coverage probability and interval length.

2. Background

2.1. Weibull distribution

The Weibull distribution is a continuous probability distribution characterized by two parameters: Shape (β) and scale (α). The probability density function (pdf) of the Weibull distribution is given by:

$$f(x;\alpha,\beta)=\left(\frac{\beta}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\beta-1}e^{-\left(\frac{x}{\alpha}\right)^{\beta}},\quad x,\alpha,\beta>0. \tag{1}$$

The distribution is denoted as $X\sim W(\beta,\alpha)$. Given a random sample $X_1,X_2,\ldots,X_n$, the log of the joint likelihood function is:

$$l(\alpha,\beta)=\log\left(\left(\frac{\beta}{\alpha}\right)^n\prod_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta-1}e^{-\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta}}\right)$$
$$=n\log(\beta)-n\log(\alpha)+\beta\sum_{i=1}^{n}\log(x_i)-\sum_{i=1}^{n}\log(x_i)-n(\beta-1)\log(\alpha)-\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta}. \tag{2}$$

As shown in Figure 1, the shape parameter influences the distribution's form, affecting its tail and peak behavior. When $\beta < 1$, the distribution has a decreasing probability density function as x increases, indicating a high initial failure rate that decreases over time. For $\beta = 1$, the Weibull distribution reduces to the exponential distribution, which is suitable for modeling events with a constant failure rate over time. This could represent components with a random chance of failure that does not change with age. When $\beta > 1$, the distribution shows an increasing failure rate, which could model situations where the likelihood of failure increases with age or use, often seen in wear-out failure modes [12,13].

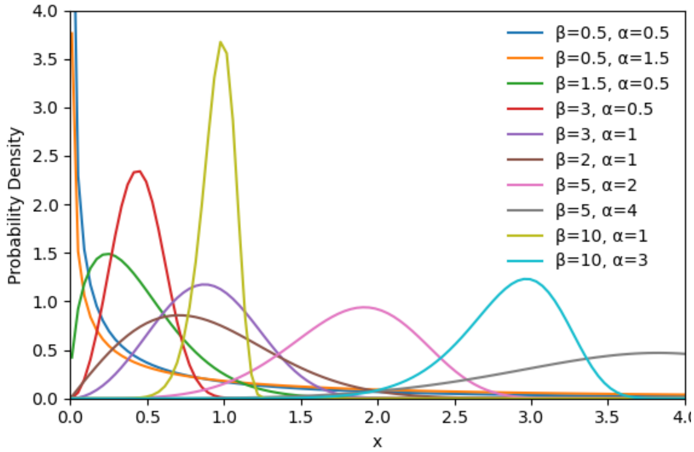

Figure 1. Weibull distribution for various parameter settings.

2.2. Point estimation

2.2.1. Method of moments

The method of moments (MM) is a classical approach for estimating parameters of statistical distributions, including the Weibull distribution. This method equates the first k theoretical moments of a distribution with the first k empirical moments from sample data to solve for the unknown parameters. For the two-parameter Weibull distribution, the MM approach yields two equations [14]:

$$\frac{\Gamma\left(2/\hat{\beta}_{MM}+1\right)}{\Gamma^2\left(2/\hat{\beta}_{MM}+1\right)}+\frac{\hat{M}_2-M_1^2}{\hat{M}_1}-1=0 \quad \text{and} \quad \hat{\alpha}_{MM}=\frac{\hat{M}_1}{\Gamma\left(1/\hat{\beta}_{MM}+1\right)},$$

where $\hat{M}_1$ and $\hat{M}_2$ are the first and second noncentral sample moments, respectively:

$$\hat{M}_1 = \sum_{i=1}^{n} X_i \Big/ n \text{ and } \hat{M}_2 = \sum_{i=1}^{n} X_i^2 \Big/ n,$$

and $\Gamma(\cdot)$ denotes the gamma function.

2.2.2. Method of maximum likelihood

The maximum likelihood estimator (MLE) is a predominant method for estimating the parameters of the two-parameter Weibull distribution. The estimation is derived from two equations [15]:

$$\hat{\alpha}_{MLE} = \left(n^{-1} \sum_{i=1}^{n} X_i^{\hat{\beta}_{MLE}} \right)^{1/\hat{\beta}_{MLE}} \text{ and } \hat{\beta}_{MLE} = \left[\left(\sum_{i=1}^{n} X_i^{\hat{\beta}_{MLE}} \ln(X_i) \right) \left(\sum_{i=1}^{n} X_i^{\hat{\beta}_{MLE}} \right)^{-1} - n^{-1} \sum_{i=1}^{n} \ln(X_i) \right]^{-1}. \quad (3)$$

Since MLE does not yield a closed-form solution for the Weibull parameters, numerical optimization techniques are typically employed to find these estimates. The elliptical shape of the contour in Figure 2 illustrates the correlation between the shape and scale parameters. Therefore, a Wald confidence interval, which typically relies on a joint likelihood function, may not be suitable for interval construction. This is because the Wald method does not account for the dependency between the two parameters, which can result in misleading confidence levels. Instead, using the profile likelihood approach for the shape parameter is a more reliable way to estimate the interval in this case because it relies on the observed data and marginalizes over the scale parameter, which better takes into account how the shape and scale parameters are connected in the Weibull model [16,17].

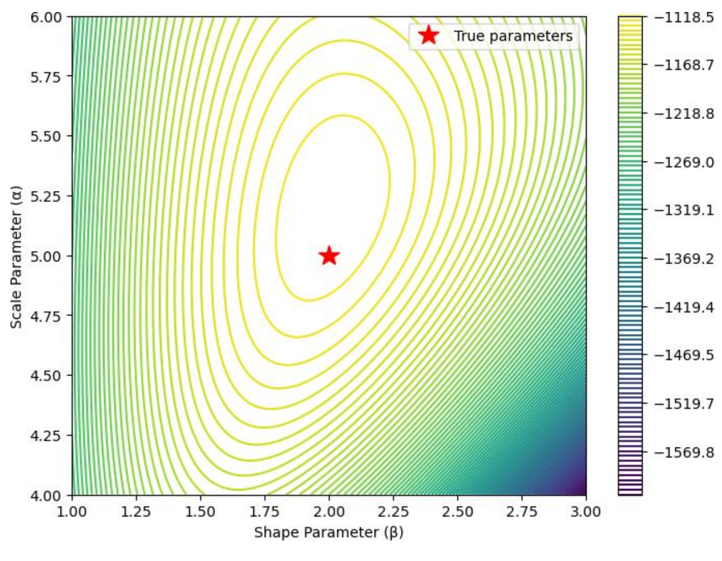

Figure 2. Log-Likelihood contour plot.

2.3. Profile likelihood function of the shape parameter

Consider a random sample $X_1, X_2, \ldots, X_n$ from a Weibull distribution with shape *β* and scale *α*. If we consider *β* as a fixed value, the MLE of *α* based on the profile likelihood is given by

$$\tilde{\alpha} = \left(n^{-1} \sum_{i=1}^{n} X_i^{\beta} \right)^{1/\beta}.$$

Plugging in α in (2) by $\tilde{\alpha}$, the log profile likelihood function becomes:

$$l_{PF}(\beta) = n\log(\beta) - n\log(\tilde{\alpha}(\beta)) + \beta\sum_{i=1}^{n}\log(x_i) - \sum_{i=1}^{n}\log(x_i) - n(\beta-1)\log(\tilde{\alpha}(\beta)) - \sum_{i=1}^{n}\left(\frac{x_i}{\tilde{\alpha}(\beta)}\right)^{\beta}.$$

Simplifying, this expression becomes:

$$l_{PF}(\beta) = n\log(\beta) - n\log\left(\sum_{i=1}^{n} (x_i)^{\beta}\right) + \beta\sum_{i=1}^{n}\log(x_i) + n(\log(n)-1) - \sum_{i=1}^{n}\log(x_i). \tag{4}$$

The profile likelihood function is denoted as $L_{PF}(\tilde{\alpha}, \beta) = \exp\left[l_{PF}(\beta)\right]$. The function $L_{PF}(\tilde{\alpha}, \beta)$ can indeed be used as a likelihood function in statistical analysis. This approach serves as a method for approximating the likelihood function by profiling out nuisance parameters.

2.4. Parameter orthogonalization

In the Weibull distribution context, parameter orthogonalization is a technique used to mitigate the potential high correlation between the MLEs for the shape and scale parameters. Reparameterizing to orthogonal parameters can lead to asymptotic independence of the MLEs. This reparameterization involves introducing a new parameter in line with the conditions set by the expected Fisher information matrix [18]. Through a differential equation that incorporates Euler's constant, the orthogonality condition can be expressed as:

$$j_{\alpha\alpha}\frac{\partial \alpha}{\partial \beta} + j_{\beta\alpha} = 0, \tag{5}$$

where $j_{\alpha\alpha} = (\beta/\alpha)^2$, $j_{\beta\alpha} = (\xi - 1)/\alpha$, and $\xi \approx 0.5772$. Solving Equation (5) to derive an orthogonal nuisance parameter yields $\lambda = \alpha\exp((1-\xi)/\beta)$. Cox and Reid [18] described a modified profile likelihood method for estimating the shape parameter, which is adjusted for the nuisance scale parameter as follows:

$$l_{MPF}(\beta) = l_{PF}(\beta) - \frac{1}{2}\log\det\left[J_{\lambda\lambda}(\beta, \tilde{\lambda}(\beta))\right], \tag{6}$$

where $l_{PF}\left(\beta\right)$ is shown in Eq (4). The $J_{\lambda\lambda}\left(\beta,\tilde{\lambda}\left(\beta\right)\right)$ is the observed information matrix, $\tilde{\lambda}\left(\beta\right)$ is the restricted MLE of α for the specified β [19]. Yang and Xie [19] showed that Eq (6) can be derived as:

$$l_{MPF}\left(\beta\right)=\left(n-2\right)\log\left(\beta\right)-n\log\left(\tilde{\alpha}\right)+\beta\sum_{i=1}^{n}\log\left(x_i\right)-\sum_{i=1}^{n}\log\left(x_i\right)-n\left(\beta-1\right)\log\left(\tilde{\alpha}\right)-\sum_{i=1}^{n}\left(\frac{x_i}{\tilde{\alpha}}\right)^{\beta}, \quad (7)$$

and the modified profile likelihood of β is denoted as $L_{MPF}\left(\tilde{\alpha},\beta\right)=\exp\left[l_{MPF}\left(\beta\right)\right]$.

To solve for the maximum modified profile likelihood estimator for the Weibull shape parameter, the function $l_{MPF}\left(\beta\right)$ can be set to zero. This modified profile likelihood approach is straightforward yet effective, as evidenced by the Monte Carlo simulations [20].

2.5. Interval estimation in literature

In the field of interval estimation, the analysis of censored data has garnered significant attention from scholars. The Wald method, for example, has been applied to derive approximate confidence intervals for distribution parameters when data is subject to Type-2 censoring [21]. Mweleli et al. [22] developed approximate confidence intervals for the two-parameter Weibull distribution, focusing on small Type-2 censored samples by employing the profile likelihood approach. The intervals obtained through this method are contingent on the shape of the profile likelihood function and lack explicit formulas. Heo et al. [23] introduced methods for constructing confidence limits and intervals for the quantiles of Weibull distributions, utilizing techniques such as MM, probability-weighted moments (PWM), and MLE. Silva and Peiris [24] examined the modeling of rainfall percentiles within the Weibull distribution framework, emphasizing the coverage probability of confidence intervals—a crucial factor for accurate inference regarding rainfall patterns. Vander Wiel and Meeker [25] assessed the precision of s-confidence intervals derived from the likelihood ratio, highlighting their superiority over intervals based on asymptotic normal theory, despite increased computational requirements. Mahdi [15] addressed one-sided conditional and unconditional interval estimation for the scale and shape parameters in a two-parameter Weibull model, drawing inferences from pivotal quantities suggested by Bain and Engelhardt, along with the likelihood ratio method and the Birnbaum statistic.

Researchers have expanded these approaches: Niaki et al. [26] developed Bayesian joint confidence intervals for Weibull parameters, accommodating both complete and censored data. Jana and Bera [27] focused on stress–strength reliability in k-out-of-n systems with inverse Weibull-distributed stress and strength components, proposing asymptotic, bootstrap, and HPD credible intervals. Park [28] examined interval-censored Weibull data estimation, and Yang et al. [29] studied interval estimation for the location parameter in three-parameter Weibull models with a known shape parameter, comparing coverage probability and average length across methods through simulations and a real-world example. Somsamai and Srisuradetchai [30] investigated the coverage probability and average length of confidence intervals for the shape parameter in the Weibull distribution when the scale parameter is unknown. They specifically examined the modified profile likelihood (MPF) and standard profile likelihood (PF) methods. The following expressions present the PF and MPF intervals in terms of the normalized profile likelihood for both MPF and PF:

$$\left\{\beta \left| \frac{L_{PF}(\tilde{\alpha},\beta)}{\max L_{PF}(\tilde{\alpha},\beta)} \geq \exp\left(-\frac{1}{2}\chi^2_{1-\alpha,1}\right)\right.\right\} \quad (8)$$

and

$$\left\{\beta \left| \frac{L_{MPF}(\tilde{\alpha},\beta)}{\max L_{MPF}(\tilde{\alpha},\beta)} \geq \exp\left(-\frac{1}{2}\chi^2_{1-\alpha,1}\right)\right.\right\}. \quad (9)$$

Since there are no closed-form solutions for Eqs (8) and (9), numerical methods are required for their computation. The normalized profile likelihood function of a sample size of 50 from $W(\beta=5,\alpha=1)$ is shown in Figure 3, with the interval bounds determined to be (3.85, 6.02).

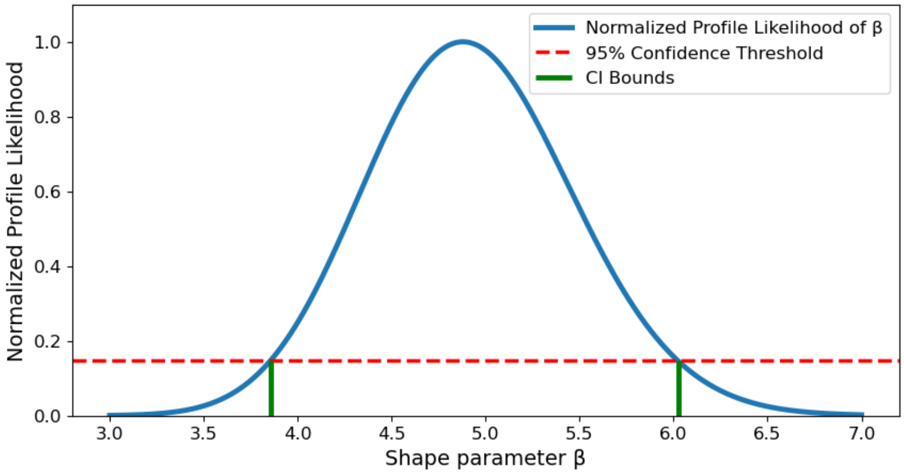

Figure 3. Normalized profile likelihood function of a sample of size 50 drawn from $W(\beta=5,\alpha=5)$ along with the corresponding confidence interval.

2.6. Performance criteria

2.6.1. Coverage probability

Let θ represent the true parameter, with (L_i, U_i) as the lower and upper bounds of the confidence interval for the *i*-th sample. Then, the coverage probability (CP) can be defined as:

$$CP = \frac{1}{N}\sum_{i=1}^{N} I(L_i \leq \theta \leq U_i),$$

where $I(\cdot)$ is the indicator function, equaling 1 if θ lies within the interval and 0 otherwise, and N represents the total number of simulated samples. This measure indicates the proportion of intervals that successfully capture the true parameter value [29,31].

2.6.2. Average length

Average length (AL) refers to the mean width of the confidence intervals across repeated samples in a simulation study [29]. A smaller AL generally indicates more precise intervals, as the range of values within each confidence interval is narrower [32,33]. The AL can be defined as:

$$AL = \frac{1}{N}\sum_{i=1}^{N}\left(U_i - L_i\right).$$

3. Theoretical results

Building on methods discussed in the literature, in this section, we introduce closed-form solutions for Wald-type intervals constructed using the profile likelihood function and the modified profile likelihood under the Weibull distribution. The following results provide explicit formulas for confidence intervals for the shape parameter when both the shape and scale parameters are unknown. These closed-form intervals offer a practical, readily applicable solution in contrast to the PF and MPF intervals presented in Eqs (8) and (9) in the literature, which lack explicit solutions.

Theorem 3.1. Consider a random sample $X_1, X_2, \ldots, X_n$ from a Weibull distribution with shape β and scale α, both of which are unknown. The corresponding maximum likelihood estimators for these parameters have been obtained. The Wald-type interval using the profile likelihood function (WPF) has a closed form as follows:

$$\hat{\beta}_{MLE}^{PF} \pm z_{(1+\gamma)/2}\hat{\beta}_{MLE}^{PF}\sqrt{\left(\frac{n+1}{n}\right)\left(\frac{1}{1.6449n - 0.4904 - 2\hat{\beta}_{MLE}^{PF}\log\left(\hat{\alpha}_{MLE}\right)}\right)}, \tag{10}$$

where $z_{(1+\gamma)/2}$ is the $(1+\gamma)/2$ quantile of the standard normal distribution, corresponding to a confidence level of γ, $\gamma \in (0,1)$.

Proof. First, consider β a fixed value. By taking into account Eq (3) and the MLE derived from the profile likelihood, we have: $\tilde{\alpha}(\beta) = \left(\frac{\sum_{i=1}^{n} x_i^{\beta}}{n}\right)^{\frac{1}{\beta}}$. Substituting α in Eq (2) with $\tilde{\alpha}(\beta)$, we obtain the log profile likelihood function:

$$l_{PF}(\beta) = n\log(\beta) - n\log\left(\tilde{\alpha}(\beta)\right) + \beta\sum_{i=1}^{n}\log(x_i) - \sum_{i=1}^{n}\log(x_i) - n(\beta-1)\log\left(\tilde{\alpha}(\beta)\right) - \sum_{i=1}^{n}\left(\frac{x_i}{\tilde{\alpha}(\beta)}\right)^{\beta}.$$

Simplifying further:

$$l_{PF}(\beta) = n\log(\beta) - n\log\left(\sum_{i=1}^{n}(x_i)^{\beta}\right) + \beta\sum_{i=1}^{n}\log(x_i) + n\left(\log(n) - 1\right) - \sum_{i=1}^{n}\log(x_i). \tag{11}$$

Note that the last two terms of Eq (4) are independent of the parameter β. The maximum profile likelihood estimator of β is defined as $\beta_{MLE}^{PF} = \arg\max l_{PF}(\beta)$. Next, we find the score function $S_{PF}(\beta)$:

$$S_{PF}(\beta) = \frac{\partial l_{PF}(\beta)}{\partial \beta} = \frac{\partial}{\partial \beta}\left(n\log(\beta) - n\log\left(\sum\nolimits_{i=1}^{n}(x_i)^{\beta}\right) + \beta\sum_{i=1}^{n}\log(x_i) + n(\log(n)-1) - \sum_{i=1}^{n}\log(x_i)\right)$$

$$= \frac{n}{\beta} - \frac{n\sum\nolimits_{i=1}^{n} x_i^{\beta}\log(x_i)}{\sum\nolimits_{i=1}^{n} x_i^{\beta}} + \sum_{i=1}^{n}\log(x_i). \tag{12}$$

The value of β_{MLE}^{PF} can also be determined by setting Eq (12) to zero. However, deriving an analytical solution for β is complicated, as β appears in both the exponent and outside the summation. Therefore, numerical methods must be employed to solve for β.

The observed Fisher information can be derived from the following expression:

$$I_{PF}(\beta) = -\frac{\partial S_{PF}(\beta)}{\partial \beta} = -\frac{n}{\beta^2} - n\frac{\left(\sum\nolimits_{i=1}^{n} x_i^{\beta}\frac{\partial}{\partial\beta}\left(\sum\nolimits_{i=1}^{n} x_i^{\beta}\log(x_i)\right) - \sum\nolimits_{i=1}^{n} x_i^{\beta}\log(x_i)\frac{\partial}{\partial\beta}\left(\sum\nolimits_{i=1}^{n} x_i^{\beta}\right)\right)}{\left(\sum\nolimits_{i=1}^{n} x_i^{\beta}\right)^2}$$

$$= -\frac{n}{\beta^2} - n\frac{\sum\nolimits_{i=1}^{n} x_i^{\beta}\left(\log(x_i)\right)^2}{\sum\nolimits_{i=1}^{n} x_i^{\beta}} + n\left(\frac{\sum\nolimits_{i=1}^{n} x_i^{\beta}\log(x_i)}{\sum\nolimits_{i=1}^{n} x_i^{\beta}}\right)^2. \tag{13}$$

Then, the expected Fisher information is derived as follows:

$$J_{PF}(\beta) = E\left[\frac{n}{\beta^2} + n\frac{\sum\nolimits_{i=1}^{n} X_i^{\beta}\left(\log(X_i)\right)^2}{\sum\nolimits_{i=1}^{n} X_i^{\beta}} - n\left(\frac{\sum\nolimits_{i=1}^{n} X_i^{\beta}\log(X_i)}{\sum\nolimits_{i=1}^{n} X_i^{\beta}}\right)^2\right]$$

$$= \frac{n}{\beta^2} + E\left[n\frac{\sum\nolimits_{i=1}^{n} X_i^{\beta}\left(\log(X_i)\right)^2}{\sum\nolimits_{i=1}^{n} X_i^{\beta}}\right] - E\left[n\left(\frac{\sum\nolimits_{i=1}^{n} X_i^{\beta}\log(X_i)}{\sum\nolimits_{i=1}^{n} X_i^{\beta}}\right)^2\right]$$

$$= \frac{n}{\beta^2} + n\frac{\left(\left(\frac{n\alpha^{\beta}}{\beta^2}\right)\left(\Gamma''(2) + 2\beta\Gamma'(2)\log(\alpha) + \beta^2\log^2(\alpha)\right)\right)}{n\alpha^{\beta}}$$
$$-n\frac{\left(\left(\frac{n\alpha^{2\beta}}{\beta^2}\right)\left(\Gamma''(3) + (n-1)\left(\Gamma'(2)\right)^2 + 2\beta\log(\alpha)\left(\Gamma'(3) + (n-1)\Gamma'(2)\right) + (n+1)\beta^2\log^2(\alpha)\right)\right)}{n(n+1)\alpha^{2\beta}},$$

where $\Gamma^{(n)}(X)$ is the partial derivative of the gamma function, defined as $\Gamma^{(n)}(X) = \int_0^\infty t^{x-1}e^{-t}\log^k(t)\,dt$. For ease of calculation, we have evaluated the following terms: $\Gamma'(2) = 0.4228$, $\Gamma''(2) = 0.8237$, $\Gamma'(3) = 1.8456$, and $\Gamma''(3) = 2.4929$. Thus, the expected Fisher information will be:

$$
\begin{aligned}
J_{PF}(\beta) &= \frac{n}{\beta^2} + \left(\frac{n}{\beta^2}\right)\left(0.8237 + 2(0.4228)\beta\log(\alpha) + \beta^2\log^2(\alpha)\right) \\
&\quad - \left(\frac{n}{(n+1)\beta^2}\right)\left(2.4929 + (n-1)(0.4228)^2 + 2\beta\log(\alpha)\left(1.8456 + (n-1)0.4228\right) + (n+1)\beta^2\log^2(\alpha)\right) \\
&= \left(\frac{n}{\beta^2}\right)\left\{\begin{array}{l} 1.8237 + 0.8456\beta\log(\alpha) - \dfrac{2.4929}{(n+1)} - 0.1788\dfrac{(n-1)}{(n+1)} - \\ 3.6912\dfrac{\beta\log(\alpha)}{(n+1)} - 0.8456\dfrac{(n-1)}{(n+1)}\beta\log(\alpha) \end{array}\right\} \\
&= \frac{n}{(n+1)\beta^2}\left(1.6449n - 0.4904 - 2\beta\log(\alpha)\right).
\end{aligned}
$$

The inverse Fisher information of the profile likelihood of β is

$$
J_{PF}(\beta)^{-1} = \frac{(n+1)\beta^2}{n\left(1.6449n - 0.4904 - 2\beta\log(\alpha)\right)}. \tag{14}
$$

Therefore, the theorem is proved.

Theorem 3.2. Consider a random sample $X_1, X_2, \ldots, X_n$ from a Weibull distribution with unknown shape β and scale α parameters. The Wald-type interval constructed using the Modified Profile Likelihood (WMPF) function has a closed form as follows:

$$
\hat{\beta}_{MLE}^{MPF} \pm z_{\frac{(1+\gamma)}{2}}\hat{\beta}_{MLE}^{MPF}\sqrt{\frac{n+1}{1.6449n^2 - 2.4904n - 2 - 2n\hat{\beta}_{MLE}^{MPF}\log\left(\hat{\alpha}_{MLE}\right)}}, \tag{15}
$$

where $z_{(1+\gamma)/2}$ is the $(1+\gamma)/2$ quantile of the standard normal distribution, corresponding to a confidence level of γ, $\gamma \in (0,1)$.

Proof. From Eq (7), the score function, $S_{MPF}(\beta)$, can be derived as follows:

$$S_{MPF}\left(\beta\right)=\frac{\partial l_{MPF}\left(\beta\right)}{\partial\beta}=\frac{n-2}{\beta}-\frac{n\sum_{i=1}^{n}x_i^{\beta}\log\left(x_i\right)}{\sum_{i=1}^{n}x_i^{\beta}}+\sum_{i=1}^{n}\log\left(x_i\right).$$

The observed Fisher information will be:

$$I_{MPF}(\beta)=\frac{\partial S_{MPF}\left(\beta\right)}{\partial\beta}=\frac{\partial^2 l_{MPF}\left(\beta\right)}{\partial\beta^2}=\frac{\partial}{\partial\beta}\left(\frac{n-2}{\beta}-\frac{n\sum_{i=1}^{n}x_i^{\beta}\log\left(x_i\right)}{\sum_{i=1}^{n}x_i^{\beta}}+\sum_{i=1}^{n}\log\left(x_i\right)\right)$$

$$=-\frac{n-2}{\beta^2}-n\frac{\sum_{i=1}^{n}x_i^{\beta}\left(\log\left(x_i\right)\right)^2}{\sum_{i=1}^{n}x_i^{\beta}}+n\left(\frac{\sum_{i=1}^{n}x_i^{\beta}\log\left(x_i\right)}{\sum_{i=1}^{n}x_i^{\beta}}\right)^2.$$

Then, the expected Fisher information is as follows:

$$J_{MPF}\left(\beta\right)=E\left[\frac{n-2}{\beta^2}+n\frac{\sum_{i=1}^{n}x_i^{\beta}\left(\log\left(X_i\right)\right)^2}{\sum_{i=1}^{n}x_i^{\beta}}-n\left(\frac{\sum_{i=1}^{n}X_i^{\beta}\log\left(X_i\right)}{\sum_{i=1}^{n}X_i^{\beta}}\right)^2\right]$$

$$=\frac{n-2}{\beta^2}+E\left[n\frac{\sum_{i=1}^{n}x_i^{\beta}\left(\log\left(X_i\right)\right)^2}{\sum_{i=1}^{n}X_i^{\beta}}\right]-E\left[n\left(\frac{\sum_{i=1}^{n}X_i^{\beta}\log\left(X_i\right)}{\sum_{i=1}^{n}X_i^{\beta}}\right)^2\right]$$

$$=\frac{n-2}{\beta^2}+\left(\frac{n}{\beta^2}\right)\left(\Gamma''\left(2\right)+2\beta\Gamma'\left(2\right)\log\left(\alpha\right)+\beta^2\log^2\left(\alpha\right)\right)-\left(\frac{n}{\left(n+1\right)\beta^2}\right)\left(\Gamma''\left(3\right)\right.$$

$$\left.+\left(n-1\right)\left(\Gamma'\left(2\right)\right)^2+2\beta\log\left(\alpha\right)\left(\Gamma'\left(3\right)+\left(n-1\right)\Gamma'\left(2\right)\right)+\left(n+1\right)\beta^2\log^2\left(\alpha\right)\right),$$

where the gamma function's partial derivatives are evaluated as follows: $\Gamma'\left(2\right)=0.4228,$ $\Gamma''\left(2\right)=0.8237, \Gamma'\left(3\right)=1.8456, \Gamma''\left(3\right)=2.4929$. Thus, the expected Fisher information is:

$$J_{MPF}\left(\beta\right)=\frac{n-2}{\beta^2}+\left(\frac{n}{\beta^2}\right)\left(0.8237+2\left(0.4228\right)\beta\log\left(\alpha\right)+\beta^2\log^2\left(\alpha\right)\right)-\left(\frac{n}{\left(n+1\right)\beta^2}\right)\left(2.4929\right.$$

$$\left.+\left(n-1\right)\left(0.4228\right)^2+2\beta\log\left(\alpha\right)\left(1.8456+\left(n-1\right)0.4228\right)+\left(n+1\right)\beta^2\log^2\left(\alpha\right)\right)$$

$$=\frac{n}{\beta^2\left(n+1\right)}\left(-\frac{2}{n}+1.6449n-2.4904-2\beta\log\left(\alpha\right)\right).$$

The inverse Fisher information of the modified profile likelihood of β is:

$$J_{MPF}(\beta)^{-1} = \frac{(n+1)\beta^2}{1.6449n^2 - 2.4904n - 2 - 2n\hat{\beta}_{MLE}\log(\hat{\alpha}_{MLE})}. \quad (16)$$

Therefore, the theorem is completely proved.

4. Simulation study

In the simulation study, we evaluated five interval estimation methods—W, PF, MPF, and the proposed WPF and WMPF—across a diverse set of scenarios to assess their performance. The samples were generated from a Weibull distribution with varying shape and scale parameters and multiple sample sizes to provide a comprehensive analysis. Specifically, shape parameters of 0.5, 1, 5, and 10 were paired with scale parameters of 0.5, 1, and 5, and sample sizes of 5, 10, 20, 30, 50, 80, 100, and 200 were examined. This setup resulted in 96 scenarios for each method, allowing for an extensive comparison. The results are illustrated in Figure 4.

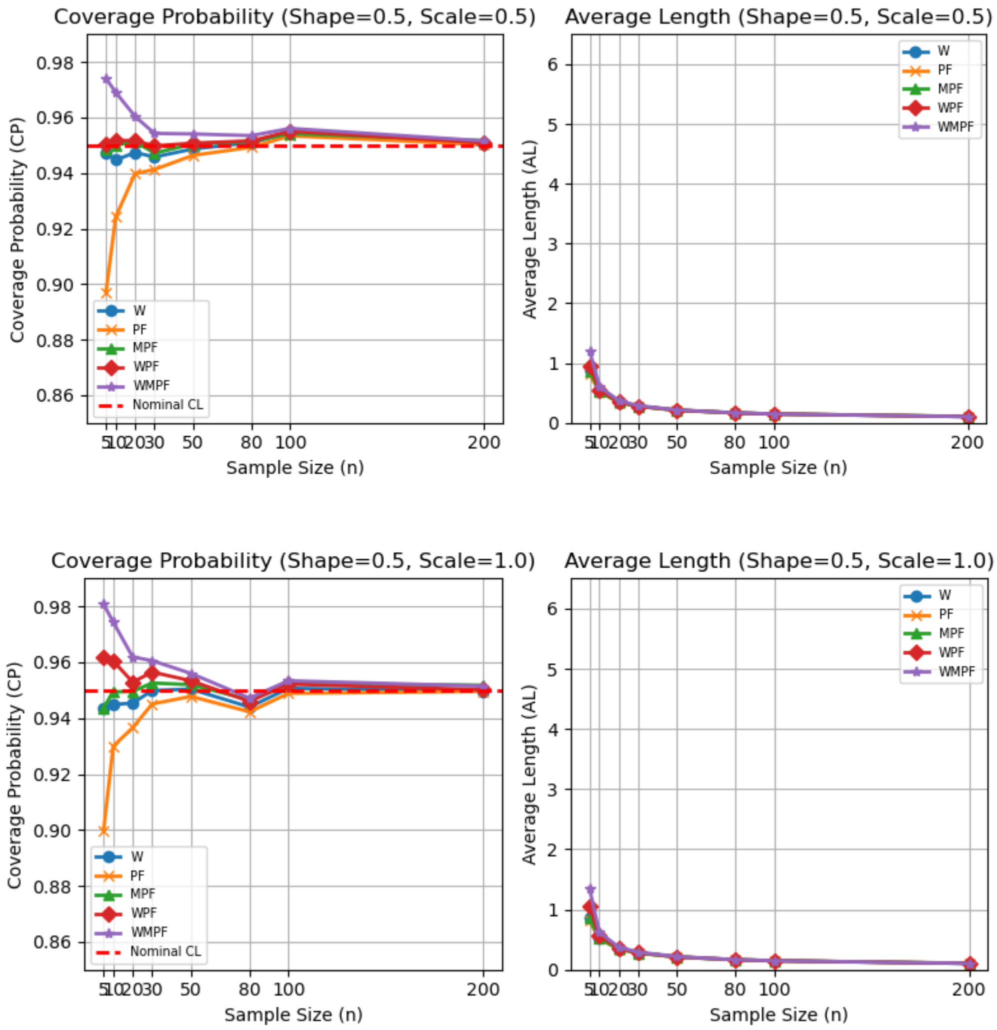

Figure 4. Coverage probability (left) and average length (right) for varying sample sizes under different shape and scale parameters of the Weibull distribution (Continued on next page).

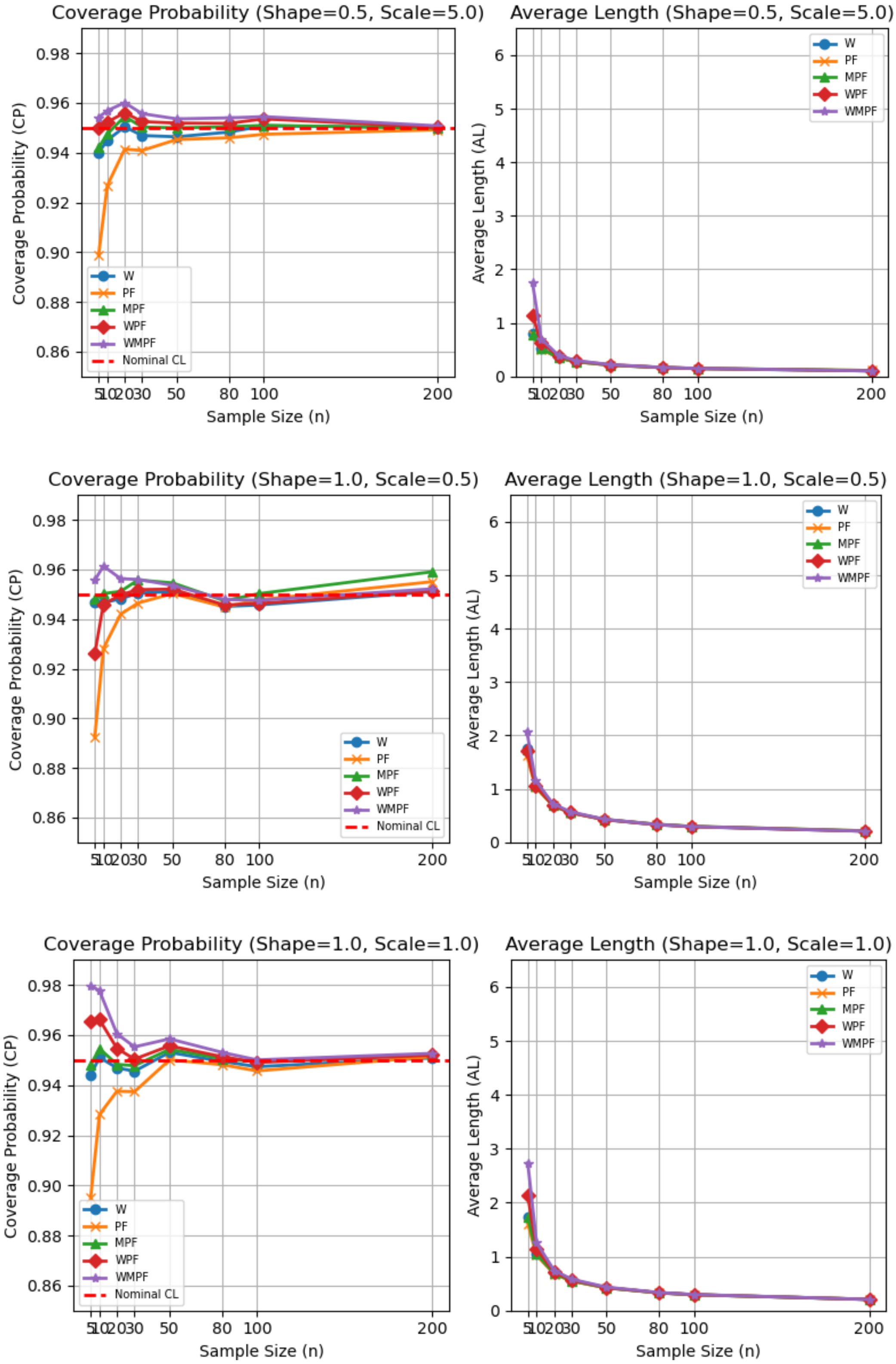

Figure 4. Coverage probability (left) and average length (right) for varying sample sizes under different shape and scale parameters of the Weibull distribution (Continued on next page).

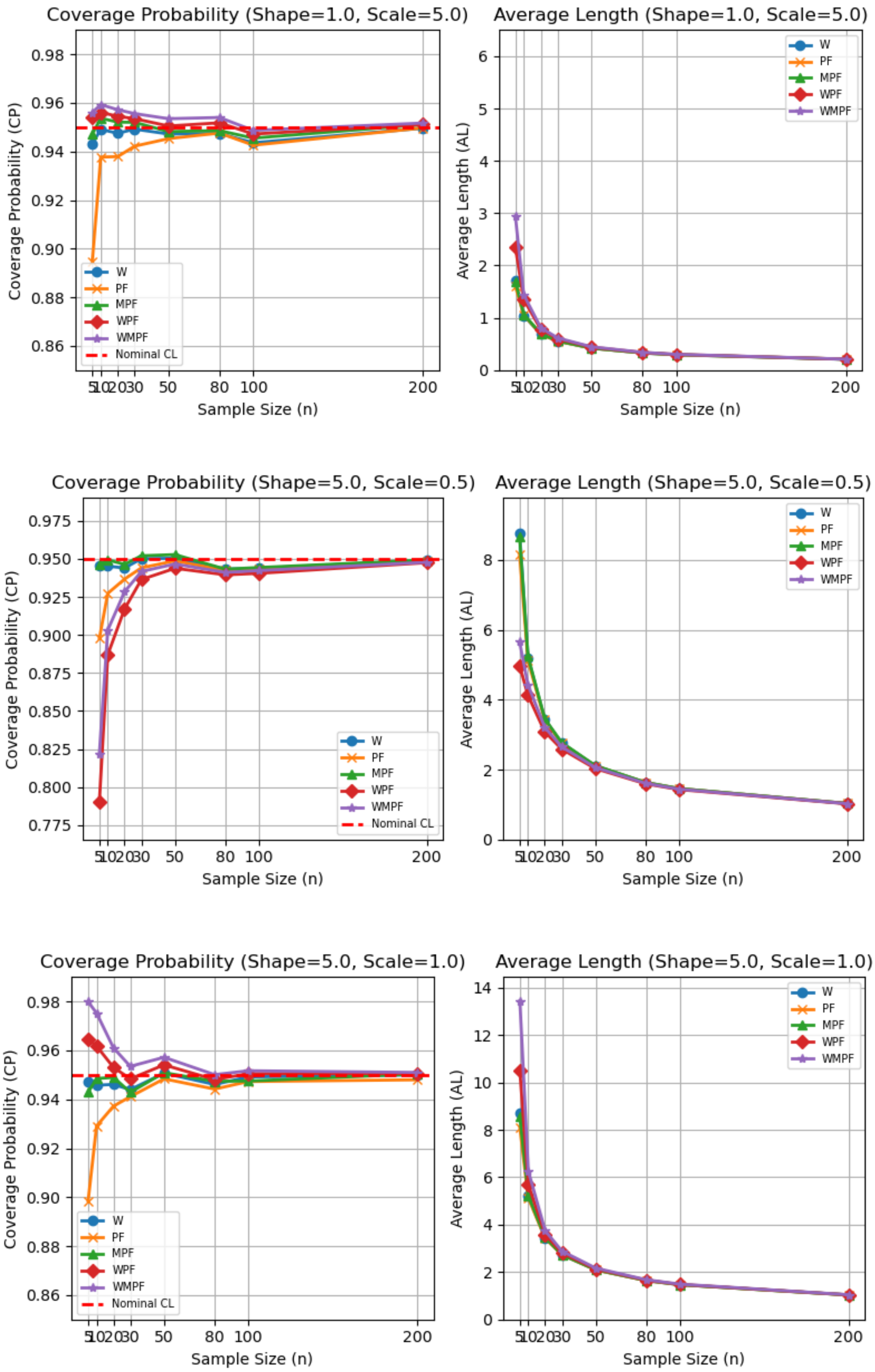

Figure 4. Coverage probability (left) and average length (right) for varying sample sizes under different shape and scale parameters of the Weibull distribution (Continued on next page).

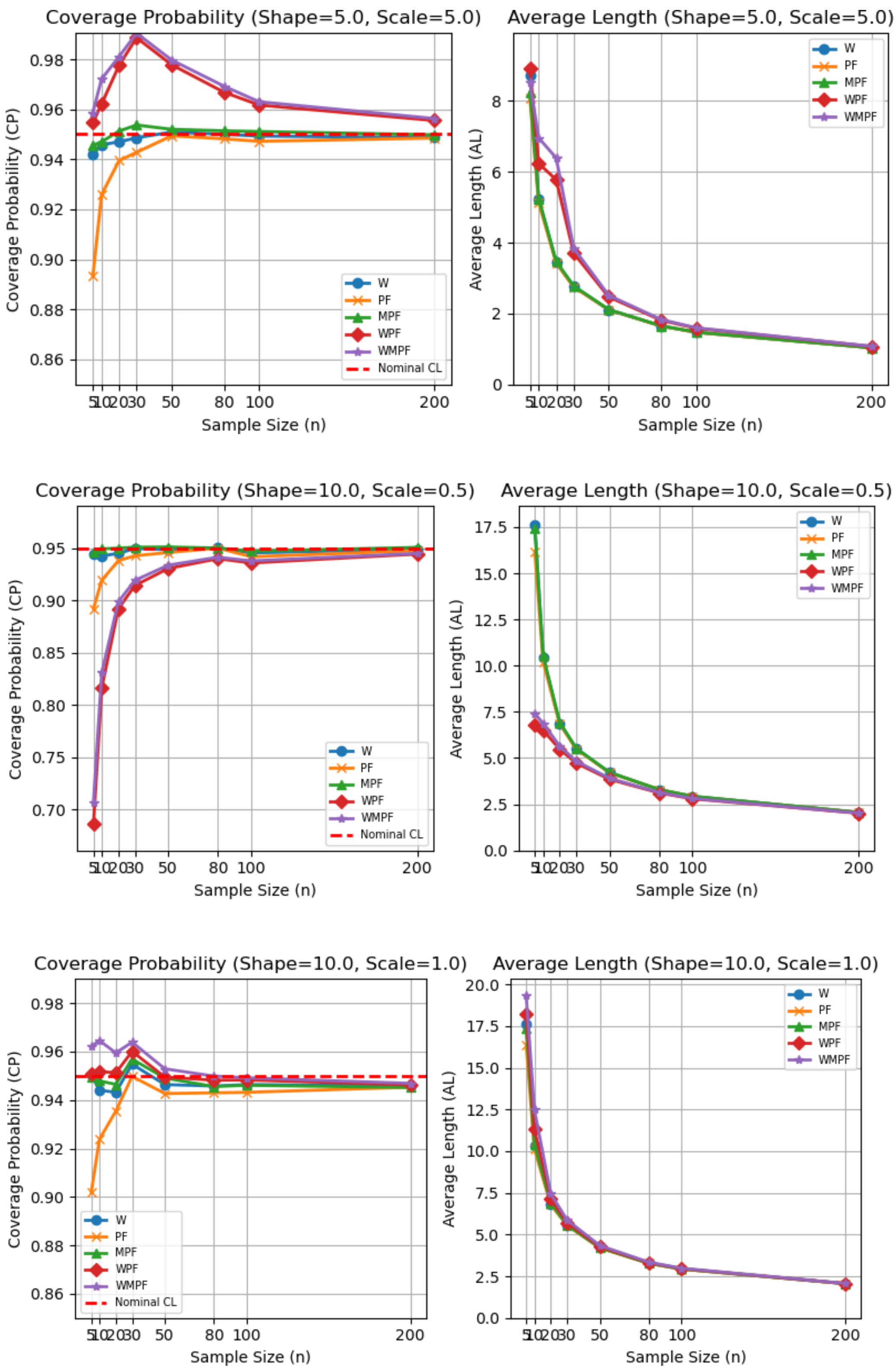

Figure 4. Coverage probability (left) and average length (right) for varying sample sizes under different shape and scale parameters of the Weibull distribution (Continued on next page).

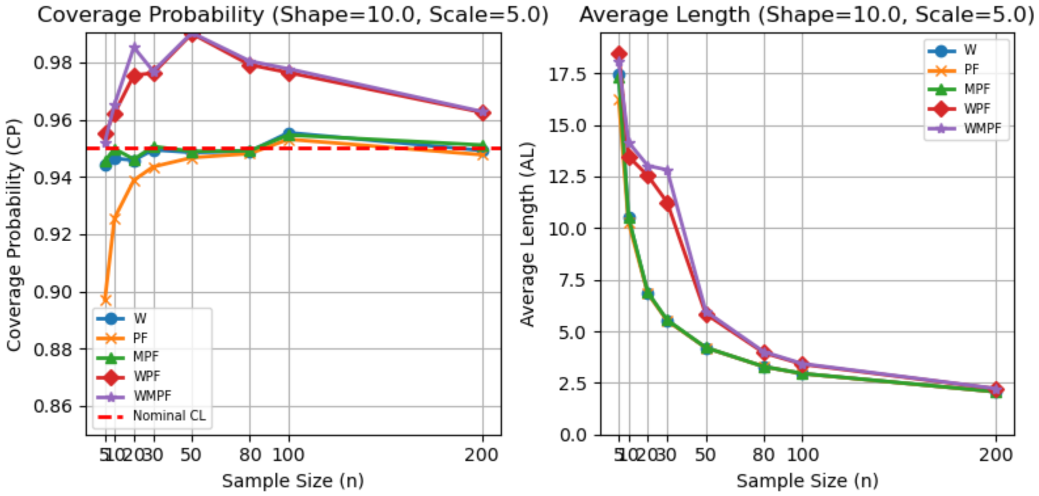

Figure 4. Coverage probability (left) and average length (right) for varying sample sizes under different shape and scale parameters of the Weibull distribution.

The shape parameter has a noticeable effect on coverage probability (CP) and average length (AL) of interval estimates, though its impact varies by method. When the scale and sample size are fixed, the shape parameter's influence on CP is minimal for the W, PF, and MPF methods. However, the shape parameter has a stronger effect on CP in the WPF and WMPF methods. For example, with a shape of 0.5 and a sample size of 5, the CPs for the PF method are 0.8971, 0.8925, 0.8976, and 0.8920 for shape values of 0.5, 1, 5, and 10, respectively. By contrast, the WPF method under these same conditions yields CPs of 0.9506, 0.9260, 0.7905, and 0.6858, showing a clear trend: as the shape parameter increases, CP decreases for both WPF and WMPF. This effect diminishes as sample sizes grow beyond 30, at which point the influence of the shape parameter on CP becomes negligible. In terms of AL, an increase in the shape parameter generally results in longer intervals across all methods.

The scale parameter, on the other hand, does not significantly affect CP when the shape parameter and sample size are held constant, especially at lower shape values. However, higher scale values tend to increase AL. For instance, with a shape of 0.5 and a sample size of 5 using the WMPF method, AL values are 1.1951, 1.3590, and 1.7530 for scale values of 0.5, 1, and 5, respectively. Once the sample size exceeds 30, the scale parameter's effect on both CP and AL becomes minimal.

Sample size itself has a substantial influence on both CP and AL. As sample size increases, CP generally stabilizes near the nominal level. For instance, as the sample size grows from 5 to 200, the CP for the W method levels out at the nominal confidence level of 0.95. Similarly, AL decreases as sample size increases, leading to narrower intervals, a trend that holds true across all methods.

When comparing interval methods, performance becomes largely similar when the sample size exceeds 30. For smaller sample sizes (30 or less), the proposed WPF and WMPF methods generally provide better coverage than the others. This advantage is consistent across most scenarios, except when shape values are high (5 or 10), scale is low (0.5), and sample size remains below 30. Under these specific conditions, the MPF and W intervals achieve nominal CP.

5. Application to real data

To demonstrate the applicability of the proposed interval estimation methods, data from Santiago and Smith [34] concerning hospital-acquired urinary tract infections (UTIs) was analyzed, as shown in Figure 5 (left panel). This dataset tracks 54 male patients who acquired UTIs during their hospital stay, providing essential insights into infection frequency over time—a metric critical for monitoring and intervention within healthcare settings.

The dataset includes time intervals (in days) between UTI cases, with an average interval of approximately 0.21 days, or about 5 hours. A goodness-of-fit test, conducted to assess the fit of the data to a Weibull distribution, yielded a high sample correlation of 0.9909 and a p-value of 0.6614, indicating that the Weibull distribution is an appropriate model [35,36].

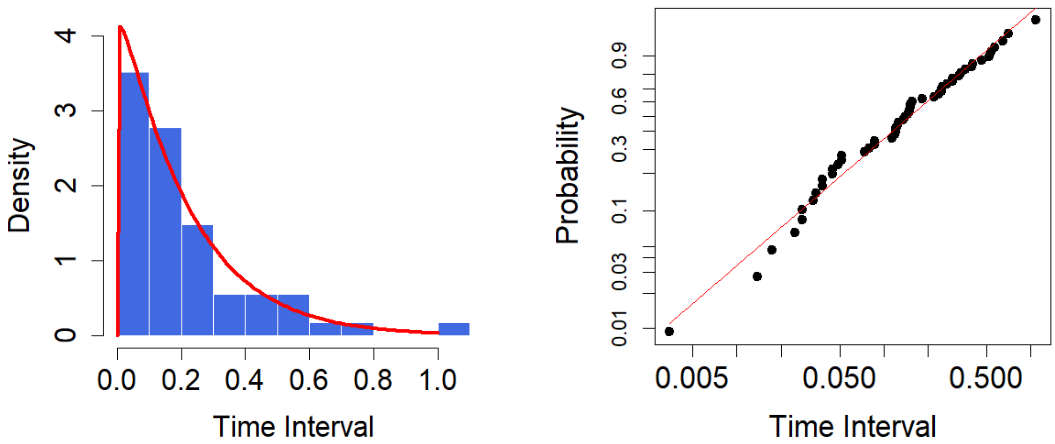

Figure 5. Histogram of the time intervals (in days) between UTI cases (left) and a quantile plot (right).

Maximizing the joint likelihood function in Eq (2) yielded an estimated shape parameter of approximately 1.0401 and a scale parameter of 0.2138. Based on these estimates, confidence intervals for the shape parameter were constructed using both the proposed methods and traditional approaches (W, PF, and MPF). The resulting interval estimates for each method are as follows: W method (0.8238, 1.2564), PF method (0.8380, 1.2631), MPF method (0.8163, 1.2394), WPF method (0.8251, 1.2551), and WMPF method (0.8226, 1.2576). The closest scenario in the simulation study to this data setup used a shape parameter of 1, a scale parameter of 0.5, and a sample size of 50. In this case, the simulation results showed that WMPF had the highest CP as well as AL, which aligns with the real data findings, where the AL for WMPF is 0.435—the largest among the intervals.

6. Conclusions and future work

We derived explicit formulas for Wald-type intervals using modified and non-modified profile likelihoods, providing a practical approach for interval estimation. Simulations showed that the WMPF method generally performs best, particularly for small sample sizes (under 30). Additionally, as the shape parameter increases, the CP of the proposed intervals slightly decreases, while changes in the scale parameter have minimal effect. For larger sample sizes (50 or more), all interval methods exhibit similar performance, confirming the reliability of the derived formulas across various settings. Furthermore, the real data application to hospital-acquired urinary tract infections demonstrated the practical value of these intervals in healthcare, supporting early infection monitoring and response.

Future research could extend these Weibull interval estimation methods to high-dimensional data contexts, which often involve multiple predictors or response variables. While we focused on a single response variable, high-dimensionality could arise in similar healthcare settings if multiple factors, such as patient demographics, treatment types, or environmental conditions, are considered alongside infection times. Recent advances, like those by Chaipitak and Choopradit [37] in high-dimensional covariance testing, suggest that adaptive approaches—such as dimension reduction or regularization—may help extend Weibull-based interval estimation to complex, multi-variable datasets. Such extensions could broaden the applicability of Weibull methods in fields requiring robust interval estimation across many variables, particularly in healthcare and engineering reliability.

Author contributions

P. Srisuradetchai: Conceptualization, formal analysis, investigation, methodology, validation, software, writing-original draft, writing–review and editing; J. Somsamai: Data curation, formal analysis, methodology, software, visualization; W. Phaphan: Formal analysis, investigation, project administration, resources, supervision, validation, writing – original draft. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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