



Research article

A multi-player game equilibrium problem based on stochastic variational inequalities

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Abstract: In this paper, we studied a generalized Nash equilibrium problem where the constraint conditions were limited to a certain probability. The existence of an equilibrium solution for the vector-valued optimization problem was verified using Ky Fan's inequality and Lusin's theorem, considering the conditions of lower semi-continuity and concavity. Based on the study of the variational inequality method, we proposed a new algorithm to solve the problem. Furthermore, we analyzed the convergence of the algorithm. Finally, we applied the model to examine the economic benefits of digital currency issuance, corroborating the algorithm's effectiveness with a concrete numerical example.

Keywords: multi-player game; nonlinear programming; random variational inequality

Mathematics Subject Classification: 91A10, 90C30, 49J53

1. Introduction

With the increasing interest in comprehending the decision-making process of multi-player games, numerous scholars have endeavored to develop theoretical frameworks for describing player outcomes. They aim to construct mathematical models that accurately depict interactions between participants. In this context, Nash's work has garnered widespread attention. He introduced a game model known as the "Nash equilibrium" (1951, [1–3]), in which each player assumes that the strategies of other players remain constant and makes an optimal choice accordingly. This state of balance is crucial for ensuring stability and predictability during the game, thus attracting significant attention and

discussion. Furthermore, by utilizing variational inequalities as a mathematical tool, researchers can more precisely characterize the constraints and interactions within a system.

Nash equilibrium theory has not only profoundly influenced economics and game theory but also has a wide range of applications in computer science, mathematics, and other fields. For example, Yao [4] uses a heuristic method combining backward induction and exhaustion to solve the two-stage decision-making process of family car purchase. Deligiannis [5] utilizes convex optimization methods and noncooperative game-theoretic techniques to investigate the power distribution scheme of a multi-static system and a multi-input multi-output radar network, based on the estimation of the signal-to-interference-plus-noise ratio (SINR). Tominac [6] formulates a non-convex nonlinear programming problem (NLP) and extends it to non-convex mixed integer nonlinear programming (MINLP) for refinery strategic production planning. Glizer et al. [7] studied a regularization approach to solving singular two-person Nash equilibrium games with state and control delays. Passacantando and Raciti [8] investigated the continuity of the Nash equilibrium in network games. Dechboon et al. [9] proposed a generalized F-contraction mapping for coupled fixed point theorems and applied it to a three-player game.

The scope of variational inequality theory is extensive, encompassing fields such as management science and engineering sciences. Over the years, there has been substantial focus on the theory and algorithms of deterministic variational inequalities [10]. This paper addresses establishing an equilibrium traffic flow distribution within a deterministic urban network, given the known traffic demand on a specified set of routes. Additionally, [11] determines the impact of expected equilibrium distributions based on the valuation of assets and liabilities across various financial instruments, leading to a quasi-variational formula characterizing equilibrium in dynamic financial models. Furthermore, [12] studies the impact of new competitors on market structure and consumer choice by analyzing strategic adjustments and market dynamics of existing platforms. For additional instances of deterministic variational inequalities, please consult [13–16].

In reality, we frequently encounter situations with uncertain properties. For instance, in business dealings, our goal is to achieve a certain return probability within the range of $[0, 1]$, subject to uncertainty influenced by market policies, distributors, consumers, and other factors. However, there is no clear formula for calculating this specific return probability. This paper addresses this issue by studying a class of Nash equilibrium models with uncertain constraint sets to establish a more flexible model. Drawing from the classical mathematical tools discussed by Cavazzuti and Pappalardo et al. [17], we transform the multi-player game problem into a vector variational inequality problem and define its equilibrium. Additionally, we introduce the Knaster-Kuratowski-Mazurkiewicz (KKM) theorem, Lusin's theorem, and their related definitions to consider the existence of equilibrium in transformation problems, accounting for conditions of lower semi-continuity and concavity based on the established model. Using this relationship, we derive the existence of a weak Pareto-Nash equilibrium for the multi-player vector optimization problem under consideration. To solve this problem, we propose a new algorithm based on the Lagrange multiplier method, allowing us to obtain a set of probabilities corresponding to different objective returns, thus ensuring that achieving expected returns in a multi-player game is reasonable.

The structure of this paper is as follows: The second part presents the generalized Nash equilibrium model of multi-player games through a concrete example and defines its nonlinear programming maximization problem. The third part defines weak Pareto-Nash equilibrium and other related

definitions and deduces the existence of the weak Pareto-Nash equilibrium. In the fourth section, we introduce a new algorithm based on the Lagrange function for solving constrained uncertain multi-player game problems and analyze the convergence of this algorithm. In the conclusion, we apply this model to the economic benefit analysis of digital currency issuance, demonstrating its feasibility and effectiveness through concrete examples. Finally, in the sixth part, we summarize our research results and describe future tasks.

2. The model

In real life, conflicts between objectives and behaviors are quite common. Typically, these issues involve multiple participants and multiple objectives. The primary problem we need to solve is how to maximize our benefits without compromising the interests of other players. The Nash equilibrium theory has become a powerful tool for scholars to address this issue. Next, let us first review the classical generalized Nash equilibrium (GNEP).

Definition 1. [18] Suppose there are $N = \{1, \dots, m\}$ players, each denoted by player i with control variable $x_i \in R$. We represent the vector composed of all decision variables as $x := (x_1, x_2, \dots, x_m)$, and x_{-i} denotes the vector composed of decision variables of all players except player i . To emphasize player i 's strategy, we use (x_i, x_{-i}) instead of x . Player i 's strategy is $x_i \in V_i$, which relies on the strategies of other players. Given the strategies x_{-i} of the other players, if player i 's payoff function is expressed by $R(x_i, x_{-i})$, then player i aims to choose a strategy x_i , along with strategies x_{-i}^* of the other players that maximize their payoff:

$$\begin{cases} \max & R_i(x_i, x_{-i}^*), \\ \text{s.t.} & x_i \in V_i(x_{-i}^*). \end{cases} \quad (2.1)$$

Given any x_{-i} , let $S(x_{-i}^*)$ represent the solution set of this problem. The GNEP aims to identify a vector $x^* := (x_1^*, \dots, x_m^*)$ such that $x_i^* \in S(x_{-i}^*), \forall i \in N$.

Next, in conjunction with the definition of the generalized Nash equilibrium, we will use an example to show how to rewrite a multi-player game problem with uncertain constraints into a variational inequality problem with random variables.

With the rapid ascent of digital currency, central banks have begun actively exploring the issuance of their own central bank digital currency (CBDC) to align with the developmental trajectory of the digital economy. Each country wants to secure its monetary sovereignty by issuing its digital currency, while also trying to take a leading position in the global digital currency market. However, due to the rapid changes in the technological, regulatory, and market environment in the digital currency space, competition among central banks has become more intense.

Assume the i -th ($i = 1, \dots, m$) central bank chosen strategy is denoted by x_i , where it is assumed that the maximum issuance for each central bank per day is t . When the issuance reaches t , the profit for that day will reach its maximum value, but excessive issuance will affect the economic benefits of the next day. Therefore, to prevent excessive issuance of virtual currency leading to inflation, it is necessary to find an optimal issuance amount x^* . Let u represent the relationship between the issuance amount and the benefits generated. Thus, $u(x_i)(i = 1, \dots, m)$ denotes the economic benefit when the issuance volume is x for the i -th central bank, where x is a random variable constrained to the interval $[0, t]$.

Next, we can construct a game process, assuming that the return per unit of virtual currency issued is

$$r_i(x_i, x_{-i}),$$

and the total operating costs and marketing expenses are

$$m_i(x_i),$$

so, the total revenue for the enterprise in the end is

$$R_i(x_i, x_{-i}) = r_i(x_i, x_{-i}) \cdot x_i - m_i(x_i), i = 1, \dots, m.$$

This problem of the game can be rewritten as follows:

$$R_i(x_i, x_{-i}) \geq R_i(y_i, x_{-i}). \quad (2.2)$$

Therefore, according to Proposition 1.4.2 in Facchinei and Pang (2003) [19], the generalized Nash equilibrium problem for problem (2.2) can be reformulated as the following variational inequality: Find an $x \in V_l$ such that

$$\langle H(x), y - x \rangle \geq 0, \forall y \in V_l, \quad (2.3)$$

where V_l represents the feasible domain of the processing capacity that satisfies the constraint condition. In this model, the firm aims to maximize $P_i(R(x_i) \geq e_{l_i})$, ensuring it meets its expected payoff $e = (e_{l_1}, \dots, e_{l_m})$ by combining various factors, where P represents a probability function. Thus, the following set is defined:

$$V_l = P_i(R(x_i) \geq e_{l_i}) \geq l_i. \quad (2.4)$$

Then, problem (2.2) can be reformulated as follows:

$$\begin{cases} \max & \langle H(x), y - x \rangle \geq 0 \\ \text{s.t.} & P_i(R(x_i) \geq e_{l_i}) \geq 0 \end{cases}, \quad (2.5)$$

where $x = (x_1, x_2, \dots, x_m)$, H represents the marginal benefit, which is denoted as the partial derivative of R_i with respect to x_i , and there is

$$H(x) = \frac{\partial R}{\partial x} = \left(-\frac{\partial R_1(x_1, x_{-1})}{\partial x_1}, \dots, -\frac{\partial R_m(x_m, x_{-m})}{\partial x_m} \right).$$

This variational inequality formulation encapsulates the equilibrium condition by ensuring that no player can unilaterally improve their payoff by deviating from their current strategy, given the strategies of the other players and the constraints. The function $H(x)$ integrates the gradients of the Lagrangians associated with each player's optimization problem, ensuring that the equilibrium respects both the maximization of payoffs and the constraints influenced by uncertainty.

3. Analysis of the existence of the equilibrium

In this section, we define the weak Pareto-Nash equilibrium of vector-variational inequality and introduce several other theorems and definitions to construct the model.

Definition 2. Let $F = (f_1, \dots, f_m) : R^m \rightarrow R^m$ represent the target revenue function for the multi-player game. If there is $x^* = (x_1^*, \dots, x_m^*)$, such that

$$F(y_i, x_{-i}^*) - F(x_i^*, x_{-i}^*) \notin \text{int}R_+^m, \forall y_i \in V_{l_i},$$

then x^* is called a weak Pareto-Nash equilibrium of the multi-player game.

Definition 3. Let V_l be a nonempty subset of R^n , and $\psi : V_l \rightarrow R^k$ be a vector-valued function. If for $\forall x_1, x_2 \in V_l, \forall \lambda \in (0, 1), \forall y_0 \in R^k$, and $\psi(x_1) \in y_0 + R_+^k, \psi(x_2) \in y_0 + R_+^k$, there is

$$\psi(\lambda x_1 + (1 - \lambda)x_2) \in y_0 + R_+^k,$$

then ψ is said to be quasi-concave on x in R_+^k .

Definition 4. Consider V_l as a nonempty subset of R^n , and let $\phi : V_l \rightarrow R^m$ be a vector-valued function. $\forall x \in V_l$, if for any open neighborhood B of 0 in R^m , there is an open neighborhood $O(x)$ of $x \in V_l$ such that for all $x' \in O(x)$, there exists

$$\phi(x') \in \phi(x) + B + R_+^m,$$

then ϕ is said to be lower semicontinuous on V_l in R_+^m .

Definition 5. Generalized KKM mapping: Let V_l denote a nonempty compact set and G denote a Banach space. Let $F : V_l \rightarrow 2^G$ be a set-valued mapping and a nonempty compact set for every $x \in V_l$, and if for any finite set $\{x_1, x_2, \dots, x_n\} \subset V_l$ there exists a corresponding finite set $\{y_1, y_2, \dots, y_n\} \subset G$ such that for any subset $\{y_{i_1}, y_{i_2}, \dots, y_{i_k}\} \subset \{y_1, y_2, \dots, y_n\}, 1 \leq k \leq n$ there is $\text{Co}\{y_{i_1}, y_{i_2}, \dots, y_{i_k}\} \subset \cup_{j=1}^k F(x_{i_j})$ (where $\text{Co}\{y_{i_1}, y_{i_2}, \dots, y_{i_k}\}$ is the convex hull of $\{y_{i_1}, y_{i_2}, \dots, y_{i_k}\}$), then $F : V_l \rightarrow 2^G$ is said to be a generalized KKM mapping.

Next, in order to prove the existence of the solution, we introduce the Lusin's theorem and KKM theorem.

Theorem 1. [20] Let "meas" represent the Lebesgue measure on R^m , let $h : R^m \rightarrow R^m$ be a measurable function, and B is a measurable set in R^m with finite measure, i.e., $\text{meas}(B) < \infty$. For any $\varepsilon > 0$, there exists a closed set $B_1 \subseteq B$ with $\text{meas}(B \setminus B_1) < \varepsilon$, such that the restriction of h to B_1 is continuous.

Theorem 2. [21] KKM theorem: Let V_l denote a nonempty set and G denote a topological space. Assume for every $N = t_0, t_1, \dots, t_n \in \langle T \rangle$ there is a mapping $F : K \rightarrow 2^T$ that is a closed-valued generalized KKM mapping. If there exists $M \in \langle V_l \rangle$, such that $\cap_{x \in M} F(x)$ is a compact subset of G , then

$$\cap_{x \in K} F(x) \neq \emptyset.$$

In Theorem 3, we will establish an existence of a solution to a multi-player vector optimization problem.

Theorem 3. Let V_{l_i} be a nonempty bounded closed convex set in R^m , and $V_l = \prod_{i \in N} V_{l_i}$. Consider the mapping $F_i = (f_1, \dots, f_m) : V_l \rightarrow R^m$, which satisfies the following conditions:

(1) The vector-valued mapping $F_i = \{f_1, \dots, f_m\} : V_l \rightarrow R^m$ is continuous for any $i = 1, \dots, m$.

(2) The function $f_i(y_i, x_{-i})$ is concave on V_l for every fixed $x_{-i} \in V_{l_{-i}}, i = 1, \dots, m$.

Then, there exists a weak Pareto-Nash equilibrium for the multi-player vector optimization problem (2.5).

Proof. To begin, consider the function $F(x, y)$ defined by:

$$F(x, y) = \left[\sum_{i=1}^m f_i(y_i, x_{-i}) - \sum_{i=1}^m f_i(x_i, x_{-i}) \right], i = 1, \dots, m,$$

where $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_m)$ are points in V_l . Since $f_i(y_i, x_{-i})$ is a concave function on V_{l_i} and concave functions are lower semicontinuous on their domains, for any fixed x_{-i} , $f_i(y_i, x_{-i})$ as a function of y_i is lower semicontinuous. Specifically, let $B = ((-\varepsilon, \varepsilon)_1, \dots, (-\varepsilon, \varepsilon)_m)$ be the set of m open neighborhoods in R^m , for any $y_i \in V_{l_i}$, and for $\varepsilon > 0$ there exists an open neighborhood $O(y_i)$ such that

$$\begin{aligned} F(x', y) &\in F(x, y) + B + R_+^m \\ &= ((f_1(y_i, x_{-i}) - \varepsilon, +\infty), \dots, (f_m(y_i, x_{-i} - \varepsilon, +\infty))). \end{aligned}$$

For any $i = 1, \dots, m, x' \in O(x)$, there exists $f_i(y_i', x_{-i}) > f_i(y_i, x_{-i}) - \varepsilon$, so $F(x, y)$ is lower semicontinuous on V_l .

Next, we prove that $F(x, y)$ is concave on V_l .

For each $f_i(y_i, x_{-i})$, since it is concave with respect to y_i , it satisfies:

$$f_i(\lambda y_{i_1} + (1 - \lambda)y_{i_2}, x_{-i}) \geq \lambda f_i(y_{i_1}, x_{-i}) + (1 - \lambda)f_i(y_{i_2}, x_{-i}),$$

where $y_{i_1}, y_{i_2} \in V_{l_i}$. Since $F(x, y)$ is a linear combination of the concave functions $f_i(y_i, x_{-i})$, it also inherits this concavity. Thus, $F(x, y)$ is concave on V_l .

To further demonstrate the existence of the weak Pareto-Nash equilibrium of problem (2.5), next we use the KKM theorem and Ky Fan's inequality to prove the existence of a weak Pareto-Nash equilibrium. For each $y \in V_l$, define the set $\Gamma(y)$ as

$$\Gamma(y) = \{x \in V_l | F(x, y) \notin \text{int}R_+^m\}, \quad (3.1)$$

where $\text{int}R_+^m$ denotes the interior of R_+^m , which is the set of vectors in \mathbb{R}^m with all positive coordinates.

Suppose $\bigcap_{y \in V_l} \Gamma(y) = \emptyset$. By the KKM theorem, if $\Gamma(y)$ is a closed set and satisfies certain conditions, so the $\bigcap_{y \in V_l} \Gamma(y)$ cannot be empty, then there exists a $x \in V_l$ such that $F(x, y) \notin \text{int}R_+^m$. According to Lusin's theorem, there exists a set $\tilde{V} \subseteq V_l$ with $m(\tilde{V}) > 0$, where $m(\tilde{V}) > 0$ represents the Lebesgue measure. For any $\varepsilon > 0$, there is a subset $\tilde{V}_0 \subset \tilde{V}$ such that $m(\tilde{V} \setminus \tilde{V}_0) \leq \varepsilon$, and F is also a continuous function on \tilde{V}_0 . Since $F(x, y)$ is lower semicontinuous on V_l and \tilde{V}_0 is a closed subset of V_l , the lower semicontinuity extends to the whole V_l . Next, we prove that for any $y = (y_1, \dots, y_m) \in V_l$, we have

$$\text{co}\{y_1, \dots, y_m\} \subset \bigcup_{i=1}^m \Gamma(y_i).$$

Here we use the method of proof by contradiction. Assume $\hat{y} = \sum_{i=1}^m \lambda_i x_i \in V_l$ such that $\hat{y} \notin \Gamma(y_i)$ for some $\lambda_i \geq 0, \sum_{i=1}^m \lambda_i = 1$. Then, for any $y \in V_l$, there exists $H(\hat{y}, y_i) \in \text{int}R_+^m$. For $x_1, x_2 \in V_l$, then

similarly we have $F(\hat{y}, x_1) \in \text{int}R_+^m, F(\hat{y}, x_2) \in \text{int}R_+^m$, then for $\lambda \in (0, 1)$, we have

$$F(\hat{y}, \lambda x_1 + (1 - \lambda)x_2) \in \text{int}R_+^m.$$

Let \bar{B} be an open neighborhood which contains 0 in R_+^m such that $F(\hat{y}, x_1) + \bar{B} \in \text{int}R_+^m, F(\hat{y}, x_2) + \bar{B} \in \text{int}R_+^m$. Then, we take any open neighborhood B of 0 in R_+^m again, so there is a $\delta > 0$ such that $O(0, \delta) \subset B$. Let $\bar{B} = O(0, \delta)$, then $\bar{B} \subset B$ and there is $\bar{B} = -\bar{B}$. For any $z \in \text{int}R_+^m$, when c_0 is sufficiently small, there exists $c_0z \in \text{int}R_+^m$ and $c_0z \in \bar{B}$, then we can obtain that $-c_0z \in V_l$. Let $c = -c_0z$, then $F(\hat{y}, x_1) + c \in \text{int}R_+^m, F(\hat{y}, x_2) + c \in \text{int}R_+^m$, and given that F is quasi-convex on V_l , there is

$$\begin{aligned} F(\hat{y}, \lambda x_1 + (1 - \lambda)x_2) &\in -c + R_+^m \\ &= c_0z + R_+^m \subset \text{int}R_+^m + R_+^m = \text{int}R_+^m. \end{aligned}$$

This implies that if $\hat{y} \notin \Gamma(y_i)$, then

$$F(\hat{y}, x_1) + c \in \text{int}R_+^m,$$

for some $c \in \text{int}R_+^m$, contradicting $\hat{y} \notin \Gamma(y_i)$.

So, the intersection $\cap_{y \in V_l} \Gamma(y) \neq \emptyset$, and there exists a point $x^* = (x_1^*, \dots, x_m^*) \in \cap_{y \in V_l} \Gamma(y)$. Thus for all $y \in V_l, F(x^*, y) \notin \text{int}R_+^m$. Hence, there exists a weak Pareto-Nash equilibrium x^* such that

$$F(x^*, y) = \left[\sum_{i=1}^m f_i(y_i, x_{-i}) - \sum_{i=1}^m f_i(x_i^*, x_{-i}) \right] \notin \text{int}R_+^m.$$

Thus, we can conclude that the vector-valued function $F(x, y)$ has a weak Pareto-Nash equilibrium such that (2.5) holds. \square

4. Algorithm and convergence analysis for problem (2.5)

Then, by proving Theorem 3, we establish that problem (2.5) has a weak Pareto-Nash equilibrium. Next, for a given e_l , we aim to find the optimal point (x^*, l^*) . Thus, we consider the nonlinear programming problem with only inequality constraints:

$$\begin{cases} \max & f(x, l), \\ \text{s.t.} & q(x, l) \leq 0, \end{cases} \quad (4.1)$$

where $f(x, l) : V_l \times R \rightarrow R^m$ and $q(x, l) : V_l \times R \rightarrow R^m$ are second-order continuously differentiable scalar and vector functions, respectively. Let (x^*, l^*) satisfy the constraint, and $I_m^i(x^*, l^*)$ represents the indicator set of i corresponding to $I_m^i(x^*, l^*)$, i. e.,

$$I_m^i(x^*, l^*) = \{i | q_i(x^*, l^*) = 0, i = 1, \dots, m\}.$$

If the gradient $\nabla q_i(x^*, l^*), i \in I_m^i(x^*, l^*)$ is linearly independent, (x^*, l^*) is called a regular point of problem (4.1).

The classical Lagrange function is

$$L(x, l, \lambda) = f(x, l) + \lambda q(x, l),$$

where $\lambda = \zeta^2$, and ζ represents the Lagrange multiplier.

Theorem 4. (Karush-Kuhn-Tucker optimality condition) [22] Let (x^*, l^*) be the local optimal solution and the regular point of the problem (4.1), then there exists a unique vector λ^* , such that

$$\begin{aligned}\nabla L(x^*, l^*, \lambda^*) &= 0, \\ \lambda_i^* q_i(x^*, l^*) &= 0, \\ q(x^*, l^*) &\leq 0, i = 1, \dots, m.\end{aligned}$$

Next, we define the generalized Lagrange function as

$$\begin{aligned}G_\sigma(x, l, \lambda) &= f(x, l) + \lambda^T q(x, l) + \frac{1}{2} \sigma \lambda q^T(x, l) q(x, l) \\ &= L(x, l, \lambda) + \frac{1}{2} \sigma \lambda q^T(x, l) q(x, l),\end{aligned}$$

where σ is a penalty parameter, and without losing generality, we assume that the penalty factor σ is an increasing bounded sequence, i.e., $\sigma_0 \leq \sigma \leq \sigma_{max}$, in order to strengthen the punishment for violating the constraint gradually in the optimization process, so as to converge to the solution satisfying the constraint.

Then, the gradient of the generalized Lagrange function with respect to (x, l) and the Hessian matrix are respectively,

$$\begin{aligned}\nabla_x G_\sigma(x, l, \lambda) &= \nabla_x L(x, l, \lambda) + \sigma \cdot \lambda \cdot q(x, l) \cdot \nabla_x q(x, l), \\ \nabla_{xx}^2 G_\sigma(x, l, \lambda) &= \nabla_{xx}^2 L(x, l, \lambda) + \sigma \cdot \lambda \cdot \nabla_x q(x, l) \cdot \nabla_x q^T(x, l) + \sigma \cdot \lambda \cdot q(x, l) \cdot \nabla_{xx}^2 q(x, l).\end{aligned}$$

The specific iteration process is as follows:

Step 1. Initialization:

- Selects the initial point $(x_0, l_0, \lambda_0, \sigma_0)$.

Step 2. Iterative process:

- For each iteration step $w = w + 1, w = 0, 1, 2, \dots$, perform the following steps:

$$\begin{aligned}x_{w+1} &= x_w - \eta \cdot \nabla_x L(x_w, l_w, \lambda_w), \\ l_{w+1} &= l_w - \eta \cdot \nabla_l L(x_w, l_w, \lambda_w), \\ \lambda_{w+1} &= \lambda_w \cdot [1 + \sigma_{w+1} \cdot q(x_w, l_w)], \\ \sigma_{w+1} &= \rho \cdot \sigma_w,\end{aligned}$$

where $\eta > 0$ is the iterative step, $\rho > 1$ is the growth rate of the penalty factor, and $\nabla L(x_{w+1}, l_{w+1}, \lambda_{w+1})$ is the gradient of the Lagrange function with respect to the constraint L .

Step 3. Convergence judgment:

- When $w > 1$ and $|(x_{w+1}, l_{w+1}, \lambda_{w+1}) - (x_w, l_w, \lambda_w)| < \varepsilon$, we output the current $(x_{w+1}, l_{w+1}, \lambda_{w+1})$ as a weak Pareto Nash equilibrium.
- Otherwise, return to Step 2 for the next iteration.

Next, to introduce the convergence analysis of the above algorithm, we introduce the subsequent theorem:

Theorem 5. [23, 24] Let (x^*, l^*) be the regular point of problem (4.1), and there exists a vector λ_i^* satisfying

$$\begin{aligned}\nabla_x L(x^*, l^*, \lambda^*) &= 0, \\ \lambda_i^* q_i(x^*, l^*) &= 0, \\ q_i(x^*, l^*) &\leq 0, i = 1, \dots, m.\end{aligned}$$

For any nonzero vector g , such that $\nabla q_i^T(x^*, l^*)g = 0, i \in I_m^i(x^*, l^*)$, there is

$$g^T [\nabla^2 f(x^*, l^*) + \lambda^* \nabla^2 q(x^*, l^*)] g < 0.$$

In addition, λ_i^* satisfies the strict complementarity hypothesis

$$\lambda_i^* > 0, \forall i \in I_m^i(x^*, l^*).$$

Then we call (x^*, l^*) the strictly local maximum point of problem (4.1).

Theorem 6. [25] Let M be an $n \times n$ symmetric matrix, and R be a semi-negative definite symmetric matrix of the same size. If for any nonzero vector p satisfying $p^T R p = 0$, we have $p^T M p < 0$, then there exists a scalar $\sigma > 0$ such that

$$M + \sigma R < 0.$$

So, according to Theorems 5 and 6, we can know that there exists a scalar $\sigma > 0$, such that

$$\nabla_x G_\sigma(x^*, l^*, \lambda^*) = \nabla_x L(x^*, l^*, \lambda^*) + \sigma \cdot \lambda^* \cdot q(x^*, l^*) \cdot \nabla_x q(x^*, l^*) = 0, \quad (4.2)$$

and

$$\nabla_{xx}^2 G_\sigma(x^*, l^*, \lambda^*) = \nabla_{xx}^2 L(x^*, l^*, \lambda^*) + \sigma \cdot \lambda^* \cdot \nabla_x q(x^*, l^*) \cdot \nabla_x q^T(x^*, l^*) < 0. \quad (4.3)$$

Based on these two formulas, we can conclude that (x^*, l^*) is a strictly local maximum point of $G_\sigma(\cdot, \cdot, \lambda^*)$.

Assuming all the inequality constraints are active, a similar proof to the case with equalities can be applied, leading to the following conclusions.

Theorem 7. Let (x^*, l^*) be the regular point of problem (4.1), and there is $\sigma > 0$, such that

$$\nabla_{xx}^2 G_\sigma(x^*, l^*, \lambda^*) < 0,$$

so there are positive numbers δ, ε and subset $\mathbb{D} \subset R^{m+1}$ where

$$\mathbb{D} = \{(\lambda, \sigma) : \|\lambda - \lambda^*\| \leq \delta, \sigma_0 \leq \sigma \leq \sigma_{max}\}. \quad (4.4)$$

Therefore, $\forall (\lambda, \sigma) \in \mathbb{D}$, maximizes the problem

$$\begin{cases} \max & G_\sigma(x, l, \lambda), x \in V_l, \\ \text{s.t.} & (x, l) \in B(S(x^*, l^*), \varepsilon), \end{cases} \quad (4.5)$$

where $S = \{(x^*, l^*) | P(u(x^*) \geq e_l) \geq l^*\}$, and $B(S(x^*, l^*), \varepsilon)$ represents the ε neighborhood of $S(x^*, l^*)$. Thus, there is a unique maximum point (x^*, l^*) in the internal continuity of \mathbb{D} .

Proof. Let $\Xi = \text{diag}(\lambda^*)$ represent the vector diagonalization operator. So, consider the following two equations:

$$\begin{aligned}\nabla f(x^*, l^*) + \lambda^* \nabla q(x^*, l^*) &= 0, \\ \Xi(\lambda^*)q(x^*, l^*) + \frac{\lambda - \lambda^*}{\sigma} &= 0.\end{aligned}\quad (4.6)$$

Next, let $\rho = \frac{\lambda - \lambda_0}{\sigma}$, $\tau = \frac{1}{\sigma}$, then formula (4.7) can be redrafted as

$$\begin{aligned}\nabla f(x^*, l^*) + \lambda \cdot \nabla q(x^*, l^*) &= 0, \\ \Xi(\lambda^*)q(x^*, l^*) + \rho + \tau \cdot (\lambda_0 - \lambda^*) &= 0,\end{aligned}\quad (4.7)$$

and the Jacob matrix at (x^*, l^*) and λ^* is

$$\begin{bmatrix} \nabla_{xx}^2 L(x^*, l^*, \lambda^*) & \nabla q(x^*, l^*) \\ \Xi(\lambda^*)\nabla q^T(x^*, l^*) & -\tau I_m \end{bmatrix}, \quad (4.8)$$

where I_m is an identity matrix of $n \times n$. Because $\frac{1}{\sigma_{max}} \leq \tau \leq \frac{1}{\sigma_0}$, so the Jacob matrix is non-singular. If not, let there be a nonzero vector $(\nu^T, \omega^T)^T$, such that

$$\begin{bmatrix} \nabla_{xx}^2 L(x^*, l^*, \lambda^*) & \nabla q(x^*, l^*) \\ \Xi(\lambda^*)\nabla q^T(x^*, l^*) & -\tau I_m \end{bmatrix} \begin{bmatrix} \nu \\ \omega \end{bmatrix} = 0. \quad (4.9)$$

So, we have

$$\nabla_{xx}^2 L(x^*, l^*, \lambda^*) \cdot \nu + \nabla q(x^*, l^*) \cdot \omega = 0, \quad (4.10)$$

$$\Xi(\lambda^*)\nabla q^T(x^*, l^*) \cdot \nu - \tau \cdot \omega = 0, \quad (4.11)$$

and we can get the ω expression obtained in Eq (4.11) as

$$\omega = \frac{\Xi(\lambda^*)\nabla q^T(x^*, l^*) \cdot \nu}{\tau I_m^i}. \quad (4.12)$$

By substituting (4.12) into formula (4.10), we get

$$\nabla_{xx}^2 L(x^*, l^*, \lambda^*) \cdot \nu + \sigma \cdot \nabla q(x^*, l^*) \cdot \Xi(\lambda^*) \cdot \nabla q^T(x^*, l^*) \cdot \omega = 0, \quad (4.13)$$

which means

$$\nabla_{xx}^2 G_\sigma(x^*, l^*, \lambda^*) \cdot \nu = 0. \quad (4.14)$$

Because $\sigma \geq \sigma_0$, $\nabla_{xx}^2 G_\sigma(x^*, l^*, \lambda^*)$ is negative definite, therefore, so we get $\nu = 0$ for every $\sigma \in [\sigma_0, \sigma_{max}]$. Then, combine with Eq (4.11), and it is easy to obtain $\omega = 0$, which contradicts the previous nonzero hypothesis.

Next, define the closed set $K = \{(0, \tau) | \tau \in [\frac{1}{\sigma_{max}}, \frac{1}{\sigma_0}]\}$ and its neighborhood $B(K, \delta)$. According to the implicit function theorem, there are ε, δ and the unique continuous differentiable functions $(\hat{x}(\rho, \tau), \hat{l}(\rho, \tau))$, and $\hat{\lambda}(\rho, \tau)$ for any $(\rho, \tau) \in B(K, \delta)$, which defined on $B(K, \delta)$ satisfy

$$\|((\hat{x}(\rho, \tau), \hat{l}(\rho, \tau)), \hat{\lambda}(\rho, \tau)) - ((x^*, l^*), \lambda^*)\| \leq \varepsilon.$$

Combined with Eq (4.7), we can get

$$\begin{aligned} f(\hat{x}(\rho, \tau), \hat{l}(\rho, \tau)) + \hat{\lambda}(\rho, \tau) \cdot \nabla q(\hat{x}(\rho, \tau), \hat{l}(\rho, \tau)) &= 0, \\ \Xi(\lambda)q(x(\rho, \tau), l(\rho, \tau)) + \rho + \tau \cdot (\lambda^* - \hat{\lambda}(\rho, \tau)) &= 0. \end{aligned} \quad (4.15)$$

The expressions of $\rho = \frac{\lambda - \lambda_0}{\sigma}$, and $\tau = \frac{1}{\sigma}$ into $(\hat{x}(\rho, \tau), \hat{l}(\rho, \tau))$, $\hat{\lambda}(\rho, \tau)$ are defined by

$$\begin{aligned} (\hat{x}(\rho, \tau), \hat{l}(\rho, \tau)) &= (\hat{x}(\frac{\lambda - \lambda_0}{\sigma}, \frac{1}{\sigma}), \hat{l}(\frac{\lambda - \lambda_0}{\sigma}, \frac{1}{\sigma})) = (x(\lambda, \sigma), l(\lambda, \sigma)), \\ \hat{\lambda}(\rho, \tau) &= \hat{\lambda}(\frac{\lambda - \lambda_0}{\sigma}, \frac{1}{\sigma}) = \lambda(\lambda, \sigma), \end{aligned}$$

then by (4.14), $\forall(\lambda, \sigma) \in \mathbb{D}$, we get

$$\begin{aligned} f(x(\lambda, \sigma), l(\lambda, \sigma)) + \lambda(\lambda, \sigma) \nabla q(x(\lambda, \sigma), l(\lambda, \sigma)) &= 0, \\ \lambda(\lambda, \sigma) &= \lambda_w [1 + \sigma_{w+1} \cdot q_i(x_w(\lambda, \sigma), l_w(\lambda, \sigma))]. \end{aligned} \quad (4.16)$$

By the continuity assumption, appropriate ε and δ can be selected to ensure $\nabla_{xx}^2 G_\sigma(x(\lambda, \sigma), l(\lambda, \sigma), \lambda(\lambda, \sigma))$ negative definite.

To sum up, we can conclude that $\forall(\lambda, \sigma) \in \mathbb{D}$. In the ε neighborhood of the optimal solution (x^*, l^*) , the generalized Lagrange function $G_\sigma(\cdot, \cdot, \lambda)$ has a unique maximal point.

5. Numerical examples

Now, consider three businesses that issue digital currencies. Here are their launch profits and operating development costs:

$$\begin{aligned} r_1(x_1, x_2) &= 140 - 3x_1 - 2x_2 - x_3, \\ m_1(x_1) &= 2x_1^2 + x_1, \\ r_2(x_1, x_2) &= 130 - 2x_1 - 2x_2 + 2x_3, \\ m_2(x_2) &= 2x_2^2 - x_2, \\ r_3(x_1, x_2) &= 120 - 3x_1 - 5x_2 + x_3, \\ m_3(x_1) &= 2x_3^2 - x_3. \end{aligned}$$

The total revenue is

$$\begin{aligned} R_1(x_1, x_2, x_3) &= (140 - 3x_1 - 2x_2 - x_3)x_1 - (2x_1^2 + x_1), \\ R_2(x_1, x_2, x_3) &= (130 - 2x_1 - 2x_2 + 2x_3)x_2 - (2x_2^2 - x_2), \\ R_3(x_1, x_2, x_3) &= (120 - 3x_1 - 5x_2 + x_3)x_3 - (2x_3^2 - x_3). \end{aligned}$$

Let $R_i(x_1, x_2, x_3)$ be a random variable following a uniform distribution on $[80, 90]$, and e_{l_i} be a random variable following a uniform distribution on $[0, 5]$. Under these assumptions, we can determine the distribution of $R_i(x_1, x_2, x_3) - e_{l_i}$ as follows:

$$f_{R_i - e_{l_i}} = \begin{cases} 0, & x < 75, x > 90, \\ \frac{x - 75}{50}, & 75 < x < 80, \\ \frac{90 - x}{50}, & 80 < x < 90. \end{cases}$$

Assuming that the expected earnings of the three enterprises are all $e_{l_i} = 400$ for $i = 1, 2, 3$, it can be concluded that enterprise 1 needs to issue $x_1 = 3.652$ units to achieve the expected earnings, enterprise 2 needs to issue $x_2 = 3.353$ units, and enterprise 3 needs to issue $x_3 = 4.505$ units. Furthermore, it was determined that $P(R_1(x_1, x_2, x_3) - e_{l_1} \geq 0) = 0.762$, $P(R_1(x_1, x_2, x_3) - e_{l_2} \geq 0) = 0.799$, and $P(R_2(x_1, x_2, x_3) - e_{l_3} \geq 0) = 0.518$. Based on these findings, it is believed that the expected earnings of both enterprises 1 and 2 are reasonable. As shown in Figure 1, the horizontal axis represents the expected earnings of the three enterprises (x), and the vertical axis represents the probability that actual earnings exceed the expected earnings (P). The blue, yellow, and green lines respectively represent the probability trend of enterprises 1–3 reaching the expected income (P).

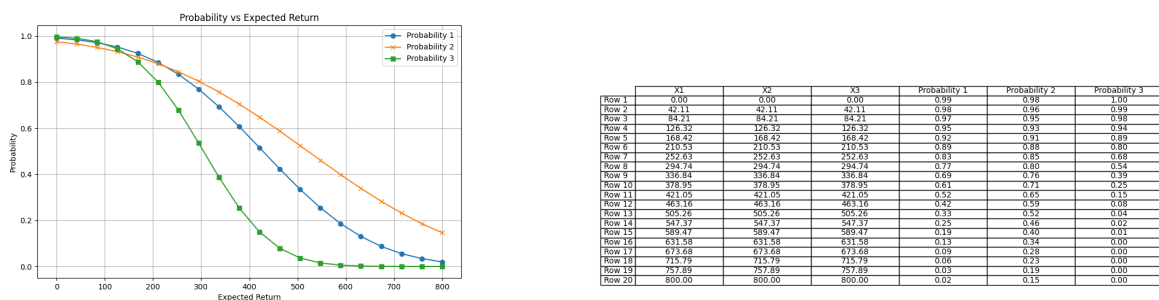


Figure 1. Trend graph of the probability of expected and actual returns.

6. Conclusions

In summary, this paper introduces a model for a multi-player game with uncertain constraint sets. By employing variational analysis tools, the problem of multi-player games is transformed into a vector variational inequality problem. The paper illustrates a class of vector variational inequality optimization problems established under lower semicontinuous conditions and defines its solution. It also considers the existence of weak Pareto-Nash equilibrium after transformation, utilizing the KKM theorem, Lusin’s theorem, and related definitions. Additionally, an algorithm is developed using the improved Lagrange function, followed by a convergence analysis. All results demonstrate the feasibility of our model and algorithm. Finally, a numerical example is provided to validate the effectiveness of the model in addressing specific problems.

The integration of these concepts not only facilitates the study of players’ strategy choices in complex systems but also advances the application of variational inequality theory in multi-player games. This integration offers a more robust analytical tool for solving real-world problems and holds promise for yielding profound insights and more effective results. By exploring the intersections of theoretical frameworks, we can gain a better understanding of system interactions, providing crucial insights for promoting sustainable development and optimizing complex systems.

Author contributions

Wenyu Zhang: Research’s conception and design, Writing and revision of the manuscript; Baojun Yang: Handled data collection and processing; Cunlin Li: Conducted data analysis and interpretation, Writing and revision of the manuscript; Hooi Min Yee: Supervised and guided the entire study. All

authors discussed and reviewed the research findings and approved the final version of the manuscript.

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Conflict of interest

The authors declare that they have no conflict of interest.

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