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Research article

Existence and stability results for a coupled system of Hilfer-Hadamard sequential fractional differential equations with multi-point fractional integral boundary conditions

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Abstract: In this paper, we study the existence and uniqueness of solutions for a coupled system of Hilfer-Hadamard sequential fractional differential equations with multi-point Riemann-Liouville fractional integral boundary conditions via standard fixed point theorems. The existence of solutions is proved using Krasnoselskii's fixed point theorem, while the existence and uniqueness of solutions is established using the Banach fixed point theorem. We also discuss the stability of the problem in terms of Ulam-Hyers, Ulam-Hyers-Rassias, generalized Ulam-Hyers, and generalized Ulam-Hyers-Rassias stability. As an application, some examples are presented to illustrate our theoretical results.

Keywords: fractional differential equations; Hilfer-Hadamard fractional derivative; boundary value problems; fixed point theory; Ulam-Hyers-Rassias stability
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1. Introduction

Fractional differential equations (FDEs) extend the concept of classical integer-order differential equations to non-integer orders, providing a powerful framework for modeling complex systems exhibiting memory and hereditary properties. The origins of fractional calculus can be traced back to the late of 17th century when Gottfried Wilhelm Leibniz and Guillaume de L'Hôpital exchanged letters discussing the possibility of derivatives of non-integer order. Over the centuries, the theory of fractional calculus has been developed and formalized by mathematicians such as Joseph Liouville, Bernhard Riemann, Hermann Weyl, and Marcel Riesz, among others.

In the modern era, fractional calculus has found applications in various scientific and engineering disciplines. Unlike classical differential equations, FDEs are capable of capturing the dynamics of processes that exhibit anomalous diffusion, non-local behavior, and long-range temporal correlations. This makes them particularly suitable for modeling phenomena in fields such as physics, control theory, biology, finance, and engineering; see the monographs [1–6].

The mathematical foundation of FDEs involves several definitions of fractional derivatives and integrals, each suited for different types of problems. The most commonly used definitions include the Riemann-Liouville, Caputo, and Hadamard fractional derivatives. These derivatives are integral operators that generalize the concept of differentiation to fractional orders, providing a flexible tool for describing the evolution of systems over time. Within this framework, several types of fractional derivatives have been developed, each tailored to capture different aspects of these systems. Among these, the Hilfer-Hadamard fractional derivative is a new fractional derivative introduced in 2012 by M. D. Qasim [7]. It has emerged as a notable concept, blending the features of the Hilfer and Hadamard fractional derivatives. The unique combination of Hilfer and Hadamard characteristics makes it particularly versatile for a wide range of applications. As research in this area continues to evolve, the Hilfer-Hadamard fractional derivative is expected to play an increasingly important role in the mathematical modeling of complex systems.

The study of existence and uniqueness of solutions to FDEs is crucial, as it ensures that the models are mathematically well-posed and their solutions are reliable for practical applications. These properties are typically established using fixed point theorems, which are foundational tools in functional analysis; see the monographs [8–10]. The stability analysis of FDEs is essential to understand how solutions behave under small perturbations, which is crucial for the robustness of models in real-world applications. Various concepts of stability have been developed, each suited to different types of perturbations and scenarios. In the context of FDEs, concepts such as Lyapunov stability, asymptotic stability, Mittag-Leffler stability, Ulam-Hyers stability, and Ulam-Hyers-Rassias stability are commonly used, see [11–17] and references cited therein.

The research since 2017 has focused on exploring the theoretical properties of the Hilfer-Hadamard fractional derivative and its applications to differential equations. The study of the existence and uniqueness of solutions to differential equations involving this derivative has been a vibrant area of investigation. Researchers have also examined the stability of such systems, employing various fixed point theorems and stability concepts, including Ulam-Hyers stability, Ulam-Hyers-Rassias stability, and their generalizations. For more detailed discussions, see [18–21].

Research on sequential fractional differential equations (FDEs) with boundary value problems has made significant progress, primarily focusing on single sequential FDEs with various types of boundary conditions regarding the existence and uniqueness of solutions. Notable examples can be found in the monographs [22–26]. These studies utilize a range of mathematical techniques and contribute to a broader understanding of the applications of fractional calculus in various fields, including physics, engineering, and applied mathematics.

Among the various applications of FDEs, coupled systems involving sequential fractional differential equations with boundary value problems have also garnered significant attention due to their ability to describe complex interactions between multiple components or variables within a system. Recent results on this topic are limited and include the following.

In 2018, Zada et al. [27] studied the nonlinear sequential coupled system of Caputo fractional differential equations with Riemann-Liouville fractional integral boundary conditions of the form

$$\begin{cases} ({}^cD^q + k{}^cD^{q-1})x(t) = f(t, x(t), y(t)), \quad t \in [0, T],\ 2 < q \leq 3, \\ ({}^cD^p + k{}^cD^{p-1})y(t) = g(t, x(t), y(t)), \quad t \in [0, T],\ 2 < p \leq 3, \\ x(0) = 0,\ \ x(T) = \displaystyle\sum_{i=1}^{n} \alpha_i I^{\rho_i} y(\eta_i), \\ y(0) = 0,\ \ y(T) = \displaystyle\sum_{j=1}^{m} \beta_j I^{\gamma_j} y(\theta_j), \end{cases}$$

where ${}^cD^{(.)}$ denotes the caputo fractional derivatives of order p and q, $I^{(.)}$ denotes the Riemann-Liouville fractional integral of order ρ_i and $\gamma_j > 0$, $\eta_i, \theta_j \in (0, T)$, $k \in \mathbb{R}^+$, $f, g : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$, and $\alpha_i, \beta_j \in \mathbb{R}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$ are real constants such that

$$\sum_{i=1}^{n} \frac{\alpha_i \eta_i^{\rho_i+1}}{\Gamma(\rho_i + 2)} . \sum_{j=1}^{m} \frac{\beta_j \theta_j^{\gamma_j+1}}{\Gamma(\gamma_j + 2)} \neq T^2.$$

The authors in this paper demonstrated the existence of solutions by applying the Leray-Schauder alternative criterion, while uniqueness is established using the Banach fixed point theorem. They also presented the Ulam-Hyers stability of the mentioned system.

A few years later, in 2022, Zada and Yar [28] studied the existence, uniqueness, and Ulam-Hyers stability of a sequential coupled system of Hadamard fractional differential equations with nonlocal Hadamard fractional integral boundary conditions as follows:

$$\begin{cases} (D^q + kD^{q-1})u(t) = f(t, u(t), v(t)), \quad k > 0,\ 1 < q \leq 2,\ t \in (1, e), \\ (D^p + kD^{p-1})v(t) = g(t, u(t), v(t)), \quad k > 0,\ 1 < p \leq 2,\ t \in (1, e), \\ u(1) = 0,\ \ \displaystyle\sum_{i=1}^{m} \lambda_i I^{\alpha_i} u(\eta_i) = \sum_{j=1}^{n} \mu_j (I^{\beta_j} u(e) - I^{\beta_j} u(\xi_j)), \\ v(1) = 0,\ \ \displaystyle\sum_{i=1}^{m} \rho_i I^{\gamma_i} v(\theta_i) = \sum_{j=1}^{n} \kappa_j (I^{\delta_j} v(e) - I^{\delta_j} v(\zeta_j)), \end{cases}$$

where $D^{(.)}$ denotes Hadamard fractional derivative of order p and q, $f, g : [1, e] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions, $\eta_i, \theta_i, \xi_j, \zeta_j \in (1, e)$, $\lambda_i, \rho_i, \mu_j, \kappa_j \in \mathbb{R}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, and $I^{(\phi)}$ is the Hadamard fractional integral of order $\phi > 0, \phi = \alpha_i, \gamma_i, \beta_j, \delta_j$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$. The existence of solutions was derived from Leray-Schauder's alternative, whereas the uniqueness was established by the Banach fixed point theorem.

In the same year, Agarwal et al. [29] studied the existence and Ulam-Hyers stability results for a fully coupled system of nonlinear sequential Hilfer fractional differential equations of the form

$$\begin{cases} ({}_HD^{\alpha_1,\beta_1} - \sigma_1 {}_HD^{\alpha_1-1,\beta_1})x(t) = f_1(t, x(t), y(t)), \quad t \in [a, b], \quad a \geq 0, \\ ({}_HD^{\alpha_2,\beta_2} - \sigma_2 {}_HD^{\alpha_2-1,\beta_2})y(t) = f_2(t, x(t), y(t)), \quad t \in [a, b], \end{cases}$$

subject to integro-multistrip-multipoint boundary conditions

$$\begin{cases} x^{(k)}(a) = y^{(k)}(a) = 0, \;\; k = 0, 1, 2, \ldots, n-2, \\ \displaystyle\int_a^b x(s)ds = \sum_{i=2}^{p} \lambda_{i-1} \int_{\eta_{i-1}}^{\eta_i} y(s)ds + \sum_{j=1}^{q} \mu_j y(\varrho_j), \\ \displaystyle\int_a^b y(s)ds = \sum_{i=2}^{p} \lambda_{i-1} \int_{\eta_{i-1}}^{\eta_i} x(s)ds + \sum_{j=1}^{q} \mu_j x(\varrho_j), \end{cases}$$

where ${}_HD^{\alpha_i,\beta_i}$ is the Hilfer fractional derivative operator of order α_i, $n-1 < \alpha_i \leq n$, $n \geq 3$, and type β_i, $0 \leq \beta_i \leq 1$, $i = 1, 2$, and $f_i : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ are continuous functions, $a < \eta_1 < \eta_2 < \cdots < \eta_p < \varrho_1 < \varrho_2 < \cdots < \varrho_q < b$, $\lambda_{i-1}, \mu_j > 0$, $i = 2, 3, \ldots, p$, $j = 1, 2, \ldots, q$ with $p, q \in \mathbb{N}$, and $\sigma_1, \sigma_2 > 0$. The authors applied the standard fixed point theorems due to Banach and Krasnoselskii, as well as the Leray-Schauder nonlinear alternative, to establish the existence and uniqueness of the solution. The paper also included a study of Ulam-Hyers stability for the given problem.

Very recently, in 2024, Sompong et al. [30] changed a coupled system of sequential Hilfer fractional differential equations from the aforementioned paper to a coupled system of sequential Hilfer-Hadamard fractional differential equations

$$\begin{cases} ({}_HD_{1^+}^{\alpha_1,\beta_1} + k_1 \, {}_HD_{1^+}^{\alpha_1-1,\beta_1})u(t) = f(t, u(t), v(t)), \;\; 1 < \alpha_1 \leq 2, \;\; t \in [1, e], \\ ({}_HD_{1^+}^{\alpha_2,\beta_2} + k_2 \, {}_HD_{1^+}^{\alpha_2-1,\beta_2})v(t) = g(t, u(t), v(t)), \;\; 1 < \alpha_2 \leq 2, \;\; t \in [1, e], \end{cases}$$

with four-point coupled boundary conditions

$$\begin{aligned} u(1) = 0, \quad u(e) = \lambda v(\theta), \;\; 1 < \theta < e, \\ v(1) = 0, \quad v(e) = \mu u(\eta), \;\; 1 < \eta < e, \end{aligned}$$

where ${}_HD_{1^+}^{\alpha_i,\beta_i}$ denotes the Hilfer-Hadamard fractional derivatives of order $\alpha_i \in (1, 2]$ and type $\beta_i \in [0, 1]$ for $i \in \{1, 2\}$, $k_1, k_2 \in \mathbb{R}^+$, $f, g : [1, e] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, and λ and μ are real constants.

Motivated by the research going on in this direction, in the present paper we extend the results of [30] to a coupled system of sequential Hilfer-Hadamard fractional differential equations

$$\begin{cases} ({}_HD_{1^+}^{\alpha_1,\beta_1} + k_1 {}_HD_{1^+}^{\alpha_1-1,\beta_1})u(t) = f(t, u(t), v(t)), \quad t \in [1, e], \\ ({}_HD_{1^+}^{\alpha_2,\beta_2} + k_2 {}_HD_{1^+}^{\alpha_2-1,\beta_2})v(t) = g(t, u(t), v(t)), \quad t \in [1, e], \end{cases} \tag{1.1}$$

with multi-point Riemann-Liouville fractional integral boundary conditions

$$\begin{aligned} u(1) = 0, \quad u(e) = \sum_{i=1}^{n} \lambda_i I_{1^+}^{\delta_i} v(\theta_i) = \sum_{i=1}^{n} \frac{\lambda_i}{\Gamma(\delta_i)} \int_1^{\theta_i} (\theta_i - s)^{\delta_i - 1} v(s)ds, \\ v(1) = 0, \quad v(e) = \sum_{j=1}^{n} \mu_j I_{1^+}^{\sigma_j} u(\eta_j) = \sum_{j=1}^{n} \frac{\mu_j}{\Gamma(\sigma_j)} \int_1^{\eta_j} (\eta_j - s)^{\sigma_j - 1} u(s)ds, \end{aligned} \tag{1.2}$$

where ${}_{H}D_{1^+}^{\alpha_i,\beta_i}$ denotes the Hilfer-Hadamard fractional derivatives of order $\alpha_i \in (1,2]$ and type $\beta_i \in [0,1]$ for $i \in \{1,2\}, k_1, k_2 \in \mathbb{R}_+$, and $f, g : [1,e] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions. I^{δ_i} and I^{σ_j} are the Riemann-Liouville fractional integral of positive order θ_i and $\eta_j \in (1,e)$, and $\lambda_i, \mu_j, 1 \leq i, j \leq n$ are real constants. Note that the Hilfer-Hadamard fractional derivative is viewed as interpolating the Hadamard fractional derivative and the Caputo-Hadamard fractional derivative. In this paper, we focus on the special case of order $\alpha_i \in (1,2]$ as most applications involve derivatives of no more than second-order. Another important highlight of this research are the boundary conditions of the problem, which are multi-point Riemann-Liouville fractional integral boundary conditions that make it applicable to a broader class of mathematical models.

Fractional differential equations continue to be a vibrant area of research, with ongoing developments enhancing our understanding and expanding their applicability. This paper contributes to this growing body of knowledge by addressing key aspects of existence, uniqueness, and stability in the context of fractional integral boundary conditions. The existence and uniqueness of solutions are established using fixed point theorems such as Krasnoselskii's and the Banach fixed point theorem. Ulam-Hyers, Ulam-Hyers-Rassias, generalized Ulam-Hyers, and generalized Ulam-Hyers-Rassias stability are also discussed for the system of Hilfer-Hadamard sequential fractional differential equations (1.1) with conditions (1.2). Illustrative examples are provided.

The remaining structure of this paper is as follows: In Section 2, we recall some definitions, notations, and theorems needed for our proofs. The main results regarding existence and uniqueness are presented in Section 3. The stability results in the sense of Ulam-Hyers, generalized Ulam-Hyers, Ulam-Hyers-Rassias, and generalized Ulam-Hyers-Rassias are discussed in Section 4. Examples illustrating the main results are provided in Section 5.

2. Preliminaries

In this section, we present some fundamental definitions and theorems. Let $L^1[a,b]$ be the Banach space of an Lebesgue integrable function. We consider $AC[a,b]$, the space of absolutely continuous function on the interval $[a,b]$, and $AC_\delta^n[a,b]$ is the space of n-times $\delta-$differentiable absolutely continuous functions on the interval $[a,b]$, as follows

$$AC[a,b] = \left\{ f : f(t) = c + \int_a^t \varphi(\tau)d\tau,\ c \in \mathbb{R},\ \varphi \in L^1[a,b] \right\},$$
$$AC_\delta^n[a,b] = \left\{ f : [a,b] \to \mathbb{R} : \delta^{(n-1)} f(t) \in AC[a,b] \right\},$$

where δ is the Euler operator $t\frac{d}{dt}$.

Definition 2.1. (The Riemann-Liouville fractional integral [2]) *The Riemann-Liouville integral of order $\alpha > 0$ of a function $f : [a,\infty) \to \mathbb{R}$ is defined as*

$$I_{a^+}^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \quad t > a.$$

Definition 2.2. (Hadamard fractional integral [2]) *The Hadamard fractional integral of order $\alpha \in \mathbb{R}_+$ for a function $f : [a,\infty) \to \mathbb{R}$ is defined as*

$${}_{H}I_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \frac{f(\tau)}{\tau} d\tau, \quad t > a,$$

provided the integral exists, where $\log(.) = \log_e(.)$.

Definition 2.3. (Hadamard fractional derivative [2]) *The Hadamard fractional derivative of order* $\alpha > 0$, *applied to the function* $f : [a, \infty) \to \mathbb{R}$, *is defined as*

$$ {}_H D^{\alpha}_{a^+} f(t) = \delta^n ({}_H I^{n-\alpha}_{a^+} f(t)), \quad n-1 < \alpha < n, \quad n = [\alpha] + 1, $$

where $\delta^n = (t\frac{d}{dt})^n$ *and* $[\alpha]$ *denotes the integer part of the real number* α.

Definition 2.4. (Hilfer-Hadamard fractional derivative [20, 31]) *Let* $n - 1 < \alpha \leq n,\ 0 \leq \beta \leq 1$, *and* $f \in L^1[a, b]$. *The Hilfer-Hadamard fractional derivative of order* α *and type* β *of* f *is defined as*

$$ \begin{aligned} ({}_H D^{\alpha,\beta}_{a^+} f)(t) &= ({}_H I^{\beta(n-\alpha)}_{a^+} \delta^n \ {}_H I^{(n-\alpha)(1-\beta)}_{a^+} f)(t) \\ &= ({}_H I^{\beta(n-\alpha)}_{a^+} \delta^n \ {}_H I^{n-\gamma}_{a^+} f)(t) \\ &= ({}_H I^{\beta(n-\alpha)}_{a^+} \ {}_H D^{\gamma}_{a^+} f)(t), \end{aligned} $$

where ${}_H I^{(.)}_{a^+}$, $\gamma = \alpha + n\beta - \alpha\beta$, *and* ${}_H D^{(.)}_{a^+}$ *are the Hadamard fractional integral and derivative defined by Definitions 2.2 and 2.3, respectively.*

The Hilfer-Hadamard fractional derivative may be viewed as interpolating the Hadamard fractional derivative and the Caputo-Hadamard fractional derivative. Indeed, for $\beta = 0$, this derivative reduces to the Hadamard fractional derivative, and when $\beta = 1$, we recover the Caputo-Hadamard fractional derivative.

We use some theorems of the Hadamard fractional integral and Hilfer-Hadamard fractional derivative by Kilbas [2].

Theorem 2.5. [2] *Let* $\alpha > 0$, $n = [\alpha] + 1$, *and* $0 < a < b < \infty$. *If* $f \in L^1[a, b]$ *and* $({}_H I^{n-\alpha}_{a+} f)(t) \in AC^n_\delta[a, b]$, *then*

$$ ({}_H I^{\alpha}_{a+} \ {}_H D^{\alpha}_{a+} f)(t) = f(t) - \sum_{j=0}^{n-1} \frac{(\delta^{(n-j-1)}({}_H I^{n-\alpha}_{a+} f))(a)}{\Gamma(\alpha - j)} \Big(\log \frac{t}{a} \Big)^{\alpha-j-1}. $$

Theorem 2.6. [20] *Let* $\alpha > 0$, $0 \leq \beta \leq 1$, $\gamma = \alpha + n\beta - \alpha\beta$, $n - 1 < \gamma \leq n$, $n = [\alpha] + 1$, *and* $0 < a < b < \infty$. *If* $f \in L^1[a, b]$ *and* $({}_H I^{n-\gamma}_{a+} f)(t) \in AC^n_\delta[a, b]$, *then*

$$ {}_H I^{\alpha}_{a+} \ ({}_H D^{\alpha,\beta}_{a+} f)(t) = \ {}_H I^{\gamma}_{a+} \ ({}_H D^{\gamma}_{a+} f)(t) = f(t) - \sum_{j=0}^{n-1} \frac{(\delta^{(n-j-1)}({}_H I^{n-\gamma}_{a+} f))(a)}{\Gamma(\gamma - j)} \Big(\log \frac{t}{a} \Big)^{\gamma-j-1}. $$

From this theorem, we notice that if $\beta = 0$, the formula reduces to the formula in the Theorem 2.5.

We will use the following well-known fixed point theorems in Banach space to prove the existence and uniqueness of solutions for the coupled system of the sequential Hilfer-Hadamard fractional differential problem.

Theorem 2.7. (Krasnoselskii's fixed point theorem [32]) *Let* Y *be a bounded, closed, convex, and nonempty subset of a Banach space* X. *Let* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *be the operators satisfying the conditions: (i)* $\mathcal{F}_1 y_1 + \mathcal{F}_2 y_2 \in Y$ *whenever* $y_1, y_2 \in Y$; *(ii)* $\mathcal{F}_1$ *is compact and continuous; (iii)* $\mathcal{F}_2$ *is a contraction mapping. Then, there exists* $y \in Y$ *such that* $y = \mathcal{F}_1 y + \mathcal{F}_2 y$.

Theorem 2.8. (Banach fixed point theorem [33]) *Let X be a Banach space, $D \subset X$, nonempty closed subset, and $\mathcal{F} : D \to D$ a strict contraction, i.e., there exists $k \in (0, 1)$ such that $\|\mathcal{F}x - \mathcal{F}y\| \leq k\|x - y\|$ for all $x, y \in D$. Then, $\mathcal{F}$ has a fixed point in D.*

In summary, Krasnoselskii's fixed point theorem is a generalization of the Banach fixed point theorem that allows for the sum of two operators, while Banach's theorem is a special case that deals with a single contraction mapping on a complete norm space.

In this paper, we also focus on the stability of the solution to the problem in the class defined by system (1.1) and conditions (1.2), ensuring it stays continuous despite changes to the equation while maintaining the boundary condition structure. We explore and analyze four types of Ulam stability: Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability, and generalized Ulam-Hyers-Rassias stability for the fractional differential problem (1.1)–(1.2).

Let $\epsilon_1, \epsilon_2 > 0$, $\alpha_1, \alpha_2 \in (1, 2]$, $\beta_1, \beta_2 \in [0, 1]$, $f, g : [1, e] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous functions, and $\varphi_1, \varphi_2 : [1, e] \to \mathbb{R}_+$. We consider the system of fractional differential problem (1.1)–(1.2) and the systems of inequalities

$$\begin{cases} |({}_HD^{\alpha_1,\beta_1}_{1^+} + k_1 {}_HD^{\alpha_1-1,\beta_1}_{1^+})x(t) - f(t, x(t), y(t))| \leq \epsilon_1, \quad t \in [1, e], \\ |({}_HD^{\alpha_2,\beta_2}_{1^+} + k_2 {}_HD^{\alpha_2-1,\beta_2}_{1^+})y(t) - g(t, x(t), y(t))| \leq \epsilon_2, \end{cases} \tag{2.1}$$

$$\begin{cases} |({}_HD^{\alpha_1,\beta_1}_{1^+} + k_1 {}_HD^{\alpha_1-1,\beta_1}_{1^+})x(t) - f(t, x(t), y(t))| \leq \varphi_1(t), \quad t \in [1, e], \\ |({}_HD^{\alpha_2,\beta_2}_{1^+} + k_2 {}_HD^{\alpha_2-1,\beta_2}_{1^+})y(t) - g(t, x(t), y(t))| \leq \varphi_2(t), \end{cases} \tag{2.2}$$

$$\begin{cases} |({}_HD^{\alpha_1,\beta_1}_{1^+} + k_1 {}_HD^{\alpha_1-1,\beta_1}_{1^+})x(t) - f(t, x(t), y(t))| \leq \epsilon_1\varphi_1(t), \quad t \in [1, e], \\ |({}_HD^{\alpha_2,\beta_2}_{1^+} + k_2 {}_HD^{\alpha_2-1,\beta_2}_{1^+})y(t) - g(t, x(t), y(t))| \leq \epsilon_2\varphi_2(t), \end{cases} \tag{2.3}$$

with multi-point Riemann-Liouville fractional integral boundary conditions

$$\begin{aligned} x(1) = 0, \quad x(e) &= \sum_{i=1}^{n} \lambda_i I^{\delta_i}_{1^+} y(\theta_i) = \sum_{i=1}^{n} \frac{\lambda_i}{\Gamma(\delta_i)} \int_1^{\theta_i} (\theta_i - s)^{\delta_i - 1} y(s) ds, \\ y(1) = 0, \quad y(e) &= \sum_{j=1}^{n} \mu_j I^{\sigma_j}_{1^+} x(\eta_j) = \sum_{j=1}^{n} \frac{\mu_j}{\Gamma(\sigma_j)} \int_1^{\eta_j} (\eta_j - s)^{\sigma_j - 1} x(s) ds, \end{aligned} \tag{2.4}$$

where $\theta_i, \eta_j \in (1, e)$, and λ_i, μ_j are real constant, for $1 \leq i, j \leq n$.

In the following Ulam stability definitions, we denote $Y = C^1([1, e], \mathbb{R})$. For a vector $\boldsymbol{v} = (v_1, v_2) > 0$, this means $v_1, v_2 > 0$.

Definition 2.9. (Ulam-Hyers stable [11]) *Problem (1.1)–(1.2) is Ulam-Hyers stable, if there exists a constant vector $\boldsymbol{c}_{f,g} = (c_f, c_g) > 0$ such that, for each $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2) > 0$ and for each solution $(x, y) \in Y \times Y$ of inequalities (2.1) with (2.4), there exists a solution $(u, v) \in Y \times Y$ of problem (1.1)–(1.2) satisfying*

$$\|(x, y) - (u, v)\| \leq \boldsymbol{c}_{f,g}\boldsymbol{\epsilon}^T, \quad t \in [1, e].$$

Definition 2.10. (Generalized Ulam-Hyers stable [11]) *Problem (1.1)–(1.2) is generalized Ulam-Hyers stable, if there exists a continuous vector function $\theta_{f,g} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ with $\theta_{f,g}(\mathbf{0}) = 0$ such that, for each solution $(x, y) \in Y \times Y$ of inequalities (2.1) with (2.4), there exists a solution $(u, v) \in Y \times Y$ of problem (1.1)–(1.2) satisfying*

$$\|(x, y) - (u, v)\| \leq \theta_{f,g}(\boldsymbol{\epsilon}), \quad t \in [1, e].$$

Definition 2.11. (Ulam-Hyers-Rassias stable [11]) *Problem (1.1)–(1.2) is Ulam-Hyer-Rassias stable with respect to* $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)$ *if there exists a constant vector* $\boldsymbol{c}_{f,g,\boldsymbol{\varphi}} = (c_{f,\varphi_1}, c_{g,\varphi_2}) > 0$ *such that for each* $\epsilon > 0$ *and for each solution* $(x, y) \in Y \times Y$ *of inequalities (2.3) with (2.4), there exists a solution* $(u, v) \in Y \times Y$ *of problem (1.1)–(1.2) satisfying*

$$\|(x, y) - (u, v)\| \leq \epsilon \boldsymbol{c}_{f,g,\boldsymbol{\varphi}}[\boldsymbol{\varphi}(t)]^T, \quad t \in [1, e].$$

Definition 2.12. (Generalized Ulam-Hyers-Rassias stable [11]) *Problem (1.1)–(1.2) is generalized Ulam-Hyer-Rassias stable with respect to* $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)$ *if there exists a constant vector* $\boldsymbol{c}_{f,g,\boldsymbol{\varphi}} = (c_{f,\varphi_1}, c_{g,\varphi_2}) > 0$ *such that, for each solution* $(x, y) \in Y \times Y$ *of inequalities (2.2) with (2.4), there exists a solution* $(u, v) \in Y \times Y$ *of problem (1.1)–(1.2) satisfying*

$$\|(x, y) - (u, v)\| \leq \boldsymbol{c}_{f,g,\boldsymbol{\varphi}}[\boldsymbol{\varphi}(t)]^T, \quad t \in [1, e].$$

In a word, the primary differences among the stability concepts lie in the types of perturbations and the generality of the stability conditions: Ulam-Hyers stability addresses basic stability under small perturbations, while Ulam-Hyers-Rassias stability extends this by allowing more flexible perturbations. Generalized Ulam-Hyers stability broadens the application of Ulam-Hyers stability to various FDEs, and generalized Ulam-Hyers-Rassias stability combines this generalization with flexible perturbations for a more comprehensive analysis. These concepts are crucial for studying stability in various mathematical models, particularly in fractional differential equations.

Remark 2.13. *It is clear that (i) Definition 2.9* $\Longrightarrow$ *Definition 2.10; (ii) Definition 2.11* $\Longrightarrow$ *Definition 2.12; (iii) Definition 2.11* $\Longrightarrow$ *Definition 2.9.*

Remark 2.14. *A function vector* $(x, y) \in Y \times Y$ *is a solution of inequalities (2.1) if and only if there exists a function* $v_1, v_2 \in C([1, e], \mathbb{R})$ *such that* $|v_1(t)| \leq \epsilon_1,\ |v_2(t)| \leq \epsilon_2,\ t \in [1, e]$*, and*

$$\begin{cases} ({}_HD_{1^+}^{\alpha_1,\beta_1} + k_1\, {}_HD_{1^+}^{\alpha_1-1,\beta_1})x(t) = f(t, x(t), y(t)) + v_1(t), & t \in [1, e], \\ ({}_HD_{1^+}^{\alpha_2,\beta_2} + k_2\, {}_HD_{1^+}^{\alpha_2-1,\beta_2})y(t) = g(t, x(t), y(t)) + v_2(t). \end{cases}$$

One can make similar observations to Remark 2.14 for inequalities (2.2) and (2.3).

3. Existence and uniqueness results

We start by proving a basic lemma concerning a linear variant of boundary value problem (1.1)–(1.2), which be used to transform boundary value problem (1.1)–(1.2) into an equivalent integral system of equations.

3.1. An auxiliary lemma

Lemma 3.1. *Let* $h_1, h_2 \in C([1, e], \mathbb{R})$ *and* $\Delta = 1 - AB \neq 0$*, where*

$$A = \sum_{i=1}^{n} \frac{\lambda_i}{\Gamma(\delta_i)} \int_1^{\theta_i} (\theta_i - s)^{\delta_i - 1} (\log s)^{\gamma_2 - 1} ds \ \ \text{and} \ \ B = \sum_{j=1}^{n} \frac{\mu_j}{\Gamma(\sigma_j)} \int_1^{\eta_j} (\eta_j - s)^{\sigma_j - 1} (\log s)^{\gamma_1 - 1} ds.$$

Then, $u, v \in C([1,e],\mathbb{R})$ are solutions of the Hilfer-Hadamard sequential fractional differential equations

$$\begin{cases} ({}_HD^{\alpha_1,\beta_1}_{1^+} + k_1{}_HD^{\alpha_1-1,\beta_1}_{1^+})u(t) = h_1(t), & 1<\alpha_1\le 2,\ 0\le\beta_1\le 1,\ t\in[1,e],\\ ({}_HD^{\alpha_2,\beta_2}_{1^+} + k_2{}_HD^{\alpha_2-1,\beta_2}_{1^+})v(t) = h_2(t), & 1<\alpha_2\le 2,\ 0\le\beta_2\le 1,\ t\in[1,e], \end{cases} \tag{3.1}$$

supplemented with integral boundary conditions (1.2) if and only if

$$\begin{aligned} u(t) =&\frac{(\log t)^{\gamma_1-1}}{\Delta}\bigg\{\bigg[k_1\int_1^e \frac{u(s)}{s}ds - \frac{1}{\Gamma(\alpha_1)}\int_1^e \frac{h_1(s)}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_1-1}ds\\ &-\sum_{i=1}^n \frac{\lambda_i}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\Big(k_2\int_1^s\frac{v(r)}{r}dr - \frac{1}{\Gamma(\alpha_2)}\int_1^s\frac{h_2(r)}{r}\Big(\log\frac{s}{r}\Big)^{\alpha_2-1}dr\Big)ds\bigg]\\ &+A\bigg[k_2\int_1^e\frac{v(s)}{s}ds - \frac{1}{\Gamma(\alpha_2)}\int_1^e\frac{h_2(s)}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_2-1}ds\\ &-\sum_{j=1}^n\frac{\mu_j}{\Gamma(\sigma_j)}\int_1^{\eta_j}(\eta_j-s)^{\sigma_j-1}\Big(k_1\int_1^s\frac{u(r)}{r}dr - \frac{1}{\Gamma(\alpha_1)}\int_1^s\frac{h_1(r)}{r}\Big(\log\frac{s}{r}\Big)^{\alpha_1-1}dr\Big)ds\bigg]\bigg\}\\ &-k_1\int_1^t\frac{u(s)}{s}ds + \frac{1}{\Gamma(\alpha_1)}\int_1^t\frac{h_1(s)}{s}\Big(\log\frac{t}{s}\Big)^{\alpha_1-1}ds \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} v(t) =&\frac{(\log t)^{\gamma_2-1}}{\Delta}\bigg\{\bigg[k_2\int_1^e \frac{v(s)}{s}ds - \frac{1}{\Gamma(\alpha_2)}\int_1^e \frac{h_2(s)}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_2-1}ds\\ &-\sum_{j=1}^n \frac{\mu_j}{\Gamma(\sigma_j)}\int_1^{\eta_j}(\eta_j-s)^{\sigma_j-1}\Big(k_1\int_1^s\frac{u(r)}{r}dr - \frac{1}{\Gamma(\alpha_1)}\int_1^s\frac{h_1(r)}{r}\Big(\log\frac{s}{r}\Big)^{\alpha_1-1}dr\Big)ds\bigg]\\ &+B\bigg[k_1\int_1^e\frac{u(s)}{s}ds - \frac{1}{\Gamma(\alpha_1)}\int_1^e\frac{h_1(s)}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_1-1}ds\\ &-\sum_{i=1}^n\frac{\lambda_i}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\Big(k_2\int_1^s\frac{v(r)}{r}dr - \frac{1}{\Gamma(\alpha_2)}\int_1^s\frac{h_2(r)}{r}\Big(\log\frac{s}{r}\Big)^{\alpha_2-1}dr\Big)ds\bigg]\bigg\}\\ &-k_2\int_1^t\frac{v(s)}{s}ds + \frac{1}{\Gamma(\alpha_2)}\int_1^t\frac{h_2(s)}{s}\Big(\log\frac{t}{s}\Big)^{\alpha_2-1}ds, \end{aligned} \tag{3.3}$$

where $\gamma_i = \alpha_i + 2\beta_i - \alpha_i\beta_i$ for $i \in \{1,2\}$.

Proof. Taking the Hadamard fractional integral of order α_1 both sides of the first equation of (3.1), we get

$$({}_HI^{\alpha_1}_{1^+}{}_HD^{\alpha_1,\beta_1}_{1^+} + k_1{}_HI^{\alpha_1}_{1^+}{}_HD^{\alpha_1-1,\beta_1}_{1^+})u(t) = {}_HI^{\alpha_1}_{1^+}h_1(t).$$

By Theorem 2.6, we have

$$u(t) - \sum_{j=0}^{1}\frac{(\delta^{(2-j-1)}({}_HI^{2-\gamma_1}_{1^+}u))(1)}{\Gamma(\gamma_1-j)}(\log t)^{\gamma_1-j-1} + k_1{}_HI^{\alpha_1}_{1^+}{}_HD^{\alpha_1-1,\beta_1}_{1^+}u(t) = {}_HI^{\alpha_1}_{1^+}h_1(t). \tag{3.4}$$

From Eq (3.4), by Definition 2.4 one has

$$u(t)-\frac{(\delta_H I_{1^+}^{2-\gamma_1}u)(1)}{\Gamma(\gamma_1)}(\log t)^{\gamma_1-1}-\frac{({}_H I_{1^+}^{2-\gamma_1}u)(1)}{\Gamma(\gamma_1-1)}(\log t)^{\gamma_1-2}+k_1{}_H I_{1^+}^{1}({}_H I_{1^+}^{\gamma_1-1}{}_H D_{1^+}^{\gamma_1-1})u(t)={}_H I_{1^+}^{\alpha_1}h_1(t), \quad (3.5)$$

where $\gamma_1,\gamma_2\in(1,2]$ for $i\in\{1,2\}$. Then, by Theorem 2.5 we get

$$\begin{aligned}&u(t)-\frac{(\delta_H I_{1^+}^{2-\gamma_1}u)(1)}{\Gamma(\gamma_1)}(\log t)^{\gamma_1-1}-\frac{({}_H I_{1^+}^{2-\gamma_1}u)(1)}{\Gamma(\gamma_1-1)}(\log t)^{\gamma_1-2}\\&+k_1{}_H I_{1^+}^{1}\Big(u(t)-\frac{({}_H I_{1^+}^{2-\gamma_1}u)(1)}{\Gamma(\gamma_1-1)}(\log t)^{\gamma_1-2}\Big)={}_H I_{1^+}^{\alpha_1}h(t).\end{aligned} \quad (3.6)$$

Equation (3.6) can be written as

$$\begin{aligned}u(t)=\ &c_0(\log t)^{\gamma_1-1}+c_1\Big((\log\ t)^{\gamma_1-2}+k_1\int_1^t\frac{(\log\ s)^{\gamma_1-2}}{s}ds\Big)-k_1\int_1^t\frac{u(s)}{s}ds\\&+\frac{1}{\Gamma(\alpha)}\int_1^t\frac{h_1(s)}{s}\Big(\log\ \frac{t}{s}\Big)^{\alpha_1-1}ds,\end{aligned} \quad (3.7)$$

where c_0 and c_1 are arbitrary constants. In a similar way, we obtain

$$\begin{aligned}v(t)=\ &d_0(\log t)^{\gamma_2-1}+d_1\Big((\log\ t)^{\gamma_2-2}+k_2\int_1^t\frac{(\log\ s)^{\gamma_2-2}}{s}ds\Big)-k_2\int_1^t\frac{v(s)}{s}ds\\&+\frac{1}{\Gamma(\alpha)}\int_1^t\frac{h_2(s)}{s}\Big(\log\ \frac{t}{s}\Big)^{\alpha_2-1}ds,\end{aligned} \quad (3.8)$$

where d_0 and d_1 are arbitrary constants. Using the first boundary conditions of (1.2), $u(1)=0$, and $v(1)=0$ together with Eqs (3.7) and (3.8), yield $c_1=0$ and $d_1=0$, respectively. Equations (3.7) and (3.8) become

$$u(t)=c_0(\log t)^{\gamma_1-1}-k_1\int_1^t\frac{u(s)}{s}ds+\frac{1}{\Gamma(\alpha_1)}\int_1^t\frac{h_1(s)}{s}\Big(\log\frac{t}{s}\Big)^{\alpha_1-1}ds, \quad (3.9)$$

and

$$v(t)=d_0(\log t)^{\gamma_2-1}-k_2\int_1^t\frac{v(s)}{s}ds+\frac{1}{\Gamma(\alpha_2)}\int_1^t\frac{h_2(s)}{s}\Big(\log\frac{t}{s}\Big)^{\alpha_2-1}ds. \quad (3.10)$$

Next, the second boundary conditions of (1.2) together with Eqs (3.9) and (3.10) yield

$$\begin{aligned}&c_0-k_1\int_1^e\frac{u(s)}{s}ds+\frac{1}{\Gamma(\alpha_1)}\int_1^e\frac{h_1(s)}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_1-1}ds\\&=\sum_{i=1}^{n}\frac{\lambda_i}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\Big(d_0(\log s)^{\gamma_2-1}-k_2\int_1^s\frac{v(r)}{r}dr+\frac{1}{\Gamma(\alpha_2)}\int_1^s\frac{h_2(r)}{r}\Big(\log\frac{s}{r}\Big)^{\alpha_2-1}dr\Big)ds\end{aligned}$$

and

$$\begin{aligned}&d_0-k_2\int_1^e\frac{v(s)}{s}ds+\frac{1}{\Gamma(\alpha_2)}\int_1^e\frac{h_2(s)}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_2-1}ds\\&=\sum_{j=1}^{n}\frac{\mu_j}{\Gamma(\sigma_j)}\int_1^{\eta_j}(\eta_j-s)^{\sigma_j-1}\Big(c_0(\log s)^{\gamma_1-1}-k_1\int_1^s\frac{u(r)}{r}dr+\frac{1}{\Gamma(\alpha_1)}\int_1^s\frac{h_1(r)}{r}\Big(\log\frac{s}{r}\Big)^{\alpha_1-1}dr\Big)ds,\end{aligned}$$

respectively. Rearranging the above equations, we get the linear system of equations:

$$c_0 - d_0 A = J_1 \quad \text{and} \quad d_0 - c_0 B = J_2, \tag{3.11}$$

where

$$\begin{aligned}
J_1 =& k_1 \int_1^e \frac{u(s)}{s} ds - \frac{1}{\Gamma(\alpha_1)} \int_1^e \frac{h_1(s)}{s} \Big(\log \frac{e}{s} \Big)^{\alpha_1 - 1} ds \\
&- \sum_{i=1}^n \frac{\lambda_i}{\Gamma(\delta_i)} \int_1^{\theta_i} (\theta_i - s)^{\delta_i - 1} \Big(k_2 \int_1^s \frac{v(r)}{r} dr - \frac{1}{\Gamma(\alpha_2)} \int_1^s \frac{h_2(r)}{r} \Big(\log \frac{s}{r} \Big)^{\alpha_2 - 1} dr \Big) ds, \\
J_2 =& k_2 \int_1^e \frac{v(s)}{s} ds - \frac{1}{\Gamma(\alpha_2)} \int_1^e \frac{h_2(s)}{s} \Big(\log \frac{e}{s} \Big)^{\alpha_2 - 1} ds \\
&- \sum_{j=1}^n \frac{\mu_j}{\Gamma(\sigma_j)} \int_1^{\eta_j} (\eta_j - s)^{\sigma_j - 1} \Big(k_1 \int_1^s \frac{u(r)}{r} dr - \frac{1}{\Gamma(\alpha_1)} \int_1^s \frac{h_1(r)}{r} \Big(\log \frac{s}{r} \Big)^{\alpha_1 - 1} dr \Big) ds.
\end{aligned}$$

Solving system (3.11), we obtain

$$c_0 = \frac{J_1 + AJ_2}{\Delta} \quad \text{and} \quad d_0 = \frac{J_2 + BJ_1}{\Delta}.$$

Substituting c_0 and d_0 back into Eqs (3.9) and (3.10), respectively, we get the integral equations (3.2) and (3.3). The converse follows by direct computation. This completes the proof. □

Let us introduce the Banach space $X = C([1,e], \mathbb{R})$ endowed with the norm defined by $\|u\| := \max_{t \in [1,e]} |u(t)|$. Thus, the product space $X \times X$ equipped with the norm $\|(u,v)\| = \|u\| + \|v\|$ is a Banach space. In view of Lemma 3.1, we define an operator $\mathcal{F} : X \times X \to X \times X$ by

$$\mathcal{F}(u,v)(t) = (\mathcal{F}_1(u,v)(t), \mathcal{F}_2(u,v)(t)), \tag{3.12}$$

where

$$\begin{aligned}
&\mathcal{F}_1(u,v)(t) \\
=& \frac{(\log t)^{\gamma_1 - 1}}{\Delta} \Big\{ \Big[k_1 \int_1^e \frac{u(s)}{s} ds - \frac{1}{\Gamma(\alpha_1)} \int_1^e \frac{f(s,u(s),v(s))}{s} \Big(\log \frac{e}{s} \Big)^{\alpha_1 - 1} ds \\
&- \sum_{i=1}^n \frac{\lambda_i}{\Gamma(\delta_i)} \int_1^{\theta_i} (\theta_i - s)^{\delta_i - 1} \Big(k_2 \int_1^s \frac{v(r)}{r} dr - \frac{1}{\Gamma(\alpha_2)} \int_1^s \frac{g(r,u(r),v(r))}{r} \Big(\log \frac{s}{r} \Big)^{\alpha_2 - 1} dr \Big) ds \Big] \\
&+ A \Big[k_2 \int_1^e \frac{v(s)}{s} ds - \frac{1}{\Gamma(\alpha_2)} \int_1^e \frac{g(s,u(s),v(s))}{s} \Big(\log \frac{e}{s} \Big)^{\alpha_2 - 1} ds \\
&- \sum_{j=1}^n \frac{\mu_j}{\Gamma(\sigma_j)} \int_1^{\eta_j} (\eta_j - s)^{\sigma_j - 1} \Big(k_1 \int_1^s \frac{u(r)}{r} dr - \frac{1}{\Gamma(\alpha_1)} \int_1^s \frac{f(r,u(r),v(r))}{r} \Big(\log \frac{s}{r} \Big)^{\alpha_1 - 1} dr \Big) ds \Big] \Big\} \\
&- k_1 \int_1^t \frac{u(s)}{s} ds + \frac{1}{\Gamma(\alpha_1)} \int_1^t \frac{f(s,u(s),v(s))}{s} \Big(\log \frac{t}{s} \Big)^{\alpha_1 - 1} ds
\end{aligned} \tag{3.13}$$

and

$$
\begin{aligned}
&\mathcal{F}_2(u,v)(t)\\
=&\frac{(\log t)^{\gamma_2-1}}{\Delta}\Bigg\{\Bigg[k_2\int_1^e\frac{v(s)}{s}ds-\frac{1}{\Gamma(\alpha_2)}\int_1^e\frac{g(s,u(s),v(s))}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_2-1}ds\\
&-\sum_{j=1}^{n}\frac{\mu_j}{\Gamma(\sigma_j)}\int_1^{\eta_j}(\eta_j-s)^{\sigma_j-1}\Big(k_1\int_1^s\frac{u(r)}{r}dr-\frac{1}{\Gamma(\alpha_1)}\int_1^s\frac{f(r,u(r),v(r))}{r}\Big(\log\frac{s}{r}\Big)^{\alpha_1-1}dr\Big)ds\Bigg]\\
&+B\Bigg[k_1\int_1^e\frac{u(s)}{s}ds-\frac{1}{\Gamma(\alpha_1)}\int_1^e\frac{f(s,u(s),v(s))}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_1-1}ds\\
&-\sum_{i=1}^{n}\frac{\lambda_i}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\Big(k_2\int_1^s\frac{v(r)}{r}dr-\frac{1}{\Gamma(\alpha_2)}\int_1^s\frac{g(r,u(r),v(r))}{r}\Big(\log\frac{s}{r}\Big)^{\alpha_2-1}dr\Big)ds\Bigg]\Bigg\}\\
&-k_2\int_1^t\frac{v(s)}{s}ds+\frac{1}{\Gamma(\alpha_2)}\int_1^t\frac{g(s,u(s),v(s))}{s}\Big(\log\frac{t}{s}\Big)^{\alpha_2-1}ds.
\end{aligned}
\tag{3.14}
$$

We use the following notations in the proofs:

$$
\omega=\sum_{i=1}^{n}|\lambda_i|\frac{(\theta_i-1)^{\delta_i}}{\Gamma(\delta_i+1)},\ \bar{\omega}=\sum_{j=1}^{n}|\mu_j|\frac{(\eta_j-1)^{\sigma_j}}{\Gamma(\sigma_j+1)},\ M=\frac{1}{|\Delta|}[1+\omega\bar{\omega}+|\Delta|],\ W_1=\frac{2\,\omega}{|\Delta|},\ W_2=\frac{2\,\bar{\omega}}{|\Delta|}.
$$

Note that $A\leq\ \omega$ and $B\leq\ \bar{\omega}$. The following hypotheses are required in the subsequent discussion :

(H_1) Let $f,g\ :\ [1,e]\times\mathbb{R}\times\mathbb{R}\ \to\ \mathbb{R}$ be continuous functions. Assume that there exist continuous nonnegative functions $\vartheta\in C([1,e],\mathbb{R}_+)$ and $\chi\in C([1,e],\mathbb{R}_+)$ such that

$$
|f(t,u(t),v(t))|\leq\vartheta(t)\quad\text{and}\quad|g(t,u(t),v(t))|\leq\chi(t),
$$

for each $t\in[1,e]$.

(H_2) Assume that there exist positive constants L and $\bar{L}$ such that, for all $t\in[1,e]$ and $u_i,v_i\in\mathbb{R}$, $i=1,2$,

$$
\begin{aligned}
|f(t,u_1,u_2)-f(t,v_1,v_2)|&\leq L(|u_1-v_1|+|u_2-v_2|),\\
|g(t,u_1,u_2)-g(t,v_1,v_2)|&\leq\bar{L}(|u_1-v_1|+|u_2-v_2|).
\end{aligned}
$$

(H_3) Assume that $\varphi:[1,e]\to\mathbb{R}_+$ is an increasing continuous function and there exists $\lambda_\varphi>0$ such that

$$
{}_HI_{1^+}^{\alpha}\varphi(t)\leq\lambda_\varphi\varphi(t),\quad t\in[1,e].
$$

3.2. Existence result via Krasnoselskii's fixed point theorem

In this subsection, we establish an existence result using Krasnoselskii's fixed point theorem.

Theorem 3.2. *Assume that* (H_1) *holds. Then, the problem defined by Eqs (1.1) and (1.2) has at least one solution on the interval* $[1,e]$*, provided that*

$$
\Pi:=k_1(M+W_2)+k_2(M+W_1)<1. \tag{3.15}
$$

Proof. By assumption (H_1), we can fix

$$R \geq \frac{\|\vartheta\|\dfrac{(M+W_2)}{\Gamma(\alpha_1+1)} + \|\chi\|\dfrac{(M+W_1)}{\Gamma(\alpha_2+1)}}{1-(k_1(M+W_2)+k_2(M+W_1))},$$

where $\|\vartheta\| = \sup_{t\in[1,e]} |\vartheta(t)|$ and $\|\chi\| = \sup_{t\in[1,e]} |\chi(t)|$, and we consider $B_R = \{(u,v) \in X \times X : \|(u,v)\| \leq R\}$. We split the operator $\mathcal{F} : X \times X \to X \times X$ defined by (3.12) as

$$(\mathcal{F}_1, \mathcal{F}_2) = (\mathcal{F}_{11}, \mathcal{F}_{21}) + (\mathcal{F}_{12}, \mathcal{F}_{22}),$$

where $\mathcal{F}_{11}, \mathcal{F}_{21}, \mathcal{F}_{12}$, and $\mathcal{F}_{22}$ are given by

$$\begin{aligned}
\mathcal{F}_{11}(u,v)(t) =& \frac{(\log t)^{\gamma_1-1}}{\Delta}\Bigg\{\Bigg[k_1\int_1^e \frac{u(s)}{s}ds - \sum_{i=1}^n \frac{\lambda_i}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\Bigg(k_2\int_1^s \frac{v(r)}{r}dr\Bigg)ds\Bigg] \\
&+ A\Bigg[k_2\int_1^e \frac{v(s)}{s}ds - \sum_{j=1}^n \frac{\mu_j}{\Gamma(\sigma_j)}\int_1^{\eta_j}(\eta_j-s)^{\sigma_j-1}\Bigg(k_1\int_1^s \frac{u(r)}{r}dr\Bigg)ds\Bigg]\Bigg\} - k_1\int_1^t \frac{u(s)}{s}ds, \\
\mathcal{F}_{21}(u,v)(t) =& \frac{(\log t)^{\gamma_2-1}}{\Delta}\Bigg\{\Bigg[k_2\int_1^e \frac{v(s)}{s}ds - \sum_{j=1}^n \frac{\mu_j}{\Gamma(\sigma_j)}\int_1^{\eta_j}(\eta_j-s)^{\sigma_j-1}\Bigg(k_1\int_1^s \frac{u(r)}{r}dr\Bigg)ds\Bigg] \\
&+ B\Bigg[k_1\int_1^e \frac{u(s)}{s}ds - \sum_{i=1}^n \frac{\lambda_i}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\Bigg(k_2\int_1^s \frac{v(r)}{r}dr\Bigg)ds\Bigg]\Bigg\} - k_2\int_1^t \frac{v(s)}{s}ds, \\
\mathcal{F}_{12}(u,v)(t) =& \frac{(\log t)^{\gamma_1-1}}{\Delta}\Bigg\{\Bigg[-\frac{1}{\Gamma(\alpha_1)}\int_1^e \frac{f(s,u(s),v(s))}{s}\Bigg(\log\frac{e}{s}\Bigg)^{\alpha_1-1}ds \\
&+ \sum_{i=1}^n \frac{\lambda_i}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\Bigg(\frac{1}{\Gamma(\alpha_2)}\int_1^s \frac{g(r,u(r),v(r))}{r}\Bigg(\log\frac{s}{r}\Bigg)^{\alpha_2-1}dr\Bigg)ds\Bigg] \\
&+ A\Bigg[-\frac{1}{\Gamma(\alpha_2)}\int_1^e \frac{g(s,u(s),v(s))}{s}\Bigg(\log\frac{e}{s}\Bigg)^{\alpha_2-1}ds \\
&+ \sum_{j=1}^n \frac{\mu_j}{\Gamma(\sigma_j)}\int_1^{\eta_j}(\eta_j-s)^{\sigma_j-1}\Bigg(\frac{1}{\Gamma(\alpha_1)}\int_1^s \frac{f(r,u(r),v(r))}{r}\Bigg(\log\frac{s}{r}\Bigg)^{\alpha_1-1}dr\Bigg)ds\Bigg]\Bigg\} \\
&+ \frac{1}{\Gamma(\alpha_1)}\int_1^t \frac{f(s,u(s),v(s))}{s}\Bigg(\log\frac{t}{s}\Bigg)^{\alpha_1-1}ds
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{F}_{22}(u,v)(t) =& \frac{(\log t)^{\gamma_2-1}}{\Delta}\Bigg\{\Bigg[-\frac{1}{\Gamma(\alpha_2)}\int_1^e \frac{g(s,u(s),v(s))}{s}\Bigg(\log\frac{e}{s}\Bigg)^{\alpha_2-1}ds \\
&+ \sum_{j=1}^n \frac{\mu_j}{\Gamma(\sigma_j)}\int_1^{\eta_j}(\eta_j-s)^{\sigma_j-1}\Bigg(\frac{1}{\Gamma(\alpha_1)}\int_1^s \frac{f(r,u(r),v(r))}{r}\Bigg(\log\frac{s}{r}\Bigg)^{\alpha_1-1}dr\Bigg)ds\Bigg] \\
&+ B\Bigg[-\frac{1}{\Gamma(\alpha_1)}\int_1^e \frac{f(s,u(s),v(s))}{s}\Bigg(\log\frac{e}{s}\Bigg)^{\alpha_1-1}ds
\end{aligned}$$

$$+\sum_{i=1}^{n}\frac{\lambda_i}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\Big(\frac{1}{\Gamma(\alpha_2)}\int_1^{s}\frac{g(r,u(r),v(r))}{r}\Big(\log\frac{s}{r}\Big)^{\alpha_2-1}dr\Big)ds\Big]\Big\}$$
$$+\frac{1}{\Gamma(\alpha_2)}\int_1^{t}\frac{g(s,u(s),v(s))}{s}\Big(\log\frac{t}{s}\Big)^{\alpha_2-1}ds.$$

Step I: First, we show that $(\mathcal{F}_{11},\mathcal{F}_{21})(u,v)+(\mathcal{F}_{12},\mathcal{F}_{22})(x,y)\in B_R$, whenever $(u,v),(x,y)\in B_R$. Let (u,v) and $(x,y)\in B_R$. Then, for any $t\in[1,e]$, we have

$$\begin{aligned}
&|\mathcal{F}_{11}(u,v)(t)+\mathcal{F}_{12}(x,y)(t)|\\
\le&\frac{1}{|\Delta|}\Big\{\Big[k_1\int_1^{e}\frac{|u(s)|}{s}ds+\frac{1}{\Gamma(\alpha_1)}\int_1^{e}\frac{|f(s,x(s),y(s))|}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_1-1}ds\\
&+\sum_{i=1}^{n}\frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\Big(k_2\int_1^{s}\frac{|v(r)|}{r}dr+\frac{1}{\Gamma(\alpha_2)}\int_1^{s}\frac{|g(r,w(r),y(r))|}{r}\Big(\log\frac{s}{r}\Big)^{\alpha_2-1}dr\Big)ds\Big]\\
&+A\Big[k_2\int_1^{e}\frac{|v(s)|}{s}ds+\frac{1}{\Gamma(\alpha_2)}\int_1^{e}\frac{|g(s,x(s),y(s))|}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_2-1}ds\\
&+\sum_{j=1}^{n}\frac{|\mu_j|}{\Gamma(\sigma_j)}\int_1^{\eta_j}(\eta_j-s)^{\sigma_j-1}\Big(k_1\int_1^{s}\frac{|u(r)|}{r}dr+\frac{1}{\Gamma(\alpha_1)}\int_1^{s}\frac{|f(r,w(r),y(r))|}{r}\Big(\log\frac{s}{r}\Big)^{\alpha_1-1}dr\Big)ds\Big]\Big\}\\
&+k_1\int_1^{t}\frac{|u(s)|}{s}ds+\frac{1}{\Gamma(\alpha_1)}\int_1^{t}\frac{|f(s,x(s),y(s))|}{s}(\log\frac{t}{s})^{\alpha_1-1}ds\\
\le&\frac{1}{|\Delta|}\Big\{\Big[k_1\|u\|+\frac{\|\vartheta\|}{\Gamma(\alpha_1+1)}+\sum_{i=1}^{n}\frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\Big(k_2\|v\|\log s+\frac{\|\chi\|(\log s)^{\alpha_2}}{\Gamma(\alpha_2+1)}\Big)ds\Big]\\
&+\omega\Big[k_2\|v\|+\frac{\|\chi\|}{\Gamma(\alpha_2+1)}+\sum_{j=1}^{n}\frac{|\mu_j|}{\Gamma(\sigma_j)}\int_1^{\eta_j}(\eta_j-s)^{\sigma_j-1}\Big(k_1\|u\|\log s+\frac{\|\vartheta\|(\log s)^{\alpha_1}}{\Gamma(\alpha_1+1)}\Big)ds\Big]\Big\}\\
&+k_1\|u\|\log t+\frac{\|\vartheta\|(\log t)^{\alpha_1}}{\Gamma(\alpha_1+1)}\\
\le&\frac{1}{|\Delta|}\Big\{\Big[k_1\|u\|+\frac{\|\vartheta\|}{\Gamma(\alpha_1+1)}+\omega\Big(k_2\|v\|+\frac{\|\chi\|}{\Gamma(\alpha_2+1)}\Big)\Big]\\
&+\omega\Big[k_2\|v\|+\frac{\|\chi\|}{\Gamma(\alpha_2+1)}+\bar{\omega}\Big(k_1\|u\|+\frac{\|\vartheta\|}{\Gamma(\alpha_1+1)}\Big)\Big]\Big\}+k_1\|u\|+\frac{\|\vartheta\|}{\Gamma(\alpha_1+1)}\\
=&\Big(\|u\|k_1+\frac{\|\vartheta\|}{\Gamma(\alpha_1+1)}\Big)M+\Big(\|v\|k_2+\frac{\|\chi\|}{\Gamma(\alpha_2+1)}\Big)W_1,
\end{aligned}$$

which, upon taking the norm for $t\in[1,e]$, yields

$$\|\mathcal{F}_{11}(u,v)+\mathcal{F}_{12}(x,y)\|\le\Big(k_1\|u\|+\frac{\|\vartheta\|}{\Gamma(\alpha_1+1)}\Big)M+\Big(k_2\|v\|+\frac{\|\chi\|}{\Gamma(\alpha_2+1)}\Big)W_1.$$

By a similar method, it can be found that

$$\|\mathcal{F}_{21}(u,v)+\mathcal{F}_{22}(x,y)\|\le\Big(k_2\|v\|+\frac{\|\chi\|}{\Gamma(\alpha_2+1)}\Big)M+\Big(k_1\|u\|+\frac{\|\vartheta\|}{\Gamma(\alpha_1+1)}\Big)W_2.$$

Hence,

$$\|(\mathcal{F}_{11},\mathcal{F}_{21})(u,v)+(\mathcal{F}_{12},\mathcal{F}_{22})(x,y)\| \le \|\mathcal{F}_{11}(u,v)+\mathcal{F}_{12}(x,y)\|+\|\mathcal{F}_{21}(u,v)+\mathcal{F}_{22}(x,y)\|$$
$$\le R(k_1(M+W_2)+k_2(M+W_1))+\|\vartheta\|\frac{(M+W_2)}{\Gamma(\alpha_1+1)}+\|\chi\|\frac{(M+W_1)}{\Gamma(\alpha_2+1)} \le R.$$

Therefore, $(\mathcal{F}_{11},\mathcal{F}_{21})(u,v)+(\mathcal{F}_{12},\mathcal{F}_{22})(x,y) \in B_R$.

Step II: Next, we show that the operator $(\mathcal{F}_{11},\mathcal{F}_{21})$ is a contraction. Let (u_1,v_1) and $(u_2,v_2) \in X\times X$. Then, for any $t\in[1,e]$, we have

$$\begin{aligned}
&|\mathcal{F}_{11}(u_2,v_2)(t)-\mathcal{F}_{11}(u_1,v_1)(t)|\\
\le&\frac{1}{|\Delta|}\bigg\{\bigg[k_1\int_1^e\frac{|u_2(s)-u_1(s)|}{s}ds+\sum_{i=1}^n\frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\Big(k_2\int_1^s\frac{|v_2(r)-v_1(r)|}{r}dr\Big)ds\bigg]\\
&+A\bigg[k_2\int_1^e\frac{|v_2(s)-v_1(s)|}{s}ds+\sum_{j=1}^n\frac{|\mu_j|}{\Gamma(\sigma_j)}\int_1^{\eta_j}(\eta_j-s)^{\sigma_j-1}\Big(k_1\int_1^s\frac{|u_2(r)-u_1(r)|}{r}dr\Big)ds\bigg]\bigg\}\\
&+k_1\int_1^t\frac{|u_2(s)-u_1(s)|}{s}ds\\
\le&\frac{1}{|\Delta|}\bigg\{\bigg[k_1\|u_2-u_1\|+k_2\|v_2-v_1\|\sum_{i=1}^n|\lambda_i|\frac{(\theta_i-1)^{\delta_i}}{\Gamma(\delta_i+1)}\bigg]\\
&+\omega\bigg[k_2\|v_2-v_1\|+k_1\|u_2-u_1\|\sum_{j=1}^n|\mu_j|\frac{(\eta_j-1)^{\sigma_j}}{\Gamma(\sigma_j+1)}\bigg]\bigg\}+k_1\|u_2-u_1\|\\
=&k_1M\|u_2-u_1\|+k_2W_1\|v_2-v_1\|\\
\le&(\|u_2-u_1\|+\|v_2-v_1\|)[k_1M+k_2W_1],
\end{aligned}$$

which, upon taking the norm for $t\in[1,e]$, yields

$$\|\mathcal{F}_{11}(u_2,v_2)-\mathcal{F}_{11}(u_1,v_1)\| \le (\|u_2-u_1\|+\|v_2-v_1\|)[k_1M+k_2W_1].$$

Similarly, one has

$$\|\mathcal{F}_{21}(u_2,v_2)-\mathcal{F}_{21}(u_1,v_1)\| \le (\|u_2-u_1\|+\|v_2-v_1\|)[k_2M+k_1W_2].$$

Hence,

$$\begin{aligned}
\|(\mathcal{F}_{11},\mathcal{F}_{21})(u_2,v_2)-(\mathcal{F}_{11},\mathcal{F}_{21})(u_1,v_1)\| =&\|\mathcal{F}_{11}(u_2,v_2)-\mathcal{F}_{11}(u_1,v_1)\|+\|\mathcal{F}_{21}(u_2,v_2)-\mathcal{F}_{21}(u_1,v_1)\|\\
\le&\Pi\,(\|u_2-u_1\|+\|v_2-v_1\|).
\end{aligned}$$

In view of (3.15), the operator $(\mathcal{F}_{11},\mathcal{F}_{21})$ is a contraction.

Step III: Finally, we will show that the operator $(\mathcal{F}_{12},\mathcal{F}_{22})$ is continuous and compact. First, we show that the operator $(\mathcal{F}_{12},\mathcal{F}_{22})$ is continuous. Let $\{(u_n,v_n)\}$ be a sequence such that $(u_n,v_n)\to(u,v)$ in $X\times X$ for any $t\in[1,e]$. Then, we have

$$\begin{aligned}
&|\mathcal{F}_{12}(u_n,v_n)(t)-\mathcal{F}_{12}(u,v)(t)|\\
\le&\frac{1}{|\Delta|}\bigg\{\bigg[\frac{1}{\Gamma(\alpha_1)}\int_1^e\frac{|f(s,u_n(s),v_n(s))-f(s,u(s),v(s))|}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_1-1}ds
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{n}\frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\Big(\frac{1}{\Gamma(\alpha_2)}\int_1^{s}\frac{|g(r,u_n(r),v_n(r))-g(r,u(r),v(r)|}{r}\Big(\log\frac{s}{r}\Big)^{\alpha_2-1}dr\Big)ds\Big]\\
&+A\Big[\frac{1}{\Gamma(\alpha_2)}\int_1^{e}\frac{|g(s,u_n(s),v_n(s))-g(s,u(s),v(s))|}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_2-1}ds\\
&+\sum_{j=1}^{n}\frac{|\mu_j|}{\Gamma(\sigma_j)}\int_1^{\eta_j}(\eta_j-s)^{\sigma_j-1}\Big(\frac{1}{\Gamma(\alpha_1)}\int_1^{s}\frac{|f(r,u_n(r),v_n(r))-f(r,u(r),v(r))|}{r}\Big(\log\frac{s}{r}\Big)^{\alpha_1-1}dr\Big)ds\Big]\\
&+\frac{1}{\Gamma(\alpha_1)}\int_1^{t}\frac{|f(s,u_n(s),v_n(s))-f(s,u(s),v(s))|}{s}\Big(\log\frac{t}{s}\Big)^{\alpha_1-1}ds.
\end{aligned}
$$

Since f and g are continuous, we get that

$$
\|\mathcal{F}_{12}(u_n,v_n)-\mathcal{F}_{12}(u,v)\|\to 0\quad\text{as}\quad\{(u_n,v_n)\}\to(u,v). \tag{3.16}
$$

Similarly, we have

$$
\|\mathcal{F}_{22}(u_n,v_n)-\mathcal{F}_{22}(u,v)\|\to 0\quad\text{as}\quad\{(u_n,v_n)\}\to(u,v). \tag{3.17}
$$

It follows from (3.16) and (3.17) that the operator $(\mathcal{F}_{12},\mathcal{F}_{22})$ is continuous.

Next, we will show that $(\mathcal{F}_{12},\mathcal{F}_{22})$ is compact by using Arzelá-Ascoli theorem. First, $(\mathcal{F}_{12},\mathcal{F}_{22})$ is uniformly bounded since

$$
\|\mathcal{F}_{12}(u,v)\|\le\frac{M}{\Gamma(\alpha_1+1)}\|\vartheta\|+\frac{W_1}{\Gamma(\alpha_2+1)}\|\chi\|
$$

and

$$
\|\mathcal{F}_{22}(u,v)\|\le\frac{M}{\Gamma(\alpha_2+1)}\|\chi\|+\frac{W_2}{\Gamma(\alpha_1+1)}\|\vartheta\|.
$$

Finally, we will show that $(\mathcal{F}_{12},\mathcal{F}_{22})$ is equicontinuous. We define

$$
\sup_{(t,u,v)\in[1,e]\times B_R\times B_R}|f(t,u,v)|=\bar{f}\quad\text{and}\quad\sup_{(t,u,v)\in[1,e]\times B_R\times B_R}|g(t,u,v)|=\bar{g}.
$$

Letting $t_1,t_2\in[1,e]$ with $t_1<t_2$, one has

$$
\begin{aligned}
&|\mathcal{F}_{12}(u,v)(t_2)-\mathcal{F}_{12}(u,v)(t_1)|\\
\le&\frac{[(\log t_2)^{\gamma_1-1}-(\log t_1)^{\gamma_1-1}]}{|\Delta|}\Big\{\Big[\frac{1}{\Gamma(\alpha_1)}\int_1^{e}\frac{|f(s,u(s),v(s))|}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_1-1}ds\\
&+\sum_{i=1}^{n}\frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\Big(\frac{1}{\Gamma(\alpha_2)}\int_1^{s}\frac{|g(r,u(r),v(r))|}{r}\Big(\log\frac{s}{r}\Big)^{\alpha_2-1}dr\Big)ds\Big]\\
&+\omega\Big[\frac{1}{\Gamma(\alpha_2)}\int_1^{e}\frac{|g(s,u(s),v(s))|}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_2-1}ds\\
&+\sum_{j=1}^{n}\frac{|\mu_j|}{\Gamma(\sigma_j)}\int_1^{\eta_j}(\eta_j-s)^{\sigma_j-1}\Big(\frac{1}{\Gamma(\alpha_1)}\int_1^{s}\frac{|f(r,u(r),v(r))|}{r}\Big(\log\frac{s}{r}\Big)^{\alpha_1-1}dr\Big)ds\Big]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha_1)}\bigg[\int_1^{t_1}\frac{|f(s,u(s),v(s))|}{s}\bigg(\Big(\log\frac{t_2}{s}\Big)^{\alpha_1-1}-\Big(\log\frac{t_1}{s}\Big)^{\alpha_1-1}\bigg)ds\\
&+\int_{t_1}^{t_2}\frac{|f(s,u(s),v(s))|}{s}\Big(\log\frac{t_2}{s}\Big)^{\alpha_1-1}ds\bigg]\\
\le&\frac{1}{|\Delta|}\bigg\{\frac{\bar f}{\Gamma(\alpha_1+1)}[1+\omega\bar\omega]+\frac{\bar g}{\Gamma(\alpha_2+1)}[2\omega]\bigg\}\Big[(\log t_2)^{\gamma_1-1}-(\log t_1)^{\gamma_1-1}\Big]\\
&+\frac{\bar f}{\Gamma(\alpha_1+1)}\Big[(\log t_2)^{\alpha_1}-(\log t_1)^{\alpha_1}\Big].
\end{aligned}\tag{3.18}
$$

In the same way, we have

$$
\begin{aligned}
&|\mathcal{F}_{22}(u,v)(t_2)-\mathcal{F}_{22}(u,v)(t_1)|\\
\le&\frac{1}{|\Delta|}\bigg\{\frac{\bar g}{\Gamma(\alpha_2+1)}[1+\omega\bar\omega]+\frac{\bar f}{\Gamma(\alpha_1+1)}[2\bar\omega]\bigg\}\Big[(\log t_2)^{\gamma_2-1}-(\log t_1)^{\gamma_2-1}\Big]\\
&+\frac{\bar g}{\Gamma(\alpha_2+1)}\Big[(\log t_2)^{\alpha_2}-(\log t_1)^{\alpha_2}\Big].
\end{aligned}\tag{3.19}
$$

From (3.18) and (3.19), we get that

$$|\mathcal{F}_{12}(u,v)(t_2)-\mathcal{F}_{12}(u,v)(t_1)|\to 0 \quad\text{and}\quad |\mathcal{F}_{22}(u,v)(t_2)-\mathcal{F}_{22}(u,v)(t_1)|\to 0 \quad\text{as}\quad t_2\to t_1.$$

Thus, $(\mathcal{F}_{12},\mathcal{F}_{22})$ is equicontinuous. By the Arzelá-Ascoli theorem, we get that $(\mathcal{F}_{12},\mathcal{F}_{22})$ is compact on B_R.

Hence, all the conditions of Krasnoselskii's fixed point theorem are satisfied, and therefore the boundary value problem defined by Eq (1.1) with conditions (1.2) has at least one solution on the interval $[1,e]$. □

3.3. Existence and uniqueness result via the Banach fixed point theorem

Next, we prove an existence and uniqueness result based on the Banach fixed point theorem.

Theorem 3.3. *Assume that (H_2) holds. Then, the boundary value problem defined by Eqs (1.1) and (1.2) has a unique solution on the interval $[1,e]$, provided that*

$$\Xi := k_1(M+W_2)+k_2(M+W_1)+\frac{L(M+W_2)}{\Gamma(\alpha_1+1)}+\frac{\bar L(M+W_1)}{\Gamma(\alpha_2+1)}<1. \tag{3.20}$$

Proof. We will use the Banach fixed point theorem to prove that $\mathcal{F}$, defined by (3.12), has a unique fixed point. Let us define

$$N_1 := \max_{t\in[1,e]}|f(t,0,0)|<\infty \quad\text{and}\quad N_2 := \max_{t\in[1,e]}|g(t,0,0)|<\infty.$$

By hypothesis (H_2), we get

$$|f(t,u(t),v(t))|\le|f(t,u(t),v(t))-f(t,0,0)|+|f(t,0,0)|\le L(\|u\|+\|v\|)+N_1=L\|(u,v)\|+N_1$$

and

$$|g(t,u(t),v(t))|\le|g(t,u(t),v(t))-g(t,0,0)|+|g(t,0,0)|\le\bar L(\|u\|+\|v\|)+N_2=\bar L\|(u,v)\|+N_2.$$

We choose

$$R \geq \frac{\dfrac{N_1(M+W_2)}{\Gamma(\alpha_1+1)} + \dfrac{N_2(M+W_1)}{\Gamma(\alpha_2+1)}}{1-\left[k_1(M+W_2)+k_2(M+W_1)+\dfrac{L(M+W_2)}{\Gamma(\alpha_1+1)}+\dfrac{\bar{L}(M+W_1)}{\Gamma(\alpha_2+1)}\right]}.$$

Step I: First, we show that $\mathcal{F}(B_R) \subset B_R$, where $B_R = \{(u,v) \in X \times X : \|(u,v)\| \leq R\}$. Let $(u,v) \in B_R$. For $t \in [1,e]$, we have

$$\begin{aligned}
&|\mathcal{F}_1(u,v)(t)| \\
\leq &\frac{1}{|\Delta|}\Bigg\{\Bigg[k_1\int_1^e \frac{|u(s)|}{s}ds + \frac{1}{\Gamma(\alpha_1)}\int_1^e \frac{|f(s,u(s),v(s))|}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_1-1}ds \\
&+\sum_{i=1}^n \frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\Big(k_2\int_1^s\frac{|v(r)|}{r}dr + \frac{1}{\Gamma(\alpha_2)}\int_1^s \frac{|g(r,u(r),v(r))|}{r}\Big(\log\frac{s}{r}\Big)^{\alpha_2-1}dr\Big)ds\Bigg] \\
&+A\Bigg[k_2\int_1^e \frac{|v(s)|}{s}ds + \frac{1}{\Gamma(\alpha_2)}\int_1^e \frac{|g(s,u(s),v(s))|}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_2-1}ds \\
&+\sum_{j=1}^n \frac{|\mu_j|}{\Gamma(\sigma_j)}\int_1^{\eta_j}(\eta_j-s)^{\sigma_j-1}\Big(k_1\int_1^s\frac{|u(r)|}{r}dr + \frac{1}{\Gamma(\alpha_1)}\int_1^s \frac{|f(r,u(r),v(r))|}{r}\Big(\log\frac{s}{r}\Big)^{\alpha_1-1}dr\Big)ds\Bigg]\Bigg\} \\
&+k_1\int_1^t \frac{|u(s)|}{s}ds + \frac{1}{\Gamma(\alpha_1)}\int_1^t \frac{|f(s,u(s),v(s))|}{s}\Big(\log\frac{t}{s}\Big)^{\alpha_1-1}ds \\
\leq &\frac{1}{|\Delta|}\Bigg\{\Bigg[k_1\|u\|\int_1^e \frac{ds}{s} + \frac{L(\|u\|+\|v\|)+N_1}{\Gamma(\alpha_1)}\int_1^e \Big(\log\frac{e}{s}\Big)^{\alpha_1-1}\frac{ds}{s} \\
&+\sum_{i=1}^n \frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\Big(k_2\|v\|\int_1^s\frac{dr}{r} + \frac{\bar{L}(\|u\|+\|v\|)+N_2}{\Gamma(\alpha_2)}\int_1^s \Big(\log\frac{s}{r}\Big)^{\alpha_2-1}\frac{dr}{r}\Big)ds\Bigg] \\
&+A\Bigg[k_2\|v\|\int_1^e \frac{ds}{s} + \frac{\bar{L}(\|u\|+\|v\|)+N_2}{\Gamma(\alpha_2)}\int_1^e \Big(\log\frac{e}{s}\Big)^{\alpha_2-1}\frac{ds}{s} \\
&+\sum_{j=1}^n \frac{|\mu_j|}{\Gamma(\sigma_j)}\int_1^{\eta_j}(\eta_j-s)^{\sigma_j-1}\Big(k_1\|u\|\int_1^s\frac{dr}{r} + \frac{L(\|u\|+\|v\|)+N_1}{\Gamma(\alpha_1)}\int_1^s \Big(\log\frac{s}{r}\Big)^{\alpha_1-1}\frac{dr}{r}\Big)ds\Bigg]\Bigg\} \\
&+k_1\|u\|\int_1^t \frac{ds}{s} + \frac{L(\|u\|+\|v\|)+N_1}{\Gamma(\alpha_1)}\int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha_1-1}\frac{ds}{s} \\
\leq &\frac{1}{|\Delta|}\Bigg\{\Bigg[k_1R + \frac{LR+N_1}{\Gamma(\alpha_1+1)} + \Big(k_2R + \frac{\bar{L}R+N_2}{\Gamma(\alpha_2+1)}\Big)\sum_{i=1}^n|\lambda_i|\frac{(\theta_i-1)^{\delta_i}}{\Gamma(\delta_i+1)}\Bigg] \\
&+A\Bigg[k_2R + \frac{\bar{L}R+N_2}{\Gamma(\alpha_2+1)} + \Big(k_1R + \frac{LR+N_1}{\Gamma(\alpha_1+1)}\Big)\sum_{j=1}^n|\mu_j|\frac{(\eta_j-1)^{\sigma_j}}{\Gamma(\sigma_j+1)}\Bigg]\Bigg\} + k_1R + \frac{LR+N_1}{\Gamma(\alpha_1+1)} \\
\leq &\frac{1}{|\Delta|}\Bigg\{\Bigg[k_1R + \frac{LR+N_1}{\Gamma(\alpha_1+1)} + \omega\Big(k_2R + \frac{\bar{L}R+N_2}{\Gamma(\alpha_2+1)}\Big)\Bigg] \\
&+\omega\Bigg[k_2R + \frac{\bar{L}R+N_2}{\Gamma(\alpha_2+1)} + \bar{\omega}\Big(k_1R + \frac{LR+N_1}{\Gamma(\alpha_1+1)}\Big)\Bigg]\Bigg\} + k_1R + \frac{LR+N_1}{\Gamma(\alpha_1+1)} \\
= &\frac{1}{|\Delta|}\Big(k_1R + \frac{LR+N_1}{\Gamma(\alpha_1+1)}\Big)[1+\omega\bar{\omega}+|\Delta|] + \frac{1}{|\Delta|}\Big(k_2R + \frac{\bar{L}R+N_2}{\Gamma(\alpha_2+1)}\Big)[2\ \omega]
\end{aligned}$$

$$=k_1MR+M\frac{LR+N_1}{\Gamma(\alpha_1+1)}+k_2W_1R+W_1\frac{\bar{L}R+N_2}{\Gamma(\alpha_2+1)}.$$

Taking the norm for $t\in[1,e]$ on the above inequality, we get

$$\|\mathcal{F}_1(u,v)\|\leq k_1MR+k_2W_1R+M\frac{LR+N_1}{\Gamma(\alpha_1+1)}+W_1\frac{\bar{L}R+N_2}{\Gamma(\alpha_2+1)}.$$

Similarly, we obtain

$$\|\mathcal{F}_2(u,v)\|\leq k_2MR+k_1W_2R+M\frac{\bar{L}R+N_2}{\Gamma(\alpha_2+1)}+W_2\frac{LR+N_1}{\Gamma(\alpha_1+1)}.$$

Hence, from (3.20), one gets

$$\begin{aligned}\|\mathcal{F}(u,v)\|=&\|(\mathcal{F}_1(u,v),\mathcal{F}_2(u,v))\|=\|\mathcal{F}_1(u,v)\|+\|\mathcal{F}_2(u,v)\|\\ \leq& R\Big[k_1(M+W_2)+k_2(M+W_1)+\frac{L(M+W_2)}{\Gamma(\alpha_1+1)}+\frac{\bar{L}(M+W_1)}{\Gamma(\alpha_2+1)}\Big]\\ &+\frac{N_1(M+W_2)}{\Gamma(\alpha_1+1)}+\frac{N_2(M+W_1)}{\Gamma(\alpha_2+1)}\\ =&\Xi R+\frac{N_1(M+W_2)}{\Gamma(\alpha_1+1)}+\frac{N_2(M+W_1)}{\Gamma(\alpha_2+1)}\leq R.\end{aligned}$$

Therefore, $\mathcal{F}(B_R)\subset B_R$.

Step II: We will show that the operator $\mathcal{F}$ is a contraction. Let $(u_2,v_2),(u_1,v_1)\in X\times X$. Then, by (H_2), for any $t\in[1,e]$, we have

$$\begin{aligned}&|\mathcal{F}_1(u_2,v_2)(t)-\mathcal{F}_1(u_1,v_1)(t)|\\ \leq&\frac{1}{|\Delta|}\Big\{\Big[k_1\int_1^e\frac{|u_2(s)-u_1(s)|}{s}ds+\frac{1}{\Gamma(\alpha_1)}\int_1^e\frac{|f(s,u_2(s),v_2(s))-f(s,u_1(s),v_1(s))|}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_1-1}ds\\ &+\sum_{i=1}^{n}\frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\Big(k_2\int_1^s\frac{|v_2(r)-v_1(r)|}{r}dr\\ &+\frac{1}{\Gamma(\alpha_2)}\int_1^s\frac{|g(r,u_2(r),v_2(r))-g(r,u_1(r),v_1(r))|}{r}\Big(\log\frac{s}{r}\Big)^{\alpha_2-1}dr\Big)ds\Big]\\ &+A\Big[k_2\int_1^e\frac{|v_2(s)-v_1(s)|}{s}ds+\frac{1}{\Gamma(\alpha_2)}\int_1^e\frac{|g(s,u_2(s),v_2(s))-g(s,u_1(s),v_1(s))|}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_2-1}ds\\ &+\sum_{j=1}^{n}\frac{|\mu_j|}{\Gamma(\sigma_j)}\int_1^{\eta_j}(\eta_j-s)^{\sigma_j-1}\Big(k_1\int_1^s\frac{|u_2(r)-u_1(r)|}{r}dr\\ &+\frac{1}{\Gamma(\alpha_1)}\int_1^s\frac{|f(r,u_2(r),v_2(r))-f(r,u_1(r),v_1(r))|}{r}\Big(\log\frac{s}{r}\Big)^{\alpha_1-1}dr\Big)ds\Big]\Big\}\\ &+k_1\int_1^t\frac{|u_2(s)-u_1(s)|}{s}ds+\frac{1}{\Gamma(\alpha_1)}\int_1^t\frac{|f(s,u_2(s),v_2(s))-f(s,u_1(s),v_1(s))|}{s}\Big(\log\frac{t}{s}\Big)^{\alpha_1-1}ds\\ \leq&\frac{1}{|\Delta|}\Big\{\Big[k_1\|u_2-u_1\|+\frac{L(\|u_2-u_1\|+\|v_2-v_1\|)}{\Gamma(\alpha_1+1)}+\Big(k_2\|v_2-v_1\|+\frac{\bar{L}(\|u_2-u_1\|+\|v_2-v_1\|)}{\Gamma(\alpha_2+1)}\Big)\end{aligned}$$

$$\times \sum_{i=1}^{n} |\lambda_i| \frac{(\theta_i - 1)^{\delta_i}}{\Gamma(\delta_i + 1)}\Big] + \omega \Big[k_2\|v_2 - v_1\| + \frac{\bar{L}(\|u_2 - u_1\| + \|v_2 - v_1\|)}{\Gamma(\alpha_2 + 1)} + \Big(k_1\|u_2 - u_1\|$$

$$+ \frac{L(\|u_2 - u_1\| + \|v_2 - v_1\|)}{\Gamma(\alpha_1 + 1)}\Big) \sum_{j=1}^{n} |\mu_j| \frac{(\eta_j - 1)^{\sigma_j}}{\Gamma(\sigma_j + 1)}\Big]\Big\} + k_1\|u_2 - u_1\| + \frac{L(\|u_2 - u_1\| + \|v_2 - v_1\|)}{\Gamma(\alpha_1 + 1)}$$

$$= M\Big(k_1\|u_2 - u_1\| + \frac{L(\|u_2 - u_1\| + \|v_2 - v_1\|)}{\Gamma(\alpha_1 + 1)}\Big) + W_1\Big(k_2\|v_2 - v_1\| + \frac{\bar{L}(\|u_2 - u_1\| + \|v_2 - v_1\|)}{\Gamma(\alpha_2 + 1)}\Big)$$

$$\leq \Big[k_1 M + \frac{ML}{\Gamma(\alpha_1 + 1)} + k_2 W_1 + \frac{W_1 \bar{L}}{\Gamma(\alpha_2 + 1)}\Big](\|u_2 - u_1\| + \|v_2 - v_1\|)$$

which, upon taking the norm for $t \in [1, e]$, yields

$$\|\mathcal{F}_1(u_2, v_2) - \mathcal{F}_1(u_1, v_1)\| \leq \Big[k_1 M + k_2 W_1 + \frac{ML}{\Gamma(\alpha_1 + 1)} + \frac{W_1 \bar{L}}{\Gamma(\alpha_2 + 1)}\Big](\|u_2 - u_1\| + \|v_2 - v_1\|). \quad (3.21)$$

Similarly,

$$\|\mathcal{F}_2(u_2, v_2) - \mathcal{F}_2(u_1, v_1)\| \leq \Big[k_2 M + k_1 W_2 + \frac{M\bar{L}}{\Gamma(\alpha_2 + 1)} + \frac{W_2 L}{\Gamma(\alpha_1 + 1)}\Big](\|u_2 - u_1\| + \|v_2 - v_1\|). \quad (3.22)$$

It follows from (3.21) and (3.22) that

$$\|\mathcal{F}(u_2, v_2) - \mathcal{F}(u_1, v_1)\| \leq \Xi(\|u_2 - u_1\| + \|v_2 - v_1\|).$$

By (3.20), it shows that the operator $\mathcal{F}$ is a contraction. Therefore, the boundary value problem defined by Eq (1.1) with conditions (1.2) has a unique solution on the interval $[1, e]$. □

4. Ulam stability results

Lastly, we study the Ulam-Hyers and Ulam-Hyers-Rassias stability of the coupled system of Hilfer-Hadamard fractional differential equations (1.1) with the integral boundary conditions (1.2).

Theorem 4.1. *If assumption (H_2) and conditions (3.20) are satisfied, then the boundary value problem defined by Eqs (1.1) and (1.2) is Ulam-Hyers stable, and hence generalized Ulam-Hyers stable.*

Proof. Let $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2) > 0$ be given. Assume (x, y) is a solution of the inequality (2.1) with the boundary conditions (2.4). Then by Remark 2.14, there exist functions $\nu_1, \nu_2 \in C^1([1, e], \mathbb{R})$ such that $|\nu_1(t)| \leq \epsilon_1,\ |\nu_2(t)| \leq \epsilon_2,\ t \in [1, e]$, and

$$\begin{cases} ({}_H D_{1^+}^{\alpha_1, \beta_1} + k_1 {}_H D_{1^+}^{\alpha_1 - 1, \beta_1}) x(t) = f(t, x(t), y(t)) + \nu_1(t), \quad t \in [1, e], \\ ({}_H D_{1^+}^{\alpha_2, \beta_2} + k_2 {}_H D_{1^+}^{\alpha_2 - 1, \beta_2}) y(t) = g(t, x(t), y(t)) + \nu_2(t), \\ x(1) = 0,\ x(e) = \sum_{i=1}^{n} \lambda_i I_{1^+}^{\delta_i} y(\theta_i) = \sum_{i=1}^{n} \frac{\lambda_i}{\Gamma(\delta_i)} \int_1^{\theta_i} (\theta_i - s)^{\delta_i - 1} y(s) ds, \\ y(1) = 0,\ y(e) = \sum_{j=1}^{n} \mu_j I_{1^+}^{\sigma_j} x(\eta_j) = \sum_{j=1}^{n} \frac{\mu_j}{\Gamma(\sigma_j)} \int_1^{\eta_j} (\eta_j - s)^{\sigma_j - 1} x(s) ds, \end{cases} \quad (4.1)$$

where $\theta_i, \eta_j \in (1, e)$. By Lemma 3.1, the solution of (4.1) can be written as

$$\begin{aligned}
x(t) =& \frac{(\log t)^{\gamma_1 - 1}}{\Delta}\bigg\{\bigg[k_1 \int_1^e \frac{x(s)}{s}ds - \frac{1}{\Gamma(\alpha_1)}\int_1^e \frac{f(s,x(s),y(s)) + \nu_1(s)}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_1-1}ds \\
& - \sum_{i=1}^n \frac{\lambda_i}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i - s)^{\delta_i-1}\Big(k_2\int_1^s \frac{y(r)}{r}dr - \frac{1}{\Gamma(\alpha_2)}\int_1^s \frac{g(r,x(r),y(r)) + \nu_2(r)}{r}\Big(\log\frac{s}{r}\Big)^{\alpha_2-1}dr\Big)ds\bigg] \\
& + A\bigg[k_2\int_1^e \frac{y(s)}{s}ds - \frac{1}{\Gamma(\alpha_2)}\int_1^e \frac{g(s,x(s),y(s)) + \nu_2(s)}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_2-1}ds \\
& - \sum_{j=1}^n \frac{\mu_j}{\Gamma(\sigma_j)}\int_1^{\eta_j}(\eta_j - s)^{\sigma_j-1}\Big(k_1\int_1^s \frac{x(r)}{r}dr - \frac{1}{\Gamma(\alpha_1)}\int_1^s \frac{f(r,x(r),y(r)) + \nu_1(r)}{r}\Big(\log\frac{s}{r}\Big)^{\alpha_1-1}dr\Big)ds\bigg]\bigg\} \\
& - k_1\int_1^t \frac{x(s)}{s}ds + \frac{1}{\Gamma(\alpha_1)}\int_1^t \frac{f(s,x(s),y(s)) + \nu_1(s)}{s}\Big(\log\frac{t}{s}\Big)^{\alpha_1-1}ds,
\end{aligned}$$

which can be rearranged as

$$\begin{aligned}
& x(t) - \frac{(\log t)^{\gamma_1 - 1}}{\Delta}\bigg\{\bigg[k_1 \int_1^e \frac{x(s)}{s}ds - \frac{1}{\Gamma(\alpha_1)}\int_1^e \frac{f(s,x(s),y(s))}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_1-1}ds \\
& - \sum_{i=1}^n \frac{\lambda_i}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i - s)^{\delta_i-1}\Big(k_2\int_1^s \frac{y(r)}{r}dr - \frac{1}{\Gamma(\alpha_2)}\int_1^s \frac{g(r,x(r),y(r))}{r}\Big(\log\frac{s}{r}\Big)^{\alpha_2-1}dr\Big)ds\bigg] \\
& + A\bigg[k_2\int_1^e \frac{y(s)}{s}ds - \frac{1}{\Gamma(\alpha_2)}\int_1^e \frac{g(s,x(s),y(s))}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_2-1}ds \\
& - \sum_{j=1}^n \frac{\mu_j}{\Gamma(\sigma_j)}\int_1^{\eta_j}(\eta_j - s)^{\sigma_j-1}\Big(k_1\int_1^s \frac{x(r)}{r}dr - \frac{1}{\Gamma(\alpha_1)}\int_1^s \frac{f(r,x(r),y(r))}{r}\Big(\log\frac{s}{r}\Big)^{\alpha_1-1}dr\Big)ds\bigg]\bigg\} \\
& + k_1\int_1^t \frac{x(s)}{s}ds - \frac{1}{\Gamma(\alpha_1)}\int_1^t \frac{f(s,x(s),y(s))}{s}\Big(\log\frac{t}{s}\Big)^{\alpha_1-1}ds \\
=& \frac{(\log t)^{\gamma_1 - 1}}{\Delta}\bigg\{\bigg[- \frac{1}{\Gamma(\alpha_1)}\int_1^e \frac{\nu_1(s)}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_1-1}ds \\
& - \sum_{i=1}^n \frac{\lambda_i}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i - s)^{\delta_i-1}\Big(- \frac{1}{\Gamma(\alpha_2)}\int_1^s \frac{\nu_2(r)}{r}\Big(\log\frac{s}{r}\Big)^{\alpha_2-1}dr\Big)ds\bigg] \\
& + A\bigg[- \frac{1}{\Gamma(\alpha_2)}\int_1^e \frac{\nu_2(s)}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_2-1}ds \\
& - \sum_{j=1}^n \frac{\mu_j}{\Gamma(\sigma_j)}\int_1^{\eta_j}(\eta_j - s)^{\sigma_j-1}\Big(- \frac{1}{\Gamma(\alpha_1)}\int_1^s \frac{\nu_1(r)}{r}\Big(\log\frac{s}{r}\Big)^{\alpha_1-1}dr\Big)ds\bigg]\bigg\} \\
& + \frac{1}{\Gamma(\alpha_1)}\int_1^t \frac{\nu_1(s)}{s}\Big(\log\frac{t}{s}\Big)^{\alpha_1-1}ds.
\end{aligned}$$

Using $|\nu_i| \le \epsilon_i,\ i = 1, 2$, for any $t \in [1, e]$, we have

$$\bigg| x(t) - \frac{(\log t)^{\gamma_1 - 1}}{\Delta}\bigg\{\bigg[k_1 \int_1^e \frac{x(s)}{s}ds - \frac{1}{\Gamma(\alpha_1)}\int_1^e \frac{f(s,x(s),y(s))}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_1-1}ds$$

$$
\begin{aligned}
&-\sum_{i=1}^{n}\frac{\lambda_i}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\Big(k_2\int_1^s\frac{y(r)}{r}dr-\frac{1}{\Gamma(\alpha_2)}\int_1^s\frac{g(r,x(r),y(r))}{r}\Big(\log\frac{s}{r}\Big)^{\alpha_2-1}dr\Big)ds\Big]\\
&+A\Big[k_2\int_1^e\frac{y(s)}{s}ds-\frac{1}{\Gamma(\alpha_2)}\int_1^e\frac{g(s,x(s),y(s))}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_2-1}ds\\
&-\sum_{j=1}^{n}\frac{\mu_j}{\Gamma(\sigma_j)}\int_1^{\eta_j}(\eta_j-s)^{\sigma_j-1}\Big(k_1\int_1^s\frac{x(r)}{r}dr-\frac{1}{\Gamma(\alpha_1)}\int_1^s\frac{f(r,x(r),y(r))}{r}\Big(\log\frac{s}{r}\Big)^{\alpha_1-1}dr\Big)ds\Big]\Big\}\\
&+k_1\int_1^t\frac{x(s)}{s}ds-\frac{1}{\Gamma(\alpha_1)}\int_1^t\frac{f(s,x(s),y(s))}{s}\Big(\log\frac{t}{s}\Big)^{\alpha_1-1}ds\Big|\\
\le&\frac{\epsilon_1 M}{\Gamma(\alpha_1+1)}+\frac{\epsilon_2 W_1}{\Gamma(\alpha_2+1)}.
\end{aligned}
$$

By virtue of Theorem 3.3, we denote by (u,v) the unique solution of problem (1.1)–(1.2). Then we have $u(t)=\mathcal{F}_1(u,v)(t)$, where $\mathcal{F}_1$ is defined by (3.12). From the above inequality, it follows that

$$
\begin{aligned}
&|x(t)-u(t)|=|x(t)-\mathcal{F}_1(u,v)(t)|\\
\le&\frac{\epsilon_1 M}{\Gamma(\alpha_1+1)}+\frac{\epsilon_2 W_1}{\Gamma(\alpha_2+1)}+\frac{(\log t)^{\gamma_1-1}}{|\Delta|}\Big\{\Big[k_1\int_1^e\frac{|x(s)-u(s)|}{s}ds\\
&+\frac{1}{\Gamma(\alpha_1)}\int_1^e\frac{|f(s,x(s),y(s))-f(s,u(s),v(s))|}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_1-1}ds\\
&+\sum_{i=1}^{n}\frac{\lambda_i}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\Big(k_2\int_1^s\frac{|y(r)-v(r)|}{r}dr\\
&+\frac{1}{\Gamma(\alpha_2)}\int_1^s\frac{|g(r,x(r),y(r))-g(r,u(r),v(r))|}{r}\Big(\log\frac{s}{r}\Big)^{\alpha_2-1}dr\Big)ds\Big]\\
&+A\Big[k_2\int_1^e\frac{|y(s)-v(s)|}{s}ds+\frac{1}{\Gamma(\alpha_2)}\int_1^e\frac{|g(s,x(s),y(s))-g(s,u(s),v(s))|}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_2-1}ds\\
&+\sum_{j=1}^{n}\frac{\mu_j}{\Gamma(\sigma_j)}\int_1^{\eta_j}(\eta_j-s)^{\sigma_j-1}\Big(k_1\int_1^s\frac{|x(r)-u(r)|}{r}dr\\
&+\frac{1}{\Gamma(\alpha_1)}\int_1^s\frac{|f(r,x(r),y(r))-f(r,u(r),v(r))|}{r}\Big(\log\frac{s}{r}\Big)^{\alpha_1-1}dr\Big)ds\Big]\Big\}\\
&+k_1\int_1^t\frac{|x(s)-u(s)|}{s}ds+\frac{1}{\Gamma(\alpha_1)}\int_1^t\frac{|f(s,x(s),y(s))-f(s,x(s),y(s))|}{s}\Big(\log\frac{t}{s}\Big)^{\alpha_1-1}ds.
\end{aligned}
$$

It follows by assumption (H_2) that

$$
\begin{aligned}
\|x-u\|\le&\frac{\epsilon_1 M}{\Gamma(\alpha_1+1)}+\frac{\epsilon_2 W_1}{\Gamma(\alpha_2+1)}+\|x-u\|\Big(k_1M+\frac{ML}{\Gamma(\alpha_1+1)}+\frac{W_1\bar{L}}{\Gamma(\alpha_2+1)}\Big)\\
&+\|y-v\|\Big(k_2W_1+\frac{ML}{\Gamma(\alpha_1+1)}+\frac{W_1\bar{L}}{\Gamma(\alpha_2+1)}\Big).
\end{aligned}\tag{4.2}
$$

Similarly, we have

$$
y(t)=\frac{(\log t)^{\gamma_2-1}}{\Delta}\Big\{\Big[k_2\int_1^e\frac{y(s)}{s}ds-\frac{1}{\Gamma(\alpha_2)}\int_1^e\frac{g(s,x(s),y(s))+\nu_2(s)}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_2-1}ds
$$

$$
\begin{aligned}
&-\sum_{j=1}^{n}\frac{\mu_j}{\Gamma(\sigma_j)}\int_1^{\eta_j}(\eta_j-s)^{\sigma_j-1}\Big(k_1\int_1^s\frac{x(r)}{r}dr-\frac{1}{\Gamma(\alpha_1)}\int_1^s\frac{f(r,x(r),y(r))+\nu_1(r)}{r}\Big(\log\frac{s}{r}\Big)^{\alpha_1-1}dr\Big)ds\Big]\\
&+B\Big[k_1\int_1^e\frac{x(s)}{s}ds-\frac{1}{\Gamma(\alpha_1)}\int_1^e\frac{f(s,x(s),y(s))+\nu_1(s)}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_1-1}ds\\
&-\sum_{i=1}^{n}\frac{\lambda_i}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\Big(k_2\int_1^s\frac{y(r)}{r}dr-\frac{1}{\Gamma(\alpha_2)}\int_1^s\frac{g(r,x(r),y(r))+\nu_2(r)}{r}\Big(\log\frac{s}{r}\Big)^{\alpha_2-1}dr\Big)ds\Big]\Big\}\\
&-k_2\int_1^t\frac{y(s)}{s}ds+\frac{1}{\Gamma(\alpha_2)}\int_1^t\frac{g(s,x(s),y(s))+\nu_2(s)}{s}\Big(\log\frac{t}{s}\Big)^{\alpha_2-1}ds.
\end{aligned}
$$

We get that

$$
\begin{aligned}
\|y-v\|\le&\frac{\epsilon_1W_2}{\Gamma(\alpha_1+1)}+\frac{\epsilon_2M}{\Gamma(\alpha_2+1)}+\|x-u\|\Big(k_1W_2+\frac{W_2L}{\Gamma(\alpha_1+1)}+\frac{M\bar{L}}{\Gamma(\alpha_2+1)}\Big)\\
&+\|y-v\|\Big(k_2M+\frac{W_2L}{\Gamma(\alpha_1+1)}+\frac{M\bar{L}}{\Gamma(\alpha_2+1)}\Big).
\end{aligned}
\tag{4.3}
$$

Solving inequalities (4.2) and (4.3) simultaneously for $\|x-u\|$ and $\|y-v\|$, we obtain

$$
\|x-u\|\le\frac{G_1\epsilon_1+G_2\epsilon_2}{D}\quad\text{and}\quad\|y-v\|\le\frac{B_1\epsilon_1+B_2\epsilon_2}{D},
$$

where

$$
\begin{aligned}
G_1&=\frac{1}{\Gamma(\alpha_1+1)}\Big[M\Big(1-k_2M-\frac{W_2L}{\Gamma(\alpha_1+1)}-\frac{M\bar{L}}{\Gamma(\alpha_2+1)}\Big)+W_2\Big(k_2W_1+\frac{ML}{\Gamma(\alpha_1+1)}+\frac{W_1\bar{L}}{\Gamma(\alpha_2+1)}\Big)\Big],\\
G_2&=\frac{1}{\Gamma(\alpha_2+1)}\Big[W_1\Big(1-k_2M-\frac{W_2L}{\Gamma(\alpha_1+1)}-\frac{M\bar{L}}{\Gamma(\alpha_2+1)}\Big)+M\Big(k_2W_1+\frac{ML}{\Gamma(\alpha_1+1)}+\frac{W_1\bar{L}}{\Gamma(\alpha_2+1)}\Big)\Big],\\
B_1&=\frac{1}{\Gamma(\alpha_1+1)}\Big[W_2\Big(1-k_1M-\frac{ML}{\Gamma(\alpha_1+1)}-\frac{W_1\bar{L}}{\Gamma(\alpha_2+1)}\Big)+M\Big(k_1W_2+\frac{W_2L}{\Gamma(\alpha_1+1)}+\frac{M\bar{L}}{\Gamma(\alpha_2+1)}\Big)\Big],\\
B_2&=\frac{1}{\Gamma(\alpha_2+1)}\Big[M\Big(1-k_1M-\frac{ML}{\Gamma(\alpha_1+1)}-\frac{W_1\bar{L}}{\Gamma(\alpha_2+1)}\Big)+W_1\Big(k_1W_2+\frac{W_2L}{\Gamma(\alpha_1+1)}+\frac{M\bar{L}}{\Gamma(\alpha_2+1)}\Big)\Big],
\end{aligned}
$$

and

$$
\begin{aligned}
D=&\Big(1-k_1M-\frac{ML}{\Gamma(\alpha_1+1)}-\frac{W_1\bar{L}}{\Gamma(\alpha_2+1)}\Big)\Big(1-k_2M-\frac{W_2L}{\Gamma(\alpha_1+1)}-\frac{M\bar{L}}{\Gamma(\alpha_2+1)}\Big)\\
&-\Big(k_1W_2+\frac{W_2L}{\Gamma(\alpha_1+1)}+\frac{M\bar{L}}{\Gamma(\alpha_2+1)}\Big)\Big(k_2W_1+\frac{ML}{\Gamma(\alpha_1+1)}+\frac{W_1\bar{L}}{\Gamma(\alpha_2+1)}\Big).
\end{aligned}
$$

In consequence, we get that

$$
\|x-u\|+\|y-v\|\le\frac{G_1+B_1}{D}\epsilon_1+\frac{G_2+B_2}{D}\epsilon_2.
$$

By virtue of condition (3.20), we note that $\frac{G_1+B_1}{D}>0$ and $\frac{G_2+B_2}{D}>0$. Letting $\boldsymbol{c}_{f,g}=(c_f,c_g):=\Big(\frac{G_1+B_1}{D},\frac{G_2+B_2}{D}\Big)>0$, for each $\boldsymbol{\epsilon}=(\epsilon_1,\epsilon_2)>0$ we get that

$$
\|(x,y)-(u,v)\|=\|x-u\|+\|y-v\|\le c_f\epsilon_1+c_g\epsilon_2=\boldsymbol{c}_{f,g}\boldsymbol{\epsilon}^T.
$$

Hence problem (1.1)–(1.2) is Ulam-Hyers stable. Moreover, it is generalized Ulam-Hyers stable, as $\|v-u\|\le\theta_{f,g}(\boldsymbol{\epsilon})$, with $\theta_{f,g}(\boldsymbol{\epsilon})=\boldsymbol{c}_{f,g}\boldsymbol{\epsilon}^{\boldsymbol{T}}$, $\theta_{f,g}(\mathbf{0})=0$. This completes the proof. □

Theorem 4.2. *Assume that* (H_2) *and conditions (3.20) hold, and that there exists a function* $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)$*, where* $\varphi_i \in C([1,e], \mathbb{R}+)$ *for* $i = 1, 2$*, satisfying* $(H3)$*. Then, the problem defined by Eqs (1.1) and (1.2) is Ulam-Hyers-Rassias stable, and hence generalized Ulam-Hyers-Rassias stable with respect to* $\boldsymbol{\varphi}$*.*

Proof. Let $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2) > 0$ and (x, y) be a solution of the inequality (2.2) with the boundary conditions (2.4). By integration of (2.2) and using (H_3), for any $t \in [1, e]$, one has

$$\begin{aligned}&\bigg|x(t) - c_0(\log t)^{\gamma_1-1} - c_1\Big((\log\ t)^{\gamma_1-2} + k\int_1^t \frac{(\log\ s)^{\gamma-2}}{s}ds\Big) + k\int_1^t \frac{x(s)}{s}ds \\ &- \frac{1}{\Gamma(\alpha)}\int_1^t \frac{f(s,x(s),y(s))}{s}\Big(\log\ \frac{t}{s}\Big)^{\alpha-1}ds\bigg| \leq\ \epsilon_H I_{1^+}^{\alpha}\varphi_1(t) \leq \epsilon\lambda_{\varphi_1}\varphi_1(t),\end{aligned}$$

for all $c_0 = \frac{(\delta_H I_{1^+}^{2-\gamma_1}x)(1)}{\Gamma(\gamma_1)}$, $c_1 = \frac{({}_H I_{1^+}^{2-\gamma_1}x)(1)}{\Gamma(\gamma_1-1)} \in \mathbb{R}$. By virtue of the proof of Lemma 3.1, we will choose c_0 and c_1 such that x in the above inequality also satisfies boundary conditions (2.4) as follows that

$$c_1 = 0 \quad \text{and} \quad c_0 = \frac{J_1 + AJ_2}{\Delta},$$

where A and Δ are defined as in Lemma 3.1, and

$$\begin{aligned}J_1 =& k_1\int_1^e \frac{x(s)}{s}ds - \frac{1}{\Gamma(\alpha_1)}\int_1^e \frac{f(s,x(s),y(s))}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_1-1}ds \\ &- \sum_{i=1}^{n}\frac{\lambda_i}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i - s)^{\delta_i-1}\Big(k_2\int_1^s \frac{y(r)}{r}dr - \frac{1}{\Gamma(\alpha_2)}\int_1^s \frac{g(r,x(r),y(r))}{r}\Big(\log\frac{s}{r}\Big)^{\alpha_2-1}dr\Big)ds, \\ J_2 =& k_2\int_1^e \frac{y(s)}{s}ds - \frac{1}{\Gamma(\alpha_2)}\int_1^e \frac{g(s,x(s),y(s))}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_2-1}ds \\ &- \sum_{j=1}^{n}\frac{\mu_j}{\Gamma(\sigma_j)}\int_1^{\eta_j}(\eta_j - s)^{\sigma_j-1}\Big(k_1\int_1^s \frac{x(r)}{r}dr - \frac{1}{\Gamma(\alpha_1)}\int_1^s \frac{f(r,x(r),y(r))}{r}\Big(\log\frac{s}{r}\Big)^{\alpha_1-1}dr\Big)ds.\end{aligned}$$

Then we have the inequality

$$\begin{aligned}&\bigg|x(t) - \frac{(\log t)^{\gamma_1-1}}{\Delta}\bigg\{\bigg[k_1\int_1^e \frac{x(s)}{s}ds - \frac{1}{\Gamma(\alpha_1)}\int_1^e \frac{f(s,x(s),y(s))}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_1-1}ds \\ &- \sum_{i=1}^{n}\frac{\lambda_i}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i - s)^{\delta_i-1}\Big(k_2\int_1^s \frac{y(r)}{r}dr - \frac{1}{\Gamma(\alpha_2)}\int_1^s \frac{g(r,x(r),y(r))}{r}\Big(\log\frac{s}{r}\Big)^{\alpha_2-1}dr\Big)ds\bigg] \\ &+ A\bigg[k_2\int_1^e \frac{y(s)}{s}ds - \frac{1}{\Gamma(\alpha_2)}\int_1^e \frac{g(s,x(s),y(s))}{s}\Big(\log\frac{e}{s}\Big)^{\alpha_2-1}ds \\ &- \sum_{j=1}^{n}\frac{\mu_j}{\Gamma(\sigma_j)}\int_1^{\eta_j}(\eta_j - s)^{\sigma_j-1}\Big(k_1\int_1^s \frac{x(r)}{r}dr - \frac{1}{\Gamma(\alpha_1)}\int_1^s \frac{f(r,x(r),y(r))}{r}\Big(\log\frac{s}{r}\Big)^{\alpha_1-1}dr\Big)ds\bigg]\bigg\} \\ &+ k_1\int_1^t \frac{x(s)}{s}ds - \frac{1}{\Gamma(\alpha_1)}\int_1^t \frac{f(s,x(s),y(s))}{s}\Big(\log\frac{t}{s}\Big)^{\alpha_1-1}ds\bigg| \\ &< \epsilon_H I_{1^+}^{\alpha}\varphi_1(t) \leq \epsilon\lambda_{\varphi_1}\varphi_1(t), \quad t \in [1, e].\end{aligned}$$

Now, by virtue of Theorem 3.3, we let (u, v) be the unique solution of problem (1.1)–(1.2). That is defined as $u(t) = \mathcal{F}_1(u, v)(t)$, where $\mathcal{F}_1$ is defined by (3.13). From the above inequality, the same method as in the proof of Theorem 4.1, it follows that

$$
\begin{aligned}
|x(t) - u(t)| =& |x(t) - \mathcal{F}_1(u, v)(t)| \le \epsilon \lambda_{\varphi_1} \varphi_1(t) + \frac{(\log t)^{\gamma_1 - 1}}{|\Delta|} \Bigg\{ \Bigg[k_1 \int_1^e \frac{|x(s) - u(s)|}{s} ds \\
&+ \frac{1}{\Gamma(\alpha_1)} \int_1^e \frac{|f(s, x(s), y(s)) - f(s, u(s), v(s))|}{s} \Big(\log \frac{e}{s} \Big)^{\alpha_1 - 1} ds \\
&+ \sum_{i=1}^{n} \frac{\lambda_i}{\Gamma(\delta_i)} \int_1^{\theta_i} (\theta_i - s)^{\delta_i - 1} \Big(k_2 \int_1^s \frac{|y(r) - v(r)|}{r} dr \\
&+ \frac{1}{\Gamma(\alpha_2)} \int_1^s \frac{|g(r, x(r), y(r)) - g(r, u(r), v(r))|}{r} \Big(\log \frac{s}{r} \Big)^{\alpha_2 - 1} dr \Big) ds \Bigg] \\
&+ A \Bigg[k_2 \int_1^e \frac{|y(s) - v(s)|}{s} ds + \frac{1}{\Gamma(\alpha_2)} \int_1^e \frac{|g(s, x(s), y(s)) - g(s, u(s), v(s))|}{s} \Big(\log \frac{e}{s} \Big)^{\alpha_2 - 1} ds \\
&+ \sum_{j=1}^{n} \frac{\mu_j}{\Gamma(\sigma_j)} \int_1^{\eta_j} (\eta_j - s)^{\sigma_j - 1} \Big(k_1 \int_1^s \frac{|x(r) - u(r)|}{r} dr \\
&+ \frac{1}{\Gamma(\alpha_1)} \int_1^s \frac{|f(r, x(r), y(r)) - f(r, u(r), v(r))|}{r} \Big(\log \frac{s}{r} \Big)^{\alpha_1 - 1} dr \Big) ds \Bigg] \Bigg\} \\
&+ k_1 \int_1^t \frac{|x(s) - u(s)|}{s} ds + \frac{1}{\Gamma(\alpha_1)} \int_1^t \frac{|f(s, x(s), y(s)) - f(s, x(s), y(s))|}{s} \Big(\log \frac{t}{s} \Big)^{\alpha_1 - 1} ds,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|x - u\| \le& \epsilon \lambda_{\varphi_1} \varphi_1(t) + \|x - u\| \Big(k_1 M + \frac{ML}{\Gamma(\alpha_1 + 1)} + \frac{W_1 \bar{L}}{\Gamma(\alpha_2 + 1)} \Big) \\
&+ \|y - v\| \Big(k_2 W_1 + \frac{ML}{\Gamma(\alpha_1 + 1)} + \frac{W_1 \bar{L}}{\Gamma(\alpha_2 + 1)} \Big).
\end{aligned}
$$

Similarly, we get that

$$
\begin{aligned}
\|y - v\| \le& \epsilon \lambda_{\varphi_2} \varphi_2(t) + \|x - u\| \Big(k_1 W_2 + \frac{W_2 L}{\Gamma(\alpha_1 + 1)} + \frac{M \bar{L}}{\Gamma(\alpha_2 + 1)} \Big) \\
&+ \|y - v\| \Big(k_2 M + \frac{W_2 L}{\Gamma(\alpha_1 + 1)} + \frac{M \bar{L}}{\Gamma(\alpha_2 + 1)} \Big).
\end{aligned}
$$

Solving the above two equations simultaneously for $\|x - u\|$ and $\|y - v\|$, and then setting D the same as in Theorem 4.1, the preceding inequalities can be written as

$$
\|x - u\| \le \frac{\epsilon (E_1 \varphi_1(t) + E_2 \varphi_2(t))}{D} \quad \text{and} \quad \|y - v\| \le \frac{\epsilon (Q_1 \varphi_1(t) + Q_2 \varphi_2(t))}{D},
$$

where

$$
E_1 = \lambda_{\varphi_1} \Big(1 - k_2 M - \frac{W_2 L}{\Gamma(\alpha_1 + 1)} - \frac{M \bar{L}}{\Gamma(\alpha_2 + 1)} \Big), \quad E_2 = \lambda_{\varphi_2} \Big(k_2 W_1 + \frac{ML}{\Gamma(\alpha_1 + 1)} + \frac{W_1 \bar{L}}{\Gamma(\alpha_2 + 1)} \Big),
$$

$$Q_1 = \lambda_{\varphi_1}\Big(k_1 W_2 + \frac{W_2 L}{\Gamma(\alpha_1+1)} + \frac{M\bar{L}}{\Gamma(\alpha_2+1)}\Big), \quad Q_2 = \lambda_{\varphi_2}\Big(1 - k_1 M - \frac{ML}{\Gamma(\alpha_1+1)} - \frac{W_1\bar{L}}{\Gamma(\alpha_2+1)}\Big).$$

In consequence, by condition (3.15), we get

$$\|x-u\| + \|y-v\| \leq \epsilon(c_{f,\varphi_1}\varphi_1(t) + c_{g,\varphi_2}\varphi_2(t)).$$

Letting $\boldsymbol{c}_{f,g,\boldsymbol{\varphi}} = (c_{f,\varphi_1}, c_{g,\varphi_2}) = \Big(\frac{E_1+Q_1}{D}, \frac{E_2+Q_2}{D}\Big)$, for each $\epsilon > 1$, we have

$$\|(x,y)-(u,v)\| \leq \epsilon \boldsymbol{c}_{f,g,\boldsymbol{\varphi}}[\boldsymbol{\varphi}(t)]^T.$$

Hence problem (1.1)–(1.2) is Ulam-Hyers-Rassias stable with respect to $\boldsymbol{\varphi}$. Moreover, it is generalized Ulam-Hyers-Rassias stable with respect to $\boldsymbol{\varphi}$ and if we take $\epsilon = 1$, then $\|(x,y)-(u,v)\| \leq \boldsymbol{c}_{f,g,\boldsymbol{\varphi}}[\boldsymbol{\varphi}(t)]^T$. This completes the proof. □

5. Examples

In this section, we give two examples to illustrate our main results.

Example 5.1. *Consider the following Hilfer-Hadamard problem:*

$$\begin{cases} \Big({}_H D^{\frac{3}{2},\frac{1}{4}} + \frac{8}{70}{}_H D^{\frac{1}{2},\frac{1}{4}}\Big)u(t) = \frac{\arctan v(t)}{1+u^2(t)}, \qquad t \in [1,e], \\ \Big({}_H D^{\frac{3}{2},\frac{3}{4}} + \frac{1}{31}{}_H D^{\frac{1}{2},\frac{3}{4}}\Big)v(t) = \frac{e^t}{2}(\sin u(t) + \cos v(t)), \\ u(1) = 0, \;\; u(e) = 3I^{\frac{7}{2}}v\Big(\frac{4}{3}\Big) + 13I^{\frac{5}{2}}v\Big(\frac{7}{6}\Big), \;\; v(1) = 0, \;\; v(e) = \frac{1}{10}I^{\frac{7}{4}}u(2) + 7I^{\frac{5}{2}}u\Big(\frac{5}{3}\Big). \end{cases} \tag{5.1}$$

Here, $\alpha_1 = \frac{3}{2},\ \alpha_2 = \frac{3}{2},\ \beta_1 = \frac{1}{4},\ \beta_2 = \frac{3}{4},\ \gamma_1 = \frac{13}{8},\ \gamma_2 = \frac{15}{8},\ k_1 = \frac{8}{70},\ k_2 = \frac{1}{31},\ \lambda_1 = 3,\ \lambda_2 = 13,$ $\delta_1 = \frac{7}{2},\ \delta_2 = \frac{5}{2},\ \theta_1 = \frac{4}{3},\ \theta_2 = \frac{7}{6},\ \mu_1 = \frac{1}{10},\ \mu_2 = 7,\ \sigma_1 = \frac{7}{4},\ \sigma_2 = \frac{5}{2},\ \eta_1 = 2$*, and* $\eta_2 = \frac{5}{3}$*. For each* $t \in [1,e]$*, there exists continuous nonnegative functions* $\vartheta(t) = \frac{\pi}{2}$ *and* $\chi(t) = e^t$ *such that*

$$|f(t,u(t),v(t))| = \left|\frac{\arctan v(t)}{1+u^2(t)}\right| \leq \vartheta(t) \quad \textit{and} \quad |g(t,u(t),v(t))| = \left|\frac{e^t}{2}(\sin u(t) + \cos v(t))\right| \leq \chi(t),$$

and (H_1) *is satisfied. Using the given data, we find that* $M = 2.0421,\ W_1 = 0.0998,$ *and* $W_2 = 1.6545.$ *Therefore, we have*

$$\Pi = k_1(M + W_2) + k_2(M + W_1) \approx 0.4916 < 1.$$

Thus, all the conditions of Theorem 3.2 are satisfied. Therefore, the boundary value problem (5.1) has at least one solution on the interval $[1,e]$*.*

Example 5.2. *Consider the following Hilfer-Hadamard problem:*

$$\begin{cases} \Big({}_H D^{\frac{5}{4},\frac{1}{2}} + \frac{4}{55}{}_H D^{\frac{1}{4},\frac{1}{2}}\Big)u(t) = \frac{|u(t)|}{\sqrt{143+t^2}(5+|u(t)|)} + \frac{1}{100}(1+\log t)|v(t)|, \quad t \in [1,e], \\ \Big({}_H D^{\frac{3}{2},1} + \frac{1}{26}{}_H D^{\frac{1}{2},1}\Big)v(t) = \frac{|u(t)|}{(4+t)^3(1+|u(t)|)} + \frac{\sin v(t)}{(5+t)^2}, \\ u(1) = 0, \;\; u(e) = \frac{5}{9}I^{\frac{5}{2}}v(2) + 6I^{\frac{3}{2}}v\Big(\frac{3}{2}\Big), \;\; v(1) = 0, \;\; v(e) = 5I^{\frac{19}{7}}u\Big(\frac{5}{3}\Big) + \frac{1}{21}I^{\frac{7}{2}}u\Big(\frac{3}{2}\Big). \end{cases} \tag{5.2}$$

Here, $\alpha_1 = \frac{5}{4},\ \alpha_2 = \frac{3}{2},\ \beta_1 = \frac{1}{2},\ \beta_2 = 1,\ \gamma_1 = \frac{13}{8},\ \gamma_2 = 2,\ k_1 = \frac{4}{55},\ k_2 = \frac{1}{26},\ \lambda_1 = \frac{5}{9},\ \lambda_2 = 6,$ $\delta_1 = \frac{5}{2},\ \delta_2 = \frac{3}{2},\ \theta_1 = 2,\ \theta_2 = \frac{3}{2},\ \mu_1 = 5,\ \mu_2 = \frac{1}{21},\ \sigma_1 = \frac{19}{7},\ \sigma_2 = \frac{7}{2},\ \eta_1 = \frac{5}{3},$ *and* $\eta_2 = \frac{3}{2}$. *Consider the functions*

$$f(t,u,v) = \frac{|u|}{\sqrt{143+t^2}(5+|u|)} + \frac{1}{100}(1+\log t)|v| \quad \textit{and} \quad g(t,u,v) = \frac{|u|}{(4+t)^3(1+|u|)} + \frac{\sin v}{(5+t)^2}.$$

We see that hypothesis (H_2) *holds, because, for any* $u_i, v_i \in \mathbb{R}$ *for* $i = 1,2$, *one has*

$$|f(t,u_1,u_2) - f(t,v_1,v_2)| \le \frac{1}{50}(|u_1 - v_1| + |u_2 - v_2|)$$

and

$$|g(t,u_1,u_2) - g(t,v_1,v_2)| \le \frac{1}{36}(|u_1 - v_1| + |u_2 - v_2|),$$

with $L = \frac{1}{50}$ *and* $\bar{L} = \frac{1}{36}$. *Using the given data, we find that* $M = 2.7575,\ W_1 = 3.6622,$ *and* $W_2 = 0.8156$. *Therefore, we have*

$$\Xi = \left[k_1(M+W_2) + k_2(M+W_1) + \frac{L(M+W_2)}{\Gamma(\alpha_1+1)} + \frac{\bar{L}(M+W_1)}{\Gamma(\alpha_2+1)}\right] \approx 0.7040 < 1.$$

Thus, all the conditions of Theorem 3.3 are satisfied. Therefore, the problem defined by Eq (5.2) has a unique solution on the interval $[1,e]$. *Moreover, according to Theorem 4.1, boundary value problem (5.2) is both Ulam-Hyers stable and generalized Ulam-Hyers stable. In addition, by virtue of Theorem 4.2, if there exists a function* $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)$, *where* $\varphi_i \in C([1,e], \mathbb{R}+)$ *for* $i = 1,2$, *satisfying* $(H3)$, *then problem (5.2) is Ulam-Hyers-Rassias stable and generalized Ulam-Hyers-Rassias stable on the interval* $[1,e]$ *with respect to* $\boldsymbol{\varphi}$.

6. Conclusions

This paper presents existence and uniqueness results for a system of Hilfer-Hadamard sequential fractional differential equations (1.1) with multi-point Riemann-Liouville fractional integral boundary conditions (1.2). First by considering a linear variant of the given problem, we converted the nonlinear problem into a fixed point problem. Once the fixed point operator was established, the existence results were derived using Krasnoselskii's fixed point theorem. The Banach fixed point theorem was then applied to achieve the existence and uniqueness result.

Moreover, the sufficient conditions for the stability of the problem in the sense of Ulam-Hyers and Ulam-Hyers-Rassias were determined. We found that if the problem has a unique solution according to the assumptions of Theorem 3.3, it is also Ulam-Hyers stable and generalized Ulam-Hyers stable on $[1,e]$. Furthermore, by adding one more condition for the function $\boldsymbol{\varphi}$ as (H_3), we obtained Ulam-Hyers-Rassias and generalized Ulam-Hyers-Rassias stability results with respect to the function $\boldsymbol{\varphi}$. Additionally, we provide two examples that illustrate the obtained results.

In summary, we obtained existence, uniqueness, and stability results for the coupled system of Hilfer-Hadamard sequential fractional differential equations with multi-point fractional integral boundary conditions, making it applicable to a broader class of mathematical models.

Author contributions

Ugyen Samdrup Tshering: Conceptualization, methodology, writing–original draft preparation, writing–review; Ekkarath Thailert: Conceptualization, methodology, writing–original draft preparation, writing–review and editing, supervision, funding acquisition; Sotiris K. Ntouyas: Writing–review suggestions and editing. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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