Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(9): 25956–25965.
DOI: 10.3934/math.20241268
Received: 19 July 2024
Revised: 29 August 2024
Accepted: 03 September 2024
Published: 06 September 2024

Research article

Blowup criterion for the Cauchy problem of 2D compressible viscous micropolar fluids with vacuum

Dayong Huang* **and Guoliang Hou**

College of Mathematics, Changchun Normal University, Changchun, China

* **Correspondence:** Email: huangdayong@ccsfu.edu.cn.

Abstract: In this study, we establish a regular criterion for the 2D compressible micropolar viscous fluids with vacuum that is independent of the velocity of rotation of the microscopic particles. Specifically, we show that if the density verifies $\|\rho\|_{L^\infty(0,T;L^\infty)} < \infty$, then the strong solution will exist globally on $\mathbb{R}^2 \times (0, T)$. Consequently, we generalize the results of Zhong (Discrete Contin. Dyn. Syst. Ser. B, **25** (2020), no. 12, 4603–4615) to the compressible case. In particular, we don't need the additional compatibility condition.

Keywords: compressible micropolar fluids; strong solutions; vacuum; Cauchy problem; blowup criterion
Mathematics Subject Classification: 76N10, 35Q35, 35B40, 35D35

1. Introduction

The 2D micropolar equations are a special case of the 3D micropolar equations as follows:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) = (\mu + \kappa)\Delta u + (\mu + \lambda - \kappa)\nabla \operatorname{div} u + 2\kappa \nabla^\perp w, \\ (\rho w)_t + \operatorname{div}(\rho u \otimes w) + 4\kappa w = \mu' \Delta w + 2\kappa \nabla \times u, \end{cases} \tag{1.1}$$

where $\rho = \rho(x_1, x_2, t)$, $u = (u_1(x_1, x_2, t), u_2(x_1, x_2, t), 0)$, $w = w(0, 0, w_3(x_1, x_2, t))$ and $P(\rho)$ denote the density, velocity, microrotational velocity, and pressure, respectively. The pressure $P(\rho) = \rho^\gamma$ where $\gamma > 1$ is the specific heat ratio. The parameter $\kappa > 0$ is the dynamics micro-rotation viscosity. The viscous constants $\mu, \lambda, \kappa, \mu'$ satisfy the physical hypothesis: $\mu > 0, \mu' > 0$ and $\mu + \lambda - \kappa \geq 0$. Here, and in what follows, $\nabla^\perp = (\partial_2, -\partial_1), \nabla \times u = \partial_1 u_2 - \partial_2 u_1, \nabla^\perp w = (\partial_2 w, -\partial_1 w)$. In this paper, we consider an initial value problem of system (1.1) with the initial data:

$$(\rho, u, w)(x, 0) = (\rho_0, u_0, w_0) \quad x \in \mathbb{R}^2, \qquad (\rho, u, w)(x, t) \to (0, 0, 0) \quad \text{as } |x| \to \infty, t \geq 0. \tag{1.2}$$

The micropolar fluids are non-Newtonian fluids with nonsymmetric stress tensors (called polar fluids), which were first established by Eringen [3] in the 1960s. To put our results, let's introduce some results on the topics of well-posedness for the multidimensional micropolar fluids. For incompressible case, the authors [8, 10–12] proved a blowup criterion, and the local and global existence of strong solutions for the whole plane and initial boundary value problem. For the compressible case, provied that initial energy is suitably small, the existence of global strong/classical solutions can be found in [4, 13]. Recently, Huang et al. [5] investigated compressible micropolar fluids on a time-dependent domain with slip boundary conditions, which contains local strong and global weak solutions. On the other hand, some blowup criteria of the strong solutions for the 3D case are proven in [1, 2]. However, the regularity and uniqueness of the weak solution to the micropolar model for large initial data are still open and challenge problems even in the two-dimensional case. Thus, it is important to study the possible mechanism of blow-up to the compressible micropolar model in $\mathbb{R}^2$. Motivated by [9, 11], we will establish a blowup criterion of the compressible micropolar model (1.1) in terms of the $L_t^\infty L_x^\infty$ norm of ρ for the 2D case; this gives the first blowup criterion for strong solutions of the 2D compressible micropolar with a vacuum.

2. Main result and the proof

Denote $\bar{x} \triangleq (\mathrm{e} + |x|^2)^{1/2} \log^{1+\eta_0}(\mathrm{e} + |x|^2)$, with $\eta_0 > 0$.

The local well-posedness result to (1.1)–(1.2) can be similarly obtained by [7] as follows.

Proposition 2.1. *Suppose that the initial data* $(\rho_0, \rho_0 u_0, \rho_0 w_0)$ *satisfy*

$$\begin{aligned} &\rho_0 \geq 0,\ \bar{x}^a \rho_0 \in L^1 \cap H^1 \cap W^{1,q},\ \nabla u_0 \in L^2, \\ &w_0 \in H^1,\ \sqrt{\rho_0} w_0 \in L^2,\ \sqrt{\rho_0} u_0 \in L^2, \end{aligned} \tag{2.1}$$

with $q > 2$ *and* $a > 1$. *Then there exist* $T_0, N > 0$, *such that the problem* (1.1)–(1.2) *has a unique strong solution* (ρ, u, w) *on* $\mathbb{R}^2 \times (0, T_0]$ *satisfying*

$$\begin{cases} \rho \in C([0, T_0]; L^1 \cap H^1 \cap W^{1,q}), \\ \bar{x}^a \rho \in L^\infty(0, T_0; L^1 \cap H^1 \cap W^{1,q}), \\ \sqrt{\rho} u, \nabla u,\ \sqrt{t}\sqrt{\rho} u_t \in L^\infty(0, T_0; L^2), \\ \sqrt{\rho} w,\ \sqrt{t}\sqrt{\rho} w_t \in L^\infty(0, T_0; L^2),\ w \in L^\infty(0, T_0; H^1) \\ \nabla u, \nabla w \in L^2(0, T_0; H^1) \cap L^{\frac{q+1}{q}}(0, T_0; W^{1,q}), \\ \sqrt{t} \nabla u,\ \sqrt{t} \nabla w \in L^2(0, T_0; W^{1,q}), \\ \sqrt{t}\sqrt{\rho} u_t,\ \sqrt{t}\sqrt{\rho} w_t,\ \sqrt{t} \nabla u_t,\ \sqrt{t} w_t \in L^2(\mathbb{R}^2 \times (0, T_0)), \end{cases} \tag{2.2}$$

and

$$\inf_{0 \leq t \leq T_0} \int_{B_N} \rho(x, t) dx \geq \frac{1}{4} \int_{\mathbb{R}^2} \rho_0(x, t) dx. \tag{2.3}$$

Our main result of this paper is stated as the following theorem:

Theorem 2.1. *Suppose that the initial data* $(\rho_0, \rho_0 u_0, \rho_0 w_0)$ *verifies*

$$\rho_0 \geq 0,\ \bar{x}^a \rho_0 \in L^1 \cap H^1 \cap W^{1,q},\ \nabla u_0 \in L^2, w_0 \in H^1,\ \sqrt{\rho_0} w_0 \in L^2,\ \sqrt{\rho_0} u_0 \in L^2,$$

with $q > 2$ and $a > 1$. Let (ρ, u, w) be a strong solution to the Cauchy problem (1.1)–(1.2)*. If $0 < T^* < \infty$ is the maximal time of existence, then*

$$\lim_{T\to T^*} \|\rho\|_{L^\infty(0,T;L^\infty)} = \infty. \tag{2.4}$$

Remark 2.1. Chen [2] established a Serrin's blowup criterion for 3D compressible micropolar fluids, i.e.,

$$\lim_{T\to T^*} (\|\rho\|_{L^\infty(0,T;L^\infty)} + \|\sqrt{\rho}u\|_{L^s(0,T;L^r)}) = \infty, \tag{2.5}$$

where r and s satisfy $\frac{2}{s} + \frac{3}{r} \le 1$, $3 < r \le \infty$. It is clear that the blowup criterion in (2.4) for compressible micropolar fluids is weaker than the one in (2.5). Especially, we also notice that (2.4) is no additional growth condition on the micro-rotational velocity and linear velocity. On the other hand, we also extend the incompressible case in Zhong [11] to the compressible case.

Now, we introduce the viscous flux of the compressible micropolar equations as follows, $F := (2\mu + \lambda)\mathrm{div}u - P(\rho)$, $V_1 := \nabla \times u$, $V_2 := \nabla^\perp w$, which implies

$$\Delta F = \mathrm{div}(\rho\dot{u}),\ (\mu+\kappa)\Delta V_1 = \nabla^\perp(\rho\dot{u} - 2\kappa V_2),\ \mu'\Delta V_2 - 4\kappa V_2 = \nabla^\perp(\rho\dot{w} - 2\kappa V_1).$$

Similar to [2], we give the estimates of the effective viscous flux below.

Lemma 2.1. *Denote $\dot{u} = u_t + u\cdot\nabla u$, $\dot{w} = w_t + u\cdot\nabla w$. Let (ρ, u, w) be a strong solution of* (1.1)–(1.2)*. Then for $p \ge 2$ there exists a positive constant C depending only on p, μ, λ, κ and μ' such that*

$$\|\nabla F\|_{L^p} + \|\nabla V_1\|_{L^p} + \|\nabla V_2\|_{L^p} \le C(\|\rho\dot{u}\|_{L^p} + \|\rho\dot{w}\|_{L^p} + \|\nabla u\|_{L^p} + \|\nabla w\|_{L^p} + \|w\|_{L^p}), \tag{2.6}$$

$$\begin{aligned}\|F\|_{L^p(\mathbb{R}^2)} + \|V_1\|_{L^p} + \|V_2\|_{L^p} &\le C\big(\|\rho\dot{u}\|_{L^2} + \|\rho\dot{w}\|_{L^2} + \|\nabla u\|_{L^2} + \|\nabla w\|_{L^2} + \|w\|_{L^2}\big)^{1-\frac{2}{p}} \\ &\quad\cdot \big(\|\nabla u\|_{L^2} + \|\nabla w\|_{L^2} + \|P(\rho)\|_{L^2}\big)^{\frac{2}{p}},\end{aligned} \tag{2.7}$$

$$\begin{aligned}\|\nabla u\|_{L^p} + \|\nabla w\|_{L^p} &\le C\big(\|\rho\dot{u}\|_{L^2} + \|\rho\dot{w}\|_{L^2} + \|\nabla u\|_{L^2} + \|\nabla w\|_{L^2} + \|w\|_{L^2}\big)^{1-\frac{2}{p}} \\ &\quad\cdot \big(\|\nabla u\|_{L^2} + \|\nabla w\|_{L^2} + \|P(\rho)\|_{L^2}\big)^{\frac{2}{p}} + C\|P(\rho)\|_{L^p}.\end{aligned} \tag{2.8}$$

Now we shall prove Theorem 2.1 by contradiction arguments. So, we assume that the opposite holds, i.e.,

$$\lim_{T\to T^*} \|\rho\|_{L^\infty(0,T;L^\infty)} \le M_0 < \infty. \tag{2.9}$$

To begin with, we have the following basic energy estimate:

$$\sup_{0\le t\le T} (\|P(\rho)\|_{L^1} + \|\sqrt{\rho}u\|_{L^2}^2 + \|\sqrt{\rho}w\|_{L^2}^2) + \int_0^T (\|\nabla u\|_{L^2}^2 + \|w\|_{H^1}^2)dt \le C. \tag{2.10}$$

Lemma 2.2. *Under the condition* (2.9)*, it holds that for $0 \le t \le T^*$*

$$\sup_{0\le t\le T} (\|\nabla u\|_{L^2}^2 + \|w\|_{H^1}^2) + \int_0^T (\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + \|\sqrt{\rho}\dot{w}\|_{L^2}^2)dt \le C. \tag{2.11}$$

Proof. Multiplying $(1.1)_2$ and $(1.1)_3$ by $\dot{u}$ and $\dot{w}$ respectively, then integrating the resulting equations by parts over $\mathbb{R}^2$, we obtain after adding them together that

$$\begin{aligned}\int(\rho|\dot{u}|^2+\rho|\dot{w}|^2)dx=&-\int\dot{u}\cdot\nabla P(\rho)dx+\mu\int\dot{u}\cdot\Delta udx+(\mu+\lambda)\int\dot{u}\cdot\nabla\mathrm{div}udx-\kappa\int\dot{u}\cdot\mathrm{rot}(\mathrm{rot}u)dx\\&+\mu'\int\dot{w}\Delta wdx-4\kappa\int w\dot{w}dx+2\kappa\int\nabla^{\perp}w\cdot\dot{u}dx+2\kappa\int\mathrm{rot}u\dot{w}dx=:\sum_{i=1}^{8}I_i.\end{aligned}\tag{2.12}$$

Similar to the estimates of [9], $I_i(i=1,\dots,6)$ can be given as

$$\begin{aligned}\sum_{i=1}^{6}I_i\le&\frac{d}{dt}\int\left(P(\rho)\mathrm{div}u-\frac{\mu}{2}|\nabla u|^2-\frac{\mu'}{2}|\nabla w|^2-\frac{\mu+\lambda}{2}(\mathrm{div}u)^2-\frac{\kappa}{2}|\mathrm{rot}u|^2-2\kappa|w|^2\right)dx\\&+C(\|\nabla w\|_{L^3}^3+\|\nabla u\|_{L^3}^3+\|w\|_{L^3}^3),\end{aligned}$$

and that

$$I_7+I_8=2\kappa\int(w\mathrm{rot}u_t+w_t\mathrm{rot}u)dx+2\kappa\int(\nabla^{\perp}w\cdot(u\cdot\nabla u)+\mathrm{rot}u\cdot(u\cdot\nabla w))dx=2\kappa\frac{d}{dt}\int w\mathrm{rot}udx,$$

where we have used the fact that

$$\begin{aligned}&2\kappa\int(\nabla^{\perp}w\cdot(u\cdot\nabla u)+\mathrm{rot}u\cdot(u\cdot\nabla w))dx\\&=2\kappa\int(u_1\partial_1u_1\partial_2w+u_2\partial_2u_1\partial_2w-u_1\partial_1u_2\partial_1w-u_2\partial_2u_2\partial_1w)dx\\&\quad+2\kappa\int(u_1\partial_1u_2\partial_1w+u_2\partial_1u_2\partial_2w-u_1\partial_2u_1\partial_1w-u_2\partial_2u_1\partial_2w)dx\\&=2\kappa\int(u_1\partial_1u_1\partial_2w-u_2\partial_2u_2\partial_1w+u_2\partial_1u_2\partial_2w-u_1\partial_2u_1\partial_1w)dx\\&=-2\kappa\int(\partial_1u_1\partial_2u_1w+u_1\partial_{21}^2u_1w)dx+2\kappa\int(\partial_1u_2\partial_2u_2w+u_2\partial_{12}^2u_2w)dx\\&\quad-2\kappa\int(\partial_1u_2\partial_2u_2w+u_2\partial_{21}^2u_2w)dx+2\kappa\int(\partial_1u_1\partial_2u_1w+u_1\partial_{12}^2u_1w)dx\\&=0.\end{aligned}$$

Now, putting I_i into (2.12), by taking advantage of (2.8), (2.9), Hölder's and Young inequalities, we are led to

$$\begin{aligned}B'(t)+\int(\rho|\dot{u}|^2+\rho|\dot{w}|^2)dx&\le C(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2)+C(\|\nabla u\|_{L^3}^3+\|\nabla w\|_{L^3}^3)+C\|w\|_{L^2}^4\\&\le\frac{1}{2}(\|\sqrt{\rho}\dot{u}\|_{L^2}^2+\|\sqrt{\rho}\dot{w}\|_{L^2}^2)+C(\|\nabla u\|_{L^2}^4+\|w\|_{H^1}^4+1),\end{aligned}\tag{2.13}$$

where

$$B(t):=\frac{\mu}{2}\|\nabla u\|_{L^2}^2+\frac{\mu+\lambda}{2}\|\mathrm{div}u\|_{L^2}^2+\frac{\mu'}{2}\|\nabla w\|_{L^2}^2+\frac{\kappa}{2}\|\mathrm{rot}u-2w\|_{L^2}^2-\int\mathrm{div}uP(\rho)dx$$

$$\geq \frac{\mu}{4}\|\nabla u\|_{L^2}^2 + \frac{\mu+\lambda}{2}\|\mathrm{div} u\|_{L^2}^2 + \frac{\mu'}{2}\|\nabla w\|_{L^2}^2 + \frac{\kappa}{2}\|\mathrm{rot} u - 2w\|_{L^2}^2 - C.$$

Then, using (2.10) and Gronwall's inequality leads to (2.11), the conclusion follows. □

Lemma 2.3. *Under the condition* (2.9)*, it holds that*

$$\sup_{0\leq t\leq T} t(\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + \|\sqrt{\rho}\dot{w}\|_{L^2}^2) + \int_0^T (t\|\nabla\dot{u}\|_{L^2}^2 + t\|\dot{w}\|_{H^1}^2)dt \leq C. \tag{2.14}$$

Proof. Operating $\dot{u}_j[\frac{\partial}{\partial t} + \mathrm{div}(u\cdot)]$ and $\dot{w}[\frac{\partial}{\partial t} + \mathrm{div}(u\cdot)]$ to $(1.1)_2$ and $(1.1)_3$ respectively, summing up and integrating by parts over $\mathbb{R}^2$, one has

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int(\rho|\dot{u}|^2 + \rho|\dot{w}|^2)dx \\
&= (\mu+\kappa)\int \dot{u}_j[\Delta u_{tj} + \mathrm{div}(u\Delta u_j)]dx + \mu'\int \dot{w}[\Delta w_t + \mathrm{div}(u\Delta w)]dx \\
&\quad + (\mu+\lambda-\kappa)\int \dot{u}_j[\partial_t\partial_j \mathrm{div} u + \mathrm{div}(u\partial_j \mathrm{div} u)]dx - \int \dot{u}_j[\partial_j P_t(\rho) + \mathrm{div}(\partial_j P(\rho)u)]dx \\
&\quad + 2\kappa\int \dot{u}_j[\nabla^{\perp} w_t + \partial_i(u_i\nabla^{\perp} w)]dx + 2\kappa\int \dot{w}[\nabla\times u_t + \partial_i(u_i\nabla\times u)]dx \\
&\quad - 4\kappa\int \dot{w}[w_t + \mathrm{div}(uw)]dx =: \sum_{i=1}^{7} J_i.
\end{aligned} \tag{2.15}$$

By virtue of estimates in [9], one can similarly give $J_i (i = 1, \ldots, 4)$ as follows

$$\sum_{i=1}^{4} J_i \leq \int\Big(-\mu|\nabla\dot{u}|^2 - (\lambda+\mu)(\mathrm{div}\,\dot{u})^2 - \kappa|\nabla\times\dot{u}|^2 - \frac{\mu'}{2}|\nabla\dot{w}|^2\Big) + C(\|\nabla u\|_{L^4}^4 + \|\nabla w\|_{L^4}^4 + \|w\|_{L^4}^4)$$

and

$$\begin{aligned}
J_5 &= 2\kappa\int \dot{w}(\nabla\times\dot{u})dx - 2\kappa\int(u\cdot\nabla w(\nabla\times\dot{u}) + u\cdot\nabla\dot{u}\cdot\nabla^{\perp} w)dx \\
&\leq 2\kappa\int \dot{w}(\nabla\times\dot{u})dx + \frac{\kappa}{4}\|\nabla\dot{u}\|_{L^2}^2 + C\|\nabla u\|_{L^4}^4 + C\|w\|_{L^4}^4, \\
J_6 &= 2\kappa\int \dot{w}(\nabla\times\dot{u})dx - 2\kappa\int(u\cdot\nabla u\cdot(\nabla^{\perp}\dot{w}) + u\cdot\nabla\dot{w}(\nabla\times u))dx \\
&= 2\kappa\int \dot{w}(\nabla\times\dot{u})dx, \\
J_7 &= -4\kappa\int|\dot{w}|^2 dx + 4\kappa\int(u_1\dot{w}\partial_1 w + u_2\dot{w}\partial_2 w + u_1 w\partial_1\dot{w} + u_2 w\partial_2\dot{w})dx \\
&= -4\kappa\int|\dot{w}|^2 dx + 4\kappa\int(u_1\dot{w}\partial_1 w + u_2\dot{w}\partial_2 w - u_2\partial_2 w\dot{w} - u_1\partial_1 w\dot{w})dx \\
&\quad - 4\kappa\int(w\dot{w}\partial_2 u_2 + w\dot{w}\partial_1 u_1)dx \\
&\leq -4\kappa\int|\dot{w}|^2 dx + \delta\|\dot{w}\|_{L^2}^2 + C\|\nabla u\|_{L^4}^4 + C\|w\|_{L^4}^4,
\end{aligned}$$

where we have used the following facts that

$$
\begin{aligned}
&2\kappa\int(u\cdot\nabla w\cdot(\nabla\times\dot u)+u\cdot\nabla\dot u\cdot\nabla^{\perp}w)dx\\
&=2\kappa\int(u_1\partial_1\dot u_2\partial_1 w+u_2\partial_1\dot u_2\partial_2 w-u_1\partial_2\dot u_1\partial_1 w-u_2\partial_2\dot u_1\partial_2 w)dx\\
&\quad+2\kappa\int(u_1\partial_1\dot u_1\partial_2 w+u_2\partial_2\dot u_1\partial_2 w-u_1\partial_1\dot u_2\partial_1 w-u_2\partial_2\dot u_2\partial_1 w)dx\\
&=2\kappa\int(u_2\partial_1\dot u_2\partial_2 w-u_2\partial_2\dot u_2\partial_1 w)dx+2\kappa\int(u_1\partial_1\dot u_1\partial_2 w-u_1\partial_2\dot u_1\partial_1 w)dx\\
&=2\kappa\int(\partial_1 u_2\partial_2\dot u_2 w+u_2\partial_{21}^2\dot u_2 w-\partial_2 u_2\partial_1\dot u_2 w-u_2\partial_{12}^2\dot u_2 w)dx\\
&\quad+2\kappa\int(\partial_1 u_1\partial_2\dot u_1 w+u_1\partial_{21}^2\dot u_1 w-\partial_2 u_1\partial_1\dot u_1 w-u_1\partial_{12}^2\dot u_1 w)dx\\
&=2\kappa\int(\partial_1 u_2\partial_2\dot u_2 w-\partial_2 u_2\partial_1\dot u_2 w)dx+2\kappa\int(\partial_1 u_1\partial_2\dot u_1 w-\partial_2 u_1\partial_1\dot u_1 w)dx\\
&\le 2\kappa\int|\nabla u||\nabla\dot u||w|dx,\\
&2\kappa\int(u\cdot\nabla u\cdot\nabla^{\perp}\dot w+u\cdot\nabla\dot w(\nabla\times u))dx\\
&=2\kappa\int(u_1\partial_1 u_1\partial_2\dot w+u_2\partial_2 u_1\partial_2\dot w-u_1\partial_1 u_2\partial_1\dot w-u_2\partial_2 u_2\partial_1\dot w)dx\\
&\quad+2\kappa\int(u_1\partial_1 u_2\partial_1\dot w+u_2\partial_1 u_2\partial_2\dot w-u_1\partial_2 u_1\partial_1\dot w-u_2\partial_2 u_1\partial_2\dot w)dx\\
&=2\kappa\int(u_1\partial_1 u_1\partial_2\dot w-u_1\partial_2 u_1\partial_1\dot w)dx+2\kappa\int(u_2\partial_1 u_2\partial_2\dot w-u_2\partial_2 u_2\partial_1\dot w)dx\\
&=2\kappa\int(\partial_1 u_1\partial_2 u_1\dot w+u_1\partial_{21}^2 u_1\dot w-\partial_1 u_1\partial_2 u_1\dot w-u_1\partial_{12}^2 u_1\dot w)dx\\
&\quad+2\kappa\int(\partial_1 u_2\partial_2 u_2\dot w+u_2\partial_{21}^2 u_2\dot w-\partial_1 u_2\partial_2 u_2\dot w-u_2\partial_{12}^2 u_2\dot w)dx=0.
\end{aligned}
$$

Then, inserting $J_i (i=1,\ldots,7)$ into (2.15), and choosing δ small enough, leads to

$$
\begin{aligned}
&\frac{d}{dt}\int(\rho|\dot u|^2+\rho|\dot w|^2)dx+\frac{\mu}{2}\|\nabla\dot u\|_{L^2}^2+\frac{\mu'}{2}\|\nabla\dot w\|_{L^2}^2+2\kappa\|\dot w\|_{L^2}^2\\
&\quad\le C(\|\sqrt{\rho}\dot u\|_{L^2}^2+\|\sqrt{\rho}\dot w\|_{L^2}^2)+C\|\nabla u\|_{L^2}^2+C\|\nabla w\|_{L^2}^2+C\|w\|_{L^2}^2\|\nabla w\|_{L^2}^2+C,
\end{aligned}
\tag{2.16}
$$

where we have used (2.8), and (2.9). Then, multiplying (2.28) by t, and integrating resultant over $(0,T)$, we deduce (2.14) from (2.16), (2.10), (2.11) and Gronwall's inequality. □

Lemma 2.4. *Under the condition* (2.9)*, it holds that*

$$
\sup_{0\le t\le T}(\|\rho\|_{H^1\cap W^{1,q}}+t\|\nabla u\|_{H^1}+t\|\nabla w\|_{H^1})+\int_0^T(\|\nabla^2 u\|_{L^q}+\|\nabla^2 w\|_{L^q})dt\le C. \tag{2.17}
$$

Proof. First, $|\nabla\rho|^q(q>2)$ satisfies

$$\begin{aligned}\frac{d}{dt}\|\nabla\rho\|_{L^q} &\le C(1+\|\nabla u\|_{L^\infty})\|\nabla\rho\|_{L^q}+C\|\nabla^2 u\|_{L^q}\\ &\le C(1+\|\nabla u\|_{L^\infty})\|\nabla\rho\|_{L^q}+C\|\rho\dot{u}\|_{L^q}+C\|\nabla w\|_{L^q},\end{aligned}\tag{2.18}$$

where we used the fact that

$$\|\nabla^2 u\|_{L^q}\le C(\|\rho\dot{u}\|_{L^q}+\|\nabla P\|_{L^q}+\|\nabla w\|_{L^q}),\qquad \|\nabla^2 w\|_{L^q}\le C(\|\rho\dot{w}\|_{L^q}+\|w\|_{L^q}+\|\nabla u\|_{L^q}),\tag{2.19}$$

which follows from the standard L^p-estimate for the following elliptic systems:

$$(\mu+\kappa)\Delta u+(\mu+\lambda-\kappa)\nabla\mathrm{div}u=\rho\dot{u}+\nabla P(\rho)-2\kappa\nabla^{\perp}w,\ \mu'\Delta w-4\kappa w=\rho\dot{w}-2\kappa\nabla\times u.$$

Next, it follows from the Gargliardo–Nirenberg inequality and Lemma 2.1 that

$$\begin{aligned}\|\mathrm{div}u\|_{L^\infty}+\|V_1\|_{L^\infty} &\le C\|F_1\|_{L^\infty}+C\|P\|_{L^\infty}+C\|V_1\|_{L^\infty}\le C(q)+C(q)\|\nabla F\|_{L^q}^{\frac{q}{2(q-1)}}+C(q)\|\nabla V_1\|_{L^q}^{\frac{q}{2(q-1)}}\\ &\le C(q)+C(q)(\|\rho\dot{u}\|_{L^q}+\|\nabla w\|_{L^q})^{\frac{q}{2(q-1)}}.\end{aligned}\tag{2.20}$$

By using the Beale-Kato-Majda type inequality (cf. [6]), and (2.19), we are led to

$$\begin{aligned}\|\nabla u\|_{L^\infty} &\le C(\|\mathrm{div}u\|_{L^\infty}+\|V_1\|_{L^\infty})\log(e+\|\nabla^2 u\|_{L^q})+C\|\nabla u\|_{L^2}+C\\ &\le C(1+\|\rho\dot{u}\|_{L^q}+\|\nabla w\|_{L^q})\log(e+\|\nabla\rho\|_{L^p}).\end{aligned}\tag{2.21}$$

Next, by taking advantage of Hölder's and Gagliardo–Nirenberg inequalities, (2.11) and (2.12) that

$$\begin{aligned}\int_0^T\|\rho\dot{u}\|_{L^q}dt &\le \int_0^T\Big(\|\sqrt{\rho}\dot{u}\|_{L^q}^{\frac{q+1}{q}}+t\|\sqrt{\rho}\dot{u}\|_{L^q}^2\Big)dt\\ &\le C\int_0^T\Big(\|\sqrt{\rho}\dot{u}\|_{L^2}^2+t\|\nabla\dot{u}\|_{L^2}^2+t^{-\frac{q^3+q^2-2q-2}{q^3-q^2-2q}}+1\Big)dt\le C,\end{aligned}\tag{2.22}$$

and

$$\begin{aligned}\int_0^T(\|\nabla u\|_{L^q}^2+\|\nabla w\|_{L^q}^2)dt &\le C\int_0^T\|\nabla w\|_{L^2}^{\frac{4}{q}}\|\nabla^2 w\|_{L^2}^{2-\frac{4}{q}}dt+C\int_0^T\|\nabla u\|_{L^2}^{\frac{4}{q}}\|\nabla^2 u\|_{L^2}^{2-\frac{4}{q}}dt\\ &\le C\int_0^T(\|\rho\dot{w}\|_{L^2}^2+\|w\|_{H^1}^2+\|\nabla u\|_{L^2}^2+\|\rho\dot{u}\|_{L^2}^2)dt\le C,\end{aligned}\tag{2.23}$$

which along with Gronwall's inequality, (2.23) and (2.19) that

$$\sup_{0\le t\le T}\|\nabla\rho\|_{L^q}\le C,\qquad \int_0^T(\|\nabla^2 u\|_{L^q}^{\frac{q+1}{q}}+\|\nabla^2 w\|_{L^q}^{\frac{q+1}{q}}+t\|\nabla^2 u\|_{L^q}^2+t\|\nabla^2 w\|_{L^q}^2)dt\le C.\tag{2.24}$$

Finally, taking $q=2$ in (2.18), one gets from (2.11), (2.21), (2.24), and Gronwall's inequality that

$$\sup_{0\le t\le T}\|\nabla\rho\|_{L^2}\le C.\tag{2.25}$$

The standard L^2-estimate for the elliptic systems, (2.10), (2.12), (2.14) lead to

$$\sup_{0\le t\le T}(t\|\nabla^2 u\|_{L^2}^2+t\|\nabla^2 w\|_{L^2}^2)\le C\sup_{0\le t\le T}t(\|\rho\dot{u}\|_{L^2}^2+\|\rho\dot{w}\|_{L^2}^2+\|\nabla P\|_{L^2}^2+\|\nabla u\|_{L^2}^2+\|w\|_{H^1}^2)\le C.$$

From which, by virtue of (2.24) and (2.25), the conclusion follows. □

With the aid of (2.17), the following x-weighted estimate of $\rho(x,t)$ can be found in [9, Lemma 3.7].

Lemma 2.5. *Under the condition* (2.9)*, it holds that for* $a>1$*,* $q>2$ *and* $0\le T\le T^*$*,*

$$\sup_{0\le t\le T}\|\bar{x}^a\rho\|_{L^1\cap H^1\cap W^{1,q}}\le C. \tag{2.26}$$

Lemma 2.6. *Under the condition* (2.9)*, it holds that for* $0\le T\le T^*$*,*

$$\sup_{0\le t\le T} t(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}w_t\|_{L^2}^2)+\int_0^T(t\|\nabla u_t\|_{L^2}^2+t\|\nabla w_t\|_{L^2}^2)dt\le C. \tag{2.27}$$

Proof. It follows from (2.19), Hölder's and Young's inequalites that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int(\rho|u_t|^2+\rho|w_t|^2)dx+\int[(\mu+\kappa)|\nabla u_t|^2+\mu'|\nabla w_t|^2]dx+(\mu+\lambda-\kappa)\int(\mathrm{div}u_t)^2dx+4\kappa\int|w_t|^2dx\\
&\le 4\kappa\int|w_t|^2dx+\kappa\int|\nabla u_t|^2dx+C\int(P|\mathrm{div}u|+|\nabla P||u|)|\mathrm{div}u_t|dx+C\|\nabla u\|_{L^2}\|\sqrt{\rho}u_t\|_{L^4}^2\\
&\quad+C\|\nabla w\|_{L^2}\|\sqrt{\rho}u_t\|_{L^3}\|\sqrt{\rho}w_t\|_{L^6}+C\|\sqrt{\rho}u\|_{L^6}\|\sqrt{\rho}w_t\|_{L^3}\|\nabla w_t\|_{L^2}+C\|\nabla w\|_{L^2}\|\sqrt{\rho}u\|_{L^8}\|\nabla u\|_{L^4}\|\sqrt{\rho}w_t\|_{L^8}\\
&\quad+C\|\rho^{\frac{1}{4}}u\|_{L^{12}}^2\|\sqrt{\rho}w_t\|_{L^3}\|\nabla^2 w\|_{L^2}+C\|\sqrt{\rho}u\|_{L^8}^2\|\nabla w\|_{L^4}\|\nabla w_t\|_{L^2}+C\|\sqrt{\rho}u\|_{L^6}\|\sqrt{\rho}u_t\|_{L^3}\|\nabla u_t\|_{L^2}\\
&\quad+C\|\sqrt{\rho}u\|_{L^6}\|\sqrt{\rho}u_t\|_{L^3}\|\nabla u\|_{L^4}^2+C\|\rho^{\frac{1}{4}}u\|_{L^{12}}^2\|\sqrt{\rho}u_t\|_{L^3}\|\nabla^2 u\|_{L^2}+C\|\sqrt{\rho}u\|_{L^8}^2\|\nabla u\|_{L^4}\|\nabla u_t\|_{L^2}\\
&\le 4\kappa\|w_t\|_{L^2}^2+\kappa\|\nabla u_t\|_{l^2}^2+\delta(\|\nabla u_t\|_{L^2}^2+\|\nabla w_t\|_{L^2}^2)+C(\delta)(\|\nabla^2 u\|_{L^2}^2+\|\nabla^2 w\|_{L^2}^2)\\
&\quad+C(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}w_t\|_{L^2}^2)+C\|\bar{x}^{-a}u\|_{L^{\frac{2q}{q-2}}}^2\|\rho\|_{L^\infty}^{2(\gamma-1)}\|\bar{x}^a\nabla\rho\|_{L^q}^2+C\\
&\le 4\kappa\|w_t\|_{L^2}^2+\kappa\|\nabla u_t\|_{L^2}^2+\delta\|(\nabla u_t,\nabla w_t)\|_{L^2}^2+C\|(\sqrt{\rho}u_t,\sqrt{\rho}w_t,\sqrt{\rho}\dot{u},\sqrt{\rho}\dot{w})\|_{L^2}^2+C,
\end{aligned} \tag{2.28}$$

where we have used the following fact that for any $\sigma\in(0,1]$, and any $s>2$, see [9, (3.72)],

$$\|\rho^\sigma v\|_{L^{\frac{s}{\sigma}}}\le C\|\rho^{\frac{3\sigma}{4s}}\|_{L^{\frac{4s}{2\sigma}}}\|\rho^{\frac{\sigma}{4s}}\bar{x}^{\frac{\sigma a}{4s}}\|_{L^{\frac{4s}{\sigma}}}\|v\bar{x}^{-\frac{\sigma a}{4s}}\|_{L^{\frac{4s}{\sigma}}}\le C(\|\sqrt{\rho}v\|_{L^2}+\|\nabla v\|_{L^2}).$$

Then, multiplying (2.28) by t and choosing δ small enough, we obtain (2.27) after using Gronwall's inequality, (2.11), and (2.14). The conclusion follows. □

Proof of Theorem 2.1. Similar to [12], by taking advantage of the estimates in Lemmas 2.2–2.6, we can define $(\rho,u,w)(x,T^*)\triangleq\lim_{t\to T^*}(\rho,u,w)(x,t)$ as a new initial data, which starts from T^*. In view of the local well-posedness presented in Proposition 2.1, we can extend the local strong solution beyond T^*, which contradicts the definition of T^*. Thus, we complete the proof of Theorem 2.1. □

3. Conclusions

In this work, we show that if the density verifies $\|\rho\|_{L^\infty(0,T;L^\infty)}<\infty$, then the strong solution of the 2D compressible micropolar viscous fluids with vacuum will exist globally on $\mathbb{R}^2\times(0,T)$. Consequently, we generalize the results of Zhong (Discrete Contin. Dyn. Syst. Ser. B, **25** (2020), no. 12, 4603–4615) to the compressible case. In particular, we don't need the additional compatibility condition.

Author contributions

Dayong Huang: Responsible for the review and editing of the manuscript, as well as the supervision of the research project; Guoliang Hou: Responsible for visualization and preparation of the original draft of the manuscript. All authors have read and approved the final version of the manuscript for publication.

Conflict of interest

The authors declare that they have no competing interests.

References

1. M. Chen, Blowup criterion for viscous, compressible micropolarfluids with vacuum, *Nonlinear Anal. Real World Appl.,* **13** (2012), 850–859. https://doi.org/10.1016/j.nonrwa.2011.08.021
2. M. Chen, B. Huang, J. Zhang, Blowup criterion for the three-dimensional equations of compressible viscous micropolar fluids with vacuum, *Nonlinear Anal.,* **79** (2013), 1–11. https://doi.org/10.1016/j.na.2012.10.013
3. A. C. Eringen, Theory of micropolar fluids, *J. Math. Mech.* **16** (1966), 1–18. Available from: `https://www.jstor.org/stable/24901466`
4. X. Hou, H. Peng, Global existence for a class large solution to the three-dimensional micropolar fluid equations with vacuum, *J. Math. Anal. Appl.*, **498** (2021), 124931. https://doi.org/10.1016/j.jmaa.2021.124931
5. B. Huang, S. Nečasová, L. Zhang, On the compressible micropolar fluids in a time-dependent domain, *Ann. Mat. Pura Appl.*, **201** (2022), 2733–2795. https://doi.org/10.1007/s10231-022-01218-6
6. T. Kato, Remarks on the Euler and Navier-Stokes equations in $\mathbb{R}^2$, *Proc. Symp. Pure Math.*, **45** (1986), 1–7. https://doi.org/10.1090/pspum/045.2/843590
7. J. Li, Z. Liang, On local classical solutions to the Cauchy problem of the two-dimensional barotropic compressible Navier-Stokes equations with vacuum, *J. Math. Pures Appl.*, **102** (2014) 640–671. https://doi.org/10.1016/j.matpur.2014.02.001
8. L. Liu, X. Zhong, Global existence and exponential decay of strong solutions for 2D nonhomogeneous micropolar fluids with density-dependent viscosity, *J. Math. Phys.*, **62** (2021), 061508. https://doi.org/10.1063/5.0055689
9. T. Wang, A regularity criterion of strong solutions to the 2D compressible magnetohydrodynamic equations, *Nonlinear Anal. Real World Appl.*, **31** (2016), 100–118. https://doi.org/10.1016/j.nonrwa.2016.01.011
10. G. Wu, X. Zhong, Global strong solution and exponential decay of 3D nonhomogeneous asymmetric fluid equations with vacuum, *Acta Math. Sci. Ser. B (Engl. Ed.)*, **41** (2021), 1428–1444. https://doi.org/10.1007/s10473-021-0503-8

11. X. Zhong, Strong solutions to the Cauchy problem of two-dimensional nonhomogeneous micropolar fluid equations with nonnegative density, *Dyn. Partial Differ. Equ.*, **18** (2021), 49–69. https://dx.doi.org/10.4310/DPDE.2021.v18.n1.a4
12. X. Zhong, A blow-up criterion of strong solutions to two-dimensional nonhomogeneous micropolar fluid equations with vacuum, *Discrete Contin. Dyn. Syst. Ser. B*, **25** (2020), 4603–4615. https://doi.org/10.3934/dcdsb.2020115
13. C. Zhu, Q. Tao, Global classical solutions to the compressible micropolar viscous fluids with large oscillations and vacuum, *Math. Method. Appl. Sci.*, **46** (2023), 28–53. https://doi.org/10.1002/mma.8490