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*Research article*

# Systems of two-dimensional complex partial differential equations for bi-polyanalytic functions

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**Abstract:** A class of Schwarz problems with the conditions concerning the real and imaginary parts of high-order partial differentiations for polyanalytic functions was discussed first on the bicylinder. Then, with the particular solution to the Schwarz problem for polyanalytic functions, a Dirichlet problem for bi-polyanalytic functions was investigated on the bicylinder. From the perspective of series, the specific representation of the solution was obtained. In this article, a novel and effective method for solving boundary value problems, with the help of series expansion, was provided. This method can also be used to solve other types of boundary value problems or complex partial differential equation problems of other functions in high-dimensional complex spaces.

**Keywords:** bi-polyanalytic functions; polyanalytic functions; Schwarz problems; Dirichlet problems; complex partial differential equations
**Mathematics Subject Classification:** 30C45, 32A30

## 1. Introduction

Partial differential equations are closely related to many physical problems in real life. For example, the ninth-order linear or non-linear boundary value problems are related to the laminar viscous flow in a semi-porous channel or the hydro-magnetic stability, and the telegraph equations are related to the vibrations within objects or the propagation of waves. There are also many different types of partial differential equations in engineering and other applied sciences. Zhang et al. [1] discussed cubic spline solutions of ninth-order linear and non-linear boundary value problems using a cubic B-spline. Shah et al. [2] proposed a new and efficient operational matrix method for solving time-fractional telegraph equations with Dirichlet boundary conditions. Nisar et al. [3] proposed a hybrid mesh free framework based on Padé approximation in order to solve the numerical solutions of nonlinear partial differential equations. There are also many different types of partial differential equations in chemistry, engineering, and other applied sciences. There have been many successful conclusions about these partial differential equations.

Complex partial differential equations of analytic functions also have a wide range of applications. Bi-analytic functions, the generalizations of analytic functions, have important applications in elasticity. In 1961, Sander [4] studied the properties of pairs of functions $\{u(x,y), v(x,y)\}$ with binary real variables $(x,y)$, which satisfy the system of partial differential equations:

$$\begin{cases} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = \theta, \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \omega, \\ (k+1)\frac{\partial \theta}{\partial x} + \frac{\partial \omega}{\partial y} = 0, \quad (k+1)\frac{\partial \theta}{\partial y} - \frac{\partial \omega}{\partial x} = 0, \end{cases}$$

for the real constant $k(k \neq -1)$, where $\theta(x,y)$ and $\omega(x,y)$ are continuously differentiable functions of $x$ and $y$. Sander provided the definition of bi-analytic functions of type $k$, which are of great significance for studying some physical problems for $k > 0$, and extended some properties of analytic functions to bi-analytic functions.

In 1965, Lin and Wu [5] introduced the function class that is more extensive than Sander's function class, i.e., bi-analytic functions of the type $(\lambda, k)$ which are defined by the system of equations:

$$\begin{cases} \frac{1}{k}\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = \theta, \quad \frac{\partial u}{\partial y} + \frac{1}{k}\frac{\partial v}{\partial x} = \omega, \\ k\frac{\partial \theta}{\partial x} + \lambda\frac{\partial \omega}{\partial y} = 0, \quad k\frac{\partial \theta}{\partial y} - \lambda\frac{\partial \omega}{\partial x} = 0, \end{cases}$$

where $\theta(x,y)$ and $\omega(x,y)$ are continuously differentiable functions of $x$ and $y$, and $\lambda, k$ are real constants with $\lambda \neq 0, 1, k^2$, and $0 < k < 1$. The complex form of the system is

$$\frac{k+1}{2}\frac{\partial f}{\partial \bar{z}} - \frac{k-1}{2}\frac{\partial f}{\partial z} = \frac{\lambda - k}{4\lambda}\varphi(z) + \frac{\lambda + k}{4\lambda}\overline{\varphi(z)},$$

in which $\varphi(z) = k\vartheta - i\lambda\omega$ is analytic and is called the associate function of $f(z) = u + iv$. In [5], the general expression and the properties of bi-analytic functions of the type $(\lambda, k)$ were researched in detail.

Hua et al. [6] introduced a mechanical interpretation for bi-analytic functions and promoted the corresponding function theory. Thereafter, bi-analytic functions aroused widespread attention from many scholars [7–9]. In 1994, Kumar [10] discussed a broader class of functions, i.e., bi-polyanalytic functions, and investigated several Riemann-Hilbert problems for systems of $n$-order partial differential equations applying polyanalytic functions [11] and bi-polyanalytic functions on the unit disk. In 2005, Kumar and Prakash [12] investigated Dirichlet problems for the Poisson equation and some boundary value problems for bi-polyanalytic functions on the unit disk. They obtained the explicit representations of the solutions and the corresponding solvable conditions. In 2006, Begehr and Kumar [13] discussed some complex partial differential equations of higher order. Some boundary value problems for bi-polyanalytic functions were solved on different conditions on the unit disk.

In recent years, some other boundary value problems for bi-analytic functions were solved [14–17]. With the gradual improvements of the theory for bi-analytic functions and polyanalytic functions [18–21] in the complex plane, some scholars attempted to generalize the relevant achievements to spaces of several complex variables [22,23].

In this paper, based on the work of the former researchers, we study a class of Schwarz problems for polyanalytic functions on the bicylinder. Then, from the perspective of series and applying the particular solution to the Schwarz problem for polyanalytic functions, we discuss a Dirichlet problem for bi-polyanalytic functions on the bicylinder.

In the following, let the bicylinder $D^2 = D_1 \times D_2 = \{(z_1, z_2) : |z_1| < 1,\ |z_2| < 1\}$, and let $\partial_0 D^2$ denote the characteristic boundary of $D^2$. Let $C(G)$ represent the set of continuous functions within $G$.

## 2. The Schwarz problem

To get the main results, we need to discuss the following Schwarz problem.

**Theorem 2.1.** Let $g_{\mu\nu} \in C(\partial_0 D^2; \mathbb{R})$ for $1 \le \mu, \nu \le m-1$ $(m \ge 2)$, and let

$$\widetilde{\phi}(z) = \sum_{m_1,m_2=0}^{+\infty} \sum_{\widetilde{v}_1=\mu}^{m-1} \sum_{\widetilde{v}_2=\nu}^{m-1} \frac{\bar{z}_1^{\widetilde{v}_1} \bar{z}_2^{\widetilde{v}_2}}{\widetilde{v}_1 ! \widetilde{v}_2 !} u_{m_1,m_2}^{m-\widetilde{v}_1, m-\widetilde{v}_2} z_1^{m_1} z_2^{m_2}, \tag{2.1}$$

where

$$u_{m_1,m_2}^{m-\widetilde{v}_1,m-\widetilde{v}_2} = \begin{cases} \dfrac{1}{(2\pi i)^2} \displaystyle\int_{\partial_0 D^2} \sum_{l_1=0}^{m-\mu-1} \sum_{l_2=0}^{m-\nu-1} \frac{g_{(\mu+l_1)(\nu+l_2)}(\zeta)}{l_1 ! l_2 !} \widetilde{A} \frac{d\zeta_1 d\zeta_2}{\zeta_1 \zeta_2}, & \begin{cases} \widetilde{v}_1 = \mu \\ \widetilde{v}_2 = \nu, \end{cases} \\ \dfrac{1}{(2\pi i)^2} \displaystyle\int_{\partial_0 D^2} \sum_{l_1=0}^{m-\mu-1} \sum_{l_2=0}^{m-1-\widetilde{v}_2} \frac{g_{(\mu+l_1)(\widetilde{v}_2+l_2)}(\zeta)}{l_1 ! l_2 !} \widetilde{B}\Big|_{v_2=\widetilde{v}_2-\nu} \frac{d\zeta_1 d\zeta_2}{\zeta_1 \zeta_2}, & \begin{cases} \widetilde{v}_1 = \mu \\ \nu < \widetilde{v}_2 \le m-1, \end{cases} \\ \dfrac{1}{(2\pi i)^2} \displaystyle\int_{\partial_0 D^2} \sum_{l_1=0}^{m-1-\widetilde{v}_1} \sum_{l_2=0}^{m-\nu-1} \frac{g_{(\widetilde{v}_1+l_1)(\nu+l_2)}(\zeta)}{l_1 ! l_2 !} \widetilde{C}\Big|_{v_1=\widetilde{v}_1-\mu} \frac{d\zeta_1 d\zeta_2}{\zeta_1 \zeta_2}, & \begin{cases} \mu < \widetilde{v}_1 \le m-1 \\ \widetilde{v}_2 = \nu, \end{cases} \\ \dfrac{1}{(2\pi i)^2} \displaystyle\int_{\partial_0 D^2} \sum_{l_1=0}^{m-1-\widetilde{v}_1} \sum_{l_2=0}^{m-1-\widetilde{v}_2} \frac{g_{(\widetilde{v}_1+l_1)(\widetilde{v}_2+l_2)}(\zeta)}{l_1 ! l_2 !} \widetilde{D}\Big|_{\substack{v_1=\widetilde{v}_1-\mu \\ v_2=\widetilde{v}_2-\nu}} \frac{d\zeta_1 d\zeta_2}{\zeta_1 \zeta_2}, & \begin{cases} \mu < \widetilde{v}_1 \le m-1 \\ \nu < \widetilde{v}_2 \le m-1, \end{cases} \end{cases} \tag{2.2}$$

and

$$\begin{cases} \widetilde{A} = 2\Big[\displaystyle\sum_{j_1=0}^{m_1} \sum_{j_2=0}^{m_2} C_{l_1}^{m_1-j_1} (-\zeta_1-\bar{\zeta}_1)^{l_1-m_1+j_1} \bar{\zeta}_1^{j_1} C_{l_2}^{m_2-j_2} (-\zeta_2-\bar{\zeta}_2)^{l_2-m_2+j_2} \bar{\zeta}_2^{j_2} \\ \quad + D_{21}|_{v_1=0} \displaystyle\sum_{j_2=0}^{m_2} C_{l_2}^{m_2-j_2} (-\zeta_2-\bar{\zeta}_2)^{l_2-m_2+j_2} \bar{\zeta}_2^{j_2} + D_{31}|_{v_2=0} \sum_{j_1=0}^{m_1} C_{l_1}^{m_1-j_1} (-\zeta_1-\bar{\zeta}_1)^{l_1-m_1+j_1} \bar{\zeta}_1^{j_1}\Big] \\ \quad - (D_{11}D_{12} + D_{22}D_{23} + D_{32}D_{33})|_{v_1=v_2=0} \\ \quad + 2(-\zeta_1-\bar{\zeta}_1)^{l_1} \Big[\displaystyle\sum_{j_2=0}^{m_2} C_{l_2}^{m_2-j_2} (-\zeta_2-\bar{\zeta}_2)^{l_2-m_2+j_2} \bar{\zeta}_2^{j_2} \bar{\zeta}_1^{m_1} + B_{21}D_{31} - \frac{D_{23}+B_{22}}{2} B_{21}\Big]\Big|_{v_2=0} \\ \quad + 2(-\zeta_2-\bar{\zeta}_2)^{l_2} \Big[\displaystyle\sum_{j_1=0}^{m_1} C_{l_1}^{m_1-j_1} (-\zeta_1-\bar{\zeta}_1)^{l_1-m_1+j_1} \bar{\zeta}_1^{j_1} \bar{\zeta}_2^{m_2} + D_{21}C_{21} - \frac{D_{11}+C_{22}}{2} C_{21}\Big]\Big|_{v_1=0} \\ \quad + 2(\bar{\zeta}_1^{\,m_1} C_{21} + B_{21} \bar{\zeta}_2^{\,m_2} - B_{21}C_{21})(-\zeta_1-\bar{\zeta}_1)^{l_1} (-\zeta_2-\bar{\zeta}_2)^{l_2}, \\ \widetilde{B} = 2\Big\{\Big[\displaystyle\sum_{j_1=0}^{m_1} C_{l_1}^{m_1-j_1} (-\zeta_1-\bar{\zeta}_1)^{l_1-m_1+j_1} \bar{\zeta}_1^{j_1} - (-\zeta_1-\bar{\zeta}_1)^{l_1} \bar{\zeta}_1^{m_1}\Big] \sum_{j_2=0}^{m_2} C_{l_2}^{m_2-j_2} (-\zeta_2-\bar{\zeta}_2)^{l_2-m_2+j_2} \bar{\zeta}_2^{j_2} \\ \quad + D_{21}|_{v_1=0} \displaystyle\sum_{j_2=0}^{m_2} C_{l_2}^{m_2-j_2} (-\zeta_2-\bar{\zeta}_2)^{l_2-m_2+j_2} \bar{\zeta}_2^{j_2} + D_{31} \sum_{j_1=0}^{m_1} C_{l_1}^{m_1-j_1} (-\zeta_1-\bar{\zeta}_1)^{l_1-m_1+j_1} \bar{\zeta}_1^{j_1} \\ \quad - B_{21}D_{31}\Big\} - (D_{11}D_{12} + D_{22}|_{v_1=0} D_{23} + D_{32}D_{33} - B_{21}D_{23} - B_{21}B_{22}), \end{cases}$$

$$\begin{cases} \widetilde{C} = 2\Big\{\sum_{j_1=0}^{m_1} C_{l_1}^{m_1-j_1}(-\zeta_1-\bar{\zeta}_1)^{l_1-m_1+j_1}\bar{\zeta}_1^{j_1}\Big[\sum_{j_2=0}^{m_2} C_{l_2}^{m_2-j_2}(-\zeta_2-\bar{\zeta}_2)^{l_2-m_2+j_2}\bar{\zeta}_2^{j_2} - (-\zeta_2-\bar{\zeta}_2)^{l_2}\bar{\zeta}_2^{m_2}\Big] \\ \quad + D_{21}\sum_{j_2=0}^{m_2} C_{l_2}^{m_2-j_2}(-\zeta_2-\bar{\zeta}_2)^{l_2-m_2+j_2}\bar{\zeta}_2^{j_2} + D_{31}|_{\nu_2=0}\sum_{j_1=0}^{m_1} C_{l_1}^{m_1-j_1}(-\zeta_1-\bar{\zeta}_1)^{l_1-m_1+j_1}\bar{\zeta}_1^{j_1} \\ \quad - D_{21}C_{21}\Big\} - (D_{11}D_{12} + D_{22}D_{23} + D_{32}D_{33}|_{\nu_2=0} - D_{11}C_{21} - C_{22}C_{21}), \\ \widetilde{D} = 2\Big[\sum_{j_1=0}^{m_1}\sum_{j_2=0}^{m_2} C_{l_1}^{m_1-j_1}(-\zeta_1-\bar{\zeta}_1)^{l_1-m_1+j_1}\bar{\zeta}_1^{j_1} C_{l_2}^{m_2-j_2}(-\zeta_2-\bar{\zeta}_2)^{l_2-m_2+j_2}\bar{\zeta}_2^{j_2} \\ \quad + D_{21}\sum_{j_2=0}^{m_2} C_{l_2}^{m_2-j_2}(-\zeta_2-\bar{\zeta}_2)^{l_2-m_2+j_2}\bar{\zeta}_2^{j_2} + D_{31}\sum_{j_1=0}^{m_1} C_{l_1}^{m_1-j_1}(-\zeta_1-\bar{\zeta}_1)^{l_1-m_1+j_1}\bar{\zeta}_1^{j_1}\Big] \\ \quad - (D_{11}D_{12} + D_{22}D_{23} + D_{32}D_{33}), \end{cases}$$

in which

$$D_{11} = \begin{cases} C_{l_1}^{m_1}(-\zeta_1-\bar{\zeta}_1)^{l_1-m_1},\ 0 \le m_1 \le l_1, \\ 0,\ \ m_1 > l_1, \end{cases} \qquad D_{12} = \begin{cases} C_{l_2}^{m_2}(-\zeta_2-\bar{\zeta}_2)^{l_2-m_2},\ 0 \le m_2 \le l_2, \\ 0,\ \ m_2 > l_2, \end{cases}$$

$$D_{21} = \begin{cases} 0,\ 0 \le m_1 < l_1, \\ (-1)^{l_1+\nu_1}\bar{\zeta}_1^{m_1-l_1}, m_1 \ge l_1, \end{cases} \qquad D_{22} = \begin{cases} (-1)^{m_1+\nu_1}, m_1 = l_1, \\ 0,\ m_1 \ne l_1, \end{cases}$$

$$D_{23} = \begin{cases} C_{l_2}^{m_2}(-\zeta_2 - \bar{\zeta}_2)^{l_2-m_2}, 0 \le m_2 \le l_2, \\ 0,\ m_2 > l_2, \end{cases} \qquad D_{31} = \begin{cases} 0,\ 0 \le m_2 < l_2, \\ (-1)^{l_2+\nu_2}\bar{\zeta}_2^{m_2-l_2}, m_2 \ge l_2, \end{cases}$$

$$D_{32} = \begin{cases} C_{l_1}^{m_1}(-\zeta_1 - \bar{\zeta}_1)^{l_1-m_1}, 0 \le m_1 \le l_1, \\ 0,\ m_1 > l_1, \end{cases} \qquad D_{33} = \begin{cases} (-1)^{m_2+\nu_2}, m_2 = l_2, \\ 0,\ m_2 \ne l_2, \end{cases}$$

and

$$C_{21} = \begin{cases} 0,\ \ m_2 \ge 1, \\ 1,\ \ m_2 = 0, \end{cases} \qquad C_{22} = \begin{cases} (-1)^{m_1+\nu_1},\ \ m_1 = l_1, \\ 0,\ \ m_1 \ne l_1, \end{cases}$$

$$B_{21} = \begin{cases} 0,\ \ m_1 \ge 1, \\ 1,\ \ m_1 = 0, \end{cases} \qquad B_{22} = \begin{cases} (-1)^{m_2+\nu_2},\ \ m_2 = l_2, \\ 0,\ \ m_2 \ne l_2. \end{cases}$$

Then, $\widetilde{\phi}(z)$ satisfies

$$\Re\partial_{\bar{z}_1}^{\mu}\partial_{\bar{z}_2}^{\nu}\widetilde{\phi}(z) = g_{\mu\nu}(z)\ (z \in \partial_0 D^2),\ \Im\partial_{\bar{z}_1}^{\mu}\partial_{\bar{z}_2}^{\nu}\widetilde{\phi}(0, z_2) = 0 = \Im\partial_{\bar{z}_1}^{\mu}\partial_{\bar{z}_2}^{\nu}\widetilde{\phi}(z_1, 0)\ (z_1 \in D_1, z_2 \in D_2).$$

*Proof:* 1) Let

$$\widetilde{\phi}(z) = \sum_{\widetilde{\nu}_1=\mu}^{m-1}\sum_{\widetilde{\nu}_2=\nu}^{m-1}\frac{\bar{z}_1^{\widetilde{\nu}_1}\bar{z}_2^{\widetilde{\nu}_2}}{\widetilde{\nu}_1!\widetilde{\nu}_2!}u_{\widetilde{\nu}_1\widetilde{\nu}_2}(z), \tag{2.3}$$

in which

$$u_{\widetilde{v}_1\widetilde{v}_2}(z)=\begin{cases}\dfrac{1}{(2\pi i)^2}\displaystyle\int_{\partial_0 D^2}\sum_{l_1=0}^{m-\mu-1}\sum_{l_2=0}^{m-\nu-1}\frac{g_{(\mu+l_1)(\nu+l_2)}(\zeta)}{l_1!l_2!}(A_1-A_2-A_3+A_4)\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}, & \{{}^{\widetilde{v}_1=\mu}_{\widetilde{v}_2=\nu},\\ \dfrac{1}{(2\pi i)^2}\displaystyle\int_{\partial_0 D^2}\sum_{l_1=0}^{m-\mu-1}\sum_{l_2=0}^{m-1-\widetilde{v}_2}\frac{g_{(\mu+l_1)(\widetilde{v}_2+l_2)}(\zeta)}{l_1!l_2!}(B_1-B_2)\Big|_{v_2=\widetilde{v}_2-\nu}\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}, & \{{}^{\widetilde{v}_1=\mu}_{\nu<\widetilde{v}_2\le m-1},\\ \dfrac{1}{(2\pi i)^2}\displaystyle\int_{\partial_0 D^2}\sum_{l_1=0}^{m-1-\widetilde{v}_1}\sum_{l_2=0}^{m-\nu-1}\frac{g_{(\widetilde{v}_1+l_1)(\nu+l_2)}(\zeta)}{l_1!l_2!}(C_1-C_2)\Big|_{v_1=\widetilde{v}_1-\mu}\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}, & \{{}^{\mu<\widetilde{v}_1\le m-1}_{\widetilde{v}_2=\nu},\\ \dfrac{1}{(2\pi i)^2}\displaystyle\int_{\partial_0 D^2}\sum_{l_1=0}^{m-1-\widetilde{v}_1}\sum_{l_2=0}^{m-1-\widetilde{v}_2}\frac{g_{(\widetilde{v}_1+l_1)(\widetilde{v}_2+l_2)}(\zeta)}{l_1!l_2!}D\Big|_{\substack{v_1=\widetilde{v}_1-\mu\\ v_2=\widetilde{v}_2-\nu}}\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}, & \{{}^{\mu<\widetilde{v}_1\le m-1}_{\nu<\widetilde{v}_2\le m-1},\end{cases} \tag{2.4}$$

is analytic on $D^2$, and

$$\begin{cases} A_1=[(z_1-\zeta_1-\bar{\zeta}_1)^{l_1}(z_2-\zeta_2-\bar{\zeta}_2)^{l_2}+(-z_1)^{l_1}(z_2-\zeta_2-\bar{\zeta}_2)^{l_2}+(z_1-\zeta_1-\bar{\zeta}_1)^{l_1}(-z_2)^{l_2}]\Big[\frac{2\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}-1\Big],\\ A_2=(-\zeta_1-\bar{\zeta}_1)^{l_1}\Big\{(z_2-\zeta_2-\bar{\zeta}_2)^{l_2}[\frac{2\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}-1]+(-z_2)^{l_2}[\frac{2\zeta_2}{\zeta_2-z_2}-1]\Big\},\\ A_3=(-\zeta_2-\bar{\zeta}_2)^{l_2}\Big\{(z_1-\zeta_1-\bar{\zeta}_1)^{l_2}[\frac{2\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}-1]+(-z_1)^{l_1}[\frac{2\zeta_1}{\zeta_1-z_1}-1]\Big\},\\ A_4=(-\zeta_1-\bar{\zeta}_1)^{l_1}(-\zeta_2-\bar{\zeta}_2)^{l_2}[\frac{2\zeta_1}{\zeta_1-z_1}+\frac{2\zeta_2}{\zeta_2-z_2}-2],\\ B_1=[(z_1-\zeta_1-\bar{\zeta}_1)^{l_1}(z_2-\zeta_2-\bar{\zeta}_2)^{l_2}+(-z_1)^{l_1}(z_2-\zeta_2-\bar{\zeta}_2)^{l_2}+(z_1-\zeta_1-\bar{\zeta}_1)^{l_1}(-z_2)^{l_2}(-1)^{v_2}]\\ \quad\cdot\Big[\frac{2\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}-1\Big],\\ B_2=(-\zeta_1-\bar{\zeta}_1)^{l_1}\Big\{(z_2-\zeta_2-\bar{\zeta}_2)^{l_2}[\frac{2\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}-1]+(-z_2)^{l_2}(-1)^{v_2}[\frac{2\zeta_2}{\zeta_2-z_2}-1]\Big\},\\ C_1=[(z_1-\zeta_1-\bar{\zeta}_1)^{l_1}(z_2-\zeta_2-\bar{\zeta}_2)^{l_2}+(-z_1)^{l_1}(-1)^{v_1}(z_2-\zeta_2-\bar{\zeta}_2)^{l_2}+(z_1-\zeta_1-\bar{\zeta}_1)^{l_1}(-z_2)^{l_2}]\\ \quad\cdot\Big[\frac{2\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}-1\Big],\\ C_2=(-\zeta_2-\bar{\zeta}_2)^{l_2}\Big\{(z_1-\zeta_1-\bar{\zeta}_1)^{l_2}[\frac{2\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}-1]+(-z_1)^{l_1}(-1)^{v_1}[\frac{2\zeta_1}{\zeta_1-z_1}-1]\Big\},\\ D=[(z_1-\zeta_1-\bar{\zeta}_1)^{l_1}(z_2-\zeta_2-\bar{\zeta}_2)^{l_2}+(-z_1)^{l_1}(-1)^{v_1}(z_2-\zeta_2-\bar{\zeta}_2)^{l_2}+(z_1-\zeta_1-\bar{\zeta}_1)^{l_1}(-z_2)^{l_2}(-1)^{v_2}]\\ \quad\cdot\Big[\frac{2\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}-1\Big].\end{cases}$$

In the following, we will show that $\widetilde{\phi}(z)$ satisfies

$$\Re\partial^{\mu}_{\bar{z}_1}\partial^{\nu}_{\bar{z}_2}\widetilde{\phi}(z)=g_{\mu\nu}(z)\ (z\in\partial_0 D^2),\ \ \Im\partial^{\mu}_{\bar{z}_1}\partial^{\nu}_{\bar{z}_2}\widetilde{\phi}(0,z_2)=0=\Im\partial^{\mu}_{\bar{z}_1}\partial^{\nu}_{\bar{z}_2}\widetilde{\phi}(z_1,0)\ (z_1\in D_1,z_2\in D_2).$$

By (2.3), we get that

$$\begin{aligned}\partial^{\mu}_{\bar{z}_1}\partial^{\nu}_{\bar{z}_2}\widetilde{\phi}(z)&=\sum_{\widetilde{v}_1=\mu}^{m-1}\sum_{\widetilde{v}_2=\nu}^{m-1}\frac{\bar{z}_1^{\widetilde{v}_1-\mu}\bar{z}_2^{\widetilde{v}_2-\nu}}{(\widetilde{v}_1-\mu)!(\widetilde{v}_2-\nu)!}u_{\widetilde{v}_1\widetilde{v}_2}(z)\\ &=\sum_{v_1=0}^{m-1-\mu}\sum_{v_2=0}^{m-1-\nu}\frac{\bar{z}_1^{v_1}\bar{z}_2^{v_2}}{v_1!v_2!}u_{(v_1+\mu)(v_2+\nu)}(z),\end{aligned} \tag{2.5}$$

where

$$u_{(v_1+\mu)(v_2+\nu)}=\begin{cases}\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\sum_{l_1=0}^{m-\mu-1}\sum_{l_2=0}^{m-\nu-1}\frac{g_{(\mu+l_1)(\nu+l_2)}(\zeta)}{l_1!l_2!}(A_1-A_2-A_3+A_4)\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}, & \{^{v_1=0}_{v_2=0},\\ \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\sum_{l_1=0}^{m-\mu-1}\sum_{l_2=0}^{m-1-v_2-\nu}\frac{g_{(\mu+l_1)(v_2+\nu+l_2)}(\zeta)}{l_1!l_2!}(B_1-B_2)\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}, & \{^{v_1=0}_{0<v_2\le m-1-\nu},\\ \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\sum_{l_1=0}^{m-1-v_1-\mu}\sum_{l_2=0}^{m-\nu-1}\frac{g_{(v_1+\mu+l_1)(\nu+l_2)}(\zeta)}{l_1!l_2!}(C_1-C_2)\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}, & \{^{0<v_1\le m-1-\mu}_{v_2=0},\\ \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\sum_{l_1=0}^{m-1-v_1-\mu}\sum_{l_2=0}^{m-1-v_2-\nu}\frac{g_{(v_1+\mu+l_1)(v_2+\nu+l_2)}(\zeta)}{l_1!l_2!}D\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}, & \{^{0<v_1\le m-1-\mu}_{0<v_2\le m-1-\nu}.\end{cases}$$

Therefore, for $1\le k_1,k_2\le m-1$,

$$\partial_{\bar z_1}^{m-k_1}\partial_{\bar z_2}^{m-k_2}\widetilde{\phi}(z)=\sum_{v_1=0}^{k_1-1}\sum_{v_2=0}^{k_2-1}\frac{\bar z_1^{v_1}\bar z_2^{v_2}}{v_1!v_2!}u_{(v_1+m-k_1)(v_2+m-k_2)}(z),\tag{2.6}$$

in which

$$u_{(v_1+m-k_1)(v_2+m-k_2)}=\begin{cases}\int_{\partial_0 D^2}\sum_{l_1=0}^{k_1-1}\sum_{l_2=0}^{k_2-1}\frac{g_{(m-k_1+l_1)(m-k_1+l_2)}(\zeta)}{(2\pi i)^2 l_1!l_2!}(A_1-A_2-A_3+A_4)\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}, & \{^{v_1=0}_{v_2=0},\\ \int_{\partial_0 D^2}\sum_{l_1=0}^{k_1-1}\sum_{l_2=0}^{k_2-1-v_2}\frac{g_{(m-k_1+l_1)(m-k_2+v_2+l_2)}(\zeta)}{(2\pi i)^2 l_1!l_2!}(B_1-B_2)\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}, & \{^{v_1=0}_{0<v_2\le k_2-1},\\ \int_{\partial_0 D^2}\sum_{l_1=0}^{k_1-1-v_1}\sum_{l_2=0}^{k_2-1}\frac{g_{(m-k_1+v_1+l_1)(m-k_2+l_2)}(\zeta)}{(2\pi i)^2 l_1!l_2!}(C_1-C_2)\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}, & \{^{0<v_1\le k_1-1}_{v_2=0},\\ \int_{\partial_0 D^2}\sum_{l_1=0}^{k_1-1-v_1}\sum_{l_2=0}^{k_2-1-v_2}\frac{g_{(m-k_1+v_1+l_1)(m-k_2+v_2+l_2)}(\zeta)}{(2\pi i)^2 l_1!l_2!}D\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}, & \{^{0<v_1\le k_1-1}_{0<v_2\le k_2-1}.\end{cases}$$

Let

$$\phi_1(z)=\frac{\bar z_1^{v_1}\bar z_2^{v_2}}{v_1!v_2!}\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\sum_{l_1=0}^{k_1-1-v_1}\sum_{l_2=0}^{k_2-1-v_2}\frac{g_{(m-k_1+v_1+l_1)(m-k_2+v_2+l_2)}(\zeta)}{l_1!l_2!}\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}.$$

For $1\le k_1,k_2\le m-1$, (2.6) follows that

$$\begin{aligned}\partial_{\bar z_1}^{m-k_1}\partial_{\bar z_2}^{m-k_2}\widetilde{\phi}(z)&=\sum_{v_1=0}^{0}\sum_{v_2=0}^{0}\phi_1(z)(A_1-A_2-A_3+A_4)+\sum_{v_1=0}^{0}\sum_{v_2=1}^{k_2-1}\phi_1(z)(B_1-B_2)\\&+\sum_{v_1=1}^{k_1-1}\sum_{v_2=0}^{0}\phi_1(z)(C_1-C_2)+\sum_{v_1=1}^{k_1-1}\sum_{v_2=1}^{k_2-1}\phi_1(z)D\\&=\sum_{v_1=0}^{k_1-1}\sum_{v_2=0}^{k_2-1}\phi_1(z)D-\sum_{v_1=0}^{0}\sum_{v_2=0}^{k_2-1}\phi_1(z)B_2-\sum_{v_1=0}^{k_1-1}\sum_{v_2=0}^{0}\phi_1(z)C_2+\sum_{v_1,v_2=0}^{0}\phi_1(z)A_4\\&=\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\sum_{v_1=0}^{k_1-1}\sum_{l_1=0}^{k_1-1-v_1}\sum_{v_2=0}^{k_2-1}\sum_{l_2=0}^{k_2-1-v_2}\frac{\bar z_1^{v_1}\bar z_2^{v_2}}{v_1!v_2!}\frac{g_{(m-k_1+v_1+l_1)(m-k_2+v_2+l_2)}(\zeta)}{l_1!l_2!}[(z_1-\zeta_1-\bar\zeta_1)^{l_1}(z_2-\zeta_2\end{aligned}$$

$$-\bar{\zeta}_2)^{l_2}+(-z_1)^{l_1}(-1)^{v_1}(z_2-\zeta_2-\bar{\zeta}_2)^{l_2}+(z_1-\zeta_1-\bar{\zeta}_1)^{l_1}(-z_2)^{l_2}(-1)^{v_2}]\Big[\frac{2\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}-1\Big]\frac{d\zeta}{\zeta}$$

$$-\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\sum_{\lambda_1=0}^{k_1-1}\sum_{v_2=0}^{k_2-1}\sum_{l_2=0}^{k_2-1-v_2}\frac{\bar{z}_2^{v_2}}{v_2!}\frac{g_{(m-k_1+\lambda_1)(m-k_2+v_2+l_2)}(\zeta)}{\lambda_1!l_2!}(-\zeta_1-\bar{\zeta}_1)^{\lambda_1}$$

$$\cdot\Big\{(z_2-\zeta_2-\bar{\zeta}_2)^{l_2}[\frac{2\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}-1]+(-z_2)^{l_2}(-1)^{v_2}[\frac{2\zeta_2}{\zeta_2-z_2}-1]\Big\}\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}$$

$$-\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\sum_{v_1=0}^{k_1-1}\sum_{l_1=0}^{k_1-1-v_1}\sum_{\lambda_2=0}^{k_2-1}\frac{\bar{z}_1^{v_1}}{v_1!}\frac{g_{(m-k_1+v_1+l_1)(m-k_2+\lambda_2)}(\zeta)}{l_1!\lambda_2!}(-\zeta_2-\bar{\zeta}_2)^{\lambda_2}$$

$$\cdot\Big\{(z_1-\zeta_1-\bar{\zeta}_1)^{l_1}[\frac{2\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}-1]+(-z_1)^{l_1}(-1)^{v_1}[\frac{2\zeta_1}{\zeta_1-z_1}-1]\Big\}\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}$$

$$+\int_{\partial_0 D^2}\sum_{\lambda_1=0}^{k_1-1}\sum_{\lambda_2=0}^{k_2-1}\frac{g_{(m-k_1+\lambda_1)(m-k_2+\lambda_2)}(\zeta)}{(2\pi i)^2\lambda_1!\lambda_2!}(-\zeta_1-\bar{\zeta}_1)^{\lambda_1}(-\zeta_2-\bar{\zeta}_2)^{\lambda_2}[\frac{2\zeta_1}{\zeta_1-z_1}+\frac{2\zeta_2}{\zeta_2-z_2}-2]\frac{d\zeta}{\zeta}$$

$$=\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\sum_{\lambda_1=0}^{k_1-1}\sum_{\lambda_2=0}^{k_2-1}\sum_{v_1=0}^{\lambda_1}\sum_{v_2=0}^{\lambda_2}\frac{\bar{z}_1^{v_1}\bar{z}_2^{v_2}}{\lambda_1!\lambda_2!}C_{\lambda_1}^{v_1}C_{\lambda_2}^{v_2}g_{(m-k_1+\lambda_1)(m-k_2+\lambda_2)}[(z_1-\zeta_1-\bar{\zeta}_1)^{\lambda_1-v_1}(z_2-\zeta_2-\bar{\zeta}_2)^{\lambda_2-v_2}$$

$$+(-z_1)^{\lambda_1-v_1}(-1)^{v_1}(z_2-\zeta_2-\bar{\zeta}_2)^{\lambda_2-v_2}+(z_1-\zeta_1-\bar{\zeta}_1)^{\lambda_1-v_1}(-z_2)^{\lambda_2-v_2}(-1)^{v_2}]\Big[\frac{2\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}-1\Big]\frac{d\zeta}{\zeta}$$

$$-\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\sum_{\lambda_1=0}^{k_1-1}\sum_{\lambda_2=0}^{k_2-1}\frac{(-\zeta_1-\bar{\zeta}_1)^{\lambda_1}}{\lambda_1!\lambda_2!}\sum_{v_2=0}^{\lambda_2}C_{\lambda_2}^{v_2}\bar{z}_2^{v_2}g_{(m-k_1+\lambda_1)(m-k_2+\lambda_2)}(\zeta)$$

$$\cdot\Big\{(z_2-\zeta_2-\bar{\zeta}_2)^{\lambda_2-v_2}[\frac{2\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}-1]+(-z_2)^{\lambda_2-v_2}(-1)^{v_2}[\frac{2\zeta_2}{\zeta_2-z_2}-1]\Big\}\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}$$

$$-\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\sum_{\lambda_1=0}^{k_1-1}\sum_{\lambda_2=0}^{k_2-1}\frac{(-\zeta_2-\bar{\zeta}_2)^{\lambda_2}}{\lambda_1!\lambda_2!}\sum_{v_1=0}^{\lambda_1}C_{\lambda_1}^{v_1}\bar{z}_1^{v_1}g_{(m-k_1+\lambda_1)(m-k_2+\lambda_2)}(\zeta)$$

$$\cdot\Big\{(z_1-\zeta_1-\bar{\zeta}_1)^{\lambda_1-v_1}[\frac{2\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}-1]+(-z_1)^{\lambda_1-v_1}(-1)^{v_1}[\frac{2\zeta_1}{\zeta_1-z_1}-1]\Big\}\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}$$

$$+\int_{\partial_0 D^2}\sum_{\lambda_1=0}^{k_1-1}\sum_{\lambda_2=0}^{k_2-1}\frac{g_{(m-k_1+\lambda_1)(m-k_2+\lambda_2)}(\zeta)}{(2\pi i)^2\lambda_1!\lambda_2!}(-\zeta_1-\bar{\zeta}_1)^{\lambda_1}(-\zeta_2-\bar{\zeta}_2)^{\lambda_2}[\frac{2\zeta_1}{\zeta_1-z_1}+\frac{2\zeta_2}{\zeta_2-z_2}-2]\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}$$

$$=\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\sum_{\lambda_1=0}^{k_1-1}\sum_{\lambda_2=0}^{k_2-1}\frac{1}{\lambda_1!\lambda_2!}\Big\{\sum_{v_1=0}^{\lambda_1}C_{\lambda_1}^{v_1}(z_1-\zeta_1-\bar{\zeta}_1)^{\lambda_1-v_1}\bar{z}_1^{v_1}\sum_{v_2=0}^{\lambda_2}C_{\lambda_2}^{v_2}(z_2-\zeta_2-\bar{\zeta}_2)^{\lambda_2-v_2}\bar{z}_2^{v_2}$$

$$-(-\zeta_1-\bar{\zeta}_1)^{\lambda_1}\sum_{v_2=0}^{\lambda_2}C_{\lambda_2}^{v_2}(z_2-\zeta_2-\bar{\zeta}_2)^{\lambda_2-v_2}\bar{z}_2^{v_2}+\sum_{v_1=0}^{\lambda_1}C_{\lambda_1}^{v_1}(-z_1)^{\lambda_1-v_1}(-\bar{z}_1)^{v_1}\sum_{v_2=0}^{\lambda_2}C_{\lambda_2}^{v_2}(z_2-\zeta_2-\bar{\zeta}_2)^{\lambda_2-v_2}\bar{z}_2^{v_2}$$

$$-\sum_{v_1=0}^{\lambda_1}C_{\lambda_1}^{v_1}(z_1-\zeta_1-\bar{\zeta}_1)^{\lambda_1-v_1}\bar{z}_1^{v_1}(-\zeta_2-\bar{\zeta}_2)^{\lambda_2}+\sum_{v_1=0}^{\lambda_1}C_{\lambda_1}^{v_1}(z_1-\zeta_1-\bar{\zeta}_1)^{\lambda_1-v_1}\bar{z}_1^{v_1}\sum_{v_2=0}^{\lambda_2}C_{\lambda_2}^{v_2}(-z_2)^{\lambda_2-v_2}(-\bar{z}_2)^{v_2}\Big\}$$

$$\cdot g_{(m-k_1+\lambda_1)(m-k_2+\lambda_2)}(\zeta)\Big[\frac{2\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}-1\Big]\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}$$

$$-\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\sum_{\lambda_1=0}^{k_1-1}\sum_{\lambda_2=0}^{k_2-1}\frac{g_{(m-k_1+\lambda_1)(m-k_2+\lambda_2)}(\zeta)}{\lambda_1!\lambda_2!}\Big\{(-\zeta_1-\bar{\zeta}_1)^{\lambda_1}\sum_{v_2=0}^{\lambda_2}C_{\lambda_2}^{v_2}(-z_2)^{\lambda_2-v_2}(-\bar{z}_2)^{v_2}[\frac{2\zeta_2}{\zeta_2-z_2}-1]$$

$$
\begin{aligned}
&+\sum_{\nu_1=0}^{\lambda_1} C_{\lambda_1}^{\nu_1}(-z_1)^{\lambda_1-\nu_1}(-\bar{z}_1)^{\nu_1}(-\zeta_2-\bar{\zeta}_2)^{\lambda_2}[\frac{2\zeta_1}{\zeta_1-z_1}-1]-(-\zeta_1-\bar{\zeta}_1)^{\lambda_1}(-\zeta_2-\bar{\zeta}_2)^{\lambda_2}[\frac{2\zeta_1}{\zeta_1-z_1}+\frac{2\zeta_2}{\zeta_2-z_2}-2]\Big\}\frac{d\zeta}{\zeta}\\
&=\int_{\partial_0 D^2}\sum_{\lambda_1=0}^{k_1-1}\sum_{\lambda_2=0}^{k_2-1}\frac{g_{(m-k_1+\lambda_1)(m-k_2+\lambda_2)}}{(2\pi i)^2\lambda_1!\lambda_2!}\Big\{(z_1+\bar{z}_1-\zeta_1-\bar{\zeta}_1)^{\lambda_1}(z_2+\bar{z}_2-\zeta_2-\bar{\zeta}_2)^{\lambda_2}-[(-\zeta_1-\bar{\zeta}_1)^{\lambda_1}-(-z_1-\bar{z}_1)^{\lambda_1}]\\
&\cdot(z_2+\bar{z}_2-\zeta_2-\bar{\zeta}_2)^{\lambda_2}-(z_1+\bar{z}_1-\zeta_1-\bar{\zeta}_1)^{\lambda_1}[(-\zeta_2-\bar{\zeta}_2)^{\lambda_2}-(-z_2-\bar{z}_2)^{\lambda_2}]\Big\}\Big[\frac{2\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}-1\Big]\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}\\
&+\int_{\partial_0 D^2}\sum_{\lambda_1=0}^{k_1-1}\sum_{\lambda_2=0}^{k_2-1}\frac{g_{(m-k_1+\lambda_1)(m-k_2+\lambda_2)}}{(2\pi i)^2\lambda_1!\lambda_2!}(-\zeta_1-\bar{\zeta}_1)^{\lambda_1}[(-\zeta_2-\bar{\zeta}_2)^{\lambda_2}-(-z_2-\bar{z}_2)^{\lambda_2}][\frac{2\zeta_2}{\zeta_2-z_2}-1]\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}\\
&+\int_{\partial_0 D^2}\sum_{\lambda_1=0}^{k_1-1}\sum_{\lambda_2=0}^{k_2-1}\frac{g_{(m-k_1+\lambda_1)(m-k_2+\lambda_2)}}{(2\pi i)^2\lambda_1!\lambda_2!}[(-\zeta_1-\bar{\zeta}_1)^{\lambda_1}-(-z_1-\bar{z}_1)^{\lambda_1}](-\zeta_2-\bar{\zeta}_2)^{\lambda_2}[\frac{2\zeta_1}{\zeta_1-z_1}-1]\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}.
\end{aligned}
\tag{2.7}
$$

Applying the properties of the Cauchy kernels on $D^2$ and $D$, for $z\in\partial_0 D^2$, (2.7) leads to

$$
\begin{aligned}
&\Re\partial_{\bar{z}_1}^{m-k_1}\partial_{\bar{z}_2}^{m-k_2}\widetilde{\phi}(z)\\
&=\sum_{\lambda_1=0}^{k_1-1}\sum_{\lambda_2=0}^{k_2-1}\Big\{\frac{g_{(m-k_1+\lambda_1)(m-k_2+\lambda_2)}(\zeta)}{\lambda_1!\lambda_2!}\Big[(z_1+\bar{z}_1-\zeta_1-\bar{\zeta}_1)^{\lambda_1}(z_2+\bar{z}_2-\zeta_2-\bar{\zeta}_2)^{\lambda_2}-[(-\zeta_1-\bar{\zeta}_1)^{\lambda_1}\\
&-(-z_1-\bar{z}_1)^{\lambda_1}](z_2+\bar{z}_2-\zeta_2-\bar{\zeta}_2)^{\lambda_2}-(z_1+\bar{z}_1-\zeta_1-\bar{\zeta}_1)^{\lambda_1}[(-\zeta_2-\bar{\zeta}_2)^{\lambda_2}-(-z_2-\bar{z}_2)^{\lambda_2}]\Big]\Big\}\Big|_{\substack{\zeta_1=z_1\\ \zeta_2=z_2}}\\
&+\int_{\partial D_1}\sum_{\lambda_1=0}^{k_1-1}\sum_{\lambda_2=0}^{k_2-1}\frac{(-\zeta_1-\bar{\zeta}_1)^{\lambda_1}}{2\pi\lambda_1!\lambda_2!}\{g_{(m-k_1+\lambda_1)(m-k_2+\lambda_2)}(\zeta)[(-\zeta_2-\bar{\zeta}_2)^{\lambda_2}-(-z_2-\bar{z}_2)^{\lambda_2}]\}|_{\zeta_2=z_2}\frac{d\zeta_1}{i\zeta_1}\\
&+\int_{\partial D_2}\sum_{\lambda_1=0}^{k_1-1}\sum_{\lambda_2=0}^{k_2-1}\frac{(-\zeta_2-\bar{\zeta}_2)^{\lambda_2}}{2\pi\lambda_1!\lambda_2!}\{g_{(m-k_1+\lambda_1)(m-k_2+\lambda_2)}(\zeta)[(-\zeta_1-\bar{\zeta}_1)^{\lambda_1}-(-z_1-\bar{z}_1)^{\lambda_1}]\}|_{\zeta_1=z_1}\frac{d\zeta_2}{i\zeta_2}\\
&=g_{(m-k_1)(m-k_2)}(z),
\end{aligned}
$$

which means $\Re\partial_{\bar{z}_1}^{\mu}\partial_{\bar{z}_2}^{\nu}\widetilde{\phi}(z)=g_{\mu\nu}(z)$ for $z\in\partial_0 D^2$.

In addition, by (2.7), we get that

$$
\begin{aligned}
&\Im\partial_{\bar{z}_1}^{m-k_1}\partial_{\bar{z}_2}^{m-k_2}\widetilde{\phi}(0,z_2)\\
&=\Im\Big\{\int_{\partial_0 D^2}\sum_{\lambda_1=0}^{k_1-1}\sum_{\lambda_2=0}^{k_2-1}\frac{g_{(m-k_1+\lambda_1)(m-k_2+\lambda_2)}(\zeta)}{(2\pi i)^2\lambda_1!\lambda_2!}\Big[(-\zeta_1-\bar{\zeta}_1)^{\lambda_1}(z_2+\bar{z}_2-\zeta_2-\bar{\zeta}_2)^{\lambda_2}-(-\zeta_1-\bar{\zeta}_1)^{\lambda_1}\\
&\cdot(z_2+\bar{z}_2-\zeta_2-\bar{\zeta}_2)^{\lambda_2}-(-\zeta_1-\bar{\zeta}_1)^{\lambda_1}[(-\zeta_2-\bar{\zeta}_2)^{\lambda_2}-(-z_2-\bar{z}_2)^{\lambda_2}]\Big]\Big(\frac{2\zeta_2}{\zeta_2-z_2}-1\Big)\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}\\
&+\int_{\partial_0 D^2}\sum_{\lambda_1=0}^{k_1-1}\sum_{\lambda_2=0}^{k_2-1}\frac{g_{(m-k_1+\lambda_1)(m-k_2+\lambda_2)}(\zeta)}{(2\pi i)^2\lambda_1!\lambda_2!}(-\zeta_1-\bar{\zeta}_1)^{\lambda_1}[(-\zeta_2-\bar{\zeta}_2)^{\lambda_2}-(-z_2-\bar{z}_2)^{\lambda_2}][\frac{2\zeta_2}{\zeta_2-z_2}-1]\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}\\
&+\int_{\partial_0 D^2}\sum_{\lambda_1=0}^{k_1-1}\sum_{\lambda_2=0}^{k_2-1}\frac{g_{(m-k_1+\lambda_1)(m-k_2+\lambda_2)}(\zeta)}{(2\pi i)^2\lambda_1!\lambda_2!}(-\zeta_1-\bar{\zeta}_1)^{\lambda_1}(-\zeta_2-\bar{\zeta}_2)^{\lambda_2}\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}\Big\}
\end{aligned}
$$

$$= \Im\Big\{\int_{\partial_0 D^2}\sum_{\lambda_1=0}^{k_1-1}\sum_{\lambda_2=0}^{k_2-1}\frac{g_{(m-k_1+\lambda_1)(m-k_2+\lambda_2)}(\zeta)}{(2\pi i)^2\lambda_1!\lambda_2!}(-\zeta_1-\bar\zeta_1)^{\lambda_1}(-\zeta_2-\bar\zeta_2)^{\lambda_2}\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}\Big\}$$
$$= 0.$$

Similarly, we have that

$$\Im\partial_{\bar z_1}^{m-k_1}\partial_{\bar z_2}^{m-k_2}\widetilde{\phi}(z_1,0)$$
$$=\Im\Big\{\int_{\partial_0 D^2}\sum_{\lambda_1=0}^{k_1-1}\sum_{\lambda_2=0}^{k_2-1}\frac{g_{(m-k_1+\lambda_1)(m-k_2+\lambda_2)}(\zeta)}{(2\pi i)^2\lambda_1!\lambda_2!}\Big[(z_1+\bar z_1-\zeta_1-\bar\zeta_1)^{\lambda_1}(-\zeta_2-\bar\zeta_2)^{\lambda_2}-[(-\zeta_1-\bar\zeta_1)^{\lambda_1}$$
$$-(-z_1-\bar z_1)^{\lambda_1}](-\zeta_2-\bar\zeta_2)^{\lambda_2}-(z_1+\bar z_1-\zeta_1-\bar\zeta_1)^{\lambda_1}(-\zeta_2-\bar\zeta_2)^{\lambda_2}\Big]\Big(\frac{2\zeta_1}{\zeta_1-z_1}-1\Big)\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}$$
$$+\int_{\partial_0 D^2}\sum_{\lambda_1=0}^{k_1-1}\sum_{\lambda_2=0}^{k_2-1}\frac{g_{(m-k_1+\lambda_1)(m-k_2+\lambda_2)}(\zeta)}{(2\pi i)^2\lambda_1!\lambda_2!}(-\zeta_1-\bar\zeta_1)^{\lambda_1}(-\zeta_2-\bar\zeta_2)^{\lambda_2}\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}$$
$$+\int_{\partial_0 D^2}\sum_{\lambda_1=0}^{k_1-1}\sum_{\lambda_2=0}^{k_2-1}\frac{g_{(m-k_1+\lambda_1)(m-k_2+\lambda_2)}(\zeta)}{(2\pi i)^2\lambda_1!\lambda_2!}[(-\zeta_1-\bar\zeta_1)^{\lambda_1}-(-z_1-\bar z_1)^{\lambda_1}](-\zeta_2-\bar\zeta_2)^{\lambda_2}[\frac{2\zeta_1}{\zeta_1-z_1}-1]\frac{d\zeta}{\zeta}\Big\}$$
$$=\Im\Big\{\int_{\partial_0 D^2}\sum_{\lambda_1=0}^{k_1-1}\sum_{\lambda_2=0}^{k_2-1}\frac{g_{(m-k_1+\lambda_1)(m-k_2+\lambda_2)}(\zeta)}{(2\pi i)^2\lambda_1!\lambda_2!}(-\zeta_1-\bar\zeta_1)^{\lambda_1}(-\zeta_2-\bar\zeta_2)^{\lambda_2}\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}\Big\}$$
$$= 0.$$

Therefore,

$$\Im\partial_{\bar z_1}^{\mu}\partial_{\bar z_2}^{\nu}\widetilde{\phi}(0,z_2) = 0 = \Im\partial_{\bar z_1}^{\mu}\partial_{\bar z_2}^{\nu}\widetilde{\phi}(z_1,0).$$

2) In the expression of $u_{\widetilde{\nu_1}\widetilde{\nu_2}}(z)$ determined by (2.4),

$$D = [(z_1-\zeta_1-\bar\zeta_1)^{l_1}(z_2-\zeta_2-\bar\zeta_2)^{l_2}+(-z_1)^{l_1}(-1)^{\nu_1}(z_2-\zeta_2-\bar\zeta_2)^{l_2}+(z_1-\zeta_1-\bar\zeta_1)^{l_1}(-z_2)^{l_2}(-1)^{\nu_2}]$$
$$\cdot\Big[\frac{2\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}-1\Big]$$
$$=\Big[\sum_{p_1=0}^{l_1}C_{l_1}^{p_1}z_1^{p_1}(-\zeta_1-\bar\zeta_1)^{l_1-p_1}\sum_{q_1=0}^{l_2}C_{l_2}^{q_1}z_2^{q_1}(-\zeta_2-\bar\zeta_2)^{l_2-q_1}+(-z_1)^{l_1}(-1)^{\nu_1}\sum_{q_1=0}^{l_2}C_{l_2}^{q_1}z_2^{q_1}(-\zeta_2-\bar\zeta_2)^{l_2-q_1} \tag{2.8}$$
$$+\sum_{p_1=0}^{l_1}C_{l_1}^{p_1}z_1^{p_1}(-\zeta_1-\bar\zeta_1)^{l_1-p_1}(-z_2)^{l_2}(-1)^{\nu_2}\Big]\Big[2\sum_{j_1=0}^{\infty}\frac{z_1^{j_1}}{\zeta_1^{j_1}}\sum_{j_2=0}^{\infty}\frac{z_2^{j_2}}{\zeta_2^{j_2}}-1\Big].$$

Moreover, we have that

$$\sum_{p_1=0}^{l_1}C_{l_1}^{p_1}z_1^{p_1}(-\zeta_1-\bar\zeta_1)^{l_1-p_1}\sum_{q_1=0}^{l_2}C_{l_2}^{q_1}z_2^{q_1}(-\zeta_2-\bar\zeta_2)^{l_2-q_1}\Big[2\sum_{j_1=0}^{+\infty}\frac{z_1^{j_1}}{\zeta_1^{j_1}}\sum_{j_2=0}^{\infty}\frac{z_2^{j_2}}{\zeta_2^{j_2}}-1\Big]$$
$$=2\sum_{p_1=0}^{l_1}\sum_{j_1=0}^{+\infty}C_{l_1}^{p_1}(-\zeta_1-\bar\zeta_1)^{l_1-p_1}\bar\zeta_1^{j_1}z_1^{p_1+j_1}\sum_{q_1=0}^{l_2}\sum_{j_2=0}^{+\infty}C_{l_2}^{q_1}(-\zeta_2-\bar\zeta_2)^{l_2-q_1}\bar\zeta_2^{j_2}z_2^{q_1+j_2}$$
$$-\sum_{p_1=0}^{l_1}\sum_{q_1=0}^{l_2}C_{l_1}^{p_1}C_{l_2}^{q_1}(-\zeta_1-\bar\zeta_1)^{l_1-p_1}(-\zeta_2-\bar\zeta_2)^{l_2-q_1}z_1^{p_1}z_2^{q_1}$$

$$
\begin{aligned}
&= 2\sum_{j_1=0}^{+\infty}\sum_{m_1=j_1}^{l_1+j_1} C_{l_1}^{m_1-j_1}(-\zeta_1-\bar{\zeta}_1)^{l_1-m_1+j_1}\bar{\zeta}_1^{j_1} z_1^{m_1}\sum_{j_2=0}^{+\infty}\sum_{m_2=j_2}^{l_2+j_2} C_{l_2}^{m_2-j_2}(-\zeta_2-\bar{\zeta}_2)^{l_2-m_2+j_2}\bar{\zeta}_2^{j_2} z_2^{m_2}\\
&\quad -\sum_{m_1=0}^{l_1}\sum_{m_2=0}^{l_2} C_{l_1}^{m_1}C_{l_2}^{m_2}(-\zeta_1-\bar{\zeta}_1)^{l_1-m_1}(-\zeta_2-\bar{\zeta}_2)^{l_2-m_2}z_1^{m_1}z_2^{m_2}\\
&= 2\sum_{m_1=0}^{+\infty}\sum_{j_1=0}^{m_1} C_{l_1}^{m_1-j_1}(-\zeta_1-\bar{\zeta}_1)^{l_1-m_1+j_1}\bar{\zeta}_1^{j_1} z_1^{m_1}\sum_{m_2=0}^{+\infty}\sum_{j_2=0}^{m_2} C_{l_2}^{m_2-j_2}(-\zeta_2-\bar{\zeta}_2)^{l_2-m_2+j_2}\bar{\zeta}_2^{j_2} z_2^{m_2}\\
&\quad -\sum_{m_1=0}^{l_1}\sum_{m_2=0}^{l_2} C_{l_1}^{m_1}C_{l_2}^{m_2}(-\zeta_1-\bar{\zeta}_1)^{l_1-m_1}(-\zeta_2-\bar{\zeta}_2)^{l_2-m_2}z_1^{m_1}z_2^{m_2}\\
&= \sum_{m_1,m_2=0}^{+\infty}\Big[2\sum_{j_1=0}^{m_1}\sum_{j_2=0}^{m_2} C_{l_1}^{m_1-j_1}(-\zeta_1-\bar{\zeta}_1)^{l_1-m_1+j_1}\bar{\zeta}_1^{j_1}C_{l_2}^{m_2-j_2}(-\zeta_2-\bar{\zeta}_2)^{l_2-m_2+j_2}\bar{\zeta}_2^{j_2}-D_{11}D_{12}\Big]z_1^{m_1}z_2^{m_2},
\end{aligned}\tag{2.9}
$$

in which

$$
D_{11}=\begin{cases} C_{l_1}^{m_1}(-\zeta_1-\bar{\zeta}_1)^{l_1-m_1}, & 0\le m_1\le l_1,\\ 0, & m_1>l_1,\end{cases}\qquad D_{12}=\begin{cases} C_{l_2}^{m_2}(-\zeta_2-\bar{\zeta}_2)^{l_2-m_2}, & 0\le m_2\le l_2,\\ 0, & m_2>l_2,\end{cases}
$$

and

$$
\begin{aligned}
&(-z_1)^{l_1}(-1)^{v_1}\sum_{q_1=0}^{l_2} C_{l_2}^{q_1}z_2^{q_1}(-\zeta_2-\bar{\zeta}_2)^{l_2-q_1}\Big[2\sum_{j_1=0}^{\infty}\frac{z_1^{j_1}}{\zeta_1^{j_1}}\sum_{j_2=0}^{+\infty}\frac{z_2^{j_2}}{\zeta_2^{j_2}}-1\Big]\\
&= 2\sum_{j_1=0}^{+\infty}(-1)^{l_1+v_1}\bar{\zeta}_1^{j_1}z_1^{l_1+j_1}\sum_{j_2=0}^{+\infty}\sum_{q_1=0}^{l_2} C_{l_2}^{q_1}(-\zeta_2-\bar{\zeta}_2)^{l_2-q_1}\bar{\zeta}_2^{j_2}z_2^{q_1+j_2}\\
&\quad - z_1^{l_1}(-1)^{l_1+v_1}\sum_{q_1=0}^{l_2} C_{l_2}^{q_1}(-\zeta_2-\bar{\zeta}_2)^{l_2-q_1}z_2^{q_1}\\
&= 2\sum_{m_1=l_1}^{+\infty}(-1)^{l_1+v_1}\bar{\zeta}_1^{m_1-l_1}z_1^{m_1}\sum_{m_2=0}^{+\infty}\sum_{j_2=0}^{m_2} C_{l_2}^{m_2-j_2}(-\zeta_2-\bar{\zeta}_2)^{l_2-m_2+j_2}\bar{\zeta}_2^{j_2}z_2^{m_2}\\
&\quad -\sum_{m_2=0}^{l_2}(-1)^{l_1+v_1}C_{l_2}^{m_2}(-\zeta_2-\bar{\zeta}_2)^{l_2-m_2}z_1^{l_1}z_2^{m_2}\\
&= \sum_{m_1,m_2=0}^{+\infty}\Big[2D_{21}\sum_{j_2=0}^{m_2} C_{l_2}^{m_2-j_2}(-\zeta_2-\bar{\zeta}_2)^{l_2-m_2+j_2}\bar{\zeta}_2^{j_2}-D_{22}D_{23}\Big]z_1^{m_1}z_2^{m_2},
\end{aligned}\tag{2.10}
$$

where

$$
D_{21}=\begin{cases} 0, & 0\le m_1<l_1,\\ (-1)^{l_1+v_1}\bar{\zeta}_1^{m_1-l_1}, & m_1\ge l_1,\end{cases}\quad D_{22}=\begin{cases}(-1)^{m_1+v_1}, & m_1=l_1,\\ 0, & m_1\ne l_1,\end{cases}\quad D_{23}=\begin{cases} C_{l_2}^{m_2}(-\zeta_2-\bar{\zeta}_2)^{l_2-m_2}, & 0\le m_2\le l_2,\\ 0, & m_2>l_2.\end{cases}
$$

Similarly, we get that

$$\begin{aligned}&\sum_{p_1=0}^{l_1} C_{l_1}^{p_1} z_1^{p_1}(-\zeta_1-\bar{\zeta}_1)^{l_1-p_1}(-z_2)^{l_2}(-1)^{v_2}\Big[2\sum_{j_1=0}^{\infty}\frac{z_1^{j_1}}{\zeta_1^{j_1}}\sum_{j_2=0}^{+\infty}\frac{z_2^{j_2}}{\zeta_2^{j_2}}-1\Big]\\&=\sum_{m_1,m_2=0}^{+\infty}\Big[2D_{31}\sum_{j_1=0}^{m_1} C_{l_1}^{m_1-j_1}(-\zeta_1-\bar{\zeta}_1)^{l_1-m_1+j_1}\bar{\zeta}_1^{j_1}-D_{32}D_{33}\Big]z_1^{m_1}z_2^{m_2},\end{aligned}\tag{2.11}$$

in which

$$D_{31}=\begin{cases}0,\ 0\le m_2<l_2,\\(-1)^{l_2+v_2}\bar{\zeta}_2^{m_2-l_2}, m_2\ge l_2,\end{cases}\quad D_{32}=\begin{cases}C_{l_1}^{m_1}(-\zeta_1-\bar{\zeta}_1)^{l_1-m_1}, 0\le m_1\le l_1,\\0,\ m_1>l_1,\end{cases}\quad D_{33}=\begin{cases}(-1)^{m_2+v_2}, m_2=l_2,\\0,\ m_2\ne l_2.\end{cases}$$

Plugging (2.9)–(2.11) into (2.8) gives that

$$\begin{aligned}D=2&\sum_{m_1,m_2=0}^{+\infty}\Big\{\Big[\sum_{j_1=0}^{m_1}\sum_{j_2=0}^{m_2} C_{l_1}^{m_1-j_1}(-\zeta_1-\bar{\zeta}_1)^{l_1-m_1+j_1}\bar{\zeta}_1^{j_1}C_{l_2}^{m_2-j_2}(-\zeta_2-\bar{\zeta}_2)^{l_2-m_2+j_2}\bar{\zeta}_2^{j_2}\\&+D_{21}\sum_{j_2=0}^{m_2}C_{l_2}^{m_2-j_2}(-\zeta_2-\bar{\zeta}_2)^{l_2-m_2+j_2}\bar{\zeta}_2^{j_2}+D_{31}\sum_{j_1=0}^{m_1}C_{l_1}^{m_1-j_1}(-\zeta_1-\bar{\zeta}_1)^{l_1-m_1+j_1}\bar{\zeta}_1^{j_1}\Big]\\&-\frac{1}{2}(D_{11}D_{12}+D_{22}D_{23}+D_{32}D_{33})\Big\}z_1^{m_1}z_2^{m_2}.\end{aligned}\tag{2.12}$$

In addition, as the result of

$$\begin{aligned}&(z_1-\zeta_1-\bar{\zeta}_1)^{l_2}\Big[\frac{2\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}-1\Big]+(-z_1)^{l_1}(-1)^{v_1}\Big[\frac{2\zeta_1}{\zeta_1-z_1}-1\Big]\\&=\sum_{p_1=0}^{l_1}C_{l_1}^{p_1}z_1^{p_1}(-\zeta_1-\bar{\zeta}_1)^{l_1-p_1}\Big[2\sum_{j_1=0}^{+\infty}\frac{z_1^{j_1}}{\zeta_1^{j_1}}\sum_{j_2=0}^{+\infty}\frac{z_2^{j_2}}{\zeta_2^{j_2}}-1\Big]+z_1^{l_1}(-1)^{l_1+v_1}\Big(2\sum_{j_1=0}^{+\infty}\frac{z_1^{j_1}}{\zeta_1^{j_1}}-1\Big)\\&=2\sum_{m_1=0}^{+\infty}\sum_{j_1=0}^{m_1}C_{l_1}^{m_1-j_1}(-\zeta_1-\bar{\zeta}_1)^{l_1-m_1+j_1}\bar{\zeta}_1^{j_1}z_1^{m_1}\sum_{m_2=0}^{+\infty}\bar{\zeta}_2^{m_2}z_2^{m_2}-\sum_{p_1=0}^{l_1}C_{l_1}^{p_1}(-\zeta_1-\bar{\zeta}_1)^{l_1-p_1}z_1^{p_1}\\&\quad+2\sum_{m_1=l_1}^{+\infty}(-1)^{l_1+v_1}\bar{\zeta}_1^{m_1-l_1}z_1^{m_1}-(-1)^{l_1+v_1}z_1^{l_1}\\&=2\sum_{m_1,m_2=0}^{+\infty}\Big[\sum_{j_1=0}^{m_1}C_{l_1}^{m_1-j_1}(-\zeta_1-\bar{\zeta}_1)^{l_1-m_1+j_1}\bar{\zeta}_1^{j_1}\bar{\zeta}_2^{m_2}+D_{21}C_{21}-\frac{1}{2}(D_{11}+C_{22})C_{21}\Big]z_1^{m_1}z_2^{m_2},\end{aligned}$$

in which

$$C_{21}=\begin{cases}0,\ \ m_2\ge 1,\\1,\ \ m_2=0,\end{cases}\qquad C_{22}=\begin{cases}(-1)^{m_1+v_1},\ \ m_1=l_1,\\0,\ \ m_1\ne l_1,\end{cases}$$

we have that

$$\begin{aligned}C_2&=(-\zeta_2-\bar{\zeta}_2)^{l_2}\Big\{(z_1-\zeta_1-\bar{\zeta}_1)^{l_2}\Big[\frac{2\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}-1\Big]+(-z_1)^{l_1}(-1)^{v_1}\Big[\frac{2\zeta_1}{\zeta_1-z_1}-1\Big]\Big\}\\&=2(-\zeta_2-\bar{\zeta}_2)^{l_2}\sum_{m_1,m_2=0}^{+\infty}\Big[\sum_{j_1=0}^{m_1}C_{l_1}^{m_1-j_1}(-\zeta_1-\bar{\zeta}_1)^{l_1-m_1+j_1}\bar{\zeta}_1^{j_1}\bar{\zeta}_2^{m_2}+D_{21}C_{21}-\frac{D_{11}+C_{22}}{2}C_{21}\Big]z_1^{m_1}z_2^{m_2},\end{aligned}\tag{2.13}$$

which follows that

$$
\begin{aligned}
C_1 - C_2 &= D|_{v_2=0} - C_2 \\
&= 2\sum_{m_1,m_2=0}^{+\infty}\Big\{\sum_{j_1=0}^{m_1} C_{l_1}^{m_1-j_1}(-\zeta_1-\bar{\zeta}_1)^{l_1-m_1+j_1}\bar{\zeta}_1^{j_1}\Big[\sum_{j_2=0}^{m_2} C_{l_2}^{m_2-j_2}(-\zeta_2-\bar{\zeta}_2)^{l_2-m_2+j_2}\bar{\zeta}_2^{j_2} - (-\zeta_2-\bar{\zeta}_2)^{l_2}\bar{\zeta}_2^{m_2}\Big] \\
&\quad + D_{21}\sum_{j_2=0}^{m_2} C_{l_2}^{m_2-j_2}(-\zeta_2-\bar{\zeta}_2)^{l_2-m_2+j_2}\bar{\zeta}_2^{j_2} + D_{31}|_{v_2=0}\sum_{j_1=0}^{m_1} C_{l_1}^{m_1-j_1}(-\zeta_1-\bar{\zeta}_1)^{l_1-m_1+j_1}\bar{\zeta}_1^{j_1} \\
&\quad - D_{21}C_{21} - \frac{1}{2}(D_{11}D_{12} + D_{22}D_{23} + D_{32}D_{33}|_{v_2=0} - D_{11}C_{21} - C_{22}C_{21})\Big\} z_1^{m_1} z_2^{m_2}.
\end{aligned}
\tag{2.14}
$$

Similarly, we have that

$$
\begin{aligned}
B_2 &= (-\zeta_1-\bar{\zeta}_1)^{l_1}\Big\{(z_2-\zeta_2-\bar{\zeta}_2)^{l_2}\Big[\frac{2\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)} - 1\Big] + (-z_2)^{l_2}(-1)^{v_2}\Big[\frac{2\zeta_2}{\zeta_2 - z_2} - 1\Big]\Big\} \\
&= 2(-\zeta_1-\bar{\zeta}_1)^{l_1}\sum_{m_1,m_2=0}^{+\infty}\Big[\sum_{j_2=0}^{m_2} C_{l_2}^{m_2-j_2}(-\zeta_2-\bar{\zeta}_2)^{l_2-m_2+j_2}\bar{\zeta}_2^{j_2}\bar{\zeta}_1^{m_1} + B_{21}D_{31} - \frac{D_{23}+B_{22}}{2}B_{21}\Big] z_1^{m_1} z_2^{m_2},
\end{aligned}
\tag{2.15}
$$

and

$$
\begin{aligned}
B_1 - B_2 &= D|_{v_1=0} - B_2 \\
&= 2\sum_{m_1,m_2=0}^{+\infty}\Big\{\Big[\sum_{j_1=0}^{m_1} C_{l_1}^{m_1-j_1}(-\zeta_1-\bar{\zeta}_1)^{l_1-m_1+j_1}\bar{\zeta}_1^{j_1} - (-\zeta_1-\bar{\zeta}_1)^{l_1}\bar{\zeta}_1^{m_1}\Big]\sum_{j_2=0}^{m_2} C_{l_2}^{m_2-j_2}(-\zeta_2-\bar{\zeta}_2)^{l_2-m_2+j_2}\bar{\zeta}_2^{j_2} \\
&\quad + D_{21}|_{v_1=0}\sum_{j_2=0}^{m_2} C_{l_2}^{m_2-j_2}(-\zeta_2-\bar{\zeta}_2)^{l_2-m_2+j_2}\bar{\zeta}_2^{j_2} + D_{31}\sum_{j_1=0}^{m_1} C_{l_1}^{m_1-j_1}(-\zeta_1-\bar{\zeta}_1)^{l_1-m_1+j_1}\bar{\zeta}_1^{j_1} \\
&\quad - B_{21}D_{31} - \frac{1}{2}(D_{11}D_{12} + D_{22}|_{v_1=0}D_{23} + D_{32}D_{33} - B_{21}D_{23} - B_{21}B_{22})\Big\} z_1^{m_1} z_2^{m_2},
\end{aligned}
\tag{2.16}
$$

in which

$$
B_{21} = \begin{cases} 0, & m_1 \geq 1, \\ 1, & m_1 = 0, \end{cases} \qquad B_{22} = \begin{cases} (-1)^{m_2+v_2}, & m_2 = l_2, \\ 0, & m_2 \neq l_2. \end{cases}
$$

Therefore,

$$
\begin{aligned}
A_1 - A_2 - A_3 + A_4 &= D|_{v_1=v_2=0} - B_2|_{v_2=0} - C_2|_{v_1=0} + A_4 \\
&= D|_{v_1=v_2=0} - B_2|_{v_2=0} - C_2|_{v_1=0} \\
&\quad + 2\sum_{m_1,m_2=0}^{+\infty}(\bar{\zeta}_1^{m_1}C_{21} + B_{21}\bar{\zeta}_2^{m_2} - B_{21}C_{21})(-\zeta_1-\bar{\zeta}_1)^{l_1}(-\zeta_2-\bar{\zeta}_2)^{l_2} z_1^{m_1} z_2^{m_2}
\end{aligned}
\tag{2.17}
$$

as the result of

$$\begin{aligned}A_4 &= (-\zeta_1-\bar{\zeta}_1)^{l_1}(-\zeta_2-\bar{\zeta}_2)^{l_2}\Big[\frac{2\zeta_1}{\zeta_1-z_1}+\frac{2\zeta_2}{\zeta_2-z_2}-2\Big]\\ &= 2(-\zeta_1-\bar{\zeta}_1)^{l_1}(-\zeta_2-\bar{\zeta}_2)^{l_2}\Big[\sum_{j_1=0}^{+\infty}\frac{z_1^{j_1}}{\zeta_1^{j_1}}+\sum_{j_2=0}^{+\infty}\frac{z_2^{j_2}}{\zeta_2^{j_2}}-1\Big]\\ &= 2(-\zeta_1-\bar{\zeta}_1)^{l_1}(-\zeta_2-\bar{\zeta}_2)^{l_2}\Big[\sum_{m_1=0}^{+\infty}\bar{\zeta}_1^{\,m_1}z_1^{m_1}\sum_{m_2=0}^{+\infty}C_{21}z_2^{m_2}+\sum_{m_1=0}^{+\infty}B_{21}z_1^{m_1}\sum_{m_2=0}^{+\infty}\bar{\zeta}_2^{\,m_2}z_2^{m_2}\\ &\quad-\sum_{m_1=0}^{+\infty}B_{21}z_1^{m_1}\sum_{m_2=0}^{+\infty}C_{21}z_2^{m_2}\Big]\\ &= 2(-\zeta_1-\bar{\zeta}_1)^{l_1}(-\zeta_2-\bar{\zeta}_2)^{l_2}\sum_{m_1,m_2=0}^{+\infty}(\bar{\zeta}_1^{\,m_1}C_{21}+B_{21}\bar{\zeta}_2^{\,m_2}-B_{21}C_{21})z_1^{m_1}z_2^{m_2}.\end{aligned}$$

On the other hand, due to the analyticity of the function $u_{\widetilde{v_1}\widetilde{v_2}}(z)$, it can be expressed as

$$u_{\widetilde{v_1}\widetilde{v_2}}(z)=\sum_{m_1,m_2=0}^{+\infty}u_{m_1,m_2}^{m-\widetilde{v_1},m-\widetilde{v_2}}z_1^{m_1}z_2^{m_2},\quad \mu\le\widetilde{v_1}\le m-1,\ \nu\le\widetilde{v_2}\le m-1. \tag{2.18}$$

Plugging (2.12), (2.14), (2.16), and (2.17) into (2.4), and considering the Eq (2.18), we get (2.2). Moreover, (2.18) and (2.3) lead to (2.1). Therefore, from the result in the first part (1), it can be concluded that $\widetilde{\phi}(z)$ determined by (2.1) satisfies

$$\Re\partial_{\bar{z}_1}^{\mu}\partial_{\bar{z}_2}^{\nu}\widetilde{\phi}(z)=g_{\mu\nu}(z)\ (z\in\partial_0D^2),\ \Im\partial_{\bar{z}_1}^{\mu}\partial_{\bar{z}_2}^{\nu}\widetilde{\phi}(0,z_2)=0=\Im\partial_{\bar{z}_1}^{\mu}\partial_{\bar{z}_2}^{\nu}\widetilde{\phi}(z_1,0)\ (z_1\in D_1,z_2\in D_2).$$

## 3. The Dirichlet problem

**Theorem 3.1.** Let $\varphi\in C(\partial_0D^2;\mathbb{C})$, $\lambda\in\mathbb{R}\setminus\{-1,0,1\}$, and let $g_{\mu\nu}\in C(\partial_0D^2;\mathbb{R})$ for $1\le\mu,\nu\le m-1$ ($m\ge2$). Then, the problem

$$\partial_{\bar{z}_1}\partial_{\bar{z}_2}f(z)=\frac{\lambda-1}{4\lambda}\phi(z)+\frac{\lambda+1}{4\lambda}\bar{\phi}(z),\ \partial_{\bar{z}_1}^{m}\partial_{\bar{z}_2}^{m}\phi(z)=0\ (z\in D^2)$$

with the conditions

$$f(z)=\varphi(z),\ \Re\partial_{\bar{z}_1}^{\mu}\partial_{\bar{z}_2}^{\nu}\phi(z)=g_{\mu\nu}(z)\ (z\in\partial_0D^2),\ \Im\partial_{\bar{z}_1}^{\mu}\partial_{\bar{z}_2}^{\nu}\phi(0,z_2)=0=\Im\partial_{\bar{z}_1}^{\mu}\partial_{\bar{z}_2}^{\nu}\phi(z_1,0)$$

is solvable and the solution is

$$\begin{aligned}f(z)=&\frac{\lambda-1}{4\lambda}\Big[\sum_{m_1,m_2=0}^{+\infty}u_{m_1,m_2}^{0,0}z_1^{m_1}z_2^{m_2}\bar{z}_1\bar{z}_2+\sum_{m_1,m_2=0}^{+\infty}\sum_{v_1=\mu}^{m-1}\sum_{v_2=\nu}^{m-1}\frac{u_{m_1,m_2}^{m-v_1,m-v_2}z_1^{m_1}z_2^{m_2}\cdot\bar{z}_1^{v_1+1}\bar{z}_2^{v_2+1}}{(v_1+1)!(v_2+1)!}\Big]\\ &+\frac{\lambda+1}{4\lambda}\Big[\sum_{m_1,m_2=0}^{+\infty}\overline{u_{m_1,m_2}^{0,0}}\frac{\bar{z}_1^{m_1+1}}{m_1+1}\frac{\bar{z}_2^{m_2+1}}{m_2+1}+\sum_{m_1,m_2=0}^{+\infty}\sum_{v_1=\mu}^{m-1}\sum_{v_2=\nu}^{m-1}\frac{z_1^{v_1}z_2^{v_2}}{v_1!v_2!}\\ &\cdot\overline{u_{m_1,m_2}^{m-v_1,m-v_2}}\frac{\bar{z}_1^{m_1+1}}{m_1+1}\frac{\bar{z}_2^{m_2+1}}{m_2+1}\Big]+\sum_{m_1,m_2=0}^{+\infty}b_{m_1,m_2}z_1^{m_1}z_2^{m_2},\end{aligned}$$

where $u_{m_1,m_2}^{m-\nu_1,m-\nu_2}$ is determined by (2.2) (in which $\widetilde{\nu_1}$ and $\widetilde{\nu_2}$ are replaced by $\nu_1$ and $\nu_2$, respectively), and $u_{m_1,m_2}^{0,0}$, $b_{m_1,m_2}$ are determined by the following:

(i) for $1 \le m_1 \le m-1$ and $1 \le m_2 \le m-1$,

$$\begin{aligned} u_{m_1,m_2}^{0,0} = &(m_1+1)(m_2+1)\Big\{\frac{\lambda}{\lambda+1}\frac{1}{\pi^2}\int_0^{2\pi} e^{-it_1(m_1+1)}\int_0^{2\pi} e^{-it_2(m_2+1)}\overline{\varphi(e^{it_1},e^{it_2})}dt_2dt_1 \\ &- \frac{\lambda-1}{\lambda+1}\sum_{\nu_1=1}^{m-\mu}\sum_{\nu_2=1}^{m-\nu}\frac{\overline{u_{(m-m_1-\nu_1),(m-m_2-\nu_2)}^{\nu_1,\nu_2}}}{(m-\nu_1+1)!(m-\nu_2+1)!} \\ &- \sum_{\nu_1=1}^{m-\mu}\sum_{\nu_2=1}^{m-\nu}\frac{1}{(m-\nu_1)!(m-\nu_2)!}\frac{u_{(m+m_1-\nu_1),(m+m_2-\nu_2)}^{\nu_1,\nu_2}}{(m+m_1-\nu_1+1)(m+m_2-\nu_2+1)}\Big\}; \end{aligned} \tag{3.1}$$

(ii) for $m_1 \ge 1$ and $m_2 \ge 1$,

$$\begin{aligned} u_{0,0}^{0,0} = &\frac{\lambda+1}{(2\pi)^2}\int_0^{2\pi} e^{-it_1}\int_0^{2\pi} e^{-it_2}\overline{\varphi(e^{it_1},e^{it_2})}dt_2dt_1 - \frac{\lambda-1}{(2\pi)^2}\int_0^{2\pi} e^{it_1}\int_0^{2\pi} e^{it_2}\varphi(e^{it_1},e^{it_2})dt_2dt_1 \\ &- \sum_{\nu_1=1}^{m-\mu}\sum_{\nu_2=1}^{m-\nu}\frac{u_{(m-\nu_1),(m-\nu_2)}^{\nu_1,\nu_2}}{(m-\nu_1+1)!(m-\nu_2+1)!}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} u_{0,m_2}^{0,0} = &\frac{4\lambda}{\lambda-1}\frac{1}{(2\pi)^2}\int_0^{2\pi} e^{it_1}\int_0^{2\pi} e^{it_2(1-m_2)}\varphi(e^{it_1},e^{it_2})dt_2dt_1 - \sum_{\nu_1=1}^{m-\mu}\sum_{\nu_2=1}^{m-\nu}\frac{u_{(m-\nu_1),(m-\nu_2+m_2)}^{\nu_1,\nu_2}}{(m-\nu_1+1)!(m-\nu_2+1)!} \\ &- \frac{\lambda+1}{\lambda-1}\begin{cases} \displaystyle\sum_{\nu_1=1}^{m-\mu}\sum_{\nu_2=1}^{m-\nu}\frac{1}{(m-\nu_1+1)!(m-\nu_2)!}\frac{\overline{u_{(m-\nu_1),(m-\nu_2-m_2)}^{\nu_1,\nu_2}}}{m-\nu_2-m_2+1}, & 1\le m_2\le \nu, \\ \displaystyle\sum_{\nu_1=1}^{m-\mu}\sum_{\nu_2=1}^{m-m_2}\frac{1}{(m-\nu_1+1)!(m-\nu_2)!}\frac{\overline{u_{(m-\nu_1),(m-\nu_2-m_2)}^{\nu_1,\nu_2}}}{m-\nu_2-m_2+1}, & \nu< m_2\le m-1, \\ 0, & m_2 > m-1, \end{cases} \end{aligned} \tag{3.3}$$

$$\begin{aligned} u_{m_1,0}^{0,0} = &\frac{4\lambda}{\lambda-1}\frac{1}{(2\pi)^2}\int_0^{2\pi} e^{i(1-m_1)t_1}\int_0^{2\pi} e^{it_2}\varphi(e^{it_1},e^{it_2})dt_2dt_1 - \sum_{\nu_1=1}^{m-\mu}\sum_{\nu_2=1}^{m-\nu}\frac{u_{(m-\nu_1+m_1),(m-\nu_2)}^{\nu_1,\nu_2}}{(m-\nu_1+1)!(m-\nu_2+1)!} \\ &- \frac{\lambda+1}{\lambda-1}\begin{cases} \displaystyle\sum_{\nu_1=1}^{m-\mu}\sum_{\nu_2=1}^{m-\nu}\frac{1}{(m-\nu_1)!(m-\nu_2+1)!}\frac{\overline{u_{(m-\nu_1-m_1,(m-\nu_2))}^{\nu_1,\nu_2}}}{m-\nu_1-m_1+1}, & 1\le m_1\le \mu, \\ \displaystyle\sum_{\nu_1=1}^{m-m_1}\sum_{\nu_2=1}^{m-\nu}\frac{1}{(m-\nu_1)!(m-\nu_2+1)!}\frac{\overline{u_{(m-\nu_1-m_1,(m-\nu_2))}^{\nu_1,\nu_2}}}{m-\nu_1-m_1+1}, & \mu< m_1\le m-1, \\ 0, & m_1 > m-1; \end{cases} \end{aligned} \tag{3.4}$$

(iii) with $u_{1,1}^{0,0}$ being determined by (3.1),

$$\begin{aligned} b_{0,0} = &\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\varphi(e^{it_1},e^{it_2})dt_2dt_1 - \frac{\lambda-1}{4\lambda}\Big[u_{1,1}^{0,0} + \sum_{\nu_1=1}^{m-\mu}\sum_{\nu_2=1}^{m-\nu}\frac{u_{(m-\nu_1+1),(m-\nu_2+1)}^{\nu_1,\nu_2}}{(m-\nu_1+1)!(m-\nu_2+1)!}\Big] \\ &- \frac{\lambda+1}{4\lambda}\sum_{\nu_1=1}^{m-\mu}\sum_{\nu_2=1}^{m-\nu}\frac{1}{(m-\nu_1)!(m-\nu_2)!}\frac{\overline{u_{(m-\nu_1-1),(m-\nu_2-1)}^{\nu_1,\nu_2}}}{(m-\nu_1)(m-\nu_2)}; \end{aligned} \tag{3.5}$$

(iv) for $\{^{m_1\geq m}_{m_2\geq m}$ or $\{^{1\leq m_1\leq m-1}_{m_2\geq m}$ or $\{^{m_1\geq m}_{1\leq m_2\leq m-1}$,

$$\begin{aligned}u^{0,0}_{m_1,m_2} = {} & \frac{4\lambda}{\lambda+1}\frac{(m_1+1)(m_2+1)}{(2\pi)^2}\int_0^{2\pi} e^{-i(m_1+1)t_1}\int_0^{2\pi} e^{-i(m_2+1)t_2}\overline{\varphi(e^{it_1}, e^{it_2})}dt_2dt_1 \\ & -\sum_{\nu_1=1}^{m-\mu}\sum_{\nu_2=1}^{m-\nu}\frac{(m_1+1)(m_2+1)}{(m-\nu_1)!(m-\nu_2)!}\frac{u^{\nu_1,\nu_2}_{(m-\nu_1+m_1),(m-\nu_2+m_2)}}{(m-\nu_1+m_1+1)(m-\nu_2+m_2+1)};\end{aligned} \tag{3.6}$$

(v) for $m_1,\ m_2 \geq 1$,

$$\begin{aligned}b_{m_1,m_2} = {} & \frac{1}{(2\pi)^2}\int_0^{2\pi} e^{-im_1t_1}\int_0^{2\pi} e^{-im_2t_2}\varphi(e^{it_1}, e^{it_2})dt_2dt_1 \\ & -\frac{\lambda-1}{4\lambda}\Big[u^{0,0}_{(m_1+1),(m_2+1)}+\sum_{\nu_1=1}^{m-\mu}\sum_{\nu_2=1}^{m-\nu}\frac{u^{\nu_1,\nu_2}_{(m_1+m-\nu_1+1),(m_2+m-\nu_2+1)}}{(m-\nu_1+1)!(m-\nu_2+1)!}\Big] \\ & -\frac{\lambda+1}{4\lambda}\begin{cases}\displaystyle\sum_{\nu_1=1}^{m-1-m_1}\sum_{\nu_2=1}^{m-1-m_2}\frac{1}{(m-\nu_1)!(m-\nu_2)!}\frac{\overline{u^{\nu_1,\nu_2}_{(m-1-m_1-\nu_1),(m-1-m_2-\nu_2)}}}{(m-\nu_1-m_1)(m-\nu_2-m_2)}, & 1\leq m_1, m_2< m-1, \\ 0, \quad m_1, m_2\geq m-1,\end{cases}\end{aligned} \tag{3.7}$$

$$\begin{aligned}b_{m_1,0} = {} & \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} e^{-im_1t_1}\varphi(e^{it_1}, e^{it_2})dt_2dt_1 \\ & -\frac{\lambda-1}{4\lambda}\Big[u^{0,0}_{(m_1+1),1}+\sum_{\nu_1=1}^{m-\mu}\sum_{\nu_2=1}^{m-\nu}\frac{u^{\nu_1,\nu_2}_{(m_1+m-\nu_1+1),(m-\nu_2+1)}}{(m-\nu_1+1)!(m-\nu_2+1)!}\Big] \\ & -\frac{\lambda+1}{4\lambda}\begin{cases}\displaystyle\sum_{\nu_1=1}^{m-1-m_1}\sum_{\nu_2=1}^{m-\nu}\frac{1}{(m-\nu_1)!(m-\nu_2)!}\frac{\overline{u^{\nu_1,\nu_2}_{(m-1-m_1-\nu_1),(m-1-\nu_2)}}}{(m-\nu_1-m_1)(m-\nu_2)}, & 1\leq m_1< m-1, \\ 0, \quad m_1\geq m-1,\end{cases}\end{aligned} \tag{3.8}$$

$$\begin{aligned}b_{0,m_2} = {} & \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} e^{-im_2t_2}\varphi(e^{it_1}, e^{it_2})dt_2dt_1 \\ & -\frac{\lambda-1}{4\lambda}\Big[u^{0,0}_{1,(m_2+1)}+\sum_{\nu_1=1}^{m-\mu}\sum_{\nu_2=1}^{m-\nu}\frac{u^{\nu_1,\nu_2}_{(m-\nu_1+1),(m_2+m-\nu_2+1)}}{(m-\nu_1+1)!(m-\nu_2+1)!}\Big] \\ & -\frac{\lambda+1}{4\lambda}\begin{cases}\displaystyle\sum_{\nu_1=1}^{m-\mu}\sum_{\nu_2=1}^{m-1-m_2}\frac{1}{(m-\nu_1)!(m-\nu_2)!}\frac{\overline{u^{\nu_1,\nu_2}_{(m-1-\nu_1),(m-1-m_2-\nu_2)}}}{(m-\nu_1)(m-\nu_2-m_2)}, & 1\leq m_2< m-1, \\ 0, \quad m_2\geq m-1,\end{cases}\end{aligned} \tag{3.9}$$

in which $u^{0,0}_{(m_1+1),(m_2+1)}$, $u^{0,0}_{(m_1+1),1}$, and $u^{0,0}_{1,(m_2+1)}$ are determined by (3.6).
*Proof:* 1) By Theorem 2.1,

$$\phi(z) = \sum_{m_1,m_2=0}^{+\infty}\sum_{\nu_1=\mu}^{m-1}\sum_{\nu_2=\nu}^{m-1}\frac{\bar{z}_1^{\nu_1}\bar{z}_2^{\nu_2}}{\nu_1!\nu_2!}u^{m-\nu_1,m-\nu_2}_{m_1,m_2}z_1^{m_1}z_2^{m_2} + u_0(z),$$

where $u_{m_1,m_2}^{m-v_1,m-v_2}$ is determined by (2.2) (in which $\widetilde{v_1}$ and $\widetilde{v_2}$ are replaced by $v_1$ and $v_2$, respectively), and $u_0(z)$ is analytic on $D^2$. Thus, $u_0(z)$ can be represented as

$$u_0(z) = \sum_{m_1,m_2=0}^{+\infty} u_{m_1,m_2}^{0,0} z_1^{m_1} z_2^{m_2},$$

in which $u_{m_1,m_2}^{0,0}$ is to be determined.

Let

$$\phi_1(z) = u_0(z)\bar{z}_1\bar{z}_2 + \sum_{v_1=\mu}^{m-1}\sum_{v_2=v}^{m-1} \frac{u_{v_1v_2}(z)}{(v_1+1)!(v_2+1)!}\bar{z}_1^{v_1+1}\bar{z}_2^{v_2+1},$$

where

$$u_{v_1,v_2}(z) = \sum_{m_1,m_2=0}^{+\infty} u_{m_1,m_2}^{m-v_1,m-v_2} z_1^{m_1} z_2^{m_2}, \quad \mu \le v_1 \le m-1,\ v \le v_2 \le m-1.$$

Thus, $\partial_{\bar{z}_1}\partial_{\bar{z}_2}\phi_1(z) = \phi(z)$. Let

$$\widetilde{u_0}(z) = \sum_{m_1,m_2=0}^{+\infty} u_{m_1,m_2}^{0,0} \frac{z_1^{m_1+1}}{m_1+1}\frac{z_2^{m_2+1}}{m_2+1},$$

and let

$$\widetilde{u}_{v_1,v_2}(z) = \sum_{m_1,m_2=0}^{+\infty} u_{m_1,m_2}^{m-v_1,m-v_2} \frac{z_1^{m_1+1}}{m_1+1}\frac{z_2^{m_2+1}}{m_2+1}, \quad \mu \le v_1 \le m-1,\ v \le v_2 \le m-1.$$

Then, we get that $\partial_{z_1}\partial_{z_2}\widetilde{u_0}(z) = u_0(z)$ and $\partial_{z_1}\partial_{z_2}\widetilde{u}_{v_1,v_2}(z) = u_{v_1,v_2}(z)$. Let

$$\phi_2(z) = \overline{\widetilde{u_0}(z) + \sum_{v_1=\mu}^{m-1}\sum_{v_2=v}^{m-1} \frac{\bar{z}_1^{v_1}\bar{z}_2^{v_2}}{v_1!v_2!}\widetilde{u}_{v_1v_2}(z)},$$

which follows that

$$\begin{aligned}\partial_{\bar{z}_1}\partial_{\bar{z}_2}\phi_2(z) &= \overline{\partial_{z_1}\partial_{z_2}\overline{\phi_2}} = \overline{\partial_{z_1}\partial_{z_2}\widetilde{u_0}(z) + \sum_{v_1=\mu}^{m-1}\sum_{v_2=v}^{m-1} \frac{\bar{z}_1^{v_1}\bar{z}_2^{v_2}}{v_1!v_2!}\partial_{z_1}\partial_{z_2}\widetilde{u}_{v_1v_2}(z)} \\ &= \overline{u_0(z) + \sum_{v_1=\mu}^{m-1}\sum_{v_2=v}^{m-1} \frac{\bar{z}_1^{v_1}\bar{z}_2^{v_2}}{v_1!v_2!}u_{v_1v_2}(z)} = \overline{\phi(z)}.\end{aligned}$$

Therefore,

$$\partial_{\bar{z}_1}\partial_{\bar{z}_2}\Big[\frac{\lambda-1}{4\lambda}\phi_1(z) + \frac{\lambda+1}{4\lambda}\phi_2(z)\Big] = \frac{\lambda-1}{4\lambda}\phi(z) + \frac{\lambda+1}{4\lambda}\bar{\phi}(z),$$

which means that

$$\frac{\lambda-1}{4\lambda}\phi_1(z) + \frac{\lambda+1}{4\lambda}\phi_2(z)$$

is a special solution to

$$\partial_{\bar{z}_1}\partial_{\bar{z}_2}f(z) = \frac{\lambda-1}{4\lambda}\phi(z) + \frac{\lambda+1}{4\lambda}\bar{\phi}(z).$$

So, the solution of the problem is

$$
\begin{aligned}
f(z) &= \Big[\frac{\lambda-1}{4\lambda}\phi_1(z)+\frac{\lambda+1}{4\lambda}\phi_2(z)\Big]+\psi(z)\\
&= \frac{\lambda-1}{4\lambda}\Big[u_0(z)\bar{z}_1\bar{z}_2+\sum_{v_1=\mu}^{m-1}\sum_{v_2=v}^{m-1}\frac{u_{v_1v_2}(z)}{(v_1+1)!(v_2+1)!}\bar{z}_1^{v_1+1}\bar{z}_2^{v_2+1}\Big]\\
&\quad+\frac{\lambda+1}{4\lambda}\Big[\overline{\widetilde{u_0}(z)}+\sum_{v_1=\mu}^{m-1}\sum_{v_2=v}^{m-1}\frac{z_1^{v_1}z_2^{v_2}}{v_1!v_2!}\overline{\widetilde{u}_{v_1v_2}(z)}\Big]+\psi(z)\\
&=\frac{\lambda-1}{4\lambda}\Big[\sum_{m_1,m_2=0}^{+\infty}u^{0,0}_{m_1,m_2}z_1^{m_1}z_2^{m_2}\bar{z}_1\bar{z}_2+\sum_{m_1,m_2=0}^{+\infty}\sum_{v_1=\mu}^{m-1}\sum_{v_2=v}^{m-1}\frac{u^{m-v_1,m-v_2}_{m_1,m_2}z_1^{m_1}z_2^{m_2}\cdot\bar{z}_1^{v_1+1}\bar{z}_2^{v_2+1}}{(v_1+1)!(v_2+1)!}\Big]\\
&\quad+\frac{\lambda+1}{4\lambda}\Big[\sum_{m_1,m_2=0}^{+\infty}\overline{u^{0,0}_{m_1,m_2}}\frac{\bar{z}_1^{m_1+1}}{m_1+1}\frac{\bar{z}_2^{m_2+1}}{m_2+1}+\sum_{m_1,m_2=0}^{+\infty}\sum_{v_1=\mu}^{m-1}\sum_{v_2=v}^{m-1}\frac{z_1^{v_1}z_2^{v_2}}{v_1!v_2!}\\
&\quad\cdot\overline{u^{m-v_1,m-v_2}_{m_1,m_2}}\frac{\bar{z}_1^{m_1+1}}{m_1+1}\frac{\bar{z}_2^{m_2+1}}{m_2+1}\Big]+\sum_{m_1,m_2=0}^{+\infty}b_{m_1,m_2}z_1^{m_1}z_2^{m_2},
\end{aligned}
\tag{3.10}
$$

where $\psi(z)=\sum_{m_1,m_2=0}^{+\infty}b_{m_1,m_2}z_1^{m_1}z_2^{m_2}$ is analytic on $D^2$, and $u^{0,0}_{m_1,m_2}$ and $b_{m_1,m_2}$ are to be determined.

2) In this part, we seek the expressions of $u^{0,0}_{m_1,m_2}$ and $b_{m_1,m_2}$.

For $z\in\partial_0D^2$, let $z_1=e^{it_1}$ and $z_2=e^{it_2}$ $(t_1,t_2\in[0,2\pi])$. Then, we get that

$$
\begin{aligned}
\varphi(e^{it_1},e^{it_2}) &= f(e^{it_1},e^{it_2})=\frac{\lambda-1}{4\lambda}\Big[\sum_{m_1,m_2=0}^{+\infty}u^{0,0}_{m_1,m_2}e^{i(m_1-1)t_1}e^{i(m_2-1)t_2}\\
&\quad+\sum_{m_1,m_2=0}^{+\infty}\sum_{\widetilde{v_1}=1}^{m-\mu}\sum_{\widetilde{v_2}=1}^{m-v}\frac{u^{\widetilde{v_1},\widetilde{v_2}}_{m_1,m_2}\cdot e^{it_1(m_1-m+\widetilde{v_1}-1)}e^{it_2(m_2-m+\widetilde{v_2}-1)}}{(m-\widetilde{v_1}+1)!(m-\widetilde{v_2}+1)!}\Big]\\
&\quad+\frac{\lambda+1}{4\lambda}\Big[\sum_{m_1,m_2=0}^{+\infty}\overline{u^{0,0}_{m_1,m_2}}\frac{e^{-it_1(m_1+1)}}{m_1+1}\frac{e^{-it_2(m_2+1)}}{m_2+1}\\
&\quad+\sum_{m_1,m_2=0}^{+\infty}\sum_{\widetilde{v_1}=1}^{m-\mu}\sum_{\widetilde{v_2}=1}^{m-v}\frac{e^{-it_1(m_1+1-m+\widetilde{v_1})}e^{-it_2(m_2+1-m+\widetilde{v_2})}}{(m-\widetilde{v_1})!(m-\widetilde{v_2})!}\frac{\overline{u^{\widetilde{v_1},\widetilde{v_2}}_{m_1,m_2}}}{(m_1+1)(m_2+1)}\Big]\\
&\quad+\sum_{m_1,m_2=0}^{+\infty}b_{m_1,m_2}e^{im_1t_1}e^{im_2t_2}\\
&=\frac{\lambda-1}{4\lambda}\Big[\sum_{m_1=0}^{0}\sum_{m_2=0}^{+\infty}u^{0,0}_{0,m_2}e^{-it_1}e^{i(m_2-1)t_2}+\sum_{m_1=1}^{+\infty}\sum_{m_2=0}^{0}u^{0,0}_{m_1,0}e^{i(m_1-1)t_1}e^{-it_2}\\
&\quad+\sum_{m_1,m_2=1}^{+\infty}u^{0,0}_{m_1,m_2}e^{i(m_1-1)t_1}e^{i(m_2-1)t_2}\\
&\quad+\sum_{\widetilde{v_1}=1}^{m-\mu}\sum_{\widetilde{v_2}=1}^{m-v}\sum_{m_1=0}^{m-\widetilde{v_1}}\sum_{m_2=0}^{m-\widetilde{v_2}}\frac{u^{\widetilde{v_1},\widetilde{v_2}}_{m_1,m_2}e^{it_1(m_1-(m-\widetilde{v_1}+1))}e^{it_2(m_2-(m-\widetilde{v_2}+1))}}{(m-\widetilde{v_1}+1)!(m-\widetilde{v_2}+1)!}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{\widetilde{v_1}=1}^{m-\mu}\sum_{\widetilde{v_2}=1}^{m-\nu}\sum_{m_1,m_2=0}^{+\infty}\frac{u^{\widetilde{v_1},\widetilde{v_2}}_{(m_1+(m-\widetilde{v_1}+1)),(m_2+(m-\widetilde{v_2}+1))}e^{im_1t_1}e^{im_2t_2}}{(m-\widetilde{v_1}+1)!(m-\widetilde{v_2}+1)!}\\
&+\sum_{\widetilde{v_1}=1}^{m-\mu}\sum_{\widetilde{v_2}=1}^{m-\nu}\sum_{m_1=0}^{m-\widetilde{v_1}}\sum_{m_2=0}^{+\infty}\frac{u^{\widetilde{v_1},\widetilde{v_2}}_{m_1,(m_2+(m-\widetilde{v_2}+1))}e^{it_1(m_1-(m-\widetilde{v_1}+1))}e^{im_2t_2}}{(m-\widetilde{v_1}+1)!(m-\widetilde{v_2}+1)!}\\
&+\sum_{\widetilde{v_1}=1}^{m-\mu}\sum_{\widetilde{v_2}=1}^{m-\nu}\sum_{m_1=0}^{+\infty}\sum_{m_2=0}^{m-\widetilde{v_2}}\frac{u^{\widetilde{v_1},\widetilde{v_2}}_{(m_1+(m-\widetilde{v_1}+1)),m_2}e^{im_1t_1}e^{it_2(m_2-(m-\widetilde{v_2}+1))}}{(m-\widetilde{v_1}+1)!(m-\widetilde{v_2}+1)!}\Big]\\
&+\frac{\lambda+1}{4\lambda}\Big[\sum_{m_1,m_2=1}^{+\infty}\overline{u^{0,0}_{(m_1-1),(m_2-1)}\frac{e^{-it_1m_1}}{m_1}}\frac{e^{-it_2m_2}}{m_2}\\
&+\sum_{m_1,m_2=0}^{+\infty}\sum_{\widetilde{v_1}=1}^{m-\mu}\sum_{\widetilde{v_2}=1}^{m-\nu}\frac{e^{-it_1(m_1+1-m+\widetilde{v_1})}e^{-it_2(m_2+1-m+\widetilde{v_2})}}{(m-\widetilde{v_1})!(m-\widetilde{v_2})!}\frac{\overline{u^{\widetilde{v_1},\widetilde{v_2}}_{m_1,m_2}}}{(m_1+1)(m_2+1)}\Big]\\
&+\sum_{m_1,m_2=0}^{+\infty}b_{m_1,m_2}e^{im_1t_1}e^{im_2t_2}.
\end{aligned}
\tag{3.11}
$$

(i) In the case of $0\le r_1,r_2\le m-2$, multiplying both sides of the Eq (3.11) by $e^{it_1(m-r_1)}e^{it_2(m-r_2)}$, and then integrating with respect to $t_1,t_2\in[0,2\pi]$ yields that

$$
\begin{aligned}
&\frac{1}{(2\pi)^2}\int_0^{2\pi}e^{it_1(m-r_1)}\Big[\int_0^{2\pi}e^{it_2(m-r_2)}\varphi(e^{it_1},e^{it_2})dt_2\Big]dt_1\\
&=\frac{\lambda-1}{4\lambda}\Big[\sum_{m_2=0}^{+\infty}u^{0,0}_{0,m_2}\frac{1}{2\pi}\int_0^{2\pi}e^{it_1(m-r_1-1)}dt_1\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{it_2(m_2-1+m-r_2)}dt_2\\
&+\sum_{m_1=1}^{+\infty}u^{0,0}_{m_1,0}\frac{1}{2\pi}\int_0^{2\pi}e^{it_1(m_1-1+m-r_1)}dt_1\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{it_2(-1+m-r_2)}dt_2\\
&+\sum_{m_1,m_2=1}^{+\infty}u^{0,0}_{m_1,m_2}\frac{1}{2\pi}\int_0^{2\pi}e^{it_1(m_1-1+m-r_1)}dt_1\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{it_2(m_2-1+m-r_2)}dt_2\\
&+\sum_{v_1=1}^{m-\mu}\sum_{v_2=1}^{m-\nu}\sum_{m_1=0}^{m-v_1}\sum_{m_2=0}^{m-v_2}\frac{u^{v_1,v_2}_{m_1,m_2}\frac{1}{2\pi}\int_0^{2\pi}e^{it_1(m_1+v_1-1-r_1)}dt_1\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{it_2(m_2+v_2-1-r_2)}dt_2}{(m-v_1+1)!(m-v_2+1)!}\\
&+\sum_{v_1=1}^{m-\mu}\sum_{v_2=1}^{m-\nu}\sum_{m_1,m_2=0}^{+\infty}\frac{u^{v_1,v_2}_{(m_1+(m-v_1+1)),(m_2+(m-v_2+1))}\frac{1}{2\pi}\int_0^{2\pi}e^{it_1(m_1+m-r_1)}dt_1\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{it_2(m_2+m-r_2)}dt_2}{(m-v_1+1)!(m-v_2+1)!}\\
&+\sum_{v_1=1}^{m-\mu}\sum_{v_2=1}^{m-\nu}\sum_{m_1=0}^{m-v_1}\sum_{m_2=0}^{+\infty}\frac{u^{v_1,v_2}_{m_1,(m_2+(m-v_2+1))}\frac{1}{2\pi}\int_0^{2\pi}e^{it_1(m_1+v_1-1-r_1)}dt_1\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{it_2(m_2+m-r_2)}dt_2}{(m-v_1+1)!(m-v_2+1)!}\\
&+\sum_{v_1=1}^{m-\mu}\sum_{v_2=1}^{m-\nu}\sum_{m_1=0}^{+\infty}\sum_{m_2=0}^{m-v_2}\frac{u^{v_1,v_2}_{(m_1+(m-v_1+1)),m_2}\frac{1}{2\pi}\int_0^{2\pi}e^{it_1(m_1+m-r_1)}dt_1\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{it_2(m_2+v_2-1-r_2)}dt_2}{(m-v_1+1)!(m-v_2+1)!}\Big]\\
&+\frac{\lambda+1}{4\lambda}\Big[\sum_{m_1,m_2=1}^{+\infty}\frac{\overline{u^{0,0}_{(m_1-1),(m_2-1)}}}{m_1m_2}\frac{1}{2\pi}\int_0^{2\pi}e^{it_1(m-r_1-m_1)}dt_1\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{it_2(m-r_2-m_2)}dt_2
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{m_1,m_2=0}^{+\infty}\sum_{v_1=1}^{m-\mu}\sum_{v_2=1}^{m-\nu}\frac{\frac{1}{2\pi}\int_0^{2\pi}e^{it_1(m-r_1-(m_1+1-m+v_1))}dt_1\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{it_2(m-r_2-(m_2+1-m+v_2))}dt_2}{(m-v_1)!(m-v_2)!}\\
&\cdot\frac{\overline{u_{m_1,m_2}^{v_1,v_2}}}{(m_1+1)(m_2+1)}\Big]+\sum_{m_1,m_2=0}^{+\infty}b_{m_1,m_2}\frac{1}{2\pi}\int_0^{2\pi}e^{it_1(m_1+m-r_1)}dt_1\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{it_2(m_2+m-r_2)}dt_2\\
&=\frac{\lambda-1}{4\lambda}\Big[\sum_{v_1=1}^{m-\mu}\sum_{v_2=1}^{m-\nu}\frac{u_{(1+r_1-v_1),(1+r_2-v_2)}^{v_1,v_2}}{(m-v_1+1)!(m-v_2+1)!}\Big]\\
&+\frac{\lambda+1}{4\lambda}\Big[\frac{\overline{u_{(m-r_1-1),(m-r_2-1)}^{0,0}}}{(m-r_1)(m-r_2)}+\sum_{v_1=1}^{m-\mu}\sum_{v_2=1}^{m-\nu}\frac{1}{(m-v_1)!(m-v_2)!}\frac{\overline{u_{(2m-r_1-1-v_1),(2m-r_2-1-v_2)}^{v_1,v_2}}}{(2m-r_1-v_1)(2m-r_2-v_2)}\Big],
\end{aligned}
$$

and the last equation is due to

$$
\frac{1}{2\pi}\int_0^{2\pi}e^{imt}dt=\begin{cases}0, & m\neq 0\ (m\in\mathbb{Z}),\\ 1, & m=0.\end{cases}
$$

Therefore,

$$
\begin{aligned}
u_{(m-r_1-1),(m-r_2-1)}^{0,0}=(m-r_1)(m-r_2)\Big\{&\frac{4\lambda}{\lambda+1}\frac{1}{(2\pi)^2}\int_0^{2\pi}e^{-it_1(m-r_1)}\int_0^{2\pi}e^{-it_2(m-r_2)}\overline{\varphi(e^{it_1},e^{it_2})}dt_2dt_1\\
&-\frac{\lambda-1}{\lambda+1}\Big[\sum_{v_1=1}^{m-\mu}\sum_{v_2=1}^{m-\nu}\frac{\overline{u_{(1+r_1-v_1),(1+r_2-v_2)}^{v_1,v_2}}}{(m-v_1+1)!(m-v_2+1)!}\Big]\\
&-\sum_{v_1=1}^{m-\mu}\sum_{v_2=1}^{m-\nu}\frac{1}{(m-v_1)!(m-v_2)!}\frac{u_{(2m-r_1-1-v_1),(2m-r_2-1-v_2)}^{v_1,v_2}}{(2m-r_1-v_1)(2m-r_2-v_2)}\Big\},
\end{aligned}
$$

which leads to (3.1) for $1\leq m_1,\ m_2\leq m-1$.

(ii) Multiplying both sides of the Eq (3.11) by $e^{it_1}e^{it_2}$, and then integrating with respect to $t_1,t_2\in[0,2\pi]$ yields that

$$
\begin{aligned}
&\frac{1}{(2\pi)^2}\int_0^{2\pi}e^{it_1}\Big[\int_0^{2\pi}e^{it_2}\varphi(e^{it_1},e^{it_2})dt_2\Big]dt_1\\
&=\frac{\lambda-1}{4\lambda}\Big[\sum_{m_2=0}^{+\infty}u_{0,m_2}^{0,0}\frac{1}{2\pi}\int_0^{2\pi}e^{im_2t_2}dt_2+\sum_{m_1=1}^{+\infty}u_{m_1,0}^{0,0}\frac{1}{2\pi}\int_0^{2\pi}e^{im_1t_1}dt_1\\
&+\sum_{m_1,m_2=1}^{+\infty}u_{m_1,m_2}^{0,0}\frac{1}{2\pi}\int_0^{2\pi}e^{im_1t_1}dt_1\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{im_2t_2}dt_2\\
&+\sum_{v_1=1}^{m-\mu}\sum_{v_2=1}^{m-\nu}\sum_{m_1=0}^{m-v_1}\sum_{m_2=0}^{m-v_2}\frac{u_{m_1,m_2}^{v_1,v_2}\frac{1}{2\pi}\int_0^{2\pi}e^{it_1(m_1-m+v_1)}dt_1\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{it_2(m_2-m+v_2)}dt_2}{(m-v_1+1)!(m-v_2+1)!}\\
&+\sum_{v_1=1}^{m-\mu}\sum_{v_2=1}^{m-\nu}\sum_{m_1,m_2=0}^{+\infty}\frac{u_{(m_1+(m-v_1+1)),(m_2+(m-v_2+1))}^{v_1,v_2}\frac{1}{2\pi}\int_0^{2\pi}e^{it_1(m_1+1)}dt_1\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{it_2(m_2+1)}dt_2}{(m-v_1+1)!(m-v_2+1)!}\\
&+\sum_{v_1=1}^{m-\mu}\sum_{v_2=1}^{m-\nu}\sum_{m_1=0}^{m-v_1}\sum_{m_2=0}^{+\infty}\frac{u_{m_1,(m_2+(m-v_2+1))}^{v_1,v_2}\frac{1}{2\pi}\int_0^{2\pi}e^{it_1(m_1-m+v_1)}dt_1\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{it_2(m_2+1)}dt_2}{(m-v_1+1)!(m-v_2+1)!}
\end{aligned}
$$

$$+\sum_{v_1=1}^{m-\mu}\sum_{v_2=1}^{m-\nu}\sum_{m_1=0}^{+\infty}\sum_{m_2=0}^{m-v_2}\frac{u^{v_1,v_2}_{(m_1+(m-v_1+1)),m_2}\frac{1}{2\pi}\int_0^{2\pi}e^{it_1(m_1+1)}dt_1\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{it_2(m_2-m+v_2)}dt_2}{(m-v_1+1)!(m-v_2+1)!}\Big]$$

$$+\frac{\lambda+1}{4\lambda}\Big[\sum_{m_1,m_2=1}^{+\infty}\frac{\overline{u^{0,0}_{(m_1-1),(m_2-1)}}}{m_1m_2}\frac{1}{2\pi}\int_0^{2\pi}e^{it_1(1-m_1)}dt_1\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{it_2(1-m_2)}dt_2$$

$$+\sum_{m_1,m_2=0}^{+\infty}\sum_{v_1=1}^{m-\mu}\sum_{v_2=1}^{m-\nu}\frac{\frac{1}{2\pi}\int_0^{2\pi}e^{-it_1(m_1-m+v_1)}dt_1\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{-it_2(m_2-m+v_2)}dt_2}{(m-v_1)!(m-v_2)!}$$

$$\cdot\frac{\overline{u^{v_1,v_2}_{m_1,m_2}}}{(m_1+1)(m_2+1)}\Big]+\sum_{m_1,m_2=0}^{+\infty}b_{m_1,m_2}\frac{1}{2\pi}\int_0^{2\pi}e^{it_1(m_1+1)}dt_1\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{it_2(m_2+1)}dt_2$$

$$=\frac{\lambda-1}{4\lambda}\Big[u^{0,0}_{0,0}+\sum_{v_1=1}^{m-\mu}\sum_{v_2=1}^{m-\nu}\frac{u^{v_1,v_2}_{(m-v_1),(m-v_2)}}{(m-v_1+1)!(m-v_2+1)!}\Big]$$

$$+\frac{\lambda+1}{4\lambda}\Big[\overline{u^{0,0}_{0,0}}+\sum_{v_1=1}^{m-\mu}\sum_{v_2=1}^{m-\nu}\frac{\overline{u^{v_1,v_2}_{(m-v_1),(m-v_2)}}}{(m-v_1+1)!(m-v_2+1)!}\Big],$$

which leads to (3.2).

Additionally, for $r_2\geq 1$, multiplying both sides of the Eq (3.11) by $e^{it_1}e^{i(1-r_2)t_2}$, and then integrating with respect to $t_1,t_2\in[0,2\pi]$ yields that

$$\frac{1}{(2\pi)^2}\int_0^{2\pi}e^{it_1}\Big[\int_0^{2\pi}e^{it_2(1-r_2)}\varphi(e^{it_1},e^{it_2})dt_2\Big]dt_1$$

$$=\frac{\lambda-1}{4\lambda}\Big[\sum_{m_2=0}^{+\infty}u^{0,0}_{0,m_2}\frac{1}{2\pi}\int_0^{2\pi}e^{it_2(m_2-r_2)}dt_2+\sum_{m_1=1}^{+\infty}u^{0,0}_{m_1,0}\frac{1}{2\pi}\int_0^{2\pi}e^{im_1t_1}dt_1\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{-ir_2t_2}dt_2$$

$$+\sum_{m_1,m_2=1}^{+\infty}u^{0,0}_{m_1,m_2}\frac{1}{2\pi}\int_0^{2\pi}e^{im_1t_1}dt_1\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{it_2(m_2-r_2)}dt_2$$

$$+\sum_{v_1=1}^{m-\mu}\sum_{v_2=1}^{m-\nu}\sum_{m_1=0}^{m-v_1}\sum_{m_2=0}^{m-v_2}\frac{u^{v_1,v_2}_{m_1,m_2}\frac{1}{2\pi}\int_0^{2\pi}e^{it_1(m_1-m+v_1)}dt_1\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{it_2(m_2-m+v_2-r_2)}dt_2}{(m-v_1+1)!(m-v_2+1)!}$$

$$+\sum_{v_1=1}^{m-\mu}\sum_{v_2=1}^{m-\nu}\sum_{m_1,m_2=0}^{+\infty}\frac{u^{v_1,v_2}_{(m_1+(m-v_1+1)),(m_2+(m-v_2+1))}\frac{1}{2\pi}\int_0^{2\pi}e^{it_1(m_1+1)}dt_1\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{it_2(m_2+1-r_2)}dt_2}{(m-v_1+1)!(m-v_2+1)!}$$

$$+\sum_{v_1=1}^{m-\mu}\sum_{v_2=1}^{m-\nu}\sum_{m_1=0}^{m-v_1}\sum_{m_2=0}^{+\infty}\frac{u^{v_1,v_2}_{m_1,(m_2+(m-v_2+1))}\frac{1}{2\pi}\int_0^{2\pi}e^{it_1(m_1-m+v_1)}dt_1\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{it_2(m_2+1-r_2)}dt_2}{(m-v_1+1)!(m-v_2+1)!}$$

$$+\sum_{v_1=1}^{m-\mu}\sum_{v_2=1}^{m-\nu}\sum_{m_1=0}^{+\infty}\sum_{m_2=0}^{m-v_2}\frac{u^{v_1,v_2}_{(m_1+(m-v_1+1)),m_2}\frac{1}{2\pi}\int_0^{2\pi}e^{it_1(m_1+1)}dt_1\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{it_2(m_2-m+v_2-r_2)}dt_2}{(m-v_1+1)!(m-v_2+1)!}\Big]$$

$$+\frac{\lambda+1}{4\lambda}\Big[\sum_{m_1,m_2=1}^{+\infty}\frac{\overline{u^{0,0}_{(m_1-1),(m_2-1)}}}{m_1m_2}\frac{1}{2\pi}\int_0^{2\pi}e^{it_1(1-m_1)}dt_1\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{it_2(1-r_2-m_2)}dt_2$$

$$
\begin{aligned}
&+\sum_{m_1,m_2=0}^{+\infty}\sum_{v_1=1}^{m-\mu}\sum_{v_2=1}^{m-\nu}\frac{\frac{1}{2\pi}\int_0^{2\pi}e^{-it_1(m_1-m+v_1)}dt_1\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{-it_2(m_2-m+v_2+r_2)}dt_2}{(m-v_1)!(m-v_2)!}\\
&\cdot\frac{\overline{u_{m_1,m_2}^{v_1,v_2}}}{(m_1+1)(m_2+1)}\Big]+\sum_{m_1,m_2=0}^{+\infty}b_{m_1,m_2}\frac{1}{2\pi}\int_0^{2\pi}e^{it_1(m_1+1)}dt_1\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{it_2(m_2+1-r_2)}dt_2\\
&=\frac{\lambda-1}{4\lambda}\Big[u_{0,r_2}^{0,0}+\sum_{v_1=1}^{m-\mu}\sum_{v_2=1}^{m-\nu}\frac{u_{(m-v_1),(m-v_2+r_2)}^{v_1,v_2}}{(m-v_1+1)!(m-v_2+1)!}\Big]\\
&+\frac{\lambda+1}{4\lambda}\begin{cases}\displaystyle\sum_{v_1=1}^{m-\mu}\sum_{v_2=1}^{m-\nu}\frac{1}{(m-v_1+1)!(m-v_2)!}\frac{\overline{u_{(m-v_1),(m-v_2-r_2)}^{v_1,v_2}}}{m-v_2-r_2+1}, & 1\le r_2\le\nu,\\ \displaystyle\sum_{v_1=1}^{m-\mu}\sum_{v_2=1}^{m-r_2}\frac{1}{(m-v_1+1)!(m-v_2)!}\frac{\overline{u_{(m-v_1),(m-v_2-r_2)}^{v_1,v_2}}}{m-v_2-r_2+1}, & \nu<r_2\le m-1,\\ 0, \quad r_2>m-1,\end{cases}
\end{aligned}
$$

and the last equation is due to

$$
\begin{aligned}
&\sum_{m_1,m_2=0}^{+\infty}\sum_{v_1=1}^{m-\mu}\sum_{v_2=1}^{m-\nu}\frac{\frac{1}{2\pi}\int_0^{2\pi}e^{-it_1(m_1-m+v_1)}dt_1\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{-it_2(m_2-m+v_2+r_2)}dt_2}{(m-v_1)!(m-v_2)!}\frac{\overline{u_{m_1,m_2}^{v_1,v_2}}}{(m_1+1)(m_2+1)}\\
&=\begin{cases}\displaystyle\sum_{v_1=1}^{m-\mu}\sum_{v_2=1}^{m-\nu}\frac{1}{(m-v_1+1)!(m-v_2)!}\frac{\overline{u_{(m-v_1),(m-v_2-r_2)}^{v_1,v_2}}}{m-v_2-r_2+1}, & 1\le r_2\le\nu,\\ \displaystyle\sum_{v_1=1}^{m-\mu}\sum_{v_2=1}^{m-r_2}\frac{1}{(m-v_1+1)!(m-v_2)!}\frac{\overline{u_{(m-v_1),(m-v_2-r_2)}^{v_1,v_2}}}{m-v_2-r_2+1}, & \nu<r_2\le m-1,\\ 0, \quad r_2>m-1.\end{cases}
\end{aligned}
$$

Therefore, for $r_2\ge 1$,

$$
\begin{aligned}
u_{0,r_2}^{0,0}=&\frac{4\lambda}{\lambda-1}\frac{1}{(2\pi)^2}\int_0^{2\pi}e^{it_1}\int_0^{2\pi}e^{it_2(1-r_2)}\varphi(e^{it_1},e^{it_2})dt_2dt_1-\sum_{v_1=1}^{m-\mu}\sum_{v_2=1}^{m-\nu}\frac{u_{(m-v_1),(m-v_2+r_2)}^{v_1,v_2}}{(m-v_1+1)!(m-v_2+1)!}\\
&-\frac{\lambda+1}{\lambda-1}\begin{cases}\displaystyle\sum_{v_1=1}^{m-\mu}\sum_{v_2=1}^{m-\nu}\frac{1}{(m-v_1+1)!(m-v_2)!}\frac{\overline{u_{(m-v_1),(m-v_2-r_2)}^{v_1,v_2}}}{m-v_2-r_2+1}, & 1\le r_2\le\nu,\\ \displaystyle\sum_{v_1=1}^{m-\mu}\sum_{v_2=1}^{m-r_2}\frac{1}{(m-v_1+1)!(m-v_2)!}\frac{\overline{u_{(m-v_1),(m-v_2-r_2)}^{v_1,v_2}}}{m-v_2-r_2+1}, & \nu<r_2\le m-1,\\ 0, \quad r_2>m-1.\end{cases}
\end{aligned}
\tag{3.12}
$$

For $r_1\ge 1$, multiplying both sides of the equation (3.11) by $e^{i(1-r_1)t_1}e^{it_2}$ and then integrating with respect to $t_1, t_2 \in [0, 2\pi]$, similar to (3.12), yields that

$$u_{r_1,0}^{0,0} = \frac{4\lambda}{\lambda-1}\frac{1}{(2\pi)^2}\int_0^{2\pi} e^{i(1-r_1)t_1}\int_0^{2\pi} e^{it_2}\varphi(e^{it_1}, e^{it_2})dt_2dt_1 - \sum_{v_1=1}^{m-\mu}\sum_{v_2=1}^{m-\nu}\frac{u_{(m-v_1+r_1),(m-v_2)}^{v_1,v_2}}{(m-v_1+1)!(m-v_2+1)!}$$
$$-\frac{\lambda+1}{\lambda-1}\begin{cases}\sum_{v_1=1}^{m-\mu}\sum_{v_2=1}^{m-\nu}\frac{1}{(m-v_1)!(m-v_2+1)!}\frac{\overline{u_{(m-v_1-r_1,(m-v_2))}^{v_1,v_2}}}{m-v_1-r_1+1}, & 1\le r_1\le\mu,\\ \sum_{v_1=1}^{m-r_1}\sum_{v_2=1}^{m-\nu}\frac{1}{(m-v_1)!(m-v_2+1)!}\frac{\overline{u_{(m-v_1-r_1,(m-v_2))}^{v_1,v_2}}}{m-v_1-r_1+1}, & \mu<r_1\le m-1,\\ 0, \quad r_1>m-1.\end{cases} \tag{3.13}$$

(iii) In addition, integrating both sides of the Eq (3.11) with respect to $t_1, t_2 \in [0, 2\pi]$ yields that

$$\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\varphi(e^{it_1}, e^{it_2})dt_2dt_1$$
$$=\frac{\lambda-1}{4\lambda}\Big[\sum_{m_2=0}^{+\infty}u_{0,m_2}^{0,0}\frac{1}{2\pi}\int_0^{2\pi}e^{it_1}dt_1\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{it_2(m_2-1)}dt_2$$
$$+\sum_{m_1=1}^{+\infty}u_{m_1,0}^{0,0}\frac{1}{2\pi}\int_0^{2\pi}e^{it_1(m_1-1)}dt_1\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{-it_2}dt_2$$
$$+\sum_{m_1,m_2=1}^{+\infty}u_{m_1,m_2}^{0,0}\frac{1}{2\pi}\int_0^{2\pi}e^{it_1(m_1-1)}dt_1\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{it_2(m_2-1)}dt_2$$
$$+\sum_{v_1=1}^{m-\mu}\sum_{v_2=1}^{m-\nu}\sum_{m_1=0}^{m-v_1}\sum_{m_2=0}^{m-v_2}\frac{u_{m_1,m_2}^{v_1,v_2}\frac{1}{2\pi}\int_0^{2\pi}e^{it_1(m_1+v_1-1-m)}dt_1\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{it_2(m_2+v_2-1-m)}dt_2}{(m-v_1+1)!(m-v_2+1)!}$$
$$+\sum_{v_1=1}^{m-\mu}\sum_{v_2=1}^{m-\nu}\sum_{m_1,m_2=0}^{+\infty}\frac{u_{(m_1+(m-v_1+1)),(m_2+(m-v_2+1))}^{v_1,v_2}\frac{1}{2\pi}\int_0^{2\pi}e^{im_1t_1}dt_1\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{im_2t_2}dt_2}{(m-v_1+1)!(m-v_2+1)!}$$
$$+\sum_{v_1=1}^{m-\mu}\sum_{v_2=1}^{m-\nu}\sum_{m_1=0}^{m-v_1}\sum_{m_2=0}^{+\infty}\frac{u_{m_1,(m_2+(m-v_2+1))}^{v_1,v_2}\frac{1}{2\pi}\int_0^{2\pi}e^{it_1(m_1+v_1-1-m)}dt_1\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{im_2t_2}dt_2}{(m-v_1+1)!(m-v_2+1)!}$$
$$+\sum_{v_1=1}^{m-\mu}\sum_{v_2=1}^{m-\nu}\sum_{m_1=0}^{+\infty}\sum_{m_2=0}^{m-v_2}\frac{u_{(m_1+(m-v_1+1)),m_2}^{v_1,v_2}\frac{1}{2\pi}\int_0^{2\pi}e^{im_1t_1}dt_1\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{it_2(m_2+v_2-1-m)}dt_2}{(m-v_1+1)!(m-v_2+1)!}\Big]$$
$$+\frac{\lambda+1}{4\lambda}\Big[\sum_{m_1,m_2=1}^{+\infty}\frac{\overline{u_{(m_1-1),(m_2-1)}^{0,0}}}{m_1m_2}\frac{1}{2\pi}\int_0^{2\pi}e^{-im_1t_1}dt_1\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{-im_2t_2}dt_2$$
$$+\sum_{m_1,m_2=0}^{+\infty}\sum_{v_1=1}^{m-\mu}\sum_{v_2=1}^{m-\nu}\frac{\frac{1}{2\pi}\int_0^{2\pi}e^{-it_1(m_1+1-m+v_1)}dt_1\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{-it_2(m_2+1-m+v_2)}dt_2}{(m-v_1)!(m-v_2)!}$$
$$\cdot\frac{\overline{u_{m_1,m_2}^{v_1,v_2}}}{(m_1+1)(m_2+1)}\Big]+\sum_{m_1,m_2=0}^{+\infty}b_{m_1,m_2}\frac{1}{2\pi}\int_0^{2\pi}e^{im_1t_1}dt_1\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{im_2t_2}dt_2$$

$$= \frac{\lambda-1}{4\lambda}\Big[u_{1,1}^{0,0} + \sum_{\nu_1=1}^{m-\mu}\sum_{\nu_2=1}^{m-\nu}\frac{u_{(m-\nu_1+1),(m-\nu_2+1)}^{\nu_1,\nu_2}}{(m-\nu_1+1)!(m-\nu_2+1)!}\Big]$$
$$+ \frac{\lambda+1}{4\lambda}\sum_{\nu_1=1}^{m-\mu}\sum_{\nu_2=1}^{m-\nu}\frac{1}{(m-\nu_1)!(m-\nu_2)!}\frac{\overline{u_{(m-\nu_1-1),(m-\nu_2-1)}^{\nu_1,\nu_2}}}{(m-\nu_1)(m-\nu_2)} + b_{0,0},$$

which follows (3.5), where $u_{1,1}^{0,0}$ is determined by (3.1).

(iv) In the case of $r_1, r_2 \geq m+1$, multiplying both sides of the Eq (3.11) by $e^{ir_1t_1}e^{ir_2t_2}$, and then integrating with respect to $t_1, t_2 \in [0, 2\pi]$ yields that

$$\frac{1}{(2\pi)^2}\int_0^{2\pi} e^{ir_1t_1}\Big[\int_0^{2\pi} e^{ir_2t_2}\varphi(e^{it_1}, e^{it_2})dt_2\Big]dt_1$$
$$= \frac{\lambda-1}{4\lambda}\Big[\sum_{m_2=0}^{+\infty} u_{0,m_2}^{0,0}\frac{1}{2\pi}\int_0^{2\pi} e^{it_1(r_1-1)}dt_1 \cdot \frac{1}{2\pi}\int_0^{2\pi} e^{it_2(m_2-1+r_2)}dt_2$$
$$+ \sum_{m_1=1}^{+\infty} u_{m_1,0}^{0,0}\frac{1}{2\pi}\int_0^{2\pi} e^{it_1(m_1-1+r_1)}dt_1 \cdot \frac{1}{2\pi}\int_0^{2\pi} e^{it_2(-1+r_2)}dt_2$$
$$+ \sum_{m_1,m_2=1}^{+\infty} u_{m_1,m_2}^{0,0}\frac{1}{2\pi}\int_0^{2\pi} e^{it_1(m_1-1+r_1)}dt_1 \cdot \frac{1}{2\pi}\int_0^{2\pi} e^{it_2(m_2-1+r_2)}dt_2$$
$$+ \sum_{\nu_1=1}^{m-\mu}\sum_{\nu_2=1}^{m-\nu}\sum_{m_1=0}^{m-\nu_1}\sum_{m_2=0}^{m-\nu_2}\frac{u_{m_1,m_2}^{\nu_1,\nu_2}\frac{1}{2\pi}\int_0^{2\pi} e^{it_1(m_1-(m-\nu_1+1)+r_1)}dt_1 \cdot \frac{1}{2\pi}\int_0^{2\pi} e^{it_2(m_2-(m-\nu_2+1)+r_2)}dt_2}{(m-\nu_1+1)!(m-\nu_2+1)!}$$
$$+ \sum_{\nu_1=1}^{m-\mu}\sum_{\nu_2=1}^{m-\nu}\sum_{m_1,m_2=0}^{+\infty}\frac{u_{(m_1+(m-\nu_1+1)),(m_2+(m-\nu_2+1))}^{\nu_1,\nu_2}\frac{1}{2\pi}\int_0^{2\pi} e^{it_1(m_1+r_1)}dt_1 \cdot \frac{1}{2\pi}\int_0^{2\pi} e^{it_2(m_2+r_2)}dt_2}{(m-\nu_1+1)!(m-\nu_2+1)!}$$
$$+ \sum_{\nu_1=1}^{m-\mu}\sum_{\nu_2=1}^{m-\nu}\sum_{m_1=0}^{m-\nu_1}\sum_{m_2=0}^{+\infty}\frac{u_{m_1,(m_2+(m-\nu_2+1))}^{\nu_1,\nu_2}\frac{1}{2\pi}\int_0^{2\pi} e^{it_1(m_1-(m-\nu_1+1)+r_1)}dt_1 \cdot \frac{1}{2\pi}\int_0^{2\pi} e^{it_2(m_2+r_2)}dt_2}{(m-\nu_1+1)!(m-\nu_2+1)!}$$
$$+ \sum_{\nu_1=1}^{m-\mu}\sum_{\nu_2=1}^{m-\nu}\sum_{m_1=0}^{+\infty}\sum_{m_2=0}^{m-\nu_2}\frac{u_{(m_1+(m-\nu_1+1)),m_2}^{\nu_1,\nu_2}\frac{1}{2\pi}\int_0^{2\pi} e^{it_1(m_1+r_1)}dt_1 \cdot \frac{1}{2\pi}\int_0^{2\pi} e^{it_2(m_2-(m-\nu_2+1)+r_2)}dt_2}{(m-\nu_1+1)!(m-\nu_2+1)!}\Big]$$
$$+ \frac{\lambda+1}{4\lambda}\Big[\sum_{m_1,m_2=1}^{+\infty}\frac{\overline{u_{(m_1-1),(m_2-1)}^{0,0}}}{m_1m_2}\frac{1}{2\pi}\int_0^{2\pi} e^{it_1(-m_1+r_1)}dt_1 \cdot \frac{1}{2\pi}\int_0^{2\pi} e^{it_2(-m_2+r_2)}dt_2$$
$$+ \sum_{m_1,m_2=0}^{+\infty}\sum_{\nu_1=1}^{m-\mu}\sum_{\nu_2=1}^{m-\nu}\frac{\frac{1}{2\pi}\int_0^{2\pi} e^{-it_1(m_1+1-m+\nu_1-r_1)}dt_1 \cdot \frac{1}{2\pi}\int_0^{2\pi} e^{-it_2(m_2+1-m+\nu_2-r_2)}dt_2}{(m-\nu_1)!(m-\nu_2)!}$$
$$\cdot \frac{\overline{u_{m_1,m_2}^{\nu_1,\nu_2}}}{(m_1+1)(m_2+1)}\Big] + \sum_{m_1,m_2=0}^{+\infty} b_{m_1,m_2}\frac{1}{2\pi}\int_0^{2\pi} e^{it_1(m_1+r_1)}dt_1 \cdot \frac{1}{2\pi}\int_0^{2\pi} e^{it_2(m_2+r_2)}dt_2$$
$$= \frac{\lambda+1}{4\lambda}\Big[\frac{\overline{u_{(r_1-1),(r_2-1)}^{0,0}}}{r_1r_2} + \sum_{\nu_1=1}^{m-\mu}\sum_{\nu_2=1}^{m-\nu}\frac{1}{(m-\nu_1)!(m-\nu_2)!}\frac{\overline{u_{(m-\nu_1+r_1-1),(m-\nu_2+r_2-1)}^{\nu_1,\nu_2}}}{(m-\nu_1+r_1)(m-\nu_2+r_2)}\Big].$$

Therefore,

$$u^{0,0}_{r_1,r_2} = \frac{4\lambda}{\lambda+1}\frac{(r_1+1)(r_2+1)}{(2\pi)^2}\int_0^{2\pi} e^{-i(r_1+1)t_1}\int_0^{2\pi} e^{-i(r_2+1)t_2}\overline{\varphi(e^{it_1}, e^{it_2})}dt_2dt_1 \\ -\sum_{v_1=1}^{m-\mu}\sum_{v_2=1}^{m-\nu}\frac{(r_1+1)(r_2+1)}{(m-v_1)!(m-v_2)!}\frac{u^{v_1,v_2}_{(m-v_1+r_1),(m-v_2+r_2)}}{(m-v_1+r_1+1)(m-v_2+r_2+1)}, \tag{3.14}$$

for $r_1, r_2 \geq m$.

Similarly, in the case of $r_1 \geq m+1$ and $0 \leq r_2 \leq m-2$, multiplying both sides of the Eq (3.11) by $e^{ir_1t_1}e^{it_2(m-r_2)}$, and then integrating with respect to $t_1, t_2 \in [0, 2\pi]$ yields that $u^{0,0}_{r_1,r_2}$ $(r_1 \geq m,\ 1 \leq r_2 \leq m-1)$ has the same representation as (3.14). In the case of $0 \leq r_1 \leq m-2$ and $r_2 \geq m+1$, multiplying both sides of the Eq (3.11) by $e^{it_1(m-r_1)}e^{ir_2t_2}$, and then integrating with respect to $t_1, t_2 \in [0, 2\pi]$ yields that $u^{0,0}_{r_1,r_2}$ $(1 \leq r_1 \leq m-1,\ r_2 \geq m)$ has the same representation as (3.14).

(v) Similarly, for $r_1, r_2 \geq 1$, multiplying both sides of the Eq (3.11) by $e^{-ir_1t_1}e^{-ir_2t_2}$, and then integrating with respect to $t_1, t_2 \in [0, 2\pi]$, we can get the expression of $b_{r_1r_2}$ which leads to (3.7). For $r_1 \geq 1$, multiplying both sides of the Eq (3.11) by $e^{-ir_1t_1}$, and then integrating with respect to $t_1, t_2 \in [0, 2\pi]$ yields the expression of $b_{r_10}$ which leads to (3.8). For $r_2 \geq 1$, we can get $b_{0r_2}$ which leads to (3.9).

**Remark 3.2.** The results obtained in this article extend the existing conclusions about ployanalytic functions and bi-ployanalytic functions. On this basis, we can study other partial differential equation problems. For example, it would be interesting to discuss whether bi-polyanalytic or even ployanalytical function solutions exist for some nonlocal integrable partial differential equations (see, e.g., [24]), which need to explore the solutions to the corresponding Riemann-Hilbert problems. Additionally, the non-existence of solutions to Cauchy problems on the real line for first-order nonlocal differential equations (see, e.g., [25]) indicates that we can attempt to discuss the generalizations of analytical solutions for partial differential equation problems.

## 4. Conclusions

With the help of the series expansion of polyanalytic functions, and applying the properties of Cauchy kernels on the bicylinder and the unit disk, we first discuss a class of Schwarz problems with the conditions concerning the real and imaginary parts of high-order partial differentiation for polyanalytic functions on the bicylinder. On this basis, we investigate a type of boundary value problem for bi-polyanalytic functions with Dirichlet boundary conditions on the bicylinder. From the perspective of series, we obtain the specific representation of the solution to the Dirichlet problem. The method used in this article, with the help of series expansion, is different from the previous methods for solving boundary value problems. It is a very effective method and can be used to solve other types of problems regarding complex partial differential equations of bi-polyanalytic functions in high-dimensional complex spaces. The conclusions of this article also lay a necessary foundation for further research on polyanalytic and bi-polyanalytic functions.

## Author contributions

Yanyan Cui: conceptualization, project administration, writing original, writing-review and editing; Chaojun Wang: investigation, writing original, writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

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## Conflict of interest

The authors declare no conflict of interest.

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