AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(9): 26182–26194.
DOI: 10.3934/math.20241279
Received: 25 June 2024
Revised: 05 August 2024
Accepted: 20 August 2024
Published: 10 September 2024

Research article

Fuzzy collineations of 3-dimensional fuzzy projective space from 4-dimensional fuzzy vector space

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Abstract: In this paper, the fuzzy counterparts of the collineations defined in classical projective spaces are defined in a 3-dimensional fuzzy projective space derived from a 4-dimensional fuzzy vector space. The properties of fuzzy projective space $(\lambda, \mathcal{S})$ left invariant under the fuzzy collineations are characterized depending on the membership degrees of the given fuzzy projective space and also depending on the pointwise invariant of the lines. Moreover, some relations between membership degrees of the fuzzy projective space are presented according to which are of the base point, base line, and base plane invariant under a fuzzy collineation. Specifically, when all membership degrees of $(\lambda, \mathcal{S})$ are distinct, the base point, base line, and base plane of $(\lambda, \mathcal{S})$ are invariant under the fuzzy collineation $\bar{f}$. Conversely, if none of the base point, base line, or base plane remain invariant, then the system becomes crisp in $(\lambda, \mathcal{S})$. Additionally, some relations between the membership degrees of the fuzzy projective space, concerning the invariance of the base point, base line, and base plane, are presented.

Keywords: projective space; fuzzy projective space; collineation; isomorphism
Mathematics Subject Classification: 51A10, 51E15

1. Introduction

The fuzzy concept was first introduced by Zadeh in 1965 [22]. In fuzzy set theory, if the membership degree of an element x is $\mu(x)$, then the non-membership degree is $1 - \mu(x)$, and thus, it remains constant. This essentially means that certain elements of an object are assigned a membership degree (a number in the unit interval [0, 1]) as an alternative to the traditional black-and-white situation of belonging to or not belonging to. Fuzzy schemes are frameworks and techniques that make use of fuzzy logic and systems to deal with a variety of imprecise and uncertain challenges. Fuzzy logic, which manages imprecise or lacking data, finds use in various areas: Fuzzy rule-based systems, fuzzy control systems, fuzzy clustering, fuzzy decision-making, fuzzy optimization, fuzzy set theory and topology. Numerous disciplines, including statistics, design theory, industrial control, robotics, finance, coding theory, cryptography, network design, scheduling, communication theory, and computer science, can use these schemes because they are flexible instruments for handling uncertainty and complexity. For example, in fuzzy rule-based systems, [7, 8] papers explore advanced methods for synchronizing and controlling chaotic dynamical systems. They introduces a Takagi-Sugeno (T-S) fuzzy sliding mode control (TSFSMC) method to address input saturation and suppress undesired behaviors in fractional-order systems and also presents a T-S fuzzy model-free state feedback (MFSF) technique for stabilizing complex systems amid uncertainties and disturbances. Both approaches highlight the effectiveness of fuzzy logic in managing complex dynamical systems with improved performance and minimal chattering. Thanks to the classical link between geometry and statistics, Akca et al. define the fuzzy counterpart of a spread in fuzzy projective spaces in their paper [3], and they use this definition to address a more complex test group topic in statistics. Using combinatorial designs, spreads can express the required arrangements for addressing the problem, allowing for the shift from spreads to the Kirkman problem. Spreads essentially provide a structure for grouping and dividing sets so that every pair of components meet precisely once, satisfying the Kirkman problem's constraints. Thus, spreads are used to give systematic solutions to the combinatorial limitations of the Kirkman issue. It is possible to solve Kirkman's well-known problem using lines of PG(3, 2) [9]. Certain rules for dealing with this alternative approach have been established (see, for example, [22]), and the theory has been applied in many areas. Many mathematical theories have so-called fuzzy counterparts.

In fuzzy set theory, which is another subfield of fuzzy schemes, Azriel Rosenfeld published the first article on fuzzy groups in 1971 [21]. Katsaras and Liu proposed fuzzy vector spaces in 1977 [15]. Let V be any vector space over K with dimension $(n + 1)$. The set of all V subspaces that are different from the trivial subspaces 0 and V is then $PG(n, K)$, the n-dimensional projective space over K. The terms "points of $PG(n, K)$" refer to the 1-dimensional subspaces, "projective lines" to the 2-dimensional subspaces, and "projective planes" to the 3-dimensional subspaces. The concept of homogeneous coordinates for projective spaces follows from this: Any nonzero scalar multiple of the nonzero vector (x_1, x_2, x_3, x_4) that we use to represent a projective point is understood to represent the same projective point (the formalities concern vector equivalency classes). It is evident that the dimension decreases by one unit while moving from a vector space to the corresponding projective space. Thus, an $n-$dimensional projective space $PG(n, K)$ is generated from a $(n + 1)-$dimensional vector space V [13]. With a finite set and a well-defined representation now established, we can apply standard counting techniques to analyze and determine various properties of our space. There are $q^4 - 1$ nonzero vectors in the vector space, and each nonzero scalar multiple of a given nonzero vector corresponds to the same projective point. Therefore, the total number of points in the projective space $PG(3, q)$ can be calculated as $\frac{(q^4-1)}{(q-1)}$ which simplifies to q^3+q^2+q+1. Every line has $q+1$ points, as may be seen by counting the number of 1-dimensional subspaces that make up a 2-dimensional subspace. Examine now the case in which $q = 2$. In this case, there are 15 points in the finite projective space $PG(3, 2)$, and there are 3 points on each line. In [16], some characterizations of fuzzy vector lines, fuzzy vector planes, and fuzzy n-dimensional vector subspaces for $n > 2$ were given. In a projective plane, about the fuzzy projective points and fuzzy projective lines, some theorems were introduced [16]. In 1999, Kuijken et. al. derived a fuzzy group from such a fuzzy projective geometry, thereby establishing a relationship between fuzzy vector spaces and fuzzy groups through the use of these fuzzy projective geometries [17]. As a result of studies [16, 17], every fuzzy vector space generates a fuzzy projective space, and every fuzzy projective space generates a fuzzy group. The classifications of fuzzy vector planes, fuzzy projective lines, fuzzy lines, and fuzzy planes were introduced in a 3-dimensional fuzzy projective space derived from a 4-dimensional fuzzy vector space [5, 12].

Abdulhalikov introduced the fuzzy correspondings of vector space maps in 1996 [1]. After that in 1998, Abdulhalikov and Kim defined and investigated the properties of the concept of a fuzzy subspace of a space of fuzzy linear maps. For the finite-dimensional case, the fuzzy basis of a fuzzy subspace of fuzzy linear maps is discovered. Under natural conditions, the fuzzy subspace of fuzzy linear maps is isomorphic to the fuzzy subspace of dual maps [2]. Kuijken and Maldeghem described fuzzy projective plane collinations in 2003 [20]. The research [3] presents results on the Klein mapping's application to projective three-space of order 4 and investigates the fuzzification of the Klein quadric within a five-dimensional projective space [3]. In [6], the definitions of the fuzzy equivalents of collineations defined in classical projective planes and proofs of some properties that are invariant under the fuzzy collineations in fuzzy projective planes were introduced by Altintas and Bayar.

The reason fuzzy collineations are important is that they provide consistency in the face of imprecision by maintaining geometric relationships such as collinearity in fuzzy projective spaces. By taking into account fuzzy representations, they make it easier to model and analyze geometric transformations under uncertainty. Fuzzy collineations, which control ambiguity in visual data and spatial relationships, can improve object detection and picture analysis in computer vision. By accounting for errors in sensor data and environmental circumstances, they serve to improve the accuracy of navigation and manipulation tasks in robotics. Furthermore it is useful in modeling and comprehending data that is intrinsically ambiguous or partial are fuzzy collineations in geographic information systems (GIS) and spatial analysis. Fuzzy collineations allow practitioners to create more flexible and resilient solutions that are better able to tackle the complexity of real-world problems. Furthermore, fuzzy collineations bridge the gap between theory and real-world applications where absolute precision isn't always achievable by extending the concepts of classical geometry into fuzzy logic. In the literature, fuzzy collineations have been explored; however, the relationships among membership degrees and their implications for base points, base lines, and base planes have not been examined in detail. This study addresses this gap by providing a thorough analysis of these relationships under fuzzy collineations, offering new insights into how membership degrees interact and influence the geometric structures within fuzzy projective spaces. By investigating these relationships, our study contributes to a deeper understanding of the effects of fuzzy collineations on fundamental elements of fuzzy projective spaces. The findings may have implications for the broader application of fuzzy geometry in areas such as computer science, artificial intelligence, and theoretical mathematics, where fuzzy logic models are increasingly utilized. Thus, our research not only extends the theoretical framework but also potentially influences practical applications where understanding fuzzy structures is crucial.

It is our aim to contribute to this fuzzy theory by characterizing fuzzy collineations in a 3-dimensional fuzzy projective space derived from a 4-dimensional fuzzy vector space. Much more can and has been done. However, we will restrict the aims of this study according to some conditions as follows:

- To investigate the fuzzy counterparts of the basic properties provided by isomorphism in projective space.

• To study which one of them, the base point, base line, and base plane, invariant under the fuzzy collineation $\bar{f}$ depending on the membership degrees of the given fuzzy projective space $(\lambda, \mathcal{S})$.

• To give the relations between the membership degrees of the fuzzy projective space $(\lambda, \mathcal{S})$ according to the base point, base line, and base plane remaining invariant under the fuzzy collineation $\bar{f}$.

• To examine the properties that remain invariant under the fuzzy collineation $\bar{f}$ depending on the pointwise invariant of the lines in the fuzzy projective space $(\lambda, \mathcal{S})$.

The paper is organized as follows: After some definitions from both the fuzzy set theory and the fuzzy projective spaces (Section 2), in Section 3 we define the fuzzy counterparts of homomorphism and isomorphism defined in vector spaces in fuzzy projective spaces and apply theorems about properties of collineations in projective space to fuzzy projective space. And Section 4 contains some proofs about fuzzy collineations according to membership degree, and also the base point, the base line, and the base plane are invariant or not invariant under the fuzzy collineations. Section 5 contains a few final remarks.

2. Preliminaries

In this section, some relevant definitions of fuzzy set theory, fuzzy vector space, and fuzzy projective space are reminded. First recall that fuzzy sets were introduced by Zadeh in the fundamental paper [22].

Definition 2.1. *([22]) A fuzzy set λ of a set X is a function $\lambda : X \to [0,1] : x \to \lambda(x)$. The number $\lambda(x)$ is called the degree of membership of the point x in λ. The intersection $\lambda \wedge \mu$ of the two fuzzy sets λ and μ on X is given by the fuzzy set $\lambda \wedge \mu : X \to [0,1] : \lambda(x) \wedge \mu(x)$, where $\wedge$ denotes the minimum operator and also $\vee$ denotes the maximum operator.*

Definition 2.2. *[19] Consider a set X and fuzzy sets λ and μ on X. The Cartesian product $\lambda \times \mu$ of the two fuzzy sets is defined as follows:*

$$\begin{array}{rccc} \mu \times \lambda : & X \times X & \to & [0,1]\,, \\ & (x,y) & \to & \mu(x) \wedge \lambda(y). \end{array}$$

Definition 2.3. *[16] Let $\mu : V \to [0,1]$ be a fuzzy set on V. Then we call μ a fuzzy vector space on V if and only if $\mu(a.\bar{u} + b.\bar{v}) \geq \mu(\bar{u}) \wedge \mu(\bar{v}), \forall \bar{u}, \bar{v} \in V$ and $a, b \in K$.*

Definition 2.4. *[10] An (axiomatic) projective plane $\mathcal{P}$ is an incidence structure $(\mathcal{N}, \mathcal{D}, \circ)$ with $\mathcal{N}$ a set of points,$\mathcal{D}$ a set of lines, and $\circ$ an incidence relations, such that the following axioms are satisfied:*

i) *Every pair of distinct points is incident with a unique common line.*

ii) *Every pair of distinct lines is incident with a unique common point.*

iii) *$\mathcal{P}$ contains a set of four points with the property that no three of them are incident with a common line.*

A closed configuration $\mathcal{S}$ of $\mathcal{P}$ is a subset of $\mathcal{N} \cup \mathcal{D}$ that is closed under taking intersection points of any pair of lines in $\mathcal{S}$ and lines spanned by any pair of distinct points of $\mathcal{S}$. We denote the line in $\mathcal{P}$ spanned by the points p and q by $\langle p, q\rangle$.

Definition 2.5. *[14] An (axiomatic) projective space $\mathcal{S}$ is an incidence structure $(\mathcal{N}, \mathcal{D}, \circ)$ with $\mathcal{N}$ a set of points,$\mathcal{D}$ a set of lines, and $\circ$ an incidence relations, such that the following axioms are satisfied:*
i) *Every line is incident with at least two points.*
ii) *Every pair of distinct points is incident with a unique common line.*
iii) *Given distinct points p, q, r, s, t such that $\langle p, q\rangle = \langle p, r\rangle \neq \langle p, s\rangle = \langle p, t\rangle$, there is a point $x \circ \langle q, s\rangle \cap \langle r, t\rangle$ (Pasch's axiom).*

Definition 2.6. *[14] Suppose $\mathcal{P}$ is an $n-$dimensional projective space. A fuzzy set λ on the point set of $\mathcal{P}$ is a fuzzy $n-$dimensional projective space on $\mathcal{P}$ if $\lambda(p) \geq \lambda(q) \wedge \lambda(r)$, for all collinear pointsp, q, r of $\mathcal{P}$. We denote as $(\lambda, \mathcal{P})$. The projective space $\mathcal{P}$ is called the underlying (crisp) projective space of $(\lambda, \mathcal{P})$. If $\mathcal{P}$ is a fuzzy point, line, plane, etc., we use underlying point, underlying line, underlying plane, etc., respectively. We will sometimes briefly write λ instead of $(\lambda, \mathcal{P})$.*

In practice, this means that in the point set of a line, all elements have the same degree of membership, but may not be the same. Moreover, more generally speaking, this means that in any subspace U , all points have the same degree of membership, except that they may be in subspace $U^{'}$ of U. All points have the same degree of membership, except for those that may be in a subspace $U^{''}$ of $U^{'}$, etc. [4].

Definition 2.7. *[4] Let $(\lambda, \mathcal{P})$ be a fuzzy projective space and let U be a subspace of $\mathcal{P}$. Then (λ_U, U) is called a fuzzy subspace of $(\lambda, \mathcal{P})$ if $\lambda_U(x) \leq \lambda(x)$ for $x \in U$, and $\lambda_U(x) = 0$ for $x \notin U$.*

As already alluded to above, the following proposition gives the structure of a fuzzy projective line.

Proposition 2.8. *[4] Let (λ, L) be a fuzzy projective line. Then there are constants $a, b \in]0, 1], a \leq b$, and a point z of L such that*
i) $\lambda(z) = b$,
ii) $\lambda(x) = a$, *for all* $x \neq z$.

By the previous proposition, every fuzzy projective line admitting points with different membership degrees contains a unique point with a maximal membership degree. We will refer to such a point as the base point of the fuzzy line. More generally, the structure of a fuzzy projective space looks as follows [16].

Definition 2.9. *[4] Let $(\lambda, \mathcal{P})$ be a fuzzy projective space of dimension n. Then there are constants $a_i \in]0, 1]$, $i = 0, 1, ..., n$, with $a_i \leq a_{i+1}$, and a chain of subspaces $(U_i)_{0\leq i\leq n}$ with $U_i \leq U_{i+1}$ and $dimU_i = i$, such that*

$$\begin{aligned} \lambda: \ \mathcal{P} &\rightarrow [0,1], \\ x &\rightarrow a_0 \qquad for\ x \in U_0, \\ x &\rightarrow a_i \qquad for\ x \in U_i \setminus U_{i-1}, \quad i = 1, 2, ..., n. \end{aligned}$$

Definition 2.10. *Consider the projective space $\mathcal{P} = (\mathcal{N}, \mathcal{D}, \circ)$. Suppose $p \in \mathcal{N}$ and $\alpha \in [0,1]$. The fuzzy point (p,α) is the following fuzzy set on the point set $\mathcal{N}$ of $\mathcal{P}$:*

$$\begin{array}{rccll} (p,\alpha): & \mathcal{N} & \to & [0,1], & \\ & p & \to & \alpha, & \\ & x & \to & 0 & if\ x \in \mathcal{N} \setminus \{p\}. \end{array}$$

The point p is called the base point of the fuzzy point (p,α).

A fuzzy line (L,β) with base line L is defined in a similar way.

Two fuzzy lines (L,α) and (M,β), with $\alpha \wedge \beta > 0$, intersect in the unique fuzzy point $(L \cap M, \alpha \wedge \beta)$. Dually, the fuzzy points (p,λ) and (q,μ), with $\lambda \wedge \mu > 0$, span the unique fuzzy line $(\langle p,q \rangle, \lambda \wedge \mu)$.

Definition 2.11. *[18] Suppose $(\lambda, \mathcal{P})$ is a projective space $(\mathcal{N}, \mathcal{D}, \circ)$. The fuzzy set λ on $\mathcal{N} \cup \mathcal{D}$ is a fuzzy projective space on $(\lambda, \mathcal{P})$ if*
i) *$\lambda(L) \geq \lambda(p) \wedge \lambda(q), \forall p, q : \langle p, q \rangle = L$ and*
ii) *$\lambda(p) \geq \lambda(L) \wedge \lambda(M), \forall L, M : L \cap M = p$.*

Definition 2.12. *[1] Let E and L be vector spaces over the same field F, and let $\mu : E \to [0,1]$, $\lambda : L \to [0,1]$ be fuzzy subspaces. If $\lambda(\varphi(x)) \geq \mu(x)$ for all $x \in E$, we say that a linear map $\varphi : E \to L$ is fuzzy linear from the fuzzy subspace μ to fuzzy subspace λ. The space of fuzzy linear maps from μ to λ is denoted by $FHom(\mu, \lambda)$.*

Definition 2.13. *[1] Let (E_1, μ_1) and (E_2, μ_2) are two fuzzy vector spaces. If there exists an isomorphism $\varphi : E_1 \to E_2$ with the property $\mu_1(x) = \mu_2(\varphi(x))$ for all $x \in E_1$, $\mu_1 : E_1 \to [0,1]$ and $\mu_2 : E_2 \to [0,1]$ are isomorphic.*

3. Collineations of fuzzy projective spaces

In the present section, collineations of fuzzy projective spaces were investigated. Compared to isomorphisms, collineations of projective space have the advantages and properties. In projective spaces, a collineation is a point-to-point and line-to-line transformation that preserves the relation of incidence. Thus, it transforms ranges into ranges, pencils into pencils, quadrangles into quadrangles, and so on. Clearly, it is a self-dual concept, the inverse of a collineation, and the product of two collineations is again a collineation [11]. Our aim is now to define the fuzzy counterparts of homomorphism and isomorphism defined in vector spaces in fuzzy projective spaces and to apply theorems about properties of collineations in projective space to fuzzy projective space. Furthermore, it will be shown that each collineation can be uniquely extended to a fuzzy projective collineation.

The definitions of homomorphism, isomorphism, and collineation in projective spaces can be adopted to fuzzy projective spaces as follows:

Definition 3.1. *Let $(\lambda, \mathcal{S})$ and $(\mu, \mathcal{S}')$ be two fuzzy projective spaces with base spaces $\mathcal{S}$, $\mathcal{S}'$ respectively. Suppose that f is a homomorphism of projective space $\mathcal{S}$ into projective space $\mathcal{S}'$. $\bar{f}$ is called as fuzzy homomorphism from $(\lambda, \mathcal{S})$ into $(\mu, \mathcal{S}')$, if $\bar{f}(p,\alpha) = (f(p),\beta)$ for all $(p,\alpha) \in (\lambda, \mathcal{S})$ where $\lambda(p) = \alpha$, $\mu(f(p)) = \beta$ and $\alpha \leq \beta$. If f is an isomorphism of $\mathcal{S}$ into $\mathcal{S}'$ and $\alpha = \beta$, then $\bar{f}$ is called a fuzzy isomorphism between the fuzzy projective spaces $(\lambda, \mathcal{S})$ and $(\mu, \mathcal{S}')$. Also, if $\mathcal{S} = \mathcal{S}'$, $\bar{f}$ fuzzy isomorphism is called a fuzzy collineation between projective spaces $(\lambda, \mathcal{S})$ and $(\mu, \mathcal{S}')$.*

Theorem 3.2. *Let $\bar{f} : (\lambda, \mathcal{S}) \to (\mu, \mathcal{S}')$ be a fuzzy isomorphism, then the followings hold:* ***(i)*** *For any pair of fuzzy points (p_1, α_1) and (p_2, α_2), $p_1 \neq p_2$ in $(\lambda, \mathcal{S})$,*

$$\bar{f}(\langle (p_1, \alpha_1), (p_2, \alpha_2) \rangle) = \left\langle \bar{f}(p_1, \alpha_1), \bar{f}(p_2, \alpha_2) \right\rangle .$$

(ii) *For any pair of fuzzy lines (L, β_1) and (M, β_2), $L \neq M$ in the same fuzzy projective plane of $(\lambda, \mathcal{S})$,*

$$\bar{f}((L, \beta_1) \cap (M, \beta_2)) = \bar{f}(L, \beta_1) \cap \bar{f}(M, \beta_2).$$

(iii) *For any fuzzy point (p, α) and fuzzy line (L, β) in the same fuzzy projective plane of $(\lambda, \mathcal{S})$, if p is not on L, then the fuzzy point $\bar{f}(p, \alpha)$ is not on $\bar{f}(L, \beta)$ in the same fuzzy projective plane of $(\mu, \mathcal{S}')$.*

Proof. **(i)** Let $\bar{f}$ be a fuzzy isomorphism between $(\lambda, \mathcal{S})$ and $(\mu, \mathcal{S}')$. The fuzzy line spanned by the fuzzy points (p_1, α_1) and (p_2, α_2) with distinct base points p_1, p_2 is $\langle (p_1, \alpha_1), (p_2, \alpha_2) \rangle = (\langle p_1, p_2 \rangle, \alpha_1 \wedge \alpha_2)$. Since f is an isomorphism between the base projective spaces $\mathcal{S}$ and $\mathcal{S}'$, $f(p_1) \neq f(p_2)$. So $\bar{f}(p_1, \alpha_1) \neq \bar{f}(p_2, \alpha_2)$. Using the definitions of $\bar{f}$ and f,

$$\bar{f}(\langle (p_1, \alpha_1), (p_2, \alpha_2) \rangle) = \langle (f(p_1), \alpha_1), (f(p_2), \alpha_2) \rangle = \left\langle \bar{f}(p_1, \alpha_1), \bar{f}(p_2, \alpha_2) \right\rangle,$$

is obtained.

(ii) Let $\bar{f}$ be a fuzzy isomorphism between $(\lambda, \mathcal{S})$ and $(\mu, \mathcal{S}')$. The intersection point of the fuzzy lines (L, β_1) and (M, β_2) with distinct base lines L, M is $(L, \beta_1) \cap (M, \beta_2) = (L \cap M, \beta_1 \wedge \beta_2)$. Since f is an isomorphism between the projective spaces $\mathcal{S}$ and $\mathcal{S}'$, $f(L) \neq f(M)$. So $\bar{f}(L, \beta_1) \neq \bar{f}(M, \beta_2)$. Using the definition of $\bar{f}$ and f

$$\bar{f}((L, \beta_1) \cap (M, \beta_2)) = (f(L \cap M), \beta_1 \wedge \beta_2) = ((f(L), \beta_1) \cap (f(M), \beta_2)) = \bar{f}(L, \beta_1) \cap \bar{f}(M, \beta_2).$$

(iii) Suppose that the fuzzy point $\bar{f}((p, \alpha))$ is on the fuzzy line $\bar{f}((L, \beta))$ when the base point p is not on the base line L. Then the fuzzy point (p, α) is not on the fuzzy line (L, β). From definitions of f and $\bar{f}$, $\bar{f}((p, \alpha)) = (f(p), \alpha)$ and $\bar{f}((L, \beta)) = (f(L), \beta)$. Since the fuzzy point $\bar{f}((p, \alpha))$ is on the fuzzy line $\bar{f}((L, \beta))$ and f is isomorphism, $f(p) \circ f(L)$ and $p \circ L$ are obtained. This contradicts the hypothesis. □

Theorem 3.3. *Suppose that $\bar{f}$ is a fuzzy collineation of $(\lambda, \mathcal{S})$ defined by the collineation f of the base space $\mathcal{S}$. If two distinct points p_1 and p_2 in the base space $\mathcal{S}$ are invariant under the collineation f of $\mathcal{S}$, the fuzzy line spanned by fuzzy points (p_1, α_1) and (p_2, α_2) is invariant under the fuzzy collineation $\bar{f}$ of $(\lambda, \mathcal{S})$.*

Proof. Let the points p_1 and p_2 in $\mathcal{S}$ of (p_1, α_1) and (p_2, α_2) in $(\lambda, \mathcal{S})$ be invariant under the collineation f of $\mathcal{S}$. Then, by the definition of fuzzy collineation $\bar{f}$ in $(\lambda, \mathcal{S})$, $\bar{f}(p_1, \alpha_1) = (f(p_1), \alpha_1) = (p_1, \alpha_1)$ and $\bar{f}(p_2, \alpha_2) = (f(p_2), \alpha_2) = (p_2, \alpha_2)$.

For any pair $((p_1, \alpha_1), (p_2, \alpha_2))$ of fuzzy points, $p_1 \neq p_2$, the fuzzy line $(\langle p_1, p_2 \rangle, \alpha_1 \wedge \alpha_2)$ spanned by them, also belongs to the fuzzy projective space $(\lambda, \mathcal{S})$. By using the definition of $\bar{f}$ of $(\lambda, \mathcal{S})$ and the remaining invariant of the points p_1 and p_2 under the collineation f in $\mathcal{S}$, the image of the fuzzy line $(\langle (p_1, \alpha_1), (p_2, \alpha_2) \rangle)$ under fuzzy collineation $\bar{f}$ is

$$(f(\langle p_1, p_2 \rangle), \alpha_1 \wedge \alpha_2) = (\langle f(p_1), f(p_2) \rangle, \alpha_1 \wedge \alpha_2) = (\langle p_1, p_2 \rangle, \alpha_1 \wedge \alpha_2).$$

Hence, the fuzzy line $(\langle p_1, p_2 \rangle, \alpha_1 \wedge \alpha_2)$ is invariant under the fuzzy collineation $\bar{f}$. □

3.1. Some relationships of collineations of fuzzy projective spaces with membership degrees

In this subsection, after giving general theorems about collineations, their relations with membership degrees are analyzed.

From now on, we considered the fuzzy projective space $(\lambda, \mathcal{S})$ with the base plane $\mathcal{P}$ and λ in the following form:

$$\begin{array}{rcll} \lambda: \ \mathcal{S} & \rightarrow & [0,1], & \\ q & \rightarrow & a_0, & \\ p & \rightarrow & a_1, & p \in L \setminus \{q\}, \\ p & \rightarrow & a_2, & p \in \mathcal{P} \setminus \{L\}, \\ p & \rightarrow & a_3, & p \in \mathcal{S} \setminus \mathcal{P}, \end{array}$$

where L is the base line of $\mathcal{S}$ containing q and $\mathcal{P}$ is the base projective plane containing L with $a_0 \geq a_1 \geq a_2 \geq a_3$, $a_i \in [0,1]$, $i = 0,1,2,3$ [18].

The projective space $\mathcal{S}$ is called the base projective space of $(\lambda, \mathcal{S})$. If $\mathcal{S}$ is a fuzzy point, a fuzzy line, a fuzzy plane, ..., we use the base point, the base line, the base plane, ..., respectively. The invariant properties under any fuzzy collineation in $(\lambda, \mathcal{S})$ depending on the base point, the base line, the base plane, and the membership degrees of $(\lambda, \mathcal{S})$ are investigated in detail with the following theorems.

Theorem 3.4. *Suppose that $\bar{f}$ is a fuzzy collineation of $(\lambda, \mathcal{S})$ defined by the collineation f of the base space $\mathcal{S}$.*

(i) *If $a_0 \neq a_1 \neq a_2 \neq a_3$, then then the fuzzy collineation $\bar{f}$ leaves invariant the base point, the base line, and the base plane of $(\lambda, \mathcal{S})$.*

(ii) *If $a_1 \neq a_2$, then the base line is invariant.*

(iii) *If $a_2 \neq a_3$, then the base plane is invariant.*

(iv) *If $a_0 \neq a_1 = a_2$, $a_2 \neq a_3$, then the base point is invariant, and the base line turns into a line passing through the base point under the fuzzy collineation $\bar{f}$.*

Proof. **(i)** The membership degrees of the points in the base plane can not be a_3, so $a_0 \neq a_3$, $a_1 \neq a_3$, $a_2 \neq a_3$. The membership degrees of the image of points on the base plane can be a_0, a_1, a_2. Since $a_2 \neq a_3$, the membership degree of the image point can not be a_3. In this case, the image of the point is in the base plane. The plane spanned by these points again becomes the base plane. These points' images are again the points of the base plane. So the base plane is invariant.

(ii) Since $\bar{f}$ is fuzzy isomorphism in $(\lambda, \mathcal{S})$, $\bar{f}(q, a_0) \circ \bar{f}(L, a_1)$. Hence, the base point (q, a_0) is on the base line $(f(L), a_1)$. The points on the base line have a_1 and a_2 membership degrees. The base line consists of points with a_1 and a_2 membership degrees. Under the fuzzy collineation $\bar{f}$, the images of these points have a_0 and a_1 membership degrees. Since $a_1 \neq a_2 \neq a_3$, the image points must be on the base line. So the base line $f(L) = \langle q, f(p) \rangle$ is invariant.

(iii) The points on the base plane have a_1, a_2, or a_3 membership degrees. From the definition of $\bar{f}$, the image points of these points have the same membership degrees under the fuzzy collineation $\bar{f}$. Since $a_2 \neq a_3$, these points can not turn into points not on the base plane. Therefore, the base plane is invariant under the fuzzy collineation $\bar{f}$.

(iv) Since $a_0 \neq a_1$ and also from i), the base point is invariant. Since $a_1 = a_2$, the point on the base line other than the base point turns into a point in the base plane and does not turn into a point out of the base plane, because of $a_2 \neq a_3$. So the base line turns into a line through the base point in the base projective plane. □

The following theorem states the properties of $\bar{f}$ fuzzy collineation while the base point is invariant.

Theorem 3.5. *Suppose that $\bar{f}$ is a fuzzy collineation of $(\lambda, \mathcal{S})$ defined by the collineation f of the base space $\mathcal{S}$ and the base point (q, a_0) is invariant under the fuzzy collineation $\bar{f}$.*

***(i)** If the base line (L, a_1) is invariant under $\bar{f}$, $(\lambda, \mathcal{S})$ has at most four membership degrees.*

***(ii)** If the base line turns into a line in the base plane, there are at most three membership degrees in $(\lambda, \mathcal{S})$.*

***(iii)** If the base line turns into a line not in the base plane, there are at most two membership degrees in $(\lambda, \mathcal{S})$.*

Proof. **(i)** Let the base point (q, a_0) and the base line (L, a_1) be invariant under the fuzzy collineation $\bar{f}$. Then $\bar{f}(q, a_0) = (q, a_0)$. The image point $\bar{f}(p, a_1)$ of the fuzzy point (p, a_1) on the base line (L, a_1) is $(f(p), a_1)$ and on the base line L.

If $a_0 \neq a_1 \neq a_2 \neq a_3$ are taken, there are at most four membership degrees in $(\lambda, \mathcal{S})$.

(ii) Let the base point (q, a_0) be invariant, and the base line (L, a_1) turns into a line in the base plane under the fuzzy collineation $\bar{f}$. Since the base line turns into a line in the base projective plane and the membership degrees of the points on the base line are a_0 or a_1, then a_1 must be equal to a_2. From the relationship between membership degrees $a_0 \geq a_1 \geq a_2 \geq a_3$, there are at most three membership degrees.

(iii) Let the base point (q, a_0) be invariant, and the base line (L, a_1) turns into a line not in the base plane under the fuzzy collineation $\bar{f}$. The image point $\bar{f}(p, a_1)$ of the fuzzy point (p, a_1) on the base line (L, a_1) has a_0 or a_1 fuzzy membership degree. Since the image line $\bar{f}(L, a_1)$ of the base line (L, a_1) is not in the base plane, a_1 must be equal to a_3. From the relationship between membership degrees $a_0 \geq a_1 \geq a_2 \geq a_3$, $a_1 = a_2 = a_3$. Therefore, there are at most two membership degrees. □

As an numerical example, we consider the projective space $PG(3, 2)$ with 15 points. Let us take the projective points of $PG(3, 2)$ and represent them as: $P_1 = (0, 0, 0, 1)$, $P_2 = (0, 0, 1, 0)$, $P_3 = (0, 0, 1, 1)$, $P_4 = (0, 1, 0, 0)$, $P_5 = (0, 1, 0, 1)$, $P_6 = (0, 1, 1, 0)$, $P_7 = (0, 1, 1, 1)$, $P_8 = (1, 0, 0, 0)$, $P_9 = (1, 0, 0, 1)$, $P_{10} = (1, 0, 1, 0)$, $P_{11} = (1, 0, 1, 1)$, $P_{12} = (1, 1, 0, 0)$, $P_{13} = (1, 1, 0, 1)$, $P_{14} = (1, 1, 1, 0)$, and $P_{15} = (1, 1, 1, 1)$. If we take the base point as $P_1 = (0, 0, 0, 1)$, the base line L constructing with the fuzzy points P_1, P_2, P_3, and, the base plane as $\{P_1, P_2, P_3, P_4, P_5, P_6, P_7\}$, then a chain of subspaces $U_0 = \{P_1\}, U_1 = \{P_1, P_2, P_3\}, U_2 = \{P_1, P_2, P_3, P_4, P_5, P_6, P_7\}, U_3 = \{P_8, P_9, P_{10}, P_{11}, P_{12}, P_{13}, P_{14}, P_{15}\}$ are obtained. From Definition 2.9 and Theorem 3.5, the base point (P_1, a_0) is invariant under the fuzzy collineation $\bar{f}$. If we apply ii) of Theorem 3.5., since the base line turns into a line in the base plane, then the fuzzy point (P_2, a_1) on the base line turns into a fuzzy point with membership degree a_2 not on the base line. From the definition of fuzzy collineation, a_1 must be equal to a_2, and also due to the relationship between the membership degrees $a_0 \geq a_1 \geq a_2 \geq a_3$, there are at most three membership degrees in $(\lambda, \mathcal{S})$. Thus, in this situation, the membership degree of the six points $\{P_2, P_3, P_4, P_5, P_6, P_7\}$ on the base plane under the fuzzy collineation $\bar{f}$ is equal.

Theorem 3.6. *Suppose that $\bar{f}$ is a fuzzy collineation of $(\lambda, \mathcal{S})$ defined by the collineation f of the base space $\mathcal{S}$ and the base point (q, a_0) is invariant under the fuzzy collineation $\bar{f}$.*

(i) *If the base plane turns into a plane other than itself passing through the base line, there are at most three membership degrees in* $(\lambda, \mathcal{S})$*.*

(ii) *If the base plane turns into a plane other than itself not containing the base line, there are at most two membership degrees in* $(\lambda, \mathcal{S})$*.*

(iii) *If the base plane is invariant, there are at most four membership degrees in* $(\lambda, \mathcal{S})$*.*

Proof. **(i)** Let the base point (q, a_0) be invariant, and the base plane $(\mathcal{P}, a_2)$ turns into a plane other than itself, passing through the base line (L, a_1) under the fuzzy collineation $\bar{f}$. The membership degrees of the image points of the points in the base plane $(\mathcal{P}, a_2)$ can be a_1 or a_2. Since the base plane $(\mathcal{P}, a_2)$ turns into a plane passing through the base line (L, a_1) under the fuzzy collineation $\bar{f}$, the membership degrees of the image points of the points not on the base line (L, a_1) can be a_2 or a_3. So $a_0 \geq a_1 \geq a_2 = a_3$ and there are at most three membership degrees in $(\lambda, \mathcal{S})$.

(ii) Let the base point (q, a_0) be invariant, and the base plane $(\mathcal{P}, a_2)$ turns into a plane other than itself, not containing the base line (L, a_1) under the fuzzy collineation $\bar{f}$. So a_1 must be equal to a_3. From the relationship between membership degrees $a_0 \geq a_1 \geq a_2 \geq a_3$, $a_0 \geq a_1 = a_2 = a_3$ is obtained. So there are at most two membership degrees in $(\lambda, \mathcal{S})$.

(iii) Let the base point (q, a_0) and the base plane $(\mathcal{P}, a_2)$ be invariant under the fuzzy collineation $\bar{f}$. The membership degrees of the images of the points in the base plane are a_0, a_1, or a_2. From the relationship between membership degrees, there are at most four membership degrees in $(\lambda, \mathcal{S})$. □

Theorem 3.7. *Suppose that* $\bar{f}$ *is a fuzzy collineation of* $(\lambda, \mathcal{S})$ *defined by the collineation* f *of the base space* $\mathcal{S}$ *and the base point is not invariant under the fuzzy collineation* $\bar{f}$*.*

(i) *If the base line is invariant and the base point turns into a point on the base line under the fuzzy collineation* $\bar{f}$ *of* $(\lambda, \mathcal{S})$*, among the membership degrees* a_i, $i = 0, 1, 2, 3$*, there is a relationship* $a_0 = a_1 \geq a_2 \geq a_3$*.*

(ii) *If the base point turns into a fuzzy point not on the base line in the base plane under the fuzzy collineation* $\bar{f}$*, then there are two membership degrees in* $(\lambda, \mathcal{S})$ *such that* $a_0 = a_1 = a_2 \geq a_3$*.*

(iii) *If the base point of* (q, a_0) *turns into a fuzzy point not in the base plane under the collineation* f *in* $\mathcal{S}$*, then there is only one membership degree in* $(\lambda, \mathcal{S})$*.*

Proof. **(i)** Let the base point be not invariant and turns into another point on the base line (L, a_1), then $\bar{f}(q, a_0) = (f(q), a_1)$, $f(q) \neq q$.

Let the base line (L, a_1) be invariant under the fuzzy collineation $\bar{f}$. Then the fuzzy point (q, a_0) turns into the fuzzy point (p, a_1) with $p \circ L$, $q \neq p$. Since $\bar{f}$ is fuzzy isomorphism, $a_0 = a_1$. Hence, there are at most three membership degrees in $(\lambda, \mathcal{S})$.

(ii) Let the base point (q, a_0) turns into a fuzzy point not on the base line (L, a_1) in the base plane under the fuzzy collineation $\bar{f}$. Since the image point of the base point has a_2 membership degree, $a_0 = a_2$ and $a_0 = a_1 = a_2 \geq a_3$ are obtained. So there are at most two membership degrees in $(\lambda, \mathcal{S})$.

(iii) Let the base point (q, a_0) turns into a fuzzy point not in the base plane under the fuzzy collineation $\bar{f}$. So, the image point of the base point has a_3 membership degree, and $a_0 = a_3$ is obtained under the fuzzy collineation $\bar{f}$. From the relationship between membership degrees, $a_0 = a_1 = a_2 = a_3$ is obtained; there is only one membership degree. □

Theorem 3.8. *Suppose that $\bar{f}$ is a fuzzy collineation of $(\lambda, \mathcal{S})$ defined by the collineation f of the base space $\mathcal{P}$. In this case,*

(i) *If M is a pointwise invariant line under the collineation f in the base projective plane $\mathcal{P}$, then the corresponding fuzzy line (M,β) is also pointwise invariant under the fuzzy collineation $\bar{f}$ in $(\lambda, \mathcal{S})$.*

(ii) *If two distinct lines L_1 and L_2 are pointwise invariant under the collineation f of the underlying space $\mathcal{S}$, then the intersection point of the fuzzy lines (L_1, α_1) and (L_2, α_2) is invariant under the fuzzy collineation $\bar{f}$.*

(iii) *If two distinct lines L_1 and L_2, $L_1 \neq L_2$ are pointwise invariant lines under the collineation f of the underlying space $\mathcal{S}$, then the fuzzy plane which is spanned by $\langle (L_1, \alpha_1), (L_2, \alpha_2) \rangle$ is invariant under the fuzzy collineation $\bar{f}$.*

(iv) *If three distinct lines not in the same plane are pointwise invariant under the fuzzy collineation $\bar{f}$, then $\bar{f}$ is unit collineation.*

Proof. **(i)** Let the base line M of the fuzzy line (M,β) in $(\lambda, \mathcal{S})$ be pointwise invariant under the collineation f of $\mathcal{S}$. For every fuzzy point (p, α) on (M,β), from the definition of $\bar{f}$ and being pointwise invariant of M under collineation, f, $\bar{f}(p, \alpha) = (f(p), \alpha) = (p, \alpha)$. Hence the fuzzy line (M,β) is pointwise invariant in $(\lambda, \mathcal{S})$.

(ii) Let the distinct lines L_1 and L_2 be pointwise invariant under the collineation f of the base space $\mathcal{S}$, respectively. From (i), the fuzzy lines (L_1, α_1) and (L_2, α_2) are pointwise invariant under the fuzzy collineation $\bar{f}$ of $(\lambda, \mathcal{S})$. Since (L_i, α_i) are pointwise invariant and (p, α) is on (L_i, α_i), $i = 1, 2$, hence the intersection point of (L_1, α_1) and (L_2, α_2) is invariant.

Let the distinct lines L_1 and L_2 be pointwise invariant under the collineation f of the underlying space $\mathcal{S}$. It is well known that, if two distinct lines are pointwise invariant in a projective plane, all points of the projective plane are invariant [6]. In addition, from i), the images of these lines are pointwise invariant under the fuzzy collineation $\bar{f}$ of $(\lambda, \mathcal{S})$. Every point in the fuzzy projective plane spanned by these fuzzy lines turns into again any point in this fuzzy projective plane, and also the fuzzy plane, which is spanned by $\langle (L_1, \alpha_1), (L_2, \alpha_2) \rangle$ is invariant under the fuzzy collineation $\bar{f}$.

Let three distinct lines not in the same plane be pointwise invariant under the fuzzy collineation $\bar{f}$. Since all points are in the base plane under f collineation and the image of any fuzzy point (p, α) is $\bar{f}(p, \alpha) = (f(p), \alpha)$, $\bar{f}$ is unit collineation.

□

4. Conclusions

In this study, fuzzy collineations in 3-dimensional fuzzy projective space derived from 4-dimensional fuzzy vector space were examined. Comparing with the well-known theorems about isomorphisms and collineations in projective space, the fuzzy counterparts of these properties were introduced. The properties of the fuzzy collineation $\bar{f}$ depending on the pointwise invariant of any line of $(\lambda, \mathcal{S})$ are analyzed. It was observed that there is a close relationship between the membership degrees and the invariance of the base point, base line, and base plane. When all membership degrees of $(\lambda, \mathcal{S})$ are different from each other, the base point, the base line, and the base plane are invariant under the fuzzy collineation $\bar{f}$. If none of the base point, base line, and base plane remains invariant, then the crisp occurs in $(\lambda, \mathcal{S})$. Analyzing the basic properties of projective spaces under these collineations led to several significant findings. These observations contribute to our understanding of how the base point, the base line, and base plane are invariant under the fuzzy collineations and also how they change their membership degrees according to some conditions in projective spaces. Consequently, it is thought that the results obtained in this study contribute to the literature on fuzzy theory.

The extent of this relation's intimacy and whether it can provide more insight into the geometry of projective planes or the triangle norms in fuzzy mathematics could be revealed by further research. Future research could advance these findings by exploring several key areas: (1) Investigating fuzzy collineations in higher-dimensional fuzzy projective spaces to uncover complex relationships and invariances; (2) conducting comparative studies of different fuzzy projective spaces to identify broader patterns and properties; (3) applying results to computational models and algorithms, especially in fields like computer vision and artificial intelligence; and (4) extending theoretical insights to other mathematical structures, such as fuzzy topology or fuzzy algebraic geometry, to potentially reveal new aspects of fuzzy theory.

Conflict of interest

The authors declare no conflict of interest.

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