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Research article

Singularities of swept surfaces in Euclidean 3-space

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Abstract: This study examines the local singularities of tube surfaces, especially those of swept surfaces (SS) in Euclidean 3-space $\mathcal{E}^3$. SS is created by moving a planar curve through space such that the trajectory of any point on the surface remains perpendicular to the plane. The Type-2 Bishop frame is considered, and the singularities of these SS are analyzed. Examples are offered and illustrated.

Keywords: unfolding theory; developable surface; Bishop height function
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1. Introduction

Singularity theory (ST) is commonly discussed in physics and mathematics for analyzing curves and surfaces. Differential geometry locates curves and surfaces using functions with one and two parameters, respectively. Recently, ST has been widely used in fields such as medical imaging and computer vision (see, e.g., [1–4]).

A swept surface SS is constructed by moving a planar curve (generatrix) through space with the movement of the plane orthogonal to it. Sweeping along a line is an accurate method in geometric modeling, often involving sweeping over a spine curve (SC) (path). The significance of this evolution, which depends on the movement through space and the intrinsic deformation of shape, lies in the concept of a swept theme. The type of swept surface is determined by the choice of the generatrix and the path. For example, rotating one curve over another creates a swept surface. Various types of swept surfaces include tubular surfaces, pipe surfaces, strings, and canal surfaces [5,6]. The geometry of SS can be described employing the Serret-Frenet frame (SFF) of space curves, an orthonormal frame that is related in some studies to eigenvalue problems [7,8]. Xu et al. [9] identified the geometric attributes required for canal surfaces as a type of SS. They also derived conditions to avoid local self-intersection in canal surfaces and formulated expressions for their area and Gaussian curvature. Izumiya et al. [10]

studied conformable circular surfaces along with traditional ruled surfaces . Ro and Yoon [11] studied tubes in Euclidean 3-space, deriving expressions for Gaussian curvature, mean curvature, and second Gaussian curvature. Cui et al. [12] explored the kinematic geometry of circular surfaces with fixed radii based on Euclidean invariants.

However, the SFF is not applicable to every point on a curve. A novel frame is needed for certain types of mathematical analyses, particularly those done via computer graphics. Bishop et al. [13] developed a novel moving frame along a space curve, allowing for movement along the curve using parallel vector fields. This is known as the Bishop frame (BF) or rotation minimizing frame (RMF). The BF has been successfully applied in fields such as biology, where it can model the shape of DNA sequences by employing curves defined by the BF. The BF can also be employed to track to the positioning of cameras in computer animations [14,15]. Recently, numerous descriptions of SS and developable surfaces have been acquired using the BF [16–19]. Various methods have been used to generate new curves in different spaces, as described in [20–26].

However, to the best of our knowledge, no existing literature discusses the singularities of SS with a conjugate mate curve using the Type-2 BF. Then, utilizing the BF, we classify generic models and introduce a novel invariant associated with the singularities of the SS. The main generic singularities of these SS are the well-known cuspidal edge and swallowtail, which are illustrated by this novel invariant. We use traditional and established outcomes in ST to demonstrate our major outcomes in this paper. In addition, this paper establishes the necessary and adequate situations for when the SS with conjugate mate curve is a developable ruled surface (DRS) and discusses further epilogues. Regarding the DRS, we inspected the ST of the DRS. Finally, to highlight the major outcomes, two examples are presented and analyzed in detail. It is worth noting that Mathematica was used to generate figures in this study.

2. Preliminaries

The ambient space considered in this article is the Euclidean 3-space $\mathcal{E}^3$, with generic references taken from [5,6]. Let $\gamma(\mathfrak{s})$ be a unit speed curve with the SF systems denoted as $\{\kappa(\mathfrak{s}), \tau(\mathfrak{s}), \mathbf{t}(\mathfrak{s}), \mathbf{p}(\mathfrak{s}), \mathbf{b}(\mathfrak{s})\}$ in $\mathcal{E}^3$. Then, $\gamma(\mathfrak{s})$ is termed an SF curve if $\kappa > 0$, and $\tau \neq 0$. So, we obtain

$$\begin{pmatrix} \mathbf{t}' \\ \mathbf{p}' \\ \mathbf{b}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{p} \\ \mathbf{b} \end{pmatrix}; \ (\ ' = \frac{d}{d\mathfrak{s}}), \tag{2.1}$$

where $\mathbf{t} = \gamma'(\mathfrak{s})$, $\mathbf{p}(\mathfrak{s}) = \gamma''(\mathfrak{s})/\left\|\gamma''(\mathfrak{s})\right\|$, and $\mathbf{b}(\mathfrak{s}) = \mathbf{t}\times\mathbf{p}$ are the unit tangent vector, the principal normal vector, and the binormal vector, respectively.

Lemma 2.1. Given the above notation, there exists a unit speed conjugate mate curve $\beta(\mathfrak{s}) = \int \mathbf{b}(\mathfrak{s})ds$. The pair $\{\gamma(\mathfrak{s}), \beta(\mathfrak{s})\}$ is referred to as a conjugate couple [27].

The Type-2 BF of $\gamma(\mathfrak{s})$ is defined as [18]:

$$\begin{pmatrix} \mathbf{b}' \\ \xi_1' \\ \xi_2' \end{pmatrix} = \begin{pmatrix} 0 & \epsilon_1 & \epsilon_2 \\ -\epsilon_1 & 0 & 0 \\ -\epsilon_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{b} \\ \xi_1 \\ \xi_2 \end{pmatrix}, \tag{2.2}$$

where

$$\left.\begin{array}{l}\epsilon_1(\mathfrak{s}) = \tau\cos\psi(\mathfrak{s}),\ \epsilon_2(\mathfrak{s}) = -\tau\sin\psi(\mathfrak{s}),\ \psi(\mathfrak{s}) = -\tan^{-1}\left(\frac{\epsilon_2}{\epsilon_1}\right);\ \epsilon_1(\mathfrak{s}) \neq 0,\\ \psi(\mathfrak{s}) = \psi_0 - \int\limits_{\mathfrak{s}_0}^{\mathfrak{s}}\kappa(\mathfrak{s})d\mathfrak{s},\ \text{and}\ \psi_0 = \psi(\mathfrak{s}_0).\end{array}\right\} \tag{2.3}$$

We find that $\{\mathbf{b}, \xi_1, \xi_2\}$ is a Type-2 *BF*. The relation matrix can be expressed as:

$$\begin{pmatrix}\mathbf{b}\\ \xi_1\\ \xi_2\end{pmatrix} = \begin{pmatrix}0 & 0 & 1\\ \sin\psi & -\cos\psi & 0\\ \cos\psi & \sin\psi & 0\end{pmatrix}\begin{pmatrix}\mathbf{t}\\ \mathbf{p}\\ \mathbf{b}\end{pmatrix}. \tag{2.4}$$

The frame $\{\mathbf{b}, \xi_1, \xi_2\}$ has a Darboux vector $\omega(\mathfrak{s}) = -\epsilon_2(\mathfrak{s})\xi_1(s) + \epsilon_1(\mathfrak{s})\xi_2(s)$. The Bishop spherical Darboux indicatrix is:

$$\mathbf{d}(\mathfrak{s}) = -\frac{\epsilon_1}{\sqrt{\epsilon_1^2 + \epsilon_2^2}}\left(\frac{\epsilon_2}{\epsilon_1}\xi_1 - \xi_2\right). \tag{2.5}$$

Thus, we introduce a new geometric invariant $\sigma(\mathfrak{s}) = \epsilon_1\epsilon_2' - \epsilon_2\epsilon_1'$.

A surface $\mathfrak{M}$ can be expressed as follows:

$$\mathfrak{M} : \mathbf{y}(\mathfrak{s}, u) = (x_1(\mathfrak{s}, u), x_2(\mathfrak{s}, u), x_3(\mathfrak{s}, u)),\quad (s, u) \in \mathcal{D} \subseteq \mathbb{R}^2. \tag{2.6}$$

The unit vector normal is defined by $\zeta(\mathfrak{s}, u) = \mathbf{y}_s \times \mathbf{y}_u \, \|\mathbf{y}_s \times \mathbf{y}_u\|^{-1}$, where $\mathbf{y}_i = \frac{yy}{\partial i}$. A *RS* in $\mathcal{E}^3$ is given by:

$$\mathfrak{D}(\mathfrak{s}, u) = \alpha(\mathfrak{s}) + u\xi(\mathfrak{s}),\ u \in \mathbb{R}, \tag{2.7}$$

where $\alpha(\mathfrak{s})$ is the base (directrix) curve, and $\xi(\mathfrak{s})$ is the director curve. The straight lines $\mathfrak{s} \to \alpha(\mathfrak{s}) + u\xi(\mathfrak{s})$ are called rulings. It is well known that $\mathfrak{D}(\mathfrak{s}, u)$ is a *DRS* iff $\det(\alpha', \xi, \xi') = 0$.

3. Swept surfaces due to Type-2 $\mathcal{BF}$

This section presents the representation of *SS* using the Type-2 *BF* along the spine curve $\beta(\mathfrak{s})$ as follows: The *SS* related with $\beta(\mathfrak{s})$ corresponds to a one-parameter set of unit spheres centered at $\beta(\mathfrak{s})$. It is simple to find that the intersection between the spheres from the set and the *SS* forms a great circle on the unit sphere within the subspace $Sp\{\xi_1, \xi_2\}$, of $\beta(\mathfrak{s})$. Let $\mathbf{y}$ be the location connecting a point on the curve $\beta(\mathfrak{s})$ with a point on the surface. Then,

$$\mathfrak{M} : \mathbf{y} = \beta(\mathfrak{s}) + \mathbf{r},$$

where the unit vector $\mathbf{r} \in Sp\{\xi_1, \xi_2\}$. Let the angle u be the angle from $\mathbf{r}$ to ζ_1. Thus, we write:

$$\mathbf{r}(u) = \cos u\xi_1 + \sin u\xi_2,$$

which is the distinctive circles of *SS*. By combining the last two equations, we obtain the parameterization of the *SS*,

$$\mathfrak{M} : \mathbf{y}(\mathfrak{s}, u) = \beta(\mathfrak{s}) + \cos u\xi_1(\mathfrak{s}) + \sin u\xi_2(\mathfrak{s}). \tag{3.1}$$

This representation of $\mathfrak{M}$ excludes SS with a constant vector ζ, as its geometric assets are not particularly remarkable and are relatively easy to examine.

Remark 3.1. Perspicuously, if $\beta(\mathfrak{s})$ is a line, then the SS is a circular cylinder with $\beta(\mathfrak{s})$ as the symmetry axis. However, if $\beta(\mathfrak{s})$ is a circle, then SS forms a torus.

We now examine the relationship between the regularity of $\beta(\mathfrak{s})$ and the associated SS. Then,

$$\left.\begin{aligned} \mathbf{y}_s(\mathfrak{s}, u) &= (1 - \epsilon_1 \cos u + \epsilon_2 \sin u)\mathbf{t}, \\ \mathbf{y}_u(\mathfrak{s}, u) &= -\sin u\xi_1 + \cos u\xi_2. \end{aligned}\right\} \tag{3.2}$$

Then,

$$\zeta(\mathfrak{s}, u) := \frac{\mathbf{y}_u \times \mathbf{y}_s}{\|\mathbf{y}_u \times \mathbf{y}_s\|} = \cos u\xi_1 + \sin u\xi_2. \tag{3.3}$$

It can be seen from Eq (3.3) that $\zeta(\mathfrak{s}, u)$ lies in the osculating plane of the curve $\gamma(\mathfrak{s})$, where $<\zeta, \mathbf{b}>=0$. Thus, the normal of the profile curve $\mathbf{q}(u) = (0, \cos u \ \sin u)^T$ and the surface normal $\zeta(\mathfrak{s}, u)$ are identical.

Proposition 3.1. Let $\mathbf{q}$ be a point in the osculating plane of the SC $\gamma(\mathfrak{s})$. The tangent vector of its trajectory $\gamma(\mathfrak{s}) + A(\mathfrak{s})\mathbf{q}(u)$, created by the Type-2 BF, is permanently parallel to $\mathbf{t}$.

The main aim of this article is to establish the following theorem: Note that (LD) refers to locally diffeomorphic.

Theorem 3.1. Under the above conditions, with $\epsilon_1^2 + \epsilon_2^2 \neq 0$. We have:

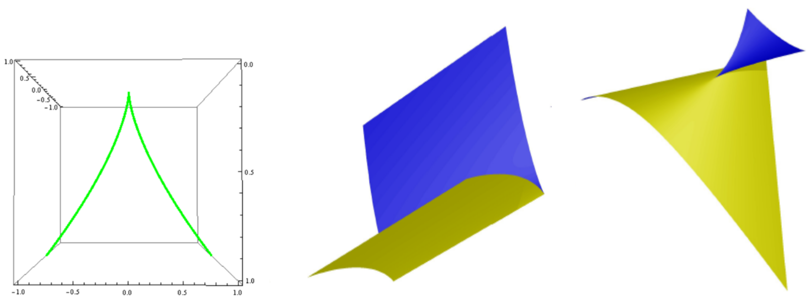
A- (1) $\mathbf{d}(\mathfrak{s})$ is LD to a line $\{\mathbf{0}\}\times\mathbb{R}$ at $\mathfrak{s}_0$ iff $\sigma(\mathfrak{s}_0) \neq 0$;

(2) $\mathbf{d}(\mathfrak{s})$ is LD to the cusp $C \times \mathbb{R}$ at $\mathfrak{s}_0$ iff $\sigma(\mathfrak{s}_0) = 0$, and $\sigma'(\mathfrak{s}_0) \neq 0$.

B- (1) $\mathfrak{M}$ is LD to the cuspidal edge (CE) at $(\mathfrak{s}_0, u_0)$ iff $\mathbf{x} = \pm\mathbf{d}(\mathfrak{s}_0)$, and $\sigma(\mathfrak{s}_0) \neq 0$;

(2) $\mathfrak{M}$ is LD to the swallowtail (SW) at $(\mathfrak{s}_0, u_0)$ iff $\mathbf{x} = \pm\mathbf{d}(\mathfrak{s}_0)$, $\sigma(\mathfrak{s}_0) = 0$, and $\sigma'(\mathfrak{s}_0) = 0$.

Here, $C \times \mathbb{R} = \left\{(x_1, x_2)|x_1^2 = x_2^3\right\} \times \mathbb{R}$, $\mathcal{CE} = \left\{(x_1, x_2, x_3)|\text{x}_1\text{=u, x}_2\text{=v}^2\text{, x}_3\text{=v}^3\right\}$, and $\mathcal{SW} = \left\{(x_1, x_2, x_3)|x_1 = u,\ x_2 = 3v^2 + uv^2,\ x_3 = 4v^3 + 2uv\right\}$ (See Figure 1).



(a) $C \times \mathbb{R}$. (b) $\mathcal{CE}$. (c) $\mathcal{SW}$.

Figure 1. (a) Cusp, (b) Cuspidal edge, and (c) Swallowtail.

3.1. Bishop height functions

Now, we will identify two families of Bishop height functions that will be useful in characterizing the singularities of $\mathfrak{M}$ as follows [1–3]: $\chi : I \times \mathcal{S}^2 \to \mathbb{R}$, by $\chi(\mathfrak{s}, \mathbf{x}) =<\beta(\mathfrak{s}), \mathbf{x}>$. We refer to this as the Bishop height function. We denote $\chi_{\mathbf{x}}(\mathfrak{s}) = \chi(\mathfrak{s}, \mathbf{x})$ for any stationary unit vector $\mathbf{x} \in \mathcal{S}^2$. Additionally, we define $\widetilde{\chi}: I \times \mathcal{S}^2 \times \mathbb{R} \to \mathbb{R}$, by $\widetilde{\chi}(\mathfrak{s}, \mathbf{x}, w) =< \beta, \mathbf{d} > -w$. This is called the extended Bishop height function of $\beta(\mathfrak{s})$. We denote $\widetilde{\chi}_{\mathbf{x}}(\mathfrak{s}) = \widetilde{\chi}(\mathfrak{s}, \mathbf{x}, w)$.

Proposition 3.2. Let $\beta(\mathfrak{s})$ be a conjugate mate curve with $\epsilon_1^2 + \epsilon_2^2 \neq 0$. Then:

(A)

(1)- $\chi_{\mathbf{x}}^{'}(\mathfrak{s}) = 0$ iff $\mathbf{x} = a_1\xi_1 + a_2\xi_2$, and $a_1^2 + a_2^2 = 1$;

(2)- $\chi_{\mathbf{x}}^{'}(\mathfrak{s}) = \chi_{x}^{''}(\mathfrak{s}) = 0$ iff $\mathbf{x} = \pm\mathbf{d}(\mathfrak{s})$;

(3)- $\chi_{\mathbf{x}}^{'}(\mathfrak{s}) = \chi_{x}^{''}(\mathfrak{s}) = \chi_{\mathbf{x}}^{'''}(\mathfrak{s}) = 0$ iff $\mathbf{x} = \pm\mathbf{d}(\mathfrak{s})$, and $\sigma(\mathfrak{s}) = 0$;

(4)- $\chi_{\mathbf{x}}^{'}(\mathfrak{s}) = \chi_{x}^{''}(\mathfrak{s}) = \chi_{\mathbf{x}}^{'''}(\mathfrak{s}) = \chi_{\mathbf{x}}^{(4)}(\mathfrak{s}) = 0$ iff $\mathbf{x} = \pm\mathbf{d}(\mathfrak{s})$, and $\sigma(\mathfrak{s}) = \sigma^{'}(\mathfrak{s}) = 0$;

(5)- $\chi_{\mathbf{x}}^{'}(\mathfrak{s}) = \chi_{x}^{''}(\mathfrak{s}) = \chi_{\mathbf{x}}^{'''}(\mathfrak{s}) = \chi_{\mathbf{x}}^{(4)}(\mathfrak{s}) = \chi_{\mathbf{x}}^{(5)}(\mathfrak{s}) = 0$ iff $\mathbf{x} = \pm\mathbf{d}(\mathfrak{s})$, and $\sigma(\mathfrak{s}) = \sigma^{'}(\mathfrak{s}) = \sigma^{''}(\mathfrak{s}) = 0$.

(B)

(1)- $\widetilde{\chi}_{\mathbf{x}}(\mathfrak{s}) = 0$ iff there exist $< \beta, \mathbf{x} >= w$;

(2)- $\widetilde{\chi}_{\mathbf{x}}(\mathfrak{s}) = \widetilde{\chi}_{x}^{'}(\mathfrak{s}) = 0$ iff there exist $a_1, a_2 \in \mathbb{R}$ such that $\mathbf{x} = \cos u\xi_1 + \sin u\beta_2$, and $< \beta, \mathbf{x} >= w$;

(3)- $\widetilde{\chi}_{\mathbf{x}}(\mathfrak{s}) = \widetilde{\chi}_{\mathbf{x}}^{'}(\mathfrak{s}) = \widetilde{\chi}_{x}^{''}(\mathfrak{s}) = \widetilde{\chi}_{x}^{''}(\mathfrak{s}) = 0$ iff $\mathbf{x} = \pm\mathbf{d}(\mathfrak{s})$, $< \beta, \mathbf{x} >= w$, and $\sigma(\mathfrak{s}) = 0$;

(4)- $\widetilde{\chi}_{\mathbf{x}}(\mathfrak{s}) = \widetilde{\chi}_{\mathbf{x}}^{'}(\mathfrak{s}) = \widetilde{\chi}_{x}^{''}(\mathfrak{s}) = \widetilde{\chi}_{x}^{''}(\mathfrak{s}) = \widetilde{\chi}_{x}^{'''}(s) = 0$ iff $\mathbf{x} = \pm\mathbf{d}(\mathfrak{s})$, $< \beta, \mathbf{x} >= w$, and $\sigma(\mathfrak{s}) = \sigma^{'}(\mathfrak{s}) = 0$;

(5)- $\widetilde{\chi}_{\mathbf{x}}(\mathfrak{s}) = \widetilde{\chi}_{\mathbf{x}}^{'}(\mathfrak{s}) = \widetilde{\chi}_{x}^{''}(\mathfrak{s}) = \widetilde{\chi}_{x}^{''}(\mathfrak{s}) = \widetilde{\chi}_{x}^{'''}(\mathfrak{s}) = \widetilde{\chi}_{x}^{(4)}(\mathfrak{s}) = 0$ iff $\mathbf{x} = \pm\mathbf{d}(\mathfrak{s})$, $< \beta, \mathbf{x} >= w$ and $\sigma(\mathfrak{s}) = \sigma^{'}(\mathfrak{s}) = \sigma^{''}(\mathfrak{s}) = 0$;

(6)- $\widetilde{\chi}_{\mathbf{x}}(\mathfrak{s}) = \widetilde{\chi}_{\mathbf{x}}^{'}(\mathfrak{s}) = \widetilde{\chi}_{x}^{''}(\mathfrak{s}) = \widetilde{\chi}_{x}^{''}(\mathfrak{s}) = \widetilde{\chi}_{x}^{'''}(\mathfrak{s}) = \widetilde{\chi}_{x}^{(4)}(\mathfrak{s}) = \widetilde{\chi}_{x}^{(5)}(\mathfrak{s}) = 0$ iff $\mathbf{x} = \pm\mathbf{d}(\mathfrak{s})$, $< \beta, \mathbf{x} >= w$, and $\sigma(\mathfrak{s}) = \sigma^{'}(\mathfrak{s}) = \sigma^{''}(\mathfrak{s}) = \sigma^{'''}(\mathfrak{s}) = 0$.

Proof. From Eq (2.2), we have $\left\|\mathbf{b}^{'}\right\|^2 \neq 0$ iff $\epsilon_1^2 + \epsilon_2^2 \neq 0$.

(A). (1)- Since $\chi_{\mathbf{x}}^{'}(\mathfrak{s}) =< \mathbf{b}, \mathbf{x} >= 0$, and $\{\mathbf{b}, \xi_1, \xi_2\}$ is an orthonormal frame along $\beta(\mathfrak{s})$, there exist a_1, $a_2 \in \mathbb{R}$ such that $\mathbf{x} = a_1\xi_1 + a_2\xi_2$. Given that $\mathbf{x} \in \mathcal{S}^2$, we get $a_1^2 + a_2^2 = 1$. The converse is also holds.

(2)- Since $\chi_{\mathbf{x}}^{''}(\mathfrak{s}) =< \mathbf{b}^{'}, \mathbf{x} >=< \epsilon_1\xi_1 - \epsilon_2\xi_2, \mathbf{x} > = 0$, we have $a_1\epsilon_1 - a_2\epsilon_2 = 0$. Given that $a_1^2 + a_2^2 = 1$ that $a_1 = \pm\epsilon_2/\sqrt{\epsilon_1^2 + \epsilon_2^2}$, and $a_2 = \mp\epsilon_1/\sqrt{\epsilon_1^2 + \epsilon_2^2}$.

Therefore, we have:

$$\mathbf{x} = \left(\mp\frac{\epsilon_1}{\sqrt{\epsilon_1^2 + \epsilon_2^2}}\left(\frac{\epsilon_2}{\epsilon_1}\xi_1 + \xi_2\right)\right)(s) = \pm\mathbf{d}(s).$$

Therefore, $\chi_{\mathbf{x}}^{'}(\mathfrak{s}) = \chi_{x}^{''}(\mathfrak{s}) = 0$ iff $\mathbf{x} = \pm\mathbf{d}(\mathfrak{s})$.

(3)- Since $\chi_{\mathbf{x}}^{'''}(\mathfrak{s}) =< \mathbf{b}^{''}, \mathbf{x} >=< -\left(\epsilon_1^2 + \epsilon_2^2\right)\mathbf{b} + \epsilon_1^{'}\xi_1 - \epsilon_2^{'}\xi_2, \mathbf{x} > = 0$

$$\mp\frac{\epsilon_1}{\sqrt{\epsilon_1^2 + \epsilon_2^2}}\left(\frac{\epsilon_2\epsilon_1^{'} - \epsilon_1\epsilon_2^{'}}{\epsilon_1}\right)(s) = \pm\left(\frac{\sigma}{\sqrt{\epsilon_1^2 + \epsilon_2^2}}\right)(\mathfrak{s}) = 0.$$

Thus, $\chi_{\mathbf{x}}^{'}(\mathfrak{s}) = \chi_{x}^{''}(\mathfrak{s}) = \chi_{\mathbf{x}}^{'''}(\mathfrak{s}) = 0$ iff $\mathbf{x} = \pm\mathbf{d}(\mathfrak{s})$, and $\sigma(\mathfrak{s}) = 0$.

(4)- Since

$$\left.\begin{array}{l}\chi_{\mathbf{x}}^{(4)}(\mathfrak{s}) =< \mathbf{b}^{'''}, \mathbf{x} >=< -3(\epsilon_1\epsilon_1^{'} + \epsilon_2\epsilon_2^{'})\mathbf{b} + \left(\epsilon_1^{''} - \epsilon_1\left(\epsilon_1^2 + \epsilon_2^2\right)\right)\xi_1 \\ +\left(-\epsilon_2^{''} + \epsilon_2\left(\epsilon_1^2 + \epsilon_2^2\right)\right)\xi_2, \mathbf{x} > = 0\end{array}\right\}$$

by the conditions in (3), we have:

$$\pm\frac{\epsilon_1}{\sqrt{\epsilon_1^2+\epsilon_2^2}}\left(\frac{\left(\epsilon_2\epsilon_1^{'}-\epsilon_1\epsilon_2^{'}\right)^{'}}{\epsilon_1}\right)(\mathfrak{s})=\pm\left(\frac{\sigma^{'}}{\sqrt{\epsilon_1^2+\epsilon_2^2}}\right)(\mathfrak{s})=0.$$

Thus, $\chi_{\mathbf{x}}^{'}(\mathfrak{s})=\chi_{x}^{''}(\mathfrak{s})=\chi_{\mathbf{x}}^{'''}(\mathfrak{s})=\chi_{\mathbf{x}}^{(4)}(\mathfrak{s})=0$ iff $\mathbf{x}=\pm\mathbf{d}(\mathfrak{s})$, and $\sigma(\mathfrak{s})=\sigma^{'}(\mathfrak{s})=0$.

(5)- Since $\overset{(5)}{\chi_{\mathbf{x}}}(s) =< \overset{(4)}{\mathbf{b}}, \mathbf{x} > 0$, we have:

$$\left.\begin{array}{l}<\left(\left(\epsilon_1^2+\epsilon_2^2\right)^2-4\left(\epsilon_2\epsilon_2^{''}+\epsilon_1\epsilon_1^{''}\right)-3\left(\epsilon_1^{'2}+\epsilon_2^{'2}\right)\right)\mathbf{b}+\\ \left(\epsilon_1^{'''}-\epsilon_1\left(\epsilon_1^{'}\epsilon_1+\epsilon_2^{'}\epsilon_2\right)-\epsilon_1^{'}\left(\epsilon_1^2+\epsilon_2^2\right)\right)\xi_1+\\ \left(-\epsilon_2^{'''}-\epsilon_2\left(\epsilon_1^{'}\epsilon_1+\epsilon_2^{'}\epsilon_2\right)+\epsilon_2^{'}\left(\epsilon_1^2+\epsilon_2^2\right)\right)\xi_2,\mathbf{x}>=0.\end{array}\right\}$$

By using the conditions in (4), we have:

$$\pm\frac{\epsilon_1}{\sqrt{\epsilon_1^2+\epsilon_2^2}}\left(\frac{\epsilon_2\epsilon_1^{'''}+\epsilon_1\epsilon_2^{'''}-\left(\epsilon_2\epsilon_1^{'}+\epsilon_1\epsilon_2^{'}\right)\left(\epsilon_1^2+\epsilon_2^2\right)}{\epsilon_1}\right)(s)=0.$$

Therefore, $\chi_{\mathbf{x}}^{'}(\mathfrak{s})=\chi_{x}^{''}(\mathfrak{s})=\chi_{\mathbf{x}}^{'''}(\mathfrak{s})=\chi_{\mathbf{x}}^{(4)}(\mathfrak{s})=\overset{(5)}{\chi_{\mathbf{x}}}=0$ iff $\mathbf{x}=\pm\mathbf{d}(\mathfrak{s})$, and $\sigma(\mathfrak{s})=\sigma^{'}(\mathfrak{s})=\sigma^{''}(\mathfrak{s})=0$.

(B). Using a similar calculation as in the proof of (A), we can get (B) (1). □

From now on, we shall often omit the parameter s.

Proposition 3.3. Let $\beta(\mathfrak{s})$ be a conjugate mate curve with $\epsilon_1(\mathfrak{s})\neq 0$. Then, we have $\sigma(\mathfrak{s})=0$ iff

$$\mathbf{d}(\mathfrak{s})=\pm\frac{\epsilon_1}{\sqrt{\epsilon_1^2+\epsilon_2^2}}\left(\frac{\epsilon_2}{\epsilon_1}\xi_1+\xi_2\right)$$

is a constant vector.

Proof. Suppose that $\epsilon_1\neq 0$. Then, we have:

$$\mathbf{d}^{'}(\mathfrak{s})=\frac{\left(\epsilon_2/\epsilon_1\right)^{'}}{\left(1+\epsilon_2^2/\epsilon_1^2\right)}\left(\frac{-\epsilon_1\xi_1+\epsilon_2\xi_2}{\sqrt{\epsilon_1^2+\epsilon_2^2}}\right)=\frac{1}{\epsilon_1^2}\frac{\sigma(s)}{\left(\sqrt{\epsilon_1^2+\epsilon_2^2}\right)^3}\left(-\epsilon_1\xi_1+\epsilon_2\xi_2\right).$$

Thus, $\mathbf{d}^{'}(\mathfrak{s})=\mathbf{0}$ iff $\sigma(\mathfrak{s})=0$. □

We can also state the following propositions.

Proposition 3.4. Let $\beta(\mathfrak{s})$ be a conjugate mate curve with $\epsilon_1(\mathfrak{s})\neq 0$, and $\sigma(\mathfrak{s})=(\epsilon_2/\epsilon_1)^{'}=0$. Then,

(a) ξ_1 lies on a circle on $\mathcal{S}^2$ centered at the constant vector $\mathbf{d}_0$.

(b) ξ_2 lies on a circle on $\mathcal{S}^2$ centered at the constant vector $\mathbf{d}_0$.

Proof. (a) Suppose that $\epsilon_1\neq 0$, and $\sigma(\mathfrak{s})=(\epsilon_2/\epsilon_1)^{'}=0$. Since

$$\mathbf{d}(\mathfrak{s})=\pm\frac{\epsilon_1}{\sqrt{\epsilon_1^2+\epsilon_2^2}}\left(\frac{\epsilon_2}{\epsilon_1}\xi_1+\xi_2\right),$$

we find that

$$< \mathbf{d}, \xi_1 >= \pm\frac{\epsilon_1}{\sqrt{\epsilon_1^2+\epsilon_2^2}} < \left(\frac{\epsilon_2}{\epsilon_1}\xi_1+\xi_2\right), \xi_1 >= \pm\frac{1}{\sqrt{1+\epsilon_2^2/\epsilon_1^2}}\left(\frac{\epsilon_2}{\epsilon_1}\right) = const.$$

This shows that ξ_2 lies on a circle on $\mathcal{S}^2$ centered at the constant vector $\mathbf{d}_0$.

(b) Suppose that $\epsilon_1 > 0$. By a similar computation, we find that

$$< \mathbf{d}, \xi_2 >= \frac{\epsilon_1}{\sqrt{\epsilon_1^2+\epsilon_2^2}} < \left(\frac{\epsilon_2}{\epsilon_1}\xi_1+\xi_2\right), \xi_2 >= \frac{1}{\sqrt{1+\epsilon_2^2/\epsilon_1^2}} = const$$

is constant. This means that the second Bishop spherical indicatrix, lies on a circle on $\mathcal{S}^2$ centered at the constant vector $\mathbf{d}_0$. □

3.2. Unfolding of functions by one-variable

We employed significant results from the singularity theory for families of function germs [1–3]. Let $\mathcal{G}$: $(\mathbb{R}\times\mathbb{R}^{\mathfrak{r}}, (\mathfrak{s}_0, \mathbf{x}_0)) \to \mathbb{R}$ be a differentiable function, and let $\mathfrak{g}(\mathfrak{s}) = \mathcal{G}_{x_0}(\mathfrak{s}, \mathbf{x}_0)$. Then, $\mathcal{G}$ is called an $\mathfrak{r}$-parameter unfolding of $\mathfrak{g}(\mathfrak{s})$. We say that $\mathfrak{g}(\mathfrak{s})$ has $\mathcal{A}_{\mathfrak{k}}$-singularity at $\mathfrak{s}_0$ if $\mathfrak{g}^{(\mathfrak{p})}(\mathfrak{s}_0) = 0$ for all $1 \le \mathfrak{p} \le \mathfrak{k}$, and $\mathfrak{g}^{(\mathfrak{k}+1)}(\mathfrak{s}_0) \neq 0$. We also consider that $\mathfrak{g}$ has $\mathcal{A}_{\geqslant\mathfrak{k}}$-singularity ($\mathfrak{k} \geqslant 1$) at $\mathfrak{s}_0$. Let the $(\mathfrak{k}-1)$-jet of the partial derivative $\frac{\partial\mathcal{F}}{\partial x_i}$ at $\mathfrak{s}_0$ be $\mathfrak{j}^{(\mathfrak{k}-1)}\left(\frac{\partial\mathcal{G}}{\partial x_i}(\mathfrak{s}, \mathbf{x}_0)\right)(\mathfrak{s}_0) = \Sigma_{\mathfrak{j}=0}^{\mathfrak{k}-1}\mathcal{L}_{ji}(\mathfrak{s}-\mathfrak{s}_0)^{\mathfrak{j}}$ (excluding the constant term), for $\mathfrak{i} = 1, ..., \mathfrak{r}$. Then $\mathcal{G}(\mathfrak{s})$ is called an $\mathfrak{p}$-versal unfolding (VU) if the $\mathfrak{k}\times\mathfrak{r}$ matrix of coefficients $\left(\mathcal{L}_{ji}\right)$ has rank $\mathfrak{k}$ ($\mathfrak{k} \le \mathfrak{r}$). Next, we define useful sets related to the unfolding with the above notations. The discriminant set of $\mathcal{G}$ is the set

$$\mathfrak{D}_{\mathcal{G}} = \left\{\mathbf{x}\in\mathbb{R}^{\mathfrak{r}} | \text{ there exists } \mathfrak{s} \text{ with } \mathcal{G}(\mathfrak{s}, \mathbf{x}) = \frac{\partial\mathcal{G}}{\partial\mathfrak{s}}(\mathfrak{s}, \mathbf{x}) = 0 \text{ at } (\mathfrak{s}, \mathbf{x})\right\}. \tag{3.4}$$

The bifurcation set of $\mathcal{G}$ is the set

$$\mathfrak{B}_{\mathcal{G}} = \left\{\mathbf{x}\in\mathbb{R}^{\mathfrak{r}} \mid \text{there exists } \mathfrak{s} \text{ with } \frac{\partial\mathcal{G}}{\partial\mathfrak{s}}(\mathfrak{s}, \mathbf{x}) = \frac{\partial^2\mathcal{G}}{\partial\mathfrak{s}^2}(\mathfrak{s}, \mathbf{x}) = 0 \text{ at } (\mathfrak{s}, \mathbf{x})\right\}. \tag{3.5}$$

Then, similar to the results of [1–3], we state the following theorem.

Theorem 3.2. Let $\mathcal{G}$: $(\mathbb{R}\times\mathbb{R}^{\mathfrak{r}}, (\mathfrak{s}_0, \mathbf{x}_0)) \to \mathbb{R}$ be an $\mathfrak{r}$-parameter unfolding of $\mathcal{G}(\mathfrak{s})$, which has the $\mathcal{A}_{\mathfrak{k}}$ singularity at $\mathfrak{s}_0$.

Suppose that $\mathcal{G}$ is a $\mathfrak{p}$-versal unfolding.

(a) If $\mathfrak{k} = 1$, then $\mathfrak{D}_{\mathcal{G}}$ is LD to $\{\mathbf{0}\}\times\mathbb{R}^{\mathfrak{r}-1}$, and $\mathfrak{B}_{\mathcal{G}} = \emptyset$;

(b) If $\mathfrak{k} = 2$, then $\mathfrak{D}_{\mathcal{G}}$ is LD to $C\times\mathbb{R}^{\mathfrak{r}-2}$, and $\mathfrak{B}_{\mathcal{G}}$ is $\mathcal{D}$ to $\{\mathbf{0}\}\times\mathbb{R}^{\mathfrak{r}-1}$;

(c) If $\mathfrak{k} = 3$, then $\mathfrak{D}_{\mathcal{G}}$ is LD to $SW\times\mathbb{R}^{\mathfrak{r}-3}$, and $\mathfrak{B}_{\mathcal{G}}$ is $\mathcal{D}$ to $C\times\mathbb{R}^{\mathfrak{r}-2}$.

Hence, we have the following main proposition.

Proposition 3.5. Let $\beta(\mathfrak{s})$ be a conjugate mate curve with $\epsilon_1^2+\epsilon_2^2 \neq 0$. (1). If $\chi_{\mathbf{x}}(\mathfrak{s}) = \chi(\mathfrak{s}, \mathbf{x})$ has an $\mathcal{A}_{\mathfrak{k}}$-singularity ($\mathfrak{k} = 2,\ 3$) at $\mathfrak{s}_0 \in \mathbb{R}$, then χ is a $\mathfrak{p}-VU$ of $\chi_{\mathbf{x}_0}(\mathfrak{s}_0)$. (2). If $\widetilde{\chi}_{\mathbf{x}}(\mathfrak{s}) = \widetilde{\chi}(\mathfrak{s}, \mathbf{x}, w)$ has an $\mathcal{A}_{\mathfrak{k}}$-singularity ($\mathfrak{k} = 2, 3$) at $\mathfrak{s}_0 \in \mathbb{R}$, then $\widetilde{\chi}$ is a $\mathfrak{p}-VU$ of $\widetilde{\chi}_{\mathbf{x}_0}(\mathfrak{s}_0)$

Proof. (1) Since $\mathbf{x} = (x_0,x_1,x_2) \in \mathcal{S}^2$, $x_0^2 + x_1^2 + x_2^2 = 1$, x_0, x_1, and x_2 cannot be all zero. Suppose that $x_2 \neq 0$. Then, by $x_2 = \pm\sqrt{1 - x_0^2 - x_1^2}$, we have that

$$\chi(\mathfrak{s},\mathbf{x}) = x_0\beta_0(\mathfrak{s}) + x_1\beta_1(\mathfrak{s}) \pm \sqrt{1 - x_0^2 - x_1^2}\beta_2(\mathfrak{s}).$$

Thus, we have:

$$\left.\begin{array}{l} \frac{\partial\chi}{\partial x_0} = \beta_0(\mathfrak{s}) \mp \frac{x_0\beta_2(\mathfrak{s})}{\sqrt{1-x_0^2-x_1^2}},\ \frac{\partial\chi}{\partial x_1} = \beta_1(\mathfrak{s}) \mp \frac{x_1\beta_2(\mathfrak{s})}{\sqrt{1-x_0^2-x_1^2}}, \\ \frac{\partial^2\chi}{\partial s\partial x_0} = \beta_0'(\mathfrak{s}) \mp \frac{x_0\beta_2'(\mathfrak{s})}{\sqrt{1-x_0^2-x_1^2}},\ \frac{\partial^2\chi}{\partial s\partial x_1} = \beta_1'(\mathfrak{s}) \mp \frac{x_1\beta_2'(\mathfrak{s})}{\sqrt{1-x_0^2-x_1^2}}. \end{array}\right\}$$

Therefore, the 2-jets of $\frac{\partial\chi}{\partial x_i}$ at $\mathfrak{s}_0$ (i=0, 1) are as follows: Let $\mathbf{x}_0 = (x_{00},x_{10},x_{20}) \in \mathcal{S}^2$, and assume that $x_{20} \neq 0$, then

$$\left.\begin{array}{l} j^1\left(\frac{\partial\chi}{\partial x_0}(\mathfrak{s},\mathbf{x}_0)\right) = \left(\beta_0'(\mathfrak{s}) \mp \frac{x_0\beta_2'(\mathfrak{s})}{\sqrt{1-x_0^2-x_1^2}}\right)(\mathfrak{s}-\mathfrak{s}_0), \\ j^1\left(\frac{\partial\chi}{\partial x_1}(\mathfrak{s},\mathbf{x}_0)\right) = \left(\beta_1'(\mathfrak{s}) \mp \frac{x_1\beta_2'(\mathfrak{s})}{\sqrt{1-x_0^2-x_1^2}}\right)(\mathfrak{s}-\mathfrak{s}_0), \end{array}\right\}$$

and

$$\left.\begin{array}{l} j^2\left(\frac{\partial\chi}{\partial x_0}(\mathfrak{s},\mathbf{x}_0)\right) = \left(\beta_0'(\mathfrak{s}) \mp \frac{x_0\beta_2'(\mathfrak{s})}{\sqrt{1-x_0^2-x_1^2}}\right)(\mathfrak{s}-\mathfrak{s}_0) \\ \quad +\frac{1}{2}\left(\beta_0''(\mathfrak{s}) \mp \frac{x_0\beta_2''(\mathfrak{s})}{\sqrt{1-x_0^2-x_1^2}}\right)(\mathfrak{s}-\mathfrak{s}_0)^2, \\ j^2\left(\frac{\partial\chi}{\partial x_1}(\mathfrak{s},\mathbf{x}_0)\right) = \left(\beta_1'(\mathfrak{s}) \mp \frac{x_1\beta_2'(\mathfrak{s})}{\sqrt{1-x_0^2-x_1^2}}\right)(\mathfrak{s}-\mathfrak{s}_0) \\ \quad \frac{1}{2}\left(\beta_1''(\mathfrak{s}) \mp \frac{x_1\beta_2''(\mathfrak{s})}{\sqrt{1-x_0^2-x_1^2}}\right)(\mathfrak{s}-\mathfrak{s}_0)^2 \end{array}\right\} \tag{3.6}$$

(i) If $\chi_{\mathbf{x}_0}(\mathfrak{s}_0)$ has the $\mathcal{A}_2$-singularity at $\mathfrak{s}_0$, then $\chi_{\mathbf{x}_0}'(\mathfrak{s}_0) = 0$. So, the 1×2 matrix of coefficients $\left(\mathcal{L}_{ji}\right)$ is:

$$\mathcal{A} = \left(\begin{array}{cc} \beta_0'(s) \mp \frac{x_0\beta_2'(s)}{\sqrt{1-x_0^2-x_1^2}} & \beta_1'(\mathfrak{s}) \mp \frac{x_1\beta_2'(s)}{\sqrt{1-x_0^2-x_1^2}} \end{array}\right). \tag{3.7}$$

Suppose that rank($\mathcal{A}$)=0, then we have:

$$\beta_0'(\mathfrak{s}) = \pm\frac{x_{00}\beta_2'(\mathfrak{s})}{\sqrt{1 - x_0^2 - x_1^2}},\ \beta_1'(s) = \pm\frac{x_{10}\beta_2'(\mathfrak{s})}{\sqrt{1 - x_0^2 - x_1^2}}. \tag{3.8}$$

Since $\left\|\beta'(\mathfrak{s}_0)\right\| = \|\mathbf{b}(\mathfrak{s}_0)\| = 1$, we have $\beta_2'(\mathfrak{s}_0) \neq 0$, leading to the following contradiction:

$$\begin{aligned} 0 &= <\left(\beta_0'(\mathfrak{s}_0),\beta_1'(\mathfrak{s}_0),\beta_2'(\mathfrak{s}_0)\right),(x_{00},x_{10},x_{20})> & (3.9) \\ &= \beta_0'(\mathfrak{s}_0)x_{00} + \beta_1'(\mathfrak{s}_0)x_{10} + \beta_2'(\mathfrak{s}_0)x_{20} \\ &= \frac{x_{00}^2\beta_2'(\mathfrak{s}_0)}{x_{20}} + \frac{x_{10}^2\beta_2'(\mathfrak{s}_0)}{x_{20}} + \beta_2'(\mathfrak{s}_0)x_{20} & (3.10) \\ &= \frac{\beta_2'(\mathfrak{s}_0)}{x_{20}}\left(x_{00}^2 + x_{10}^2 + x_{20}^2\right) \end{aligned}$$

$$= \frac{\beta_2'(\mathfrak{s}_0)}{x_{20}} \neq 0.$$

Therefor, rank($\mathcal{A}$) = 1, and χ is the $(\mathfrak{p})$ VU of $\chi_{\mathbf{x}_0}$ at s_0.
(ii) If $\chi_{\mathbf{x}_0}(\mathfrak{s}_0)$ has the $\mathcal{A}_3$-singularity at $\mathfrak{s}_0 \in \mathbb{R}$, then $\chi'_{\mathbf{x}_0}(\mathfrak{s}_0) = \chi''_{\mathbf{x}_0}(\mathfrak{s}_0) = 0$, and by Proposition 3.2:

$$\mathbf{d}(\mathfrak{s}_0) = \pm\frac{\epsilon_1}{\sqrt{\epsilon_1^2+\epsilon_2^2}}\left(\frac{\epsilon_2}{\epsilon_1}\xi_1+\xi_2\right), \tag{3.11}$$

where $\epsilon_1 > 0$, $\sigma'(\mathfrak{s}_0) = 0$, and $\sigma''(\mathfrak{s}_0) \neq 0$. Note that $x_{20} = \pm\sqrt{1-x_0^2-x_1^2}$ It is sufficient to have rank($\mathcal{B}$)= 2, where

$$\mathcal{B} = \begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{pmatrix} = \begin{pmatrix} \beta_0'(\mathfrak{s}) - \frac{x_0\beta_2'(\mathfrak{s})}{x_{20}} & \beta_1'(\mathfrak{s}) - \frac{x_1\beta_2'(\mathfrak{s})}{x_{20}} \\ \beta_0''(\mathfrak{s}) - \frac{x_0\beta_2''(\mathfrak{s})}{x_{20}} & \beta_1''(\mathfrak{s}) - \frac{x_1\beta_2''(\mathfrak{s})}{x_{20}} \end{pmatrix}. \tag{3.12}$$

Then, the determinant of this matrix at $\mathfrak{s}_0$ is

$$\begin{aligned} \det(\mathcal{B}) &= \frac{1}{x_{20}}\left((\beta_1'\beta_2'' - \beta_1''\beta_2'), -(\beta_0'\beta_2'' - \beta_0''\beta_2'), (\beta_0'\beta_1'' - \beta_0''\beta_1')\right)\begin{pmatrix} x_0 \\ x_0 \\ x_{20} \end{pmatrix} \\ &= \frac{1}{x_{20}}\left\{(\beta_0', \beta_1', \beta_2') \times (\beta_0'', \beta_1'', \beta_2'')\right\}\begin{pmatrix} x_0 \\ x_0 \\ x_{20} \end{pmatrix} \\ &= \frac{1}{x_{20}}\left\{\beta' \times \beta''\right\}\begin{pmatrix} x_0 \\ x_0 \\ x_{20} \end{pmatrix} \end{aligned}$$

Since $\beta' = \mathbf{b}$, we have $\beta'' = \epsilon_1\xi_1 - \epsilon_2\xi_2$. Substituting these into the equations, we get

$$\begin{aligned} \det(B) &= \frac{1}{x_{20}}\{\mathbf{b} \times (\epsilon_1\xi_1 - \epsilon_2\xi_2)\}\begin{pmatrix} x_0 \\ x_0 \\ x_{20} \end{pmatrix} \\ &= \frac{1}{x_{20}} < \epsilon_2\xi_1 + \epsilon_1\xi_2, \mathbf{d_0} > \\ &= \frac{1}{x_{20}} < \epsilon_2\xi_1 + \epsilon_1\xi_2, \pm\frac{\epsilon_1}{\sqrt{\epsilon_1^2+\epsilon_2^2}}\left(\frac{\epsilon_2}{\epsilon_1}\xi_1+\xi_2\right) > \\ &= = \pm\frac{\sqrt{\epsilon_1^2+\epsilon_2^2}}{x_{20}} \neq 0. \end{aligned}$$

This means that rank($\mathcal{B}$) = 2.
(2) Under the same notations as in (1), we have

$$\widetilde{\chi}(\mathfrak{s}, \mathbf{x}, x_2) = x_0\beta_0(\mathfrak{s}) + x_1\beta_1(\mathfrak{s}) \pm \sqrt{1-x_0^2-x_1^2}\beta_2(\mathfrak{s}) - x_2. \tag{3.13}$$

We require the 2×3 matrix $\mathcal{D}$ to have the maximal rank, where

$$\mathcal{D} = \begin{pmatrix} \beta_0'(\mathfrak{s}) + \frac{x_{00}\beta_2(\mathfrak{s})}{x_{20}} & \beta_1'(\mathfrak{s}) \mp \frac{x_{10}\beta_2'(\mathfrak{s})}{x_{20}} & -1 \\ \beta_0'' + \frac{x_{00}\beta_2''(\mathfrak{s})}{x_{20}} & \beta_1''(\mathfrak{s}) \mp \frac{x_{10}\beta_2''(\mathfrak{s})}{x_{20}} & 0 \end{pmatrix},$$

By case (1) in Eq. (3.10), the second row of $\mathcal{D}$ does not vanish, so rank($\mathcal{D}$) = 2. □

Proof of Theorem 3.1. (1) By Proposition 3.2, the bifurcation set of $\chi(\mathfrak{s}, \mathbf{x})$ is

$$\mathfrak{B}_\chi = \left\{ \frac{\epsilon_1}{\sqrt{\epsilon_1^2 + \epsilon_2^2}} \left(\frac{\epsilon_2}{\epsilon_1}\xi_1 + \xi_2 \right) | \mathfrak{s} \in \mathbb{R} | \mathfrak{s} \in \mathbb{R} \right\}. \tag{3.14}$$

The assertion (1) of Theorem 3.1 follows from Propositions 3.2 and 3.5, and Theorem 3.2.

The discriminant set of $\widetilde{\chi}(\mathfrak{s}, \mathbf{x})$ is specified as follows:

$$\mathfrak{D}_{\widetilde{\chi}} = \{ \mathbf{x}_0 = \beta + \cos u \xi_1 + \sin u \xi_2 | \mathfrak{s} \in \mathbb{R} \}. \tag{3.15}$$

The assertion (2) of Theorem 3.1 follows from Propositions 3.2 and 3.5, and Theorem 3.2. □

Example 3.1. Let $\gamma(\mathfrak{s})$ be

$$\gamma(\mathfrak{s}) = (\cos \frac{\mathfrak{s}}{\sqrt{2}}, \sin \frac{\mathfrak{s}}{\sqrt{2}}, \frac{\mathfrak{s}}{\sqrt{2}}), \; -\sqrt{2}\pi \le \mathfrak{s} \le \sqrt{2}\pi,$$

Then,

$$\left.\begin{array}{l} \mathbf{t}(\mathfrak{s}) = \frac{1}{\sqrt{2}}(-\sin \frac{\mathfrak{s}}{\sqrt{2}}, \cos \frac{\mathfrak{s}}{\sqrt{2}}, 1), \\ \mathbf{p}(\mathfrak{s}) = (-\cos \frac{\mathfrak{s}}{\sqrt{2}}, -\sin \frac{\mathfrak{s}}{\sqrt{2}}, 0), \\ \mathbf{b}(\mathfrak{s}) = \frac{1}{\sqrt{2}}(\sin \frac{\mathfrak{s}}{\sqrt{2}}, -\cos \frac{\mathfrak{s}}{\sqrt{2}}, 1), \\ \beta(\mathfrak{s}) = (-\cos \frac{\mathfrak{s}}{\sqrt{2}}, -\sin \frac{\mathfrak{s}}{\sqrt{2}}, \frac{\mathfrak{s}}{\sqrt{2}}), \\ \kappa(\mathfrak{s}) = \tau(\mathfrak{s}) = \frac{1}{2}. \end{array}\right\}$$

From $\kappa(\mathfrak{s}) = \frac{1}{2}$, we find $\psi(\mathfrak{s}) = \frac{\mathfrak{s}}{2}$. The transformation matrix is:

$$\begin{pmatrix} \mathbf{b} \\ \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ \sin \frac{\mathfrak{s}}{2} & -\cos \frac{\mathfrak{s}}{2} & 0 \\ \cos \frac{\mathfrak{s}}{2} & \sin \frac{\mathfrak{s}}{2} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{p} \\ \mathbf{b} \end{pmatrix}.$$

Hence, we have

$$\xi_1 = \begin{pmatrix} \xi_{11} \\ \xi_{12} \\ \xi_{13} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \sin \frac{\mathfrak{s}}{\sqrt{2}} \sin \frac{\mathfrak{s}}{2} + \cos \frac{\mathfrak{s}}{2} \cos \frac{\mathfrak{s}}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \cos \frac{\mathfrak{s}}{\sqrt{2}} \sin \frac{\mathfrak{s}}{2} + \cos \frac{\mathfrak{s}}{2} \sin \frac{\mathfrak{s}}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \sin \frac{\mathfrak{s}}{2} \end{pmatrix},$$

$$\xi_2 = \begin{pmatrix} \xi_{21} \\ \xi_{22} \\ \xi_{23} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \sin \frac{\mathfrak{s}}{\sqrt{2}} \cos \frac{\mathfrak{s}}{2} - \sin \frac{\mathfrak{s}}{2} \cos \frac{\mathfrak{s}}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \cos \frac{\mathfrak{s}}{\sqrt{2}} \cos \frac{\mathfrak{s}}{2} - \sin \frac{\mathfrak{s}}{2} \sin \frac{\mathfrak{s}}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \cos \frac{\mathfrak{s}}{2} \end{pmatrix},$$

$$\mathbf{b} = \begin{pmatrix} b_{11} \\ b_{12} \\ b_{13} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \sin \frac{\mathfrak{s}}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \cos \frac{\mathfrak{s}}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix},$$

$$\epsilon_1 = \frac{1}{2} \cos \frac{\mathfrak{s}}{2}, \epsilon_2 = \frac{1}{2} \sin \frac{\mathfrak{s}}{2}.$$

Hence, the geometric invariant $\sigma(\mathfrak{s}) = \frac{1}{8} \neq 0$. The swept surface is (Figure 2)

$$\mathfrak{M} : \mathbf{y}(\mathfrak{s}, u) = \left(-\cos \frac{\mathfrak{s}}{\sqrt{2}}, -\sin \frac{\mathfrak{s}}{\sqrt{2}}, \frac{\mathfrak{s}}{\sqrt{2}}\right) + \begin{pmatrix} \xi_{11} \\ \xi_{12} \\ \xi_{13} \end{pmatrix} \cos u + \begin{pmatrix} \xi_{21} \\ \xi_{22} \\ \xi_{23} \end{pmatrix} \sin u.$$

The Bishop spherical Darboux indicatrix is (Figure 3)

$$\mathbf{d}(\mathfrak{s}) = \sin \frac{\mathfrak{s}}{2} \begin{pmatrix} \xi_{11} \\ \xi_{12} \\ \xi_{13} \end{pmatrix} + \cos \frac{\mathfrak{s}}{2} \begin{pmatrix} \xi_{21} \\ \xi_{22} \\ \xi_{23} \end{pmatrix}.$$

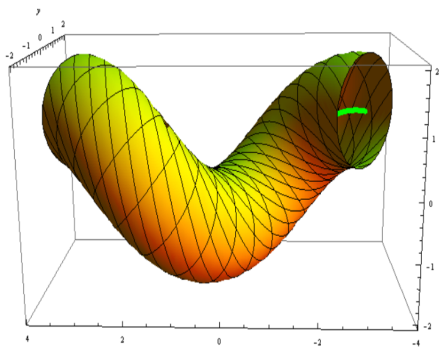


Figure 2. $\mathcal{SS}$ (green line) with conjugate mate curve of helix.

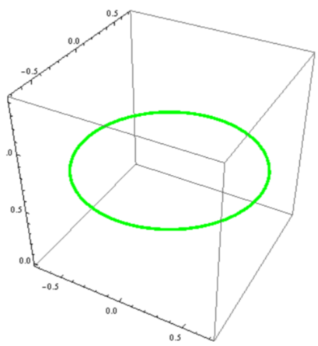


Figure 3. $\mathbf{d}(\mathfrak{s})$ has no cusp.

3.3. Developable surfaces

In the following, we resolve the issue where $\mathbf{q}(u)$ disintegrates into a line. Then, we have the following:

$$\mathbb{D}:\mathbf{P}(\mathfrak{s},u)=\beta(\mathfrak{s})+u\xi_2(\mathfrak{s}),\ u\in\mathbb{R},\tag{3.16}$$

which defines a *DRS*. Similarly, another *DRS* can be defined as:

$$\mathbb{D}^{\perp}:\mathbf{P}^{\perp}(\mathfrak{s},u)=\beta(\mathfrak{s})+u\xi_1(\mathfrak{s}),\ u\in\mathbb{R}.\tag{3.17}$$

It is clear that $\mathbf{P}(\mathfrak{s},0)=\beta(\mathfrak{s})$ (resp. $\mathbf{P}^{\perp}(\mathfrak{s},0)=\beta(\mathfrak{s})$), $0\leq\mathfrak{s}\leq L$, meaning that the surface $\mathbb{D}$ (resp. $\mathbb{D}^{\perp}$) intersects the curve $\beta(\mathfrak{s})$. We can effortlessly have:

$$\mathbb{D}:\mathbf{P}_{\mathfrak{s}}\times\mathbf{P}_u=-(1+u\epsilon_2)\,\xi_1(\mathfrak{s}),$$

and

$$\mathbb{D}^{\perp}:\mathbf{P}^{\perp}_{\mathfrak{s}}\times\mathbf{P}^{\perp}_u=(1-u\epsilon_1)\,\xi_2(\mathfrak{s}).$$

Thus, $\mathbb{D}$ (resp. $\mathbb{D}^{\perp}$) is non-singular at $(\mathfrak{s}_0,u_0)$ if and only if $1+u_0\epsilon_2(\mathfrak{s}_0)\neq0$ (resp. $(1-u_0\epsilon_1(s_0)\neq0)$. Under these conditions, the singularities of the *DRS* $\mathbb{D}$ (resp. $\mathbb{D}^{\perp}$) can be identified by employing ϵ_2(resp. ϵ_1).

Theorem 3.3. Let $\mathbb{D}$ be the *DRS* defined by Eq (3.14). Then:
(1) $\mathbb{D}$ is *LD* to *CE* at $(\mathfrak{s}_0,u_0)$ iff $\epsilon_2(\mathfrak{s}_0)=0$, and $\epsilon_2^{'}(\mathfrak{s}_0)\neq0$;
(2) $\mathbb{D}$ is *LD* to *SW* at $(\mathfrak{s}_0,u_0)$ iff $\epsilon_2(\mathfrak{s}_0)\neq0$, and $\frac{\epsilon_2^{'}(\mathfrak{s}_0)}{\epsilon_2^2(\mathfrak{s}_0)}\neq0$.

Proof. If there exists a parameter $\mathfrak{s}_0$ such that $\epsilon_2(\mathfrak{s}_0)=0$, and $u_0^{'}=\frac{\epsilon_2^{'}(\mathfrak{s}_0)}{\epsilon_2^2(\mathfrak{s}_0)}\neq0$ $(\epsilon_2^{'}(\mathfrak{s}_0)\neq0)$, then $\mathbb{D}$ is *LD* to *CE* at $(\mathfrak{s}_0,u_0)$Thus, assertion (1) holds. Also, if there exists a parameter $\mathfrak{s}_0$ such that $u_0=-\frac{1}{\epsilon_2(\mathfrak{s}_0)}\neq0$, $u_0^{'}=\frac{\epsilon_2^{'}(\mathfrak{s}_0)}{\epsilon_2^2(\mathfrak{s}_0)}=0$, and $\left(\frac{1}{\epsilon_2(\mathfrak{s}_0)}\right)^{''}\neq0$, then $\mathbb{D}$ is LD to *SW* at $(\mathfrak{s}_0,u_0)$. Assertion (2) holds.□

Example 3.2. Using the data from Example 3.1, we have the following:
(1). The equation of $\mathbb{D}$ and its singular locus $\mathfrak{C}(\mathfrak{s})$ are:

$$\mathbb{D}:\mathbf{P}(\mathfrak{s},u)=(-\cos\frac{\mathfrak{s}}{\sqrt{2}},-\sin\frac{\mathfrak{s}}{\sqrt{2}},\frac{\mathfrak{s}}{\sqrt{2}})+u\begin{pmatrix}-\frac{1}{\sqrt{2}}\sin\frac{\mathfrak{s}}{\sqrt{2}}\cos\frac{\mathfrak{s}}{2}-\sin\frac{\mathfrak{s}}{2}\cos\frac{\mathfrak{s}}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\cos\frac{\mathfrak{s}}{\sqrt{2}}\cos\frac{\mathfrak{s}}{2}-\sin\frac{\mathfrak{s}}{2}\sin\frac{\mathfrak{s}}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\cos\frac{\mathfrak{s}}{2}\end{pmatrix},u\in\mathbb{R},$$

and

$$\mathfrak{C}(\mathfrak{s})=(-\cos\frac{\mathfrak{s}}{\sqrt{2}},-\sin\frac{\mathfrak{s}}{\sqrt{2}},\frac{\mathfrak{s}}{\sqrt{2}})+2\begin{pmatrix}-\frac{1}{\sqrt{2}}\sin\frac{\mathfrak{s}}{\sqrt{2}}\cot\frac{\mathfrak{s}}{2}-\cos\frac{\mathfrak{s}}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\cos\frac{\mathfrak{s}}{\sqrt{2}}\cot\frac{\mathfrak{s}}{2}-\sin\frac{\mathfrak{s}}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\cot\frac{\mathfrak{s}}{2}\end{pmatrix}.$$

(2). The equation of $\mathbb{D}^{\perp}$ and its singular locus $\mathfrak{C}^{\perp}(\mathfrak{s})$, respectively, are:

$$\mathbb{D}^{\perp}:\mathbf{P}^{\perp}(\mathfrak{s},u)=(-\cos\frac{\mathfrak{s}}{\sqrt{2}},-\sin\frac{\mathfrak{s}}{\sqrt{2}},\frac{\mathfrak{s}}{\sqrt{2}})+u\begin{pmatrix}-\frac{1}{\sqrt{2}}\sin\frac{\mathfrak{s}}{\sqrt{2}}\sin\frac{\mathfrak{s}}{2}+\cos\frac{\mathfrak{s}}{2}\cos\frac{\mathfrak{s}}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\cos\frac{\mathfrak{s}}{\sqrt{2}}\sin\frac{\mathfrak{s}}{2}+\cos\frac{\mathfrak{s}}{2}\sin\frac{\mathfrak{s}}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\sin\frac{\mathfrak{s}}{2}\end{pmatrix},$$

and

$$\mathfrak{C}^{\perp}(\mathfrak{s}) = (-\cos\frac{\mathfrak{s}}{\sqrt{2}}, -\sin\frac{\mathfrak{s}}{\sqrt{2}}, \frac{\mathfrak{s}}{\sqrt{2}}) + 2\begin{pmatrix} -\frac{1}{\sqrt{2}}\tan\frac{\mathfrak{s}}{2}\sin\frac{\mathfrak{s}}{\sqrt{2}} + \cos\frac{\mathfrak{s}}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}\cos\frac{\mathfrak{s}}{\sqrt{2}}\tan\frac{\mathfrak{s}}{2} + \sin\frac{\mathfrak{s}}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}\tan\frac{\mathfrak{s}}{2} \end{pmatrix}.$$

We address a local part of $\beta(\mathfrak{s})$ as follows: We see that $\epsilon_1'(\mathfrak{s}) = -\frac{1}{2}\sin\frac{\mathfrak{s}}{2} \neq 0$ for $\mathfrak{s} = \pm\pi$. This shows that the *DRS* $\mathbb{D}$ is *LD* to a *CE*, and its singular locus $\mathfrak{C}(\mathfrak{s})$ is *LD* to a line (the red lines); see Figure 4. Moreover, the equation $\epsilon_2'(\mathfrak{s}) = \frac{1}{2}\cos\frac{\mathfrak{s}}{2} = 0$ gives two roots $\mathfrak{s} = \pm\pi$. We can also get that $\epsilon_2''(\pm\pi/2) = \mp\frac{1}{4} \neq 0$. This shows that the *DRS* $\mathbb{D}^{\perp}$ is *LD* to an *SW* and its singular locus $\mathfrak{C}^{\perp}(\mathfrak{s})$ is *LD* to a line (the red lines) at $\mathfrak{s} = \pm\pi$; see Figure 5. Notice that $\mathbb{D}$ and $\mathbb{D}^{\perp}$ intersect along the curve $\beta(\mathfrak{s})$ at $\pi/2$, as shown in Figure 6.

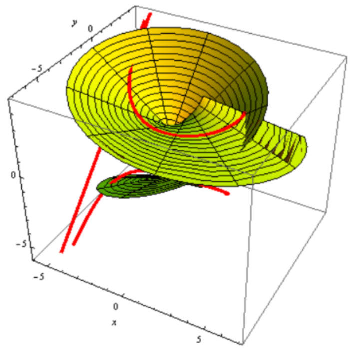


Figure 4. $\mathbf{P}(\mathfrak{s}, u)$ with its singular locus in (red) and $\mathfrak{C}(\mathfrak{s})$.

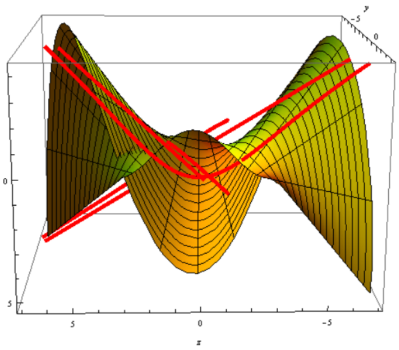


Figure 5. $\mathbf{P}^{\perp}(\mathfrak{s}, u)$ with its singular locus in (red) and $\mathfrak{C}^{\perp}(\mathfrak{s})$.

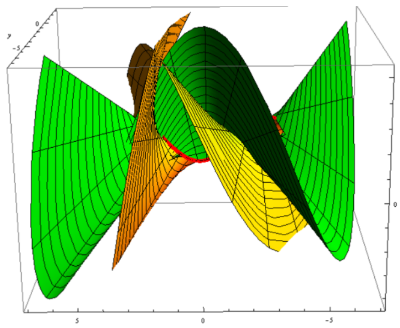


Figure 6. $\mathbf{P}^{\perp}(\mathfrak{s}, u)$ *and* $\mathbf{P}(\mathfrak{s}, u)$ Intersected along spinning curve (in red).

4. Conclusions

This study focused on researching a special type of tube surface, known as the swept surface, in Euclidean 3-space. It was created by a plane curve moving through a conjugate curve of a spatial curve such that the motion of any point on the surface remained orthogonal to the plane. We then attained the adequate and essential situations for this swept surface to be a DRS. Afterwards, the issue of singularity in a DRS was inspected. We also interpreted our main outcomes by giving some representative examples.

Author contributions

Fatemah Mofarreh: Conceptualization, investigation, methodology, and writing—original draft. Rashad A. Abdel-Baky: Investigation, methodology, writing—review, and editing. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The author declares that there is no conflict of interest.

References

1. J. W. Bruce, P. J. Giblin, Generic geometry, *Amer. Math. Mon.*, **90** (1983), 529–545. https://doi.org/10.1080/00029890.1983.11971276
2. J. W. Bruce, P. J. Giblin, *Curves and singularities, 2 Eds*, Cambridge: Cambridge University Press, 1992. https://doi.org/10.1017/CBO9781139172615
3. R. Cipolla, P. J. Giblin, *Visual motion of curves and surfaces*, Cambridge: Cambridge University Press, 2000.
4. K. Teramoto, Parallel and dual surfaces of cuspidal edges, *Differ. Geom. Appl.*, **44** (2016), 52–62. https://doi.org/10.1016/j.difgeo.2015.10.005
5. M. P. do Carmo, *Differential geometry of curves and surface*, Prentice-Hall, Englewood Cliffs, NJ, 1976.
6. H. Pottmann, J. Wallner, *Computational line geometry*, Berlin, Heidelberg: Springer-Verlag, 2001. https://doi.org/10.1007/978-3-642-04018-4
7. L. Jäntschi, The eigenproblem translated for alignment of molecules, *Symmetry*, **11** (2019), 1027. https://doi.org/10.3390/sym11081027
8. L. Jäntschi, Eigenproblem basics and algorithms, *Symmetry*, **15** (2023), 2046. https://doi.org/10.3390/sym15112046
9. Z. Xu, R. S. Feng, J. Sun, Analytic and algebraic properties of canal surfaces, *J. Comput. Appl. Math.*, **195** (2006), 220–228. https://doi.org/10.1016/j.cam.2005.08.002
10. S. Izumiya, K. Saji, N. Takeuchi, Circular surfaces, *Adv. Geom.*, **7** (2007), 295–313. https://doi.org/10.1515/ADVGEOM.2007.017
11. J. S. Ro, D. W. Yoon, Correction to: Tubes of weingarten types in Euclidean 3-space, *J. Chungcheong Math. Soc.*, **27** (2014), 403–404. https://doi.org/10.14403/jcms.2014.27.3.403
12. L. Cui, D. Wang, J. S. Dai, Kinematic geometry of circular surfaces with a fixed radius based on Euclidean invariants, *J. Mech. Des.*, **131** (2009), 101009. https://doi.org/10.1115/1.3212679
13. R. L. Bishop, There is more than one way to frame a curve, *Amer. Math. Mon.*, **82** (1975), 246–251. https://doi.org/10.1080/00029890.1975.11993807
14. N. Clauvelin, W. K. Olson, I. Tobias, Characterization of the geometry and topology of DNA pictured as a discrete collection of atoms, *J. Chem. Theory Comput.*, **8** (2012), 1092–1107. https://doi.org/10.1021/ct200657e
15. K. Shoeemake, Animating rotation with quaternion curves, In: *SIGGRAPH '85: Proceedings of the 12th annual conference on Computer graphics and interactive techniques*, 1985, 245–254. https://doi.org/10.1145/325334.325242
16. H. Zhao, G. Wang, A new method for designing a developable surface utilizing the surface pencil through a given curve, *Prog. Nat. Sci.*, **18** (2008), 105–110. https://doi.org/10.1016/j.pnsc.2007.09.001
17. C. Y. Han. Nonexistence of rational rotation-minimizing frames on cubic curves, *Comput. Aided Geom. D.*, **25** (2008), 298–304. https://doi.org/10.1016/j.cagd.2007.09.006

18. S. Yılmaz, M. Turgut, A new version of Bishop frame and an application to spherical images, *J. Math. Anal. Appl.*, **371** (2010), 764–776. https://doi.org/10.1016/j.jmaa.2010.06.012

19. R. A. Abdel-Baky, Developable surfaces through swepting surfaces, *Bull. Iran. Math. Soc.*, **45** (2019), 951–963. https://doi.org/10.1007/s41980-018-0177-8

20. Y. Li, A. Haseeb, M. Ali, LP-Kenmotsu manifolds admitting η-Ricci solitons and spacetime, *J. Math.*, **2022** (2022), 6605127. https://doi.org/10.1155/2022/6605127

21. Y. Li, F. Mofarreh, R. A. Abdel-Baky, Timelike circular surfaces and singularities in Minkowski 3-space, *Symmetry*, **14** (2022), 1914. https://doi.org/10.3390/sym14091914

22. Y. Li, N. Alluhaibi, R. A. Abdel-Baky. One-parameter Lorentzian dual spherical movements and invariants of the axodes, *Symmetry*, **14** (2022), 1930. https://doi.org/10.3390/sym14091930

23. Y. Li, K. Eren, K. H. Ayvacı, S. Ersoy. Simultaneous characterizations of partner ruled surfaces using Flc frame, *AIMS Mathematics*, **7** (2022), 20213–20229. https://doi.org/10.3934/math.20221106

24. M. D. Siddiqi, M. A. Khan, I. Al-Dayel, K. Masood, Geometrization of string cloud spacetime in general relativity, *AIMS Mathematics*, **8** (2023), 29042–29057. https://doi.org/10.3934/math.20231487

25. S. K. Nurkan, İ. A. Güven, Construction of vectorial moments via direction curves, *AIMS Mathematics*, **8** (2023), 12857–12871. https://doi.org/10.3934/math.2023648

26. Y. Li, A. A. Abdel-Salam, M. K. Saad, Primitivoids of curves in Minkowski plane, *AIMS Mathematics*, **8** (2023), 2386–2406. https://doi.org/10.3934/math.2023123

27. S. Deshmukh, B. Y. Chen, A. Alghnemi, Natural mates of Frenet curves in Euclidean 3-space, *Turk. J. Math.*, **42** (2018), 58. https://doi.org/10.3906/mat-1712-34

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