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Research article

Analytical and numerical solution of sausage MHD wave oscillation in a thin magnetic flux tube

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Abstract: The aim of the present study is to investigate the damping of slow sausage MHD waves propagating in a gravitationally-stratified magnetic cylindrical structure when the plasma is strongly partially ionised. The problem is treated as an initial value problem and the analysis deals with the temporal evolution of waves in an asymptotic sense, i.e., large values of time compared to the period of waves. The plasma is assumed to be collision-dominated, i.e., we employ a two-fluid approximation. The set of equations describing the plasma dynamics is reduced to a coupled partial differential equations. Our findings show that the slow wave of charged species is affected by the presence of a cut-off. The mode associated with the neutral fluid propagates without any cut-off and decay very quickly due to collisions between particles.

Keywords: partial differential equations; partial ionisation; solar chromosphere; MHD waves; collision
Mathematics Subject Classification: 35A09, 35A22, 35B40, 35L05, 76N30

1. Introduction

Research focusing on the plasma dynamics in the solar atmosphere has received a new connotation with the study of waves and oscillations in partially ionised solar plasmas carried out in the past decade. The temperature in the lower part of the solar atmosphere (photosphere and chromosphere) is not high enough for the complete ionisation of the plasma; instead, plasma is made up of electrons, protons, and neutral particles that interact through collisions. Solar atmospheric model used to conduct analysis on a large number of spectral lines [1–3] predict that the ratio of the number of neutrals to the number of charges changes from about 10^4 in the deep photosphere to about 1 at the top of the chromosphere. This ratio tends to zero as we approach the solar corona, where high temperatures render the plasma fully ionised.

Waves are an ideal candidate to carry energy and momentum through the solar plasma, as they can contribute to solar atmospheric heating. In the context of partially ionised plasma, previous research has shown that they have properties that are not matched in a fully ionised plasma. These waves are subject to some specific transport mechanism that can help in their attenuation, enhancing their role in the heating process. The investigation framework of waves in partially ionised plasmas (PIP) depends on the frequency regime in which they are considered. For frequencies that are much less than the collisional frequency of particles, one can use a single-fluid approximation, in which the pip aspect is described through some specific dissipative processes such as ambipolar diffusion. When the frequencies of interest are of the same order as the collisional frequency, dynamics can be investigated using the framework of a two-fluid magnetohydrodynamcs (MHD), where charged particles, strongly coupled with each other, form a single fluid (plasma) that interacts collisionally with the neutral fluid [4]. In our analysis, we are going to use a two-fluid description.

The magnetic field in the solar atmosphere plays a crucial role for channelling energy and momentum, and nowadays it is established that plasma heating has a magnetic nature, i.e., the location of high temperature plasma coincides with accumulation of the magnetic field. Very often the magnetic fields appear as well-defined structures such as sunspots, pores, magnetic flux tubes, spicules, coronal loops, etc.

The properties of MHD waves in an unbounded and homogeneous plasma as a two-fluid approximation were studied by [5], who derived (using a normal mode analysis) and numerically solved the dispersion relation of waves. Their results were compared with the single-fluid counterparts. The authors found that, while in the low-frequency regime the results were identical in both models, in the high-frequency regime the behaviour of waves is completely different. In particular, the damping rate of Alfvén and fast magneto-acoustic waves reached a maximum value at a particular frequency, after which, for higher frequencies, this damping rate is attenuated. Later, [6] reconsidered the problem of magnetoacoustic wave propagation in two-fluid pip. They determined the modifications in the wave frequencies and studied the damping of waves in terms of the collisional frequency. The authors have drawn parallels in the solutions obtained in both uncoupled and strongly coupled plasma. They found that in the uncoupled limit, acoustic waves were connected to the neutral species, while magnetoacoustic waves were connected to the charged species. In addition, the modified slow and fast magnetoacoustic waves only propagate in strongly coupled cases, and the mode connecting to neutral species becomes an entropy mode. [7] initially explained the oscillation of magnetoacoustic waves in the solar chromosphere, taking into account the influences of effect ionisation and radiative recombination. Moreover, [8] studied the propagation and damping of fasts and shocks in the chromosphere region. These authors found that the decoupling between both species resulted in the collision of damping waves, which led to an increase in temperature. [9, 10] examined the propagation of acoustic waves in multi-fluid approximation, neglecting the effect of the magnetic field. [11] studied the MHD waves in pip in non-equilibrium ionization. Their results suggest that non-equilibrium ionisation has significant implications for plasma dynamics, particularly in relation to the behaviour and dissipation of waves.

Linear acoustic waves propagating in isothermal and stratified plasma could be affected by the presence of a cut-off frequency, i.e., the stratified environment acts as a filter; only frequencies larger than a cut-off value are permitted to propagate. In a recent study [12], we considered the slow modes propagating in a two-fluid gravitationally stratified pip and found that only waves of the charged fluid were affected by the presence of a cut-off, while modes related to the neutral fluid propagate without a cut-off, and these are driven by ions due to the collisional coupling between particles. Furthermore, [13] studied numerically the effect of cut-off in two-fluid pip by simulating the region that covers the area above the convective zone up to the low corona. Their results show that the appearance and the effect of the cut-off frequency have a strong height dependence and collisions between particles affect the cut-off frequency. For more details about results of waves and oscillations in pip, we refer the reader to the recently comprehensive works by [4, 14].

In reality, the driver of a wave is rarely a harmonic, sinusoidal function; instead, sudden and transient changes (e.g., granular motion in the photosphere, energy releases following a localised energy release, etc.) give rise to a pulse-like driver. The goal of this paper is to extend the study carried out by [12] and study the initial value problem of dispersive slow MHD waves propagating in a partially ionised and stratified plasma magnetic flux tube in two-fluid plasma, where waves are driven by Γ function pulse. The structure of this paper is as follows: Section 2 presents the main mathematical framework and modelling environment of the problem, and solutions of the governing equations are presented for the two species. The initial value problem imposed on the governing equation is posed in Section 3, and in Section 4, we study the propagation characteristics of waves driven by a Γ function. Finally, in Section 5, we summarise our findings.

2. Physical background and governing equations

We consider a magnetic cylinder of cross-sectional area, A, situated along the z-axis along which the homogeneous magnetic field is aligned with the longitudinal symmetry axis. As we are interested in longitudinal waves, we consider that the longitudinal components of the velocities (here denoted by u_i and u_n are dominant and all the other velocity components will be neglected. The plasma dynamics of the lower part of the solar atmosphere can be described by two-fluid approximation, meaning that ions and neutrals are treated separately (for more details, see [15]). The set of linearised MHD equations is given as

$$\frac{\partial}{\partial t}(\varrho_i' A + \varrho_i A') = -\frac{\partial}{\partial z}(\varrho_i' A' u_i), \tag{2.1}$$

$$\frac{\partial}{\partial t}(\varrho_n' A + \rho_n A') = -\frac{\partial}{\partial z}(\varrho_n' A' u_n), \tag{2.2}$$

$$\varrho_i' \frac{\partial u_i}{\partial t} + \frac{\partial p_i}{\partial z} = -\varrho_i g - \varrho_i \nu_{in}(u_i - u_n), \tag{2.3}$$

$$\varrho_n' \frac{\partial u_n}{\partial t} + \frac{\partial p_n}{\partial z} = -\varrho_n g - \varrho_n \nu_{ni}(u_n - u_i), \tag{2.4}$$

$$\frac{\partial p_i}{\partial t} + u_i \frac{dp_i'}{dz} - c_i^2 \left(\frac{\partial \varrho_i}{\partial t} + u_i \frac{\partial \varrho_i'}{\partial z} \right) = 0, \tag{2.5}$$

$$\frac{\partial p_n}{\partial t} + u_n \frac{dp_n'}{dz} - c_n^2 \left(\frac{\partial \varrho_n}{\partial t} + u_n \frac{\partial \varrho_n'}{\partial z} \right) = 0, \tag{2.6}$$

together with the requirement that the perturbed magnetic flux is conserved and the total pressure inside the flux tube (kinetic and magnetic) is balanced on the boundary of the tube by the gas pressure outside the tube

$$B^{'}A + BA^{'} = 0, \quad p + \frac{1}{\mu}B^{'}B = p^{\star}. \tag{2.7}$$

More mathematical details of the set equations can be seen in [16,17]. In the above equation, the index (') denotes equilibrium values, and subscripts i and n refer to charged and neutral fluids, respectively. In the above equation, $\varrho^{'}$ is unperturbed density, $A^{'}$ is unperturbed cross-section area of the flux tube, $p^{'}$ is unperturbed pressure, $B^{'}$ is unperturbed magnetic field. ϱ is the density perturbation, and A is the cross-section area perturbation, $p^{'}$ is the perturbation pf the pressure, B is the perturbation magnetic field, $g = 274\ m/s^2$ the gravitational acceleration, μ is the magnetic permeability of the vacuum, $p^{\star}$ is the external pressure, and ν_{in}, ν_{ni} are the collisional frequencies between species. Since we are dealing with elastic collisions, $\rho_i^{'}\nu_{in} = \rho_n^{'}\nu_{ni}$. The perturbed density and pressure for the two constituent species are connected via the standard relation $p_{i,n} = \varrho_{i,n}c_{i,n}^2$, where the sound speed associated with ions and neutrals is given as

$$c_{i,n} = \left(\frac{\gamma p_{i,n}^{'}}{\rho_{i,n}^{'}}\right)^{1/2}.$$

We simplify our calculations by applying some assumptions. First, we assume a rigid boundary in the sense that any temporal changes in the external plasma is occurring over much longer time than the dynamics we study inside the flux tube. That means that all temporal changes of the external pressure will be very small; therefore, these will be neglected. This results in the absence of terms containing $p^{\star}$. Next, we assume the plasma is in ionisation equilibrium, meaning that during the length of the investigated dynamics no further ions and neutrals are created due to ionisation and recombination processes. The properties of linear MHD waves in a pip in ionisation non-equilibrium have beenstudied by [11]. Furthermore, we assume the flux tube is thin, which means the radius is very small compared with the wave length. [18] studied the MHD waves propagating in a thick flux tube within a plasma that is fully ionised.

Let us introduce the scaled variables $U_i = u_i e^{-\lambda_i z}$ and $U_n = u_n e^{-\lambda_n z}$, where λ_i and λ_n are given by [12]

$$\lambda_i = \frac{1}{4h_i}, \quad \lambda_n = \frac{1}{4h_i} = \frac{1}{2h_n}$$

with $h_i = c_i^2/\gamma g$ and $h_n = c_n^2/\gamma g$ being the constant gravitational scale-height for the ion and neutral particles, respectively. After a long but straightforward calculation, the governing Eqs (2.1)–(2.6) can be reduced into two partial differential equations for the velocity components

$$\frac{\partial^2 U_i}{\partial t^2} - c_T^2\frac{\partial^2 U_i}{\partial z^2} + \Omega_{ion}^2 U_i = 0, \tag{2.8}$$

$$\frac{\partial^2 U_n}{\partial t^2} - c_n^2\frac{\partial^2 U_n}{\partial z^2} + \Omega_{neutral}^2 U_n + \nu_{ni}\frac{\partial U_n}{\partial t} = f(U_i), \tag{2.9}$$

with

$$f(U_i) = \left(-\frac{c_{Sn}^2 c_T^2}{v_A^2}\frac{\partial^2 U_i}{\partial z^2} + \iota^{\star} U_i\right)\exp\left(-\frac{z}{2\gamma h}\right),$$

where c_T is phase speed of slow sausage modes and Ω_{ion}^2, $\Omega_{neutral}^2$ and $\iota^{\star}$ are defined as

$$\Omega_{ion}^2 = \left(\frac{9}{4} - \frac{2}{\gamma}\right)\omega_{Ai}^2 - \omega_{Ai}^2\frac{\beta\gamma}{2+\beta\gamma}\left(\frac{3}{2} - \frac{2}{\gamma}\right)^2,$$

$$\Omega^2_{neutral} = \frac{c_n^2}{2c_i^2}\omega^2_{Ai} + \omega^2_{An} + \frac{c_n^2}{4\gamma h_i h_n}(\gamma - 2),$$

and

$$\iota^\star = \frac{c_n^2}{C_A^2(2+\gamma\beta)}\left[\omega^2_{Ai}\left(\frac{1}{2} - \frac{2}{\gamma} + \frac{2}{\gamma^2}\right) + \frac{2c_i^2}{c_n^2}\omega^2_{An}\left(1 - \frac{2}{\gamma} + \frac{1}{\gamma^2}\right) + \frac{c_i^2}{h_i h_n}\left(\frac{1}{4} - \frac{3}{2\gamma} + \frac{1}{\gamma^2}\right)\right],$$

where C_A is the Alfvén speed, β is the plasma-β parameter, $\omega_{A_{i,n}} = c_{i,n}/(2h_{i,n})$ are the acoustics cut-off frequencies for ions and neutrals, and Ω_{ion}, $\Omega_{neutral}$ are the cut-off frequencies for the magnetoacoustic modes that propagate along the magnetic field lines. The reason why the governing equation for ions does not contain a collisional term is because this term is proportional to the ratio of the number density of neutrals to ions, which in the case of a strongly ionised plasma is very small. Consequently, the collisional term will be neglected.

The system of coupled Eqs (2.8) and (2.9) is a typical Klein-Gordon and telegrapher equation that describes the oscillation of ion and neutral modes. The parameters in the set of equations are to be constants. The couple system will be treated as an initial value problem (IVP) in order to investigate the asymptotic behaviour of slow MHD.

3. The initial value problem (IVP) in two fluid approximation

We are aiming to study the asymptotic solution of guided waves when the time begins at a large value. We solve the initial value problem (IVP) of Eqs (2.8) and (2.9). Based on the study by [12], the plasma reaches a uni-thermal state in a few collisional times; therefore, in our calculations, we consider that all species have the same temperature, i.e., $T_i = T_n$. Therefore, we can also write that $c_i = 2c_n$.

We consider the equation of ion species, which is given in the form of the Klein-Gordon equation

$$\frac{\partial^2 U_i}{\partial t^2} - c_T^2\frac{\partial^2 U_i}{\partial z^2} + \Omega^2_{ion} U_i = 0. \tag{3.1}$$

This equation will solve using initial condition (IC)

$$U_i(z,0) = \frac{\partial U_i(z,0)}{\partial t} = 0, \tag{3.2}$$

with boundary condition (BC)

$$U_i(z\to\infty, t) = 0, \quad U_i(0,t) = A_0(t). \tag{3.3}$$

The IVP of an ion fluid equation can be solved by using the Laplace transform. The Laplace transform of the $U_i(z,t)$

$$\Theta_i(z,s) = \mathcal{L}[U_i(z,t] = \int_0^\infty U_i(z,t)e^{-st}\,dt. \tag{3.4}$$

The equation of charged fluid is reduced to

$$s^2\Theta_i(z,s) - c_T^2\frac{d^2}{dz^2}\Theta_i(z,s) + \Omega^2_{ion}\Theta_i(z,s) = 0, \tag{3.5}$$

we apply the BC, hence

$$\frac{d^2}{dz^2}\Theta_i(z,s) - \frac{s^2+\Omega_{ion}^2}{c_T^2}\Theta_i(z,s) = 0, \tag{3.6}$$

and general solution is given as

$$\Theta_i(z,s) = R_1 \exp\left(\delta_i \sqrt{s^2+\Omega_{ion}^2}\right) + R_2 \exp\left(-\delta_i \sqrt{s^2+\Omega_{ion}^2}\right), \tag{3.7}$$

where $\delta_i = z/c_T$, and R_1, R_2 represent arbitrary constants. Obviously, $R_1 = 0$ because it does not satisfy the boundary condition at infinity. We assume $z = 0$, meaning that the slow MHD mode is driven by $U_i(0,t) = A_0(t)$, and its Laplace transform is $\Theta_i(0,s) = a_0(s)$. Therefore, the general solution becomes

$$\Theta_i(z,s) = a_0(s)\exp\left(-\sqrt{\frac{s^2+\Omega_{ion}^2}{c_T^2}}z\right). \tag{3.8}$$

We employ the inverse Laplace transform to the Eq (3.8) using the convolution theorem to find the function $U_i(z,t)$. We closely follow the technique done by [12, 19, 20]. In order to simplify our calculation, we employ the identity ([21])

$$\mathcal{L}^{-1}\left[\frac{e^{-a\sqrt{s^2+\Omega_{ion}^2}}}{\sqrt{s^2+\Omega_{ion}^2}}\right] = \begin{cases} J_0\left(\Omega_{ion}\sqrt{t^2-a^2}\right), & t > a \\ 0, & 0 < t < a, \end{cases} \tag{3.9}$$

where J_0 is the zeroth-order Bessel function. We consider the function

$$\Lambda = \frac{e^{-a\sqrt{s^2+\Omega_{ion}^2}}}{\sqrt{s^2+\Omega_{ion}^2}} = \int_a^\infty J_0\left(\Omega_{ion}\sqrt{t^2-a^2}\right)e^{-st}dt, \tag{3.10}$$

after differentiate with respect a, yields

$$\begin{aligned}\frac{d\Lambda}{da} &= -\Omega_{ion}a\int_a^\infty \frac{J_0'\left(\Omega_{ion}\sqrt{t^2-a^2}\right)}{\sqrt{t^2-a^2}}e^{-st}dt - e^{-as} \\ &= -\exp\left(-a\sqrt{\Omega_{ion}^2+s^2}\right). \end{aligned} \tag{3.11}$$

By using the identity $J_0'(x) = -J_1(x)$, and $a = \delta$, we arrive

$$\exp(-s\delta_i) - \exp\left(-\delta_i\sqrt{\Omega_{ion}^2+s^2}\right) = \Omega_{ion}\delta\int_{\delta_i}^\infty \frac{J_1\left(\Omega_{ion}\sqrt{t^2-\delta_i^2}\right)}{\sqrt{t^2-\delta_i^2}}e^{-st}dt. \tag{3.12}$$

We substitute Eq (3.12) to Eq (3.8), we read

$$\Theta_i(z,s) = a_0(s)\exp(-s\delta_i) - a_0(s)\Omega_{ion}\delta_i\int_\delta^\infty \frac{J_1\left(\Omega_{ion}\sqrt{t^2-\delta_i^2}\right)}{\sqrt{t^2-\delta_i^2}}e^{-st}dt. \tag{3.13}$$

We define the following function:

$$\Pi_i(z,t) = -\Omega_{ion}\delta_i \frac{J_1\left(\Omega_{ion}\sqrt{t^2-\delta_i^2}\right)}{\sqrt{t^2-\delta_i^2}} H\left(t-\delta_i\right), \tag{3.14}$$

here $H(t-\delta_i)$ is the Heaviside step function.

With the help of second shifting theorem and convolution theorem to Eq (3.13), we arrive at

$$U_i(z,t) = A_0\left(t-\delta_i\right)H\left(t-\delta_i\right) + \int_0^t A_0(t-\tau)\Pi_i(z,\tau)d\ \tau. \tag{3.15}$$

This equation describes a spatial and temporal evolution of charged species.

Now, let us return to the neutral species equation. We will follow the above method, and after long calculation (for more details, see the study by [12] we obtain the homogeneous solution as

$$U_n(z,s) = A_0\left(t-\delta_n\right)H\Big(t-\delta_n\Big) + \int_0^t A_0(t-\tau)\Pi_n(z,\tau)d\tau, \tag{3.16}$$

where,

$$\Pi_n(z,t) = \Omega_{neutral}\delta_n \frac{I_1(\Omega_{neutral}\sqrt{t^2-\delta_n^2})}{\sqrt{t^2-\delta_n^2}} H\left(t-\delta_n\right)),$$

$I_1(x)$ is the modified-Bessel function. Consequently, the particular solution reads

$$\begin{aligned}
U_p(z,t) = {} & i\zeta_1 V\left[\frac{\Xi_1}{\varphi_1+i\Omega_{ion}}\left(e^{\varphi_1 t}-e^{-i\omega t}\right)-\frac{\Xi_2}{\varphi_2+i\Omega_{ion}}\left(e^{\varphi_2 t}-e^{-i\omega t}\right)\right]\left(1+e^{-i\delta_i(\omega-\sqrt{\omega^2-\Omega_{ion}^2})\delta_i}\right)\\
& +i\zeta_1\zeta_2 V\Omega_{ion}\Bigg\{\frac{1}{t^{3/2}}\left(\frac{\Xi_1\varphi_1}{\varphi_1^2+\Omega_{ion}^2}-\frac{\Xi_2\varphi_2}{\varphi_2^2+\Omega_{ion}^2}\right)\left[\frac{\omega}{\Omega_{ion}}\cos\left(\Omega_{ion}t-\frac{3\pi}{4}\right)-i\sin\left(\Omega_{ion}t-\frac{3\pi}{4}\right)\right]\\
& -\frac{1}{\delta_i^{3/2}}\left(\frac{\Xi_1\varphi_1 e^{\varphi_1 t}}{\varphi_1^2+\Omega_{ion}^2}-\frac{K_2\varphi_2 e^{\varphi_2 t}}{\varphi_2^2+\Omega_{ion}^2}\right)\left[\frac{\omega}{\Omega_{ion}}\cos\left(\Omega_{ion}t-\frac{3\pi}{4}\right)-i\sin\left(\Omega_{ion}t-\frac{3\pi}{4}\right)\right]\\
& -\frac{\Omega_{ion}}{t^{3/2}}\left(\frac{K_1}{\varphi_1^2+\Omega_{ion}^2}-\frac{K_2}{\varphi_2^2+\Omega_{ion}^2}\right)\left[\frac{\omega}{\Omega_{ion}}\cos\left(\Omega_{ion}t-\frac{3\pi}{4}\right)-i\sin\left(\Omega_{ion}t-\frac{3\pi}{4}\right)\right]\\
& +\frac{\Omega_{ion}}{\delta_i^{3/2}}\left(\frac{K_1 e^{\varphi_1 t}}{\varphi_1^2+\Omega_{ion}^2}-\frac{K_2 e^{\varphi_2 t}}{\varphi_2^2+\Omega_{ion}^2}\right)\left[\frac{\omega}{\Omega_{ion}}\cos\left(\Omega_{ion}t-\frac{3\pi}{4}\right)-i\sin\left(\Omega_{ion}t-\frac{3\pi}{4}\right)\right]\Bigg\},
\end{aligned} \tag{3.17}$$

where the expressions of the used notations are

$$\varphi_{1,2} = \frac{-\nu_{ni} \pm \left[\nu_{ni}^2-\left(2-\frac{g}{\Omega_{ion}c_T}\right)\left(\Omega_{neutral}^2-\frac{g^2}{16c_{Sn}^2}-\frac{\Omega_{ion}^2}{2}-\frac{\Omega_{ion}g}{2c_T}\right)\right]^{1/2}}{1-g/2\Omega_{ion}c_T},$$

$$K_1 = \frac{4\pi e^{-z/4\gamma H_n} i\pi\left(1-g/2\Omega_{ion}c_T\right)}{\left[\nu_{ni}^2-\left(2-\frac{g}{\Omega_{ion}c_T}\right)\left(\Omega_{neutral}^2-\frac{g^2}{16c_{Sn}^2}-\frac{\Omega_{ion}^2}{2}-\frac{\Omega_{ion}g}{2c_T}\right)\right]^{1/2}},$$

$$K_2 = \sqrt{\frac{2\Omega_{ion}}{\pi}}\frac{1}{\omega^2-\Omega_{ion}^2}\delta_i,$$

$$\Xi_{1,2} = \iota^\star - \frac{\beta\gamma}{4}\left(\Omega_{ion}^2+\varphi_{1,2}^2\right)+\nu_{ni}\varphi_{1,2}.$$

4. Wave excitation by the Γ function

In this section, we study the wave driven by Γ function source for two fluids with partially ionised plasma. The driver form is given as

$$A_0(t) = V_0\Gamma(t), \tag{4.1}$$

where V_0 is the amplitude of the driver. We define the dimensionless quantity

$$\tilde{t} = \frac{1}{2\pi}\Omega_{ion}t = \frac{t}{P_{ion}}, \tag{4.2}$$

where P_{ion} is the cut-off period associated with the ion-acoustic modes. The form can be rewritten as

$$A_0(\tilde{t}) = V_0\Gamma(\tilde{t}). \tag{4.3}$$

Introducing this expression into the ion fluid Eq (3.15), we obtain

$$U_i(z,t) = V_0\Gamma\left(t - \delta_i P_{ion}\right) H\left(\tilde{t} - \delta_i P_{ion}\right) + \int_0^{\tilde{t}} V_0\Gamma(\tilde{t} - \tilde{\tau})\Pi_i(z,\tilde{\tau})d\ \tilde{t}au, \tag{4.4}$$

where we used

$$\tau^{\sim} = \tau/P_{ion}, \quad t^{\sim} > \delta_i P_{ion},$$

and using $t = \tilde{t}P_{ion}$, yields

$$U_i(z,t) = -\pi V_0\delta_i \frac{J_1\left(\Omega_{ion}\sqrt{t^2 - \delta_i^2}\right)}{\sqrt{t^2 - \delta_i^2}}. \tag{4.5}$$

In the case of $t >> \delta_i$, the argument of Bessel function J_1 can be written as

$$J_1(y) \approx \frac{2}{\sqrt{\pi y}}\left[\cos\left(y - \frac{3\pi}{4}\right) + O\left(\frac{1}{y}\right)\right]. \tag{4.6}$$

After some algebraic calculation (see [19], Appendix B), the Eq (4.5) can be written as

$$U_i(z,t) = -V_0\sqrt{\frac{2\pi}{\Omega_{ion}}}\frac{\delta_i}{t^{3/2}}\cos\left(\Omega_{ion}t - \frac{3\pi}{4}\right). \tag{4.7}$$

Inspired by this result, the solution shows that the waves driven by the wave generated by the Γ function propagate with the cut-off frequency and decay in time as $t^{-3/2}$. In the next section, we are going to investigate this solution numerically for realistic solar chromosphere models.

On the other hand, the solutions of the neutral fluid Eqs (3.16) and (3.17) are far too long and contain exponentially collisional terms, meaning that the wave will have a very rapid decay and the asymptotic solution is not possible to plot [22].

5. Numerical investigation

Here, we are going to plot the solution of the Eq (4.7), assuming a realistic solar atmosphere model. We use the AL C model [3] to obtain the values of physical parameters.

Our investigation found that the frequency at which ion MHD waves oscillate in stratified plasma is affected by a cut-off, so only those waves can propagate that have frequencies larger than the cut-off frequency. For a typical chromospheric plasma with $T = 10^4$ K, the cut-off frequency of the magnetoacoustic modes associated with ions is $\Omega_{ion} \approx 0.015$ Hz, and it varies as $T^{1/2}$. Interestingly, the cut-off frequency for fully ionised plasma (assuming identical values of the magnetic field and temperature) is similar to the above value, and this is due to the strongly ionised case we applied in this study. The study by [23] showed that Alfvén waves with frequencies below 0.01 Hz are unaffected by dissipative effects and propagate through the partially ionised plasma with little diffusion. In contrast, Alfvén waves with frequencies above 0.6 Hz are completely damped.

Figure 1 represents the numerical solution of magneto-acoustic waves driven by the Γ function driver in partially ionised plasma. This solution is chosen for a height of $z = 4000$ km; however, the qualitative conclusion would not change if we used another value for z. The plot shows that the wave started propagating after the value of $t > \delta_i$, meaning that they are affected by the ion cut-off frequency, Ω_{ion}. An observer located at height z will see a signal travel with ion cut-off frequency, Ω_{ion} and decay at $t^{-3/2}$ as presented in Figure 1.

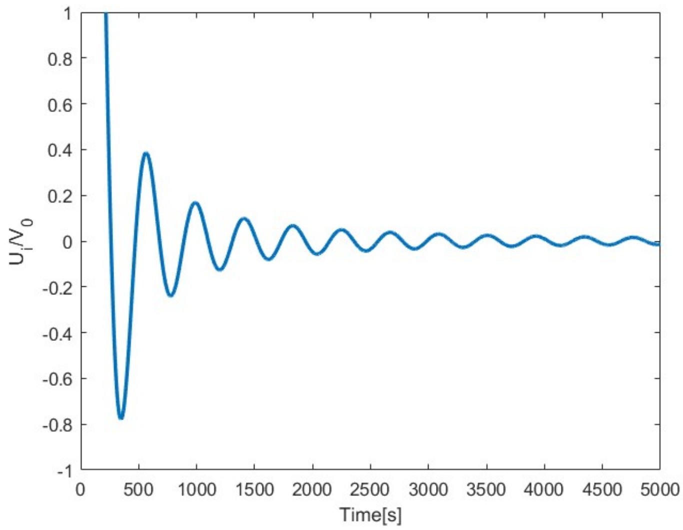

Figure 1. The temporal evolution of the solution of Eq (4.7) with normalised amplitudes its by maximum value.

The solution of neutral fluid in Eqs (3.16) and (3.17) contains a decaying collisional term, meaning that waves decay very quickly with a characteristic damping time of $2/\nu_{ni}$. In strongly ionised plasma, ν_{ni} is of the order of 0.1 kHz, which is why waves disappear very quickly and the temporal evolution of neutral modes cannot be plotted.

Finally, the quick decay that happened in neutral species means that energy is dissipated and could be converted to heat. The heat resulting from this process will balance energy and substitute for the radiative losses in the chromosphere region. This may be investigated in future studies. Our findings demonstrate that the asymptotic solution of waves propagating in multi-fluid approximation when the plasma is strongly ionised only the waves connected to the ion fluid could be observed.

6. Conclusions

In the present paper, we expanded the study conducted by [12] that addressed the propagation of slow MHD modes in a two-fluid partially ionised solar chromosphere driven by a sinusoidal pulse taking into account the effect of stratification. Here, we investigate the waves driven by the Γ function pulse.

In order to make our analysis valid we assume isothermal plasma, which means the temperature is not affected by height and all speeds are considered constant. In addition, we assume the plasma is uni-thermal, and the consequences of that are that the temperature of all species is the same.

We found the evolutionary equation that describes the propagation of slow mode. While the spatial and temporal dynamics of the ion fluid are described by a Klein-Gordon equation, and the dynamics of the neutral species are given by a telegrapher's equation. The system of partial differential equations is solved as an initial value problem (IVP) using the method of Laplace transform with the help of the convolution theorem.

We studied the temporal evolution of waves driven by the Γ function. As a result, the waves associated with ion fluid oscillate the cut-off frequency, Ω_{ion}, and decay in time as $t^{-3/2}$. This result is in contrast to the earlier findings by [12], where the sinusoidal driver generated a slow wave propagating with the speed c_T followed by a wake that oscillates with the cut-off frequency. The collision process clearly does not affect the ion mode, at least not in the leading order. This outcome is similar to the finding by [19]. However, we find that waves connected to neutral modes propagate without a cut-off frequency, and due to the collision between both species, these waves decay very quickly and are not observable.

To sum up, the observation of these waves can only be in an environment with a moderate level of ionisation.

Conflict of interest

The author declares that he has no competing interest.

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