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Research article

Weighted L^p boundedness of maximal operators with rough kernels

Hussain Al-Qassem[1,*] **and Mohammed Ali**[2]

[1] Department of Mathematics and Statistics, Qatar University, Doha, Qatar
[2] Department of Mathematics and Statistics, Faculty of Science and Arts, Jordan University of Science and Technology, P.O. Box 3030, Irbid 22110, Jordan

* **Correspondence:** Email: husseink@qu.edu.qa.

Abstract: In this paper, we study the weighted spaces $L^p(\omega, \mathbb{R}^d)$ boundedness of certain class of maximal operators when their kernels belong to the space $L^q(\mathbb{S}^{d-1})$, $q > 1$. Our results in this paper are improvements and extensions of some previously known results.

Keywords: maximal functions; L^p boundedness; rough kernels; surfaces of revolution; extrapolation
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1. Introduction and statement of results

Let $\mathbb{S}^{d-1}$ be the unit sphere in the d-dimensional Euclidean space $\mathbb{R}^d (d \geq 2)$ which is equipped with the normalized Lebesgue surface measure $d\sigma = d\sigma(\cdot)$.

Let $h : \mathbb{R}^+ \to \mathbb{C}$ be a radial function satisfying

$$\|h\|_{L^2(\mathbb{R}^+, \frac{ds}{s})} = \left(\int_0^\infty |h(s)|^2 \frac{ds}{s} \right)^{1/2} \leq 1,$$

and let $\mho$ be a homogeneous function of degree zero on $\mathbb{R}^d$ with $\mho \in L^1(\mathbb{S}^{d-1})$ and

$$\int_{\mathbb{S}^{d-1}} \mho(x') d\sigma(x') = 0, \tag{1.1}$$

where $x' = x/|x|$ for $x \in \mathbb{R}^d \setminus \{0\}$.

For a Schwartz function $f \in (\mathbb{R}^d)$, we consider the maximal operator $\mathfrak{M}_{\mho,P}$ given by

$$\mathfrak{M}_{\mho,P}(f)(x) = \sup_{h \in L^2(\mathbb{R}^+, \frac{ds}{s})} \left| \int_{\mathbb{R}^d} e^{iP(u)} f(x-u) \frac{\mho(u) h(|u|)}{|u|^d} du \right|, \tag{1.2}$$

where $P : \mathbb{R}^d \to \mathbb{R}$ is a real-valued polynomial.

We notice that if $P(y) \equiv 0$, then the operator $\mathfrak{M}_{\mho,P}$ is reduced to be the classical maximal operator denoted by $\mathfrak{M}_{\mho}$, which was introduced by Chen and Lin in [1]. The authors of [1] proved the boundedness of $\mathfrak{M}_{\mho}$ on $L^p(\mathbb{R}^d)$ for $2d/(2d-1) < p < \infty$ if $\mho \in C(\mathbb{S}^{d-1})$, and they showed that the range of p is the best possible. This result was extended in [2] in which the author confirmed the L^p boundedness of $\mathfrak{M}_{\mho}$ for all $p \in [2,\infty)$ whenever $\mho \in L(\log L)^{1/2}(\mathbb{S}^{d-1})$, and that the condition $\mho \in L(\log L)^{1/2}(\mathbb{S}^{d-1})$ is optimal in the sense that the operator $\mathfrak{M}_{\mho}$ may fail to be bounded on $L^2(\mathbb{R}^d)$ when $\mho \in L(\log L)^r(\mathbb{S}^{d-1})$ for any $r \in (0, 1/2)$. On the other hand, the author of [3] proved that $\mathfrak{M}_{\mho}$ is bounded on $L^p(\mathbb{R}^d)$ for $p \geq 2$ if $\mho$ lies in the block spaces $B_q^{(0,-1/2)}(\mathbb{S}^{d-1})$ with $q > 1$, and they also proved that if the kernel $\mho$ belongs to $B_q^{(0,r)}(\mathbb{S}^{d-1})$ for some $r \in (-1,-1/2)$, then $\mathfrak{M}_{\mho}$ may not be bounded in $L^2(\mathbb{R}^d)$. In [4], the author generalized the above results. In fact, he proved that $\mathfrak{M}_{\mho,P}$ is bounded on $L^p(\mathbb{R}^d)$ for all $p \geq 2$ provided that $\mho \in B_q^{(0,-1/2)}(\mathbb{S}^{d-1}) \cup L(\log L)^{1/2}(\mathbb{S}^{d-1})$. Subsequently, the investigation of the boundedness of $\mathfrak{M}_{\mho,P}$ on $L^p(\mathbb{R}^d)$ under various conditions has attracted the attention of many authors: For background information [5–8], importance and the development [9–11], and recent advances and studies [12, 13].

On the other hand, in [14] Y. Ding and H. Qingzheng proved the weighted L^p boundedness of $\mathfrak{M}_{\mho}$ as described in the following theorem.

Theorem A. *Let $d \geq 2$. Assume $\mho \in L^2(\mathbb{S}^{d-1})$ satisfies (1.1). Then,*

$$\|\mathfrak{M}_{\mho}(f)\|_{L^p(\omega,\, \mathbb{R}^d)} \leq C_p \|f\|_{L^p(\omega,\, \mathbb{R}^d)}, \tag{1.3}$$

if p and ω satisfy one of the following conditions:

(a) $2 \leq p < \infty$ *and* $\omega \in A_{p/2}(\mathbb{R}^d)$*;*

(b) $2d/(2d-1) < p < 2$, $\omega(x) = |x|^{\alpha}$, *and* $\frac{1}{2}(1-d)(2-p) < \alpha < \frac{1}{2}(2dp - 2d - p)$, *where* A_p is the Muckenhoupt's weight class, and the weighted $L^p(\omega, \mathbb{R}^d)$ *with* $\omega \geq 0$ *is defined by*

$$L^p(\omega, \mathbb{R}^d) = \left\{ f : \|f\|_{L^p(\omega,\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} |f(y)|^p \, \omega(y) dy \right)^{1/p} < \infty \right\}.$$

Subsequently, Al-Qassem in [15] generalized the above result as in the following theorem:

Theorem B. *Suppose that $\mho \in L^q(\mathbb{S}^{d-1})$ for some $q > 1$ and it satisfies (1.1). Then,*

$$\|\mathfrak{M}_{\mho}(f)\|_{L^p(\omega,\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\omega,\mathbb{R}^d)},$$

if p and ω satisfy one of the following conditions:

(a) $\delta \leq p < \infty$ *and* $\omega \in A_{p/\delta}$*;*

(b) $2d\delta/(2d + d\delta - 2) < p < 2$, $\omega(x) = |x|^{\alpha}$, $\frac{1}{2}(1-d)(2-p) < \alpha < \frac{1}{2}(2dp - 2d - p)$, *where* $\delta = \max\{2, q'\}$ *and q' is the dual exponent of q.*

In view of the results in [4] concerning the L^p boundedness of $\mathfrak{M}_{\mho,P}$ and of the results in [15] concerning the weighted L^p boundedness of $\mathfrak{M}_{\mho}$, it is natural to ask wether the weighted L^p boundedness of $\mathfrak{M}_{\mho,P}$ holds under the same conditions as assumed in Theorem B. We shall obtain an answer to this question in the affirmative as described in the following theorem.

Theorem 1.1. *Let $\mho \in L^q(\mathbb{S}^{d-1})$ with $q > 1$. Suppose that $P : \mathbb{R}^d \to \mathbb{R}$ is a polynomial of degree k, then the estimate*

$$\left\|\mathfrak{M}_{\mho,P}(f)\right\|_{L^p(\omega,\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\omega,\mathbb{R}^d)} \tag{1.4}$$

holds for $\delta \leq p < \infty$ and $\omega \in A_{p/\delta}$, where $\delta = \max\{2, q'\}$.

Now let us give some results which follow as a consequence of Theorem 1.1. For $\gamma \in (1, \infty)$, we let $L^\gamma(\mathbb{R}^+, \frac{ds}{s})$ be the set of all measurable functions $h : \mathbb{R}^+ \to \mathbb{R}$ such that

$$\|h\|_{L^\gamma(\mathbb{R}^+,\frac{ds}{s})} = \left(\int_0^\infty |h(s)|^\gamma \frac{ds}{s}\right)^{1/\gamma} \leq 1.$$

Consider the maximal operator $\mathfrak{M}^{(\gamma)}_{\mho,P}$ given by

$$\mathfrak{M}^{(\gamma)}_{\mho,P}(f)(x) = \sup_{h\in L^\gamma(\mathbb{R}^+,\frac{ds}{s})} \left| \int_{\mathbb{R}^d} e^{iP(u)} f(x-u) \frac{\mho(u)h(|u|)}{|u|^d} du \right|, \tag{1.5}$$

where $P : \mathbb{R}^d \to \mathbb{R}$ is a real-valued polynomial, $f \in \mathcal{S}(\mathbb{R}^d)$ and $1 \leq \gamma \leq 2$.

The study of the boundedness of the operator $\mathfrak{M}^{(\gamma)}_{\mho,P}$ started in [1] in which the authors proved that if $\mho \in C(\mathbb{S}^{d-1})$ and $h \in L^\gamma(\mathbb{R}^+, \frac{ds}{s})$ for some $1 \leq \gamma \leq 2$, then the $L^p(\mathbb{R}^d)$ boundedness of the operator $\mathfrak{M}^{(\gamma)}_{\mho,0}$ is satisfied for $(\gamma d)' < p < \infty$. For more information about the investigation of $\mathfrak{M}^{(\gamma)}_{\mho,P}$, under various conditions and some past studies, readers are referred to see [16–18] and the references therein. In this work, an extension and improvement over the result in [1] shall be obtained by proving the weighted L^p of $\mathfrak{M}^{(\gamma)}_{\mho,P}$ when the condition $\mho \in C(\mathbb{S}^{d-1})$ is replaced by the weaker condition $\mho \in L^q(\mathbb{S}^{d-1})$ with $q > 1$. Precisely, we have the following:

Theorem 1.2. *Let $\mho \in L^q(\mathbb{S}^{d-1})$ with $q > 1$. Let $\omega \in A_{p/\delta}$ and $h \in L^\gamma(\mathbb{R}^+, \frac{ds}{s})$ with $1 \leq \gamma \leq 2$. Then, we have*

$$\left\|\mathfrak{M}^{(\gamma)}_{\mho,P}(f)\right\|_{L^p(\omega,\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\omega,\mathbb{R}^d)} \tag{1.6}$$

for $(\delta\gamma')/2 \leq p < \infty$.

Concerning the boundedness of a certain class of oscillatory singular integrals, we have the following:

Theorem 1.3. *Assume that $\mho \in L^q(\mathbb{S}^{d-1})$ with $q > 1$. Let $\omega \in A_{p/\delta}$ and $h \in L^\gamma(\mathbb{R}^+, \frac{ds}{s})$ for some $1 < \gamma \leq 2$. Then, the oscillatory singular integral operator $T^{(\gamma)}_{\mho,P}$ given by*

$$T^{(\gamma)}_{\mho,P}(f)(x) = p.v. \int_{\mathbb{R}^d} e^{iP(u)} f(x-u) \frac{\mho(u)h(|u|)}{|u|^d} du,$$

is bounded on $L^p(\omega, \mathbb{R}^d)$ for $(\delta\gamma')/2 \leq p < \infty$, and it is bounded on $L^p(\omega, \mathbb{R}^d)$ for $1 < p \leq (\frac{\delta\gamma'}{2})'$ and $\omega^{1-p'} \in A_{p'/\delta}$.

For background information and related work about the operator, see [19–24].

We point out that the generalized Marcinkiewicz operator concerning the operator $\mathfrak{M}^{(\gamma)}_{\mho,P}$ is given by

$$\mathcal{M}^{(\gamma)}_{\mho,P}(f)(x) = \left(\int_{\mathbb{R}^+} \left| \frac{1}{s} \int_{|u|\le s} e^{iP(u)} f(x-u)\mho(u)|u|^{-d+1} du \right|^{\gamma'} \frac{ds}{s} \right)^{1/\gamma'} . \tag{1.7}$$

As an immediate consequence of the fact

$$\mathcal{M}^{(\gamma)}_{\mho,P}(f)(x) \le C\, \mathfrak{M}^{(\gamma)}_{\mho,P}(f)(x)$$

for $1 \le \gamma \le 2$, we obtain the following result:

Theorem 1.4. *Let $\mho$, ω, P, and γ be given as in Theorem 1.2. Then, the generalized Marcinkiewicz integral $\mathcal{M}^{(\gamma)}_{\mho,P}$ is bounded on $L^p(\omega, \mathbb{R}^d)$ for $(\delta\gamma')/2 \le p < \infty$ with $1 < \gamma \le 2$.*

It is clear that for the special case $P = 0$ and $\gamma = 2$, the operator $\mathcal{M}^{(2)}_{\mho,0}$ reduces to the classical Marcinkiewicz integral operator, which was introduced in [25], in which the author proved that the operator is bounded on $L^p(\mathbb{R}^d)$ only for $1 < p \le 2$ whenever $\mho \in Lip_\eta(\mathbb{S}^{d-1})$ for some $0 < \eta \le 2$. Thereafter, the study of the operator $\mathcal{M}^{(\gamma)}_{\mho,P}$ under several conditions has been discussed by many mathematicians (see, for instance [4, 26–30]).

Throughout the rest of the paper, the letter C stands for a positive constant which is independent of the essential variables and its value is not necessary the same at each occurrence.

2. Preliminary lemmas

In this section, we give some preliminary lemmas to prove our main results. Let us start with the following lemma, which is found in [4].

Lemma 2.1. *Let $\mho \in L^q(\mathbb{S}^{d-1})$, $q > 1$ be a homogeneous function of degree zero. Suppose that*

$$P(x) = \sum_{|\eta|\le k} \lambda_\eta x^\eta,$$

is a polynomial of degree $k > 1$ such that $|u|^k$ is not one of its terms. For $j \in \mathbb{Z}$, define $I_{j,\mho} : \mathbb{R}^d \to \mathbb{R}$ by

$$I_{j,\mho}(\xi) = \int_{2^{-(j+1)}}^{2^{-(j-1)}} \left| \int_{\mathbb{S}^{d-1}} \mho(u) e^{-i\left[P(su)+su\cdot\xi\right]} d\sigma(u) \right|^2 \frac{ds}{s}. \tag{2.1}$$

Then, there exist constants $C > 0$ and $0 < \epsilon < 1$ such that

$$\sup_{\xi\in\mathbb{R}^d} I_{j,\mho}(\xi) \le C\, 2^{(j+1)/4q'} \left(\sum_{|\eta|=m} \left|\lambda_\eta\right| \right)^{-\epsilon/q'} .$$

We need the following lemma from [15].

Lemma 2.2. *Let $\mho \in L^q(\mathbb{S}^{d-1})$ for some $q > 1$ and $\omega \in A_{p/q'}(\mathbb{R}^+)$ with $1 < p < \infty$. Assume that the maximal function $M_\mho$ is given by*

$$M_\mho f(x) = \sup_{j\in\mathbb{Z}} \int\limits_{2^j\leq|u|\leq 2^{(j+1)}} |f(x-u)| \frac{|\mho(u)|}{|u|^d} du.$$

Then there exists a positive constant C_p such that

$$\|M_\mho(f)\|_{L^p(\omega,\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\omega,\mathbb{R}^d)}$$

for any $f \in L^p(\omega, \mathbb{R}^d)$ with $q' \leq p < \infty$.

The next lemma can be proved by employing the same argument as in the proof of Theorem 1.1 in [15].

Lemma 2.3. *Let $\omega \in A_{p/\delta}$ and $\mho \in L^q(\mathbb{S}^{d-1})$ with $q > 1$. Then, there is a constant $C_p > 0$ such that*

$$\|\mathfrak{M}_\mho(f)\|_{L^p(\omega,\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\omega,\mathbb{R}^d)} \tag{2.2}$$

for all $\delta \leq p < \infty$.

Proof. Let $\left\{\psi_j\right\}_{j\in\mathbb{Z}}$ be a smooth partition of unity in $(0, \infty)$ with the following properties:

$$\begin{aligned} &\psi_j \in C^\infty, \quad \operatorname{supp}\psi_j \subseteq \left[2^{-(j+1)}, 2^{-(j-1)}\right], \quad 0 \leq \psi_j \leq 1, \\ &\sum_{j\in\mathbb{Z}} \psi_j(s) = 1, \quad \text{and} \quad \left|\frac{d^k\psi_j(s)}{ds^k}\right| \leq \frac{C_k}{s^k}. \end{aligned} \tag{2.3}$$

For $j \in \mathbb{Z}$, define the operator Υ_j in $\mathbb{R}^d$ by

$$(\widehat{\Upsilon_j(f)})(\xi) = \psi_j(|\xi|)\hat{f}(\xi) \quad \text{for } \xi \in \mathbb{R}^d.$$

Then, for $f \in \mathcal{S}(\mathbb{R}^d)$, we have that

$$\mathfrak{M}_\mho(f)(x) \leq \sum_{k\in\mathbb{Z}} \mathcal{G}_{\mho,k}(f)(x), \tag{2.4}$$

where

$$\mathcal{G}_{\mho,k}(f)(x) = \left(\sum_{j\in\mathbb{Z}} \int_{2^{-(j+1)}}^{2^{-(j-1)}} \left|\int_{\mathbb{S}^{d-1}} (\Upsilon_{k+j}f)(x-su)\mho(u)d\sigma(u))\right|^2 \frac{ds}{s}\right)^{1/2}.$$

By following the same argument utilized in the proof of Theorem 1.1 in [15], along with invoking Lemma 2.1, we obtain that

$$\left\|\mathcal{G}_{\mho,k}(f)\right\|_{L^p(\omega,\mathbb{R}^d)} \leq C_p 2^{-\tau|k|} \|f\|_{L^p(\omega,\mathbb{R}^d)}, \tag{2.5}$$

for some constant $\tau \in (0, 1)$ and for all $\delta \leq p < \infty$. Consequently, by (2.4) and (2.5), we get (2.2) for all $\delta \leq p < \infty$. □

3. Proof of the main results

Proof of Theorem 1.1. We shall use some of the ideas from [4]. Precisely, we use the induction on the degree of the polynomial P. It is clear that if the degree of P is 0, then by Lemma 2.3 we get

$$\left\|\mathfrak{M}_{\mho,P}(f)\right\|_{L^p(\omega,\mathbb{R}^d)} \le C_p \ \|f\|_{L^p(\omega,\mathbb{R}^d)} \tag{3.1}$$

for all $\delta \le p < \infty$. Now, if the degree of P is 1, then we deduce that there is $\overrightarrow{c} \in \mathbb{R}^d$ so that $P(u) = \overrightarrow{c} \cdot u$. Hence, if we set $g(u) = e^{-iP(u)} f(u)$, then by (3.1) we get that

$$\left\|\mathfrak{M}_{\mho,P}(f)\right\|_{L^p(\omega,\mathbb{R}^d)} \le \left\|\mathfrak{M}_{\mho,P}(g)\right\|_{L^p(\omega,\mathbb{R}^d)} \le C_p \ \|f\|_{L^p(\omega,\mathbb{R}^d)}.$$

Next, suppose that (1.4) holds for any polynomial P whose degree is less than or equal to $k \ge 1$. We need to prove that the inequality (1.4) is also satisfied for any polynomial of degree $k+1$. Let

$$P(u) = \sum_{|\eta| \le k+1} \lambda_\eta u^\eta$$

be a polynomial of degree $k+1$. Without loss of generality, we may assume that P does not contain $|u|^{k+1}$ as one of its terms, and $\sum_{|\eta|=k+1} \left|\lambda_\eta\right| = 1$.

For $j \in \mathbb{Z}$, let $\left\{\psi_j\right\}$ and Υ_j be chosen as those in (2.3). Set

$$\Gamma_\infty(s) = \sum_{j=-\infty}^{0} \psi_j(s) \text{ and } \quad \Gamma_0(s) = \sum_{j=1}^{\infty} \psi_j(s).$$

Then, $\Gamma_\infty(s) + \Gamma_0(s) = 1$, $\operatorname{supp}(\Gamma_\infty(s)) \subseteq [2^{-1}, \infty)$, and $\operatorname{supp}(\Gamma_0(s)) \subseteq (0, 1]$. Hence, we get by Minkowski's inequality that

$$\mathfrak{M}_{\mho,P}(f)(x) \le \mathfrak{M}_{\mho,P,\infty}(f)(x) + \mathfrak{M}_{\mho,P,0}(f)(x), \tag{3.2}$$

where

$$\mathfrak{M}_{\mho,P,\infty}(f)(x) = \left(\int_{2^{-1}}^{\infty} \left| \Gamma_\infty(s) \int_{\mathbb{S}^{d-1}} e^{iP(su)} f(x - su) \mho(u) d\sigma(u) \right|^2 \frac{ds}{s} \right)^{1/2},$$

and

$$\mathfrak{M}_{\mho,P,0}(f)(x) = \left(\int_{0}^{1} \left| \Gamma_0(s) \int_{\mathbb{S}^{d-1}} e^{iP(su)} f(x - su) \mho(u) d\sigma(u) \right|^2 \frac{ds}{s} \right)^{1/2}.$$

Let us estimate $\left\|\mathfrak{M}_{\mho,P,\infty}(f)\right\|_{L^p(\omega,\mathbb{R}^d)}$. Define

$$\mathfrak{M}_{\mho,P,\infty,j}(f)(x) = \left(\int_{2^{-(j+1)}}^{2^{-(j-1)}} \left| \int_{\mathbb{S}^{d-1}} e^{iP(su)} f(x - su) \mho(u) d\sigma(u) \right|^2 \frac{ds}{s} \right)^{1/2}.$$

Then, by the generalized Minkowski's inequality, we have

$$\mathfrak{M}_{\mho,P,\infty}(f)(x) \leq \sum_{j=-\infty}^{0} \mathfrak{M}_{\mho,P,\infty,j}(f)(x). \tag{3.3}$$

Case 1. When $q \geq 2$. In this case, we have $2 \leq p < \infty$ and $\omega \in A_{p/2}$. Let us consider first the case $p > 2$. By duality, there is $g \in L^{(p/2)'}(\omega^{1-(p/2)'}, \mathbb{R}^d)$ such that $\|g\|_{L^{(p/2)'}(\omega^{1-(p/2)'},\mathbb{R}^d)} \leq 1$ and

$$\begin{aligned}
&\left\|\mathfrak{M}_{\mho,P,\infty,j}(f)\right\|^2_{L^p(\omega,\mathbb{R}^d)} \\
= &\int_{\mathbb{R}^d} \int_1^4 \left| \int_{\mathbb{S}^{d-1}} e^{-iP(2^{-(j+1)}su)} \mho(u) f(x - 2^{-(j+1)}su) d\sigma(u) \right|^2 \frac{ds}{s} |g(x)|\, dx \\
\leq &\|\mho\|^2_{L^q(\mathbb{S}^{d-1})} \int_{\mathbb{R}^d} \int_1^4 \left(\int_{\mathbb{S}^{d-1}} \left| f(x - 2^{-(j+1)}su) \right|^{q'} d\sigma(u) \right)^{2/q'} \frac{ds}{s} |g(x)|\, dx \\
\leq &\|\mho\|^2_{L^q(\mathbb{S}^{d-1})} \int_{\mathbb{R}^d} \int_1^4 \left(\int_{\mathbb{S}^{d-1}} \left| f(x - 2^{-(j+1)}su) \right|^{2} d\sigma(u) \right) \frac{ds}{s} |g(x)|\, dx.
\end{aligned}$$

Hence, by Hölder's inequality, we get

$$\begin{aligned}
\left\|\mathfrak{M}_{\mho,P,a,j}(f)\right\|^2_{L^p(\omega,\mathbb{R}^d)} &\leq C \int_{\mathbb{R}^d} |f(y)|^2 \int_1^4 \int_{\mathbb{S}^{d-1}} \left| g(y + 2^{-(j+1)}su) \right| d\sigma(u) \frac{ds}{s} dy \\
&\leq C_p \left\| |f|^2 \right\|_{L^{(p/2)}(\omega,\mathbb{R}^d)} \|M^*(\widetilde{g})\|_{L^{(p/2)'}(\omega^{1-(p/2)'},\mathbb{R}^d)} \\
&\leq C_p \|f\|^2_{L^p(\omega,\mathbb{R}^d)} \|\widetilde{g}\|_{L^{(p/2)'}(\omega^{1-(p/2)'},\mathbb{R}^d)},
\end{aligned}$$

where $\widetilde{g}(y) = g(-y)$ and $M^*(f)$ is the Hardy-Littlewood maximal function. Thus,

$$\left\|\mathfrak{M}_{\mho,P,\infty,j}(f)\right\|_{L^p(\omega,\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\omega,\mathbb{R}^d)}, \tag{3.4}$$

for $2 < p < \infty$ and $\omega \in A_{p/2}$.

Now, for the case $p = 2$ and $\omega \in A_1$, we have

$$\begin{aligned}
\left\|\mathfrak{M}_{\mho,P,\infty,j}(f)\right\|^2_{L^2(\omega,\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \int_1^4 \left| \int_{\mathbb{S}^{d-1}} e^{-iP(2^{-(j+1)}su)} \mho(u) f(x - 2^{-(j+1)}su) d\sigma(u) \right|^2 \frac{ds}{s} \omega(x) dx \\
&\leq \|\mho\|_q^2 \int_{\mathbb{R}^n} |f(x)|^2 \left(\int_1^4 \int_{\mathbb{S}^{d-1}} \omega(x + 2^{-(j+1)}su) d\sigma(u) \frac{ds}{s} \right) \omega(x) dx \\
&\leq C \int_{\mathbb{R}^n} |f(x)|^2 M^*(\tilde{\omega})(-x) dx, \text{ with } \tilde{\omega}(x) = \omega(-x) \\
&\leq C \int_{\mathbb{R}^n} |f(x)|^2 \omega(x) dx = C \|f\|^2_{L^2(\omega,\mathbb{R}^d)},
\end{aligned} \tag{3.5}$$

where the last inequality is obtained by the fact that $M^*\omega(x) \leq C\omega(x)$ for a.e. $x \in \mathbb{R}^d$.

Since for any $\omega \in A_{p/2}$ there exists $\alpha > 0$ such that $\omega^{1+\alpha} \in A_{p/2}$, by (3.4) and (3.5), we get that

$$\left\|\mathfrak{M}_{\mho,P,\infty,j}(f)\right\|_{L^p(\omega^{1+\alpha},\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\omega^{1+\alpha},\mathbb{R}^d)}, \tag{3.6}$$

for $2 < p < \infty$ and $\omega \in A_{p/2}$.

Now we will obtain a sharp unweighted L^2 estimate of $\mathfrak{M}_{\mho,P,\infty,j}(f)$. By Fubini's theorem, Plancherel's theorem and Lemma 2.1 we get

$$\begin{aligned}\left\|\mathfrak{M}_{\mho,P,\infty,j}(f)\right\|_{L^2(\mathbb{R}^d)} &= \left(\int_{\mathbb{R}^d}\left|\widehat{f}(\xi)\right|^2 I_{j,\mho}(\xi)d\xi\right)^{1/2}\\ &\le C\,2^{\frac{(j+1)}{8q'}}\|f\|_{L^2(\mathbb{R}^d)}.\end{aligned}\tag{3.7}$$

Thus, using the Stein-Weiss interpolation theorem with change of measure [31], we may interpolate between (3.6) and (3.7) to obtain

$$\left\|\mathfrak{M}_{\mho,P,\infty,j}(f)\right\|_{L^p(\omega,\mathbb{R}^d)} \le C_p 2^{\frac{\varepsilon(j+1)}{8q'}}\|f\|_{L^p(\omega,\mathbb{R}^d)}\tag{3.8}$$

for $2 \le p < \infty$, $\omega \in A_{p/2}$, and for some $\varepsilon \in (0,1)$. Consequently, by (3.3) and (3.8), we conclude that

$$\left\|\mathfrak{M}_{\mho,P,\infty}(f)\right\|_{L^p(\omega,\mathbb{R}^d)} \le C_p\|f\|_{L^p(\omega,\mathbb{R}^d)}\tag{3.9}$$

for $2 \le p < \infty$ and $\omega \in A_{p/2}$.

Case 2. When $1 < q < 2$. In this case, we have $q' \le p < \infty$ and $\omega \in A_{p/q'}$. Since $p > 2$, by duality, there exists $F \in L^{(p/2)'}(\omega^{1-(p/2)'},\mathbb{R}^d)$ such that $\|F\|_{L^{(p/2)'}(\omega^{1-(p/2)'},\mathbb{R}^d)} \le 1$ and

$$\begin{aligned}&\left\|\mathfrak{M}_{\mho,P,\infty,j}(f)\right\|^2_{L^p(\omega,\mathbb{R}^d)}\\ =\;& \int_{\mathbb{R}^d}\int_1^4\left|\int_{\mathbb{S}^{d-1}} e^{-iP(2^{-(j+1)}su)}\mho(u)f(x-2^{-(j+1)}su)d\sigma(u)\right|^2\frac{ds}{s}|F(x)|\,dx\\ \le\;& \|\mho\|^q_{L^q(\mathbb{S}^{d-1})}\int_{\mathbb{R}^d}\int_1^4\left(\int_{\mathbb{S}^{d-1}}|\mho(u)|^{2-q}\left|f(x-2^{-(j+1)}su)\right|^2d\sigma(u)\right)\frac{ds}{s}|F(x)|\,dx.\end{aligned}$$

Hence, by Hölder's inequality, we get

$$\begin{aligned}\left\|\mathfrak{M}_{\mho,P,\infty,j}(f)\right\|^2_{L^p(\omega,\mathbb{R}^d)} &\le C\int_{\mathbb{R}^d}|f(y)|^2\int_1^4\int_{\mathbb{S}^{d-1}}|\mho(u)|^{2-q}\left|F(y+2^{-(j+1)}su)\right|d\sigma(u)\frac{ds}{s}dy\\ &\le C\left\||f|^2\right\|_{L^{(p/2)}(\omega,\mathbb{R}^d)}\left\|M_{\mho^{(2-q)}}(\widetilde{F})\right\|_{L^{(p/2)'}(\omega^{1-(p/2)'},\mathbb{R}^d)}\\ &\le C_p\|f\|^2_{L^p(\omega,\mathbb{R}^d)}\left\|\widetilde{F}\right\|_{L^{(p/2)'}(\omega^{1-(p/2)'},\mathbb{R}^d)},\end{aligned}$$

where $\widetilde{F}(y) = F(-y)$. The last inequality holds since $(p/2)' > q/(2-q)$ and by invoking Lemma 2.2. Therefore, we have

$$\left\|\mathfrak{M}_{\mho,P,\infty,j}(f)\right\|_{L^p(\omega,\mathbb{R}^d)} \le C_p\|f\|_{L^p(\omega,\mathbb{R}^d)}\tag{3.10}$$

for $q' \le p < \infty$ and $\omega \in A_{p/q'}$. By the last inequality and (3.3), we have that

$$\left\|\mathfrak{M}_{\mho,P,\infty}(f)\right\|_{L^p(\omega,\mathbb{R}^d)} \le C_p\|f\|_{L^p(\omega,\mathbb{R}^d)}\tag{3.11}$$

for $\delta \le p < \infty$ and $\omega \in A_{p/\delta}$.

Now, let us estimate the $\left\|\mathfrak{M}_{\mho,P,0}(f)\right\|_{L^p(\omega,\mathbb{R}^d)}$. Take $Q(x) = \sum_{|\eta|\le k}\lambda_\eta x^\eta$, and let $\mathfrak{M}_{\mho,Q,0}(f)$ and $\mathfrak{M}_{\mho,P,Q,0}(f)$ be given by

$$\mathfrak{M}_{\mho,Q,0}(f)(x) = \left(\int_0^1\left|\int_{\mathbb{S}^{d-1}} e^{iQ(sw)}f(x-su)\mho(u)d\sigma(u)\right|^2\frac{ds}{s}\right)^{1/2},$$

and

$$\mathfrak{M}_{\mho,P,Q,0}(f)(x)=\left(\int_0^1\left|\int_{\mathbb{S}^{d-1}}\left(e^{iP(su)}-e^{iQ(su)}\right)f(x-su)\mho(u)d\sigma(u)\right|^2\frac{ds}{s}\right)^{1/2}.$$

By Minkowski's inequality, we deduce that

$$\mathfrak{M}_{\mho,P,0}(f)(x)\leq\mathfrak{M}_{\mho,Q,0}(f)(x)+\mathfrak{M}_{\mho,P,Q,0}(f)(x). \tag{3.12}$$

Since the degree of the polynomial Q is less than or equal to k, we have that

$$\left\|\mathfrak{M}_{\mho,Q,0}(f)\right\|_{L^p(\omega,\mathbb{R}^d)}\leq C_p\ \|f\|_{L^p(\omega,\mathbb{R}^d)} \tag{3.13}$$

for all $\delta\leq p<\infty$ and $\omega\in A_{p/\delta}$. By noticing that

$$\left|e^{iP(su)}-e^{iQ(su)}\right|\leq s^{(d+1)}\left|\sum_{|\eta|=d+1}\lambda_\eta u^\eta\right|\leq s^{(d+1)}$$

and using the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}\mathfrak{M}_{\mho,P,Q,0}(f)(x) &\leq C\left(\int_0^1\int_{\mathbb{S}^{d-1}}s^{2(k+1)}\left|\mho(u)f(x-su)\right|^2 d\sigma(u)\frac{ds}{s}\right)^{1/2}\\ &\leq \left(\sum_{\ell=1}^{\infty}2^{-j\ell 2(k+1))}\int_{2^{-\ell}}^{2^{-\ell+1}}\int_{\mathbb{S}^{d-1}}\left|\mho(u)f(x-su)\right|^2 d\sigma(u)\frac{ds}{s}\right)^{1/2}.\end{aligned}$$

Therefore, by following the same arguments as above, we obtain that

$$\left\|\mathfrak{M}_{\mho,P,Q,0}(f)\right\|_{L^p(\omega,\mathbb{R}^d)}\leq C_p\ \|f\|_{L^p(\omega,\mathbb{R}^d)} \tag{3.14}$$

for all $\delta\leq p<\infty$ and $\omega\in A_{p/\delta}$. Hence, by (3.13) and (3.14), we deduce that

$$\left\|\mathfrak{M}_{\mho,P,0}(f)\right\|_{L^p(\omega,\mathbb{R}^d)}\leq C_p\ \|f\|_{L^p(\omega,\mathbb{R}^d)}. \tag{3.15}$$

Consequently, by (3.2), (3.9), (3.11) and (3.15), the proof of Theorem 1.1 is complete.

Proof of Theorem 1.2. By duality, it is easy to get that

$$\mathfrak{M}^{(\gamma)}_{\mho,P}(f)(x)=\left(\int_0^\infty\left|\int_{\mathbb{S}^{d-1}}e^{iP(sv)}f(x-sv)\mho(v)d\sigma(v)\right|^{\gamma'}\frac{ds}{s}\right)^{1/\gamma'}$$

for all $1<\gamma\leq 2$. Hence,

$$\left\|\mathfrak{M}^{(\gamma)}_{\mho,P}(f)\right\|_{L^p(\omega,\mathbb{R}^d)}=\|S(f)\|_{L^p(L^{\gamma'}(\mathbb{R}^+,\frac{ds}{s}),\omega,\mathbb{R}^d)},$$

where $S:L^p(\omega,\mathbb{R}^d)\to L^p(L^{\gamma'}(\mathbb{R}^+,\frac{ds}{s}),\omega,\mathbb{R}^d)$ is a linear operator given by

$$S(f)(x,s)=\int_{\mathbb{S}^{d-1}}e^{iP(sv)}f(x-sv)\mho(v)d\sigma(v).$$

Now, if $\gamma = 1$, $f \in L^{\infty}(\mathbb{R}^d)$ and $h \in L^1(\mathbb{R}^+, \frac{ds}{s})$, then we have that

$$\left| \int_0^{\infty} \int_{\mathbb{S}^{d-1}} e^{iP(su)} f(x - su)\, \mho(u) h(s) d\sigma(u) \frac{ds}{s} \right| \leq \|f\|_{L^{\infty}(\mathbb{R}^d)} \|\mho\|_{L^1(\mathbb{S}^{d-1})} \|h\|_{L^1(\mathbb{R}^+, \frac{ds}{s})},$$

and, hence,

$$\left\| \mathfrak{M}^{(1)}_{\mho,P}(f) \right\|_{L^{\infty}(\mathbf{R}^d)} \leq C \|f\|_{L^{\infty}(\mathbb{R}^d)}$$

which, in turn, implies

$$\left\| \mathfrak{M}^{(1)}_{\mho,P}(f) \right\|_{L^{\infty}(\mathbf{R}^d)} = \|S(f)\|_{L^{\infty}(L^{\infty}(\mathbb{R}^+, \frac{ds}{s}), \mathbb{R}^d)} \leq C \|f\|_{L^{\infty}(\mathbb{R}^d)}.$$

Since $L^{\infty}(\mathbb{R}^d, \omega) = L^{\infty}(\mathbb{R}^d)$, we have

$$\left\| \mathfrak{M}^{(1)}_{\mho,P}(f) \right\|_{L^{\infty}(\omega,\mathbb{R}^d)} = \|S(f)\|_{L^{\infty}(L^{\infty}(\mathbb{R}^+, \frac{ds}{s}), \omega, \mathbb{R}^d)} \leq C \|f\|_{L^{\infty}(\omega,\mathbb{R}^d)}. \tag{3.16}$$

On the other hand, by Theorem 1.1 we get

$$\begin{aligned} \left\| \mathfrak{M}^{(2)}_{\mho,P}(f) \right\|_{L^p(\omega,\mathbb{R}^d)} &= \left\| \mathfrak{M}_{\mho,P}(f) \right\|_{L^p(\omega,\mathbb{R}^d)} = \|S(f)\|_{L^p(L^2(\mathbb{R}^+, \frac{ds}{s}), \omega, \mathbb{R}^d)} \\ &\leq C_p \|f\|_{L^p(\omega,\mathbb{R}^d)} \end{aligned} \tag{3.17}$$

for $\delta \leq p < \infty$. Therefore, by applying the interpolation theorem for the Lebesgue mixed normed spaces to (3.16) and (3.17), we deduce that

$$\left\| \mathfrak{M}^{(\gamma)}_{\mho,P}(f) \right\|_{L^p(\omega,\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\omega,\mathbb{R}^d)}$$

for all $(\delta\gamma')/2 \leq p < \infty$ with $1 < \gamma \leq 2$.

Proof of Theorem 1.3. To begin, we notice that $\left(T^{(\gamma)}_{\mho,P} f\right)(x) = \lim_{\varepsilon \to 0} T^{(\gamma)}_{\mho,P,\varepsilon} f(x)$, where $T^{(\gamma)}_{\mho,P,\varepsilon}$ is the truncated singular integral operator given by

$$T^{(\gamma)}_{\mho,P,\varepsilon} f(x) = \int_{|u| > \varepsilon} e^{iP(u)} f(x - u) \frac{\mho(u) h(|u|)}{|u|^d} du. \tag{3.18}$$

By Hölder's inequality, we deduce

$$\begin{aligned} \left| T^{(\gamma)}_{\mho,P,\varepsilon} f(x) \right| &\leq \int_{\varepsilon}^{\infty} |h(s)| \left| \int_{\mathbb{S}^{d-1}} e^{iP(sv)} f(x - sv) \mho(v) d\sigma(v) \right| \frac{ds}{s} \\ &\leq \|h\|_{L^{\gamma}(\mathbf{R}^+, dr/r)} \left(\int_0^{\infty} \left| \int_{\mathbb{S}^{d-1}} e^{iP(sv)} f(x - sv) \mho(v) d\sigma(v) \right|^{\gamma'} \frac{ds}{s} \right)^{1/\gamma'}. \end{aligned}$$

Hence,

$$\left| T^{(\gamma)}_{\mho,P,\varepsilon}(f)(x) \right| \leq \|h\|_{L^{\gamma}(\mathbb{R}^+, \frac{ds}{s})} \mathfrak{M}^{(\gamma)}_{\mho,P}(f)(x). \tag{3.19}$$

Therefore, by Theorem 1.2, we get that $T^{(\gamma)}_{\mho,P}$ is bounded on $L^p(\omega, \mathbb{R}^d)$ for $(\delta\gamma')/2 \leq p < \infty$ and $\omega \in A_{p/\delta}$. On the other hand, by a standard duality argument, we get that $T^{(\gamma)}_{\mho,P}$ is bounded on $L^p(\omega, \mathbb{R}^d)$ for $1 < p \leq (\frac{\delta\gamma'}{2})'$ and $\omega^{1-p'} \in A_{p'/\delta}$. The proof is complete.

4. Conclusions

In this work, we studied the mapping properties of the maximal integral operators $\mathfrak{M}^{(\gamma)}_{\mho,P}$. In fact, we proved the weighted space $L^p(\omega,\mathbb{R}^d)$ boundedness of $\mathfrak{M}^{(\gamma)}_{\mho,P}$ for all $(\delta\gamma')/2 \le p < \infty$ whenever $\omega \in A_{p/\delta}$, $\mho \in L^q(\mathbb{S}^{d-1})$, and $1 \le \gamma \le 2$. Then, as consequence of the this result, we confirmed the weighted $L^p(\omega,\mathbb{R}^d)$ boundedness of the operators $T^{(\gamma)}_{\mho,P}$ and $\mathcal{M}^{(\gamma)}_{\mho,P}$. The results of this paper are substantial extensions and improvements of the main results in [4] and [15].

Author contributions

Mohammed Ali: Writing-original draft, Formal Analysis, Commenting; Hussain Al-Qassem: Writing-original draft, Commenting. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare that they have no conflicts of interest in this paper.

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