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Research article

Barrier option pricing with floating interest rate based on uncertain exponential Ornstein–Uhlenbeck model

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Abstract: A barrier option is a kind of path-dependent option whose return depends on whether the price of the underlying asset reaches a certain barrier level. This paper mainly analyzes European barrier option pricing formulas for the uncertain exponential Ornstein–Uhlenbeck model with a floating interest rate. The corresponding numerical algorithms for the knock-in and knock-out option prices are designed. Several numerical examples are given to study the relationship between barrier option prices and parameters. Finally, a real-data example is presented to illustrate the option pricing formulas.

Keywords: barrier option; exponential Ornstein–Uhlenbeck model; floating interest rate; option pricing; uncertain finance
Mathematics Subject Classification: 91G30, 91G80

1. Introduction

Barrier options are categorized as knock-in and knock-out options and have a reward that is contingent on whether the underlying asset price reaches a specified barrier level within the contract period. Barrier options are an essential tool for risk management in the financial market and are extensively applied to different domains, such as risk control, asset management, and so on. The option holders could control the potential returns and losses by setting different barriers, and the barrier options are usually cheaper than the standard options, which makes them popular with many investors.

In a standard Black–Scholes model, barrier options were first analytically valued by Merton [1], using classical results about the first passage time of Brownian motion to a point that can be traced back to Lévy [2]. Heynen and Kat [3] and Carr [4] pioneered partial and outside barrier options, while Kunitomo and Ikeda [5] first tackled double barrier options, Armstrong [6] first dealt with window barrier options, and Guillaume [7] provided the first closed form formulae for step barrier options. Other seminal contributions that cannot all be cited here tackle non-constant boundaries, either deterministic or stochastic, or have been devoted to numerical approximations of barrier option values in more general models, especially those featuring stochastic volatility or jumps. What all these papers have in common is a stochastic approach based on the martingale method of pricing.

The above studies assumed that stock prices follow a Wiener process. In fact, Liu [8] provided a paradox: the actual stock price is impossible to follow any Ito's stochastic differential equation. Additionally, the application of probability theory requires that the real frequency closely resemble the probability distribution, which means that it must be possible to acquire sufficient samples. However, it is sometimes difficult to obtain enough or no samples for an uncertain event. Therefore, Liu [9] proposed the Liu process and developed the uncertainty theory, which was refined by Liu [10].

Liu [11] applied the uncertainty theory to the field of finance, and he first proposed the uncertain stock model. Afterward, many scholars started investigating the issue of option pricing with uncertainty theory. The uncertain stock model, including a mean-reverting process, was presented by Peng and Yao [12]. Chen and Liu [13] established a new uncertain stock model that has periodic dividends. Liu et al. [14] proposed the uncertain currency model, and Deng and Qin [15] examined the pricing issue of the Parisian option within this framework. Liu et al. [16] discussed the pricing problem of the European option using the Caputo–Hadamard UFDEs to simulate the change in the stock price. Pan et al. [17] investigated the pricing problem of Bermudan options. Yao and Qin [18] investigated the European barrier option using Liu's stock model. Similarly, Yang et al. [19] and Gao et al. [20] examined American and Asian barrier options using the same model. Furthermore, Dai et al. [21] explored a non-linear stock model that is named the uncertain exponential Ornstein–Uhlenbeck model. Based on this model, Liu et al. [22] and Gao et al. [23] investigated the power option and the lookback option, respectively.

We further investigate the price functions of the barrier option with a floating interest rate under the uncertain exponential Ornstein–Uhlenbeck model. Section 2 introduces some definitions and theorems used in the paper. The price formulas of two knock-in options and two knock-out option prices are investigated in Section 3. We design the numerical algorithms to calculate the option prices and provide several numerical examples in Section 4. In Section 5, the values of the knock-in and knock-out options are calculated by utilizing the Shanghai Interbank Offered Rate (SHIBOR) and the closing price for Haitian food. Section 6 gives a concise conclusion.

2. Preliminaries

Definition 2.1. (Liu [11]) *An uncertain process C_t is called the Liu process if the following three conditions are satisfed:*
(1) $C_0 = 0$, and almost all sample paths are Lipschitz continuous,
(2) C_t has stationary and independent increments,
(3) Every increment $C_{s+t} - C_s$ is a normal uncertain variable with an expected value 0 and variance t^2.

Theorem 2.2. (Liu [9]) *Suppose M is an uncertain measure, and for events Λ_1 and Λ_2 with $\Lambda_1 \subset \Lambda_2$, we can obtain*

$$M\{\Lambda_1\} \leq M\{\Lambda_2\}.$$

Theorem 2.3. (Liu [10]) *Let ξ be an uncertain variable with a regular uncertainty distribution Φ. Then*

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha)d\alpha.$$

Definition 2.4. (Yao and Chen [24]) *Assume that $f(t,x)$ and $g(t,x)$ are two continuous functions, respectively, and C_t is a Liu process. Then*

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t$$

is called an uncertain differential equation.

Theorem 2.5. (Yao and Chen [24]) *Suppose X_t and X_t^α be the solution and α-path of the uncertain differential equation*

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t$$

respectively. Then

$$M\{X_t \le X_t^\alpha, \forall t \in [0, T]\} = \alpha,$$

$$M\{X_t > X_t^\alpha, \forall t \in [0, T]\} = 1 - \alpha,$$

and

$$\Phi^{-1}(\alpha) = X_t^\alpha$$

where $\Phi^{-1}(\alpha)$ is the inverse uncertainty distribution of the uncertain variable X_t.

3. Uncertain exponential Ornstein–Uhlenbeck model

Liu [11] first presented the uncertain stock model

$$\begin{cases} dX_t = rX_t dt, \\ dY_t = \mu Y_t dt + \sigma Y_t dC_t, \end{cases} \tag{3.1}$$

where Y_t represents the stock price, μ and σ are the drift item and the diffusion item of Y_t. X_t represents the bond price, the interest rate r is a constant, and C_t is a Liu process. Let B be the strike price and T be the expiration date. Liu [11] studied the European call option pricing formula

$$f^c = \exp(-rT)\int_0^1 \Big(Y_0 \exp\Big(\mu T + \frac{\sqrt{3}\sigma T}{\pi}\ln\frac{\alpha}{1-\alpha}\Big) - B\Big)^+ d\alpha$$

and the European put option pricing formula

$$f^p = \exp(-rT)\int_0^1 \Big(B - Y_0 \exp\Big(\mu T + \frac{\sqrt{3}\sigma T}{\pi}\ln\frac{\alpha}{1-\alpha}\Big)\Big)^+ d\alpha.$$

In the model (3.1), which considers stock price movements in the short term, it is assumed that the interest rate is a fixed constant. However, the stock price varies around a constant rather than rising or falling constantly in the long term. In order to improve the model (3.1) to reflect the real financial markets, it is vital to take into account the volatility of interest rates. Then, Sun and Su [25] presented the following model:

$$\begin{cases} dr_t = (a_1 - b_1 r_t)dt + \sigma_1 dC_{1t}, \\ dY_t = (a_2 - b_2 Y_t)dt + \sigma_2 dC_{2t}, \end{cases} \tag{3.2}$$

where r_t denotes the floating interest rate, a_1, a_2, b_1, b_2, σ_1 and σ_2 are positive constants, and $b_1 \neq 0$, $b_2 \neq 0$, C_{1t} and C_{2t} are two mutually independent Liu processes.

The above uncertain stock models are both linear. Liu [26] explored a nonlinear model with a floating interest rate that can better reflect the financial markets compared to the linear models

$$\begin{cases} dr_t = (m - ar_t)dt + \sigma_1 dC_{1t}, \\ dY_t = \mu(1 - c\ln Y_t)Y_t dt + \sigma_2 Y_t dC_{2t}, \end{cases} \tag{3.3}$$

where m, a, c, σ_1 and σ_2 are positive constants with $a \neq 0$, μ is a constant.

The model (3.3) is an uncertain exponential Ornstein–Uhlenbeck stock model that takes into account a floating interest rate. It ensures that the stock price is non-negative and does not fluctuate dramatically in a short period of time. Hence, we investigate the knock-in and knock-out options and derive the price functions under the model (3.3).

3.1. Knock-in options

This part primarily investigates the European up-and-in call option and the down-and-in put option. Suppose a barrier option has a barrier level D, a maturity date T, and an exercise price B. We define an indicator function to easily describe the barrier option

$$I_D(y) = \begin{cases} 1, & \text{if } y \geq D, \\ 0, & \text{if } y < D, \end{cases}$$

where D is a specified constant.

For an up-and-in call option, the initial asset price is below the barrier level, and the option is activated only when the price moves up to the barrier level before the expiration date.

Let C_{ui} be the option price. The investor buys the option with C_{ui} at the initial time and has a payoff

$$I_D\Big(\sup_{0\leq t\leq T} Y_t\Big)(Y_T - B)^+$$

at the expiration time T. The present value of the return is

$$\exp\Big(-\int_0^T r_t dt\Big) I_D\Big(\sup_{0\leq t\leq T} Y_t\Big)(Y_T - B)^+.$$

Since money has a time value. At the initial moment, the net income of the investor is

$$-C_{ui} + \exp\Big(-\int_0^T r_t dt\Big) I_D\Big(\sup_{0\leq t\leq T} Y_t\Big)(Y_T - B)^+.$$

The seller receives C_{ui} for selling the option and pays the investor

$$I_D\Big(\sup_{0\leq t\leq T} Y_t\Big)(Y_T - B)^+.$$

Similarly, the seller has a net income

$$C_{ui} - \exp\Big(-\int_0^T r_t dt\Big) I_D\Big(\sup_{0\leq t\leq T} Y_t\Big)(Y_T - B)^+$$

at the initial moment.

The fair option price should ensure the investor and the seller get the same expected return. Hence, the option price is

$$C_{ui} = E\Big[\exp\Big(-\int_0^T r_t dt\Big) I_D\Big(\sup_{0\le t\le T} Y_t\Big)(Y_T - B)^+\Big].$$

Theorem 3.1. *Assume that a European up-and-in call option for the uncertain exponential Ornstein–Uhlenbeck model* (3.3) *has a barrier level D, a maturity date T, and a strike price B. Then the option price is*

$$C_{ui} = \int_\theta^1 \exp\Big(\frac{r_0-\gamma}{a}\Big(\exp(-aT)-1\Big)-\gamma T\Big)\Big(Y_T^\alpha - B\Big)^+ d\alpha,$$

where

$$\theta = \Big(1+\exp\Big(\frac{\mu c\pi(\ln Y_0 \exp(-\mu cT) - \ln D)}{\sqrt{3}\sigma_2 - \sqrt{3}\sigma_2\exp(-\mu cT)} + \frac{\mu\pi}{\sqrt{3}\sigma_2}\Big)\Big)^{-1}$$

$$\gamma = \frac{m}{a} + \frac{\sqrt{3}\sigma_1}{\pi a}\ln\frac{1-\alpha}{\alpha},$$

and

$$Y_t^\alpha = \exp\Big(\Big(\frac{1}{c} + \frac{\sqrt{3}\sigma_2}{\mu c\pi}\ln\frac{\alpha}{1-\alpha}\Big)\Big(1-\exp(-\mu ct)\Big) + \ln Y_0\exp(-\mu ct)\Big)$$

is the α-path of Y_t.

Proof. First, we prove that

$$\exp\Big(-\int_0^T r_t^{1-\alpha} dt\Big) I_D\Big(\sup_{0\le t\le T} Y_t^\alpha\Big)(Y_T^\alpha - B)^+$$

is the inverse uncertainty distribution of

$$\exp\Big(-\int_0^T r_t dt\Big) I_D\Big(\sup_{0\le t\le T} Y_t\Big)(Y_T - B)^+.$$

Define two events

$$\Lambda_1 : \Big\{\exp\Big(-\int_0^T r_t dt\Big) I_D\Big(\sup_{0\le t\le T} Y_t\Big)(Y_T - B)^+ \le \exp\Big(-\int_0^T r_t^{1-\alpha} dt\Big) I_D\Big(\sup_{0\le t\le T} Y_t^\alpha\Big)(Y_T^\alpha - B)^+\Big\}$$

and

$$\Lambda_2 : \Big\{\exp\Big(-\int_0^T r_t dt\Big) I_D\Big(\sup_{0\le t\le T} Y_t\Big)(Y_T - B)^+ > \exp\Big(-\int_0^T r_t^{1-\alpha} dt\Big) I_D\Big(\sup_{0\le t\le T} Y_t^\alpha\Big)(Y_T^\alpha - B)^+\Big\}$$

where $r_t^{1-\alpha}$ is the α-path of r_t.

Since

$$\Lambda_1 \supset \Big\{r_t \ge r_t^{1-\alpha}, \sup_{0\le t\le T} Y_t \le \sup_{0\le t\le T} Y_t^\alpha, Y_T \le Y_T^\alpha\Big\} \supset \Big\{r_t \ge r_t^{1-\alpha}, Y_t \le Y_t^\alpha, \forall t\Big\},$$

we obtain

$$M\{\Lambda_1\} \geq M\left\{r_t \geq r_t^{1-\alpha}, Y_t \leq Y_t^\alpha, \forall t\right\} = M\left\{r_t \geq r_t^{1-\alpha}, \forall t\right\} \wedge M\left\{Y_t \leq Y_t^\alpha, \forall t\right\} = \alpha.$$

Similarly, because of

$$\Lambda_2 \supset \left\{r_t < r_t^{1-\alpha}, \sup_{0\leq t\leq T} Y_t > \sup_{0\leq t\leq T} Y_t^\alpha, Y_T > Y_T^\alpha\right\} \supset \left\{r_t < r_t^{1-\alpha}, Y_t > Y_t^\alpha, \forall t\right\},$$

we obtain

$$M\{\Lambda_2\} \geq M\left\{r_t < r_t^{1-\alpha}, Y_t > Y_t^\alpha, \forall t\right\} = M\left\{r_t < r_t^{1-\alpha}, \forall t\right\} \wedge M\left\{Y_t > Y_t^\alpha, \forall t\right\} = 1-\alpha.$$

According to the duality axiom, we obtain $M\{\Lambda_1\} + M\{\Lambda_2\} = 1$, which means that $M\{\Lambda_1\} = \alpha$. Hence, we obtain

$$\exp\Big(-\int_0^T r_t dt\Big) I_D\Big(\sup_{0\leq t\leq T} Y_t\Big)(Y_T - B)^+$$

has an inverse uncertainty distribution

$$\exp\Big(-\int_0^T r_t^{1-\alpha} dt\Big) I_D\Big(\sup_{0\leq t\leq T} Y_t^\alpha\Big)(Y_T^\alpha - B)^+.$$

By using the calculation formula for the expected value, we have

$$C_{ui} = \int_0^1 \exp\Big(-\int_0^T r_t^{1-\alpha} dt\Big) I_D\Big(\sup_{0\leq t\leq T} Y_t^\alpha\Big)\Big(Y_T^\alpha - B\Big)^+ d\alpha.$$

Note that r_t^α satisfies the following ordinary differential equation:

$$dr_t^\alpha = (m - ar_t^\alpha)dt + \sigma_1 \Phi^{-1}(\alpha)dt.$$

So it is easily verified that

$$r_t^{1-\alpha} = r_0 \exp(-at) + \Big(\frac{m}{a} + \frac{\sqrt{3}\sigma_1}{\pi a} \ln \frac{1-\alpha}{\alpha}\Big)\Big(1 - \exp(-at)\Big).$$

From Theorem 6 in [21], we obtain

$$Y_t^\alpha = \exp\Big(\Big(\frac{1}{c} + \frac{\sqrt{3}\sigma_2}{\mu c\pi} \ln \frac{\alpha}{1-\alpha}\Big)\Big(1 - \exp(-\mu ct)\Big) + \ln Y_0 \exp(-\mu ct)\Big).$$

Additionally, the equation

$$I_D\Big(\sup_{0\leq t\leq T} Y_t^\alpha\Big) = 1$$

is equivalent to

$$\sup_{0\leq t\leq T} Y_t^\alpha \geq D.$$

Note that $Y_0 < D$ in the up-and-in option and Y_t^α is a monotone function of t. Y_t^α increases with t, we can obtain

$$\sup_{0\leq t\leq T} Y_t^\alpha = Y_T^\alpha,$$

which implies that

$$I_D\Big(\sup_{0\le t\le T} Y_t^\alpha\Big) = 1$$

and

$$Y_T^\alpha \ge D.$$

Conversely, if Y_t^α decreases with t, we obtain

$$\sup_{0\le t\le T} Y_t^\alpha = Y_0 < D.$$

It means that

$$I_D\Big(\sup_{0\le t\le T} Y_t^\alpha\Big) = 0,$$

which contradicts with

$$I_D\Big(\sup_{0\le t\le T} Y_t^\alpha\Big) = 1.$$

Therefore, Y_t^α is an increasing function of t. We derive that

$$\exp\Big(\Big(\frac{1}{c} + \frac{\sqrt{3}\sigma_2}{\mu c\pi}\ln\frac{\alpha}{1-\alpha}\Big)\Big(1-\exp(-\mu cT)\Big) + \ln Y_0\exp(-\mu cT)\Big) \ge D,$$

which implies that

$$\alpha \ge \Big(1 + \exp\Big(\frac{\mu c\pi(\ln Y_0\exp(-\mu cT) - \ln D)}{\sqrt{3}\sigma_2 - \sqrt{3}\sigma_2\exp(-\mu cT)} + \frac{\mu\pi}{\sqrt{3}\sigma_2}\Big)\Big)^{-1} = \theta.$$

Consequently, we can rewrite the option price as

$$\begin{aligned}
C_{ui} &= \int_0^1 \exp\Big(-\int_0^T r_t^{1-\alpha}dt\Big) I_D\Big(\sup_{0\le t\le T} Y_t^\alpha\Big)\Big(Y_T^\alpha - B\Big)^+ d\alpha \\
&= \int_\theta^1 \exp\Big(-\int_0^T r_t^{1-\alpha}dt\Big)(Y_T^\alpha - B)^+ d\alpha \\
&= \int_\theta^1 \exp\Big(\frac{r_0-\gamma}{a}\Big(\exp(-aT)-1\Big) - \gamma T\Big)\Big(Y_T^\alpha - B\Big)^+ d\alpha.
\end{aligned}$$

□

For a down-and-in put option, the initial asset price is above the barrier level, and the option is not activated until the price decreases to the barrier level before the expiration date.

Let P_{di} be the option price. The investor purchases the option at the initial time for P_{di} and has a payoff of

$$\Big(1 - I_D\Big(\inf_{0\le t\le T} Y_t\Big)\Big)(B - Y_T)^+.$$

The present value of the return is

$$\exp\Big(-\int_0^T r_t dt\Big)\Big(1 - I_D\Big(\inf_{0\le t\le T} Y_t\Big)\Big)(B - Y_T)^+.$$

Then, the investor's net return is

$$-P_{di} + \exp\Big(-\int_0^T r_t dt\Big)\Big(1 - I_D\Big(\inf_{0\le t\le T} Y_t\Big)\Big)(B - Y_T)^+.$$

Similarly, the seller's net return is

$$P_{di} - \exp\Big(-\int_0^T r_t dt\Big)\Big(1 - I_D\Big(\inf_{0\le t\le T} Y_t\Big)\Big)(B - Y_T)^+.$$

Hence, the option price is

$$P_{di} = E\Big[\exp\Big(-\int_0^T r_t dt\Big)\Big(1 - I_D\Big(\inf_{0\le t\le T} Y_t\Big)\Big)\Big(B - Y_T\Big)^+\Big].$$

Corollary 3.1. *Assume that a European down-and-in put option for the uncertain exponential Ornstein–Uhlenbeck model* (3.3) *has a barrier level D, a maturity date T, and a strike price B. Then the price of this option is*

$$P_{di} = \int_0^\theta \exp\Big(\frac{r_0 - \eta}{a}\Big(\exp(-aT) - 1\Big) - \eta T\Big)\Big(B - Y_T^\alpha\Big)^+ d\alpha,$$

where

$$\theta = \Big(1 + \exp\Big(\frac{\mu c\pi(\ln Y_0 \exp(-\mu cT) - \ln D)}{\sqrt{3}\sigma_2 - \sqrt{3}\sigma_2\exp(-\mu cT)} + \frac{\mu\pi}{\sqrt{3}\sigma_2}\Big)\Big)^{-1},$$

$$\eta = \frac{m}{a} + \frac{\sqrt{3}\sigma_1}{\pi a}\ln\frac{\alpha}{1-\alpha},$$

and

$$Y_t^\alpha = \exp\Big(\Big(\frac{1}{c} + \frac{\sqrt{3}\sigma_2}{\mu c\pi}\ln\frac{\alpha}{1-\alpha}\Big)\Big(1 - \exp(-\mu ct)\Big) + \ln Y_0\exp(-\mu ct)\Big)$$

is the α-path of Y_t.

3.2. Knock-out options

This part investigates two European knock-out options under the uncertain exponential Ornstein–Uhlenbeck model (3.3), including the up-and-out option and the down-and-out option.

The down-and-out call option is a contract whose price of the asset is above the barrier level at the beginning of the transaction. The option is void until the price decreases to the barrier level before the expiration date.

Let C_{do} be the option price. The investor buys an option with C_{do} at the initial time and has a payoff

$$\exp\Big(-\int_0^T r_t dt\Big)I_D\Big(\inf_{0\le t\le T} Y_t\Big)(Y_T - B)^+.$$

Then the investor's net return is

$$-C_{do} + \exp\Big(-\int_0^T r_t dt\Big)I_D\Big(\inf_{0\le t\le T} Y_t\Big)(Y_T - B)^+.$$

And the seller receives C_{do} for selling the option at the initial time and has a net return

$$C_{do} - \exp\Big(-\int_0^T r_t dt\Big) I_D\Big(\inf_{0\le t\le T} Y_t\Big)(Y_T - B)^+.$$

Therefore, the option price is

$$C_{do} = E\Big[\exp\Big(-\int_0^T r_t dt\Big) I_D\Big(\inf_{0\le t\le T} Y_t\Big)(Y_T - B)^+\Big].$$

Theorem 3.2. *Assume that a European down-and-out call option for the uncertain exponential Ornstein–Uhlenbeck model* (3.3) *has a barrier level D, a maturity date T, and a strike price B. Then the option price is*

$$C_{do} = \int_\theta^1 \exp\Big(\frac{r_0 - \gamma}{a}\Big(\exp(-aT) - 1\Big) - \gamma T\Big)\Big(Y_T^\alpha - B\Big)^+ d\alpha,$$

where

$$\theta = \Big(1 + \exp\Big(\frac{\mu c\pi(\ln Y_0 \exp(-\mu cT) - \ln D)}{\sqrt{3}\sigma_2 - \sqrt{3}\sigma_2 \exp(-\mu cT)} + \frac{\mu\pi}{\sqrt{3}\sigma_2}\Big)\Big)^{-1},$$

$$\gamma = \frac{m}{a} + \frac{\sqrt{3}\sigma_1}{\pi a}\ln\frac{1-\alpha}{\alpha},$$

and

$$Y_t^\alpha = \exp\Big(\Big(\frac{1}{c} + \frac{\sqrt{3}\sigma_2}{\mu c\pi}\ln\frac{\alpha}{1-\alpha}\Big)\Big(1 - \exp(-\mu ct)\Big) + \ln Y_0 \exp(-\mu ct)\Big)$$

is the α-path of Y_t.

Proof. First, we prove that

$$\exp\Big(-\int_0^T r_t^{1-\alpha} dt\Big) I_D\Big(\inf_{0\le t\le T} Y_t^\alpha\Big)(Y_T^\alpha - B)^+$$

is the inverse uncertainty distribution of

$$\exp\Big(-\int_0^T r_t dt\Big) I_D\Big(\inf_{0\le t\le T} Y_t\Big)(Y_T - B)^+.$$

Define two events

$$\Lambda_1 : \Big\{\exp\Big(-\int_0^T r_t dt\Big) I_D\Big(\inf_{0\le t\le T} Y_t\Big)(Y_T - B)^+ \le \exp\Big(-\int_0^T r_t^{1-\alpha} dt\Big) I_D\Big(\inf_{0\le t\le T} Y_t^\alpha\Big)(Y_T^\alpha - B)^+\Big\}$$

and

$$\Lambda_2 : \Big\{\exp\Big(-\int_0^T r_t dt\Big) I_D\Big(\inf_{0\le t\le T} Y_t\Big)(Y_T - B)^+ > \exp\Big(-\int_0^T r_t^{1-\alpha} dt\Big) I_D\Big(\inf_{0\le t\le T} Y_t^\alpha\Big)(Y_T^\alpha - B)^+\Big\},$$

where $r_t^{1-\alpha}$ is the α-path of r_t.

Since

$$\Lambda_1 \supset \left\{r_t \geq r_t^{1-\alpha}, \inf_{0\leq t\leq T} Y_t \leq \inf_{0\leq t\leq T} Y_t^{\alpha}, Y_T \leq Y_T^{\alpha}\right\} \supset \{r_t \geq r_t^{1-\alpha}, Y_t \leq Y_t^{\alpha}, \forall t\},$$

we obtain

$$M\{\Lambda_1\} \geq M\{r_t \geq r_t^{1-\alpha}, Y_t \leq Y_t^{\alpha}, \forall t\} = M\left\{r_t \geq r_t^{1-\alpha}, \forall t\right\} \wedge M\left\{Y_t \leq Y_t^{\alpha}, \forall t\right\} = \alpha.$$

Similarly, due to

$$\Lambda_2 \supset \left\{r_t < r_t^{1-\alpha}, \inf_{0\leq t\leq T} Y_t > \inf_{0\leq t\leq T} Y_t^{\alpha}, Y_T > Y_T^{\alpha}\right\} \supset \{r_t < r_t^{1-\alpha}, Y_t > Y_t^{\alpha}, \forall t\},$$

we obtain

$$M\{\Lambda_2\} \geq M\left\{r_t < r_t^{1-\alpha}, Y_t > Y_t^{\alpha}, \forall t\right\} = M\left\{r_t < r_t^{1-\alpha}, \forall t\right\} \wedge M\left\{Y_t > Y_t^{\alpha}, \forall t\right\} = 1-\alpha.$$

According to the duality axiom, we obtain

$$M\{\Lambda_1\} + M\{\Lambda_2\} = 1,$$

which indicates that

$$M\{\Lambda_1\} = \alpha.$$

Thus, we obtain

$$\exp\Big(-\int_0^T r_t dt\Big) I_D\Big(\inf_{0\leq t\leq T} Y_t\Big)(Y_T - B)^+$$

has an inverse uncertainty distribution

$$\exp\Big(-\int_0^T r_t^{1-\alpha} dt\Big) I_D\Big(\inf_{0\leq t\leq T} Y_t^{\alpha}\Big)(Y_T^{\alpha} - B)^+.$$

From the calculation formula for the expected value, we obtain

$$C_{do} = \int_0^1 \exp\Big(-\int_0^T r_t^{1-\alpha} dt\Big) I_D\Big(\inf_{0\leq t\leq T} Y_t^{\alpha}\Big)\Big(Y_T^{\alpha} - B\Big)^+ d\alpha.$$

Additionally, note that

$$I_D\Big(\inf_{0\leq t\leq T} Y_t^{\alpha}\Big) = 1$$

is equivalent to

$$\inf_{0\leq t\leq T} Y_t^{\alpha} \geq D.$$

Note that $Y_0 > D$, and Y_t^{α} is a monotonic function of time t. If Y_t^{α} increases with t, we can obtain

$$\inf_{0\leq t\leq T} Y_t^{\alpha} = Y_0 > D,$$

which implies

$$I_D\Big(\inf_{0\leq t\leq T} Y_t^{\alpha}\Big) = 1.$$

Conversely, if Y_t^α decreases with t, we obtain

$$\inf_{0\le t\le T} Y_t^\alpha = Y_T^\alpha,$$

which means that

$$I_D\Big(\inf_{0\le t\le T} Y_t^\alpha\Big) = 1$$

and $Y_T^\alpha \ge D$.

Therefore, we deduce that

$$\exp\Big(\Big(\frac{1}{c} + \frac{\sqrt{3}\sigma_2}{\mu c\pi}\ln\frac{\alpha}{1-\alpha}\Big)\Big(1-\exp(-\mu cT)\Big) + \ln Y_0 \exp(-\mu cT)\Big) \ge D,$$

which indicates that

$$\alpha \ge \Big(1 + \exp\Big(\frac{\mu c\pi(\ln Y_0\exp(-\mu cT) - \ln D)}{\sqrt{3}\sigma_2 - \sqrt{3}\sigma_2\exp(-\mu cT)} + \frac{\mu\pi}{\sqrt{3}\sigma_2}\Big)\Big)^{-1} = \theta.$$

Consequently, we can rewrite the option price as

$$\begin{aligned} C_{do} &= \int_0^1 \exp\Big(-\int_0^T r_t^{1-\alpha}dt\Big) I_D\Big(\inf_{0\le t\le T} Y_t^\alpha\Big)\Big(Y_T^\alpha - B\Big)^+ d\alpha \\ &= \int_\theta^1 \exp\Big(-\int_0^T r_t^{1-\alpha}dt\Big)(Y_T^\alpha - B)^+ d\alpha \\ &= \int_\theta^1 \exp\Big(\frac{r_0-\gamma}{a}\Big(\exp(-aT)-1\Big) - \gamma T\Big)\Big(Y_T^\alpha - B\Big)^+ d\alpha. \end{aligned}$$

□

The up-and-out put option is a contract whose price of the asset is below the barrier level at the beginning of the transaction. The option is invalid until the price exceeds the barrier level before the expiration date.

Let P_{uo} be the option price. The investor buys an option with P_{uo} at the initial moment and has a payoff

$$\exp\Big(-\int_0^T r_t dt\Big)\Big(1 - I_D\Big(\sup_{0\le t\le T} Y_t\Big)\Big)(B - Y_T)^+.$$

Then, the investor's net return at the initial time is

$$-P_{uo} + \exp\Big(-\int_0^T r_t dt\Big)\Big(1 - I_D\Big(\sup_{0\le t\le T} Y_t\Big)\Big)(B - Y_T)^+.$$

And the seller receives P_{uo} for selling the option at the initial time and has a net return

$$P_{uo} - \exp\Big(-\int_0^T r_t dt\Big)\Big(1 - I_D\Big(\sup_{0\le t\le T} Y_t\Big)\Big)(B - Y_T)^+.$$

Therefore, the option price is

$$P_{uo} = E\Big[\exp\Big(-\int_0^T r_t dt\Big)\Big(1 - I_D\Big(\sup_{0\le t\le T} Y_t\Big)\Big)\Big(B - Y_T\Big)^+\Big].$$

Corollary 3.2. *Assume that a European up-and-out put option for the uncertain exponential Ornstein–Uhlenbeck model* (3.3) *has a barrier level D, a maturity date T, and a strike price B. Then the price of this option is*

$$P_{uo} = \int_0^{\theta} \exp\Big(\frac{r_0 - \eta}{a}\Big(\exp(-aT) - 1\Big) - \eta T\Big)\Big(B - Y_T^{\alpha}\Big)^{+} d\alpha,$$

where

$$\theta = \Big(1 + \exp\Big(\frac{\mu c\pi(\ln Y_0 \exp(-\mu cT) - \ln D)}{\sqrt{3}\sigma_2 - \sqrt{3}\sigma_2\exp(-\mu cT)} + \frac{\mu\pi}{\sqrt{3}\sigma_2}\Big)\Big)^{-1},$$

$$\eta = \frac{m}{a} + \frac{\sqrt{3}\sigma_1}{\pi a}\ln\frac{\alpha}{1-\alpha},$$

and

$$Y_t^{\alpha} = \exp\Big(\Big(\frac{1}{c} + \frac{\sqrt{3}\sigma_2}{\mu c\pi}\ln\frac{\alpha}{1-\alpha}\Big)\Big(1 - \exp(-\mu ct)\Big) + \ln Y_0 \exp(-\mu ct)\Big)$$

is the α-path of Y_t.

4. Numerical experiments

This section focuses on developing numerical methods to compute the prices of knock-in options and analyzing the effects of different parameters on the option values. The numerical algorithms for calculating the knock-out option prices are similar to those presented in this section. Furthermore, the effects of the parameters on the knock-out option prices can be analyzed in the same way.

4.1. Up-and-in call option

The algorithm for calculating the price C_{ui} is designed according to Theorem 3.1.

Step 0: Set the values of r_0, m, a , σ_1, Y_0, μ, c , σ_2, B, T, and D.

Step 1: Calculate

$$\theta = \Big(1 + \exp\Big(\frac{\mu c\pi(\ln Y_0 \exp(-\mu cT) - \ln D)}{\sqrt{3}\sigma_2 - \sqrt{3}\sigma_2\exp(-\mu cT)} + \frac{\mu\pi}{\sqrt{3}\sigma_2}\Big)\Big)^{-1}.$$

Step 2: Set $\alpha_j = \theta + j(1-\theta)/N$, $j = 1, 2, ..., N-1$, where N is a large positive integer.

Step 3: Set $j = 0$.

Step 4: Set $j \leftarrow j + 1$.

Step 5: Compute the positive deviation

$$\begin{aligned} Z_j =& (Y_T^{\alpha_j} - B)^{+} \\ =& \max(Y_T^{\alpha_j} - B, 0) \\ =& \max\Big(\exp\Big(\Big(\frac{1}{c} + \frac{\sqrt{3}\sigma_2}{\mu c\pi}\ln\frac{\alpha_j}{1-\alpha_j}\Big)\Big(1 - \exp(-\mu cT)\Big) + \ln Y_0 \exp(-\mu cT)\Big) - B, 0\Big). \end{aligned}$$

Step 6: Calculate

$$G_j = \exp\Big(\Big(\frac{r_0}{a} - \frac{m}{a^2} - \frac{\sigma_1\sqrt{3}}{a^2\pi}\ln\frac{1-\alpha_j}{\alpha_j}\Big)\Big(\exp(-aT) - 1\Big) - \Big(\frac{m}{a} + \frac{\sigma_1\sqrt{3}}{a\pi}\ln\frac{1-\alpha_j}{\alpha_j}\Big)T\Big)$$

and $W_j = Z_j \times G_j$. Return to **Step 4** if $j < N - 1$.
Step 7: The option price function is

$$C_{ui} = \frac{1-\theta}{N-1} \sum_{j=1}^{N-1} W_j.$$

Example 4.1. *Assume the initial interest rate* $r_0 = 0.03$*, the initial stock price* $Y_0 = 16$*, and other parameters of model* (3.3) *are* $m = 0.01, a = 0.8, \sigma_1 = 0.01, \mu = 0.9, c = 0.35$*, and* $\sigma_2 = 0.1$*, and the parameters of the option are* $B = 18, T = 5$*, and* $D = 20$*, respectively. Then the price* C_{ui} *is* 1.3657.

It is noted that there are many parameters in the pricing formula of C_{ui} in Theorem 3.1. Next, we investigate the influence of the parameters on the price C_{ui} through numerical experiments. Several examples are given to illustrate the change of C_{ui} on one parameter, in which the other parameters are consistent with Example 4.1.

First, the strike price B and the parameter m are discussed.

Example 4.2. *Let the strike price B change from 18 to 23 with step 0.01, and the other parameters remain unchanged. Figure 1 displays the results.*

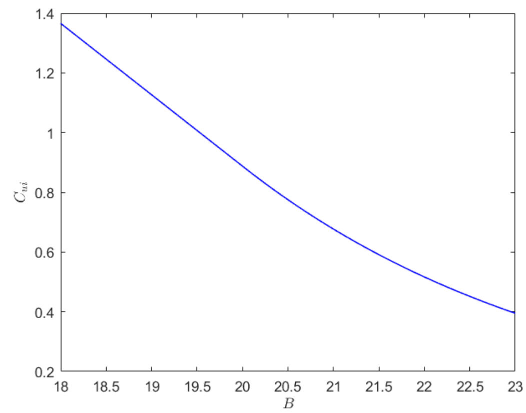

Figure 1. Variation of the price C_{ui} with B.

It illustrates a negative correlation between the price C_{ui} and B in Figure 1. The result can be explained intuitively from the option pricing formula in Theorem 3.1. The strike price B appears only in the positive deviation, which demonstrates that C_{ui} has a monotonically decreasing relationship with B. Similarly, it can be shown that the price C_{ui} demonstrates a monotonically decreasing relationship with m.

Then, we study the parameter σ_2, the maturity time T, and the barrier level D.

Example 4.3. *Let the parameter* σ_2 *change from 0.01 to 0.2 with step 0.01, and the other parameters remain unchanged. Figure 2 displays the results.*

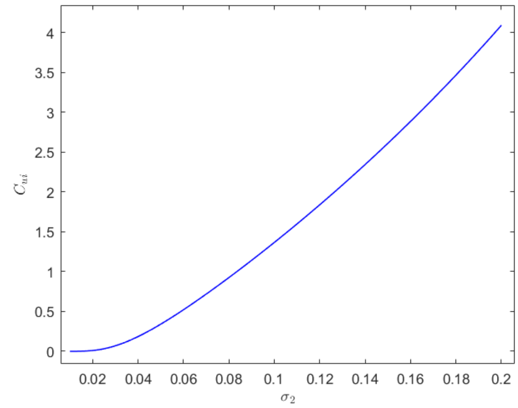

Figure 2. Variation of the price C_{ui} with σ_2.

It displays that the price C_{ui} ascends with the parameter σ_2 in Figure 2. The asset price is more likely to move up to the barrier level if σ_2 increases. Thus, the option price C_{ui} increases with σ_2.

Example 4.4. *Let the maturity time T change from 1 to 6 with step 0.01, and the other parameters remain unchanged. Figure 3 displays the results.*

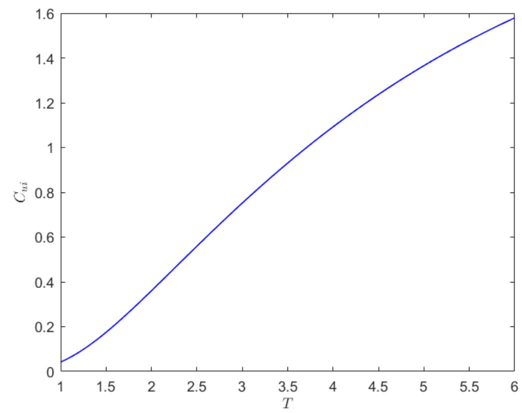

Figure 3. Variation of the price C_{ui} with T.

As it is illustrated in Figure 3, the price C_{ui} grows when the maturity time T gets longer. The investor possibly gets more profits, and the seller takes more risks when T ascends. Therefore, the option price C_{ui} increases.

Example 4.5. *Let the barrier level D change from 18 to 25 with step 0.01, and the other parameters remain unchanged. Figure 4 displays the results.*

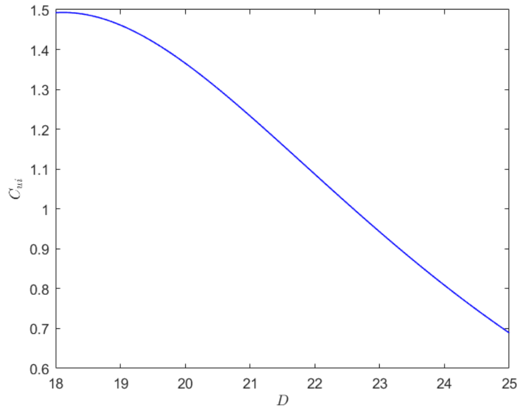

Figure 4. Variation of the price C_{ui} with D.

Figure 4 demonstrates that the price C_{ui} decreases with D. This result may be attributed to the fact that the price of stock is less likely to move up to the barrier level as D increases, and the option is less likely to get activated.

Finally, we consider the parameters σ_1, a, μ, and c.

Example 4.6. *Let the parameter σ_1 change from 0 to 0.1 with step 0.01. Figure 5 displays the results.*

Figure 5 illustrates that the price C_{ui} exhibits a positive correlation with the parameter σ_1. It is difficult to investigate the relationship between the price C_{ui} and σ_1 from the pricing formula. Next, we investigate the impact of σ_1 on C_{ui} with different values of σ_2, μ, c, Y_0, D, and T. The curves are illustrated in Figure 6.

In all cases, it reveals that the price C_{ui} increases with the parameter σ_1. Thus, the price C_{ui} is a monotonically increasing function of σ_1 as the remaining parameters remain within an acceptable range. Analogously, the changes in the price C_{ui} on the parameters a, μ, and c can be analyzed in the same way.

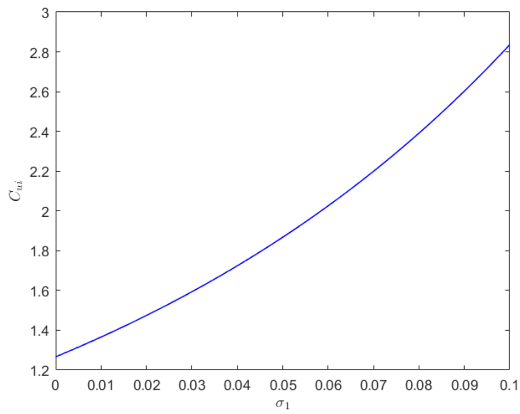

Figure 5. Variation of the price C_{ui} with σ_1.

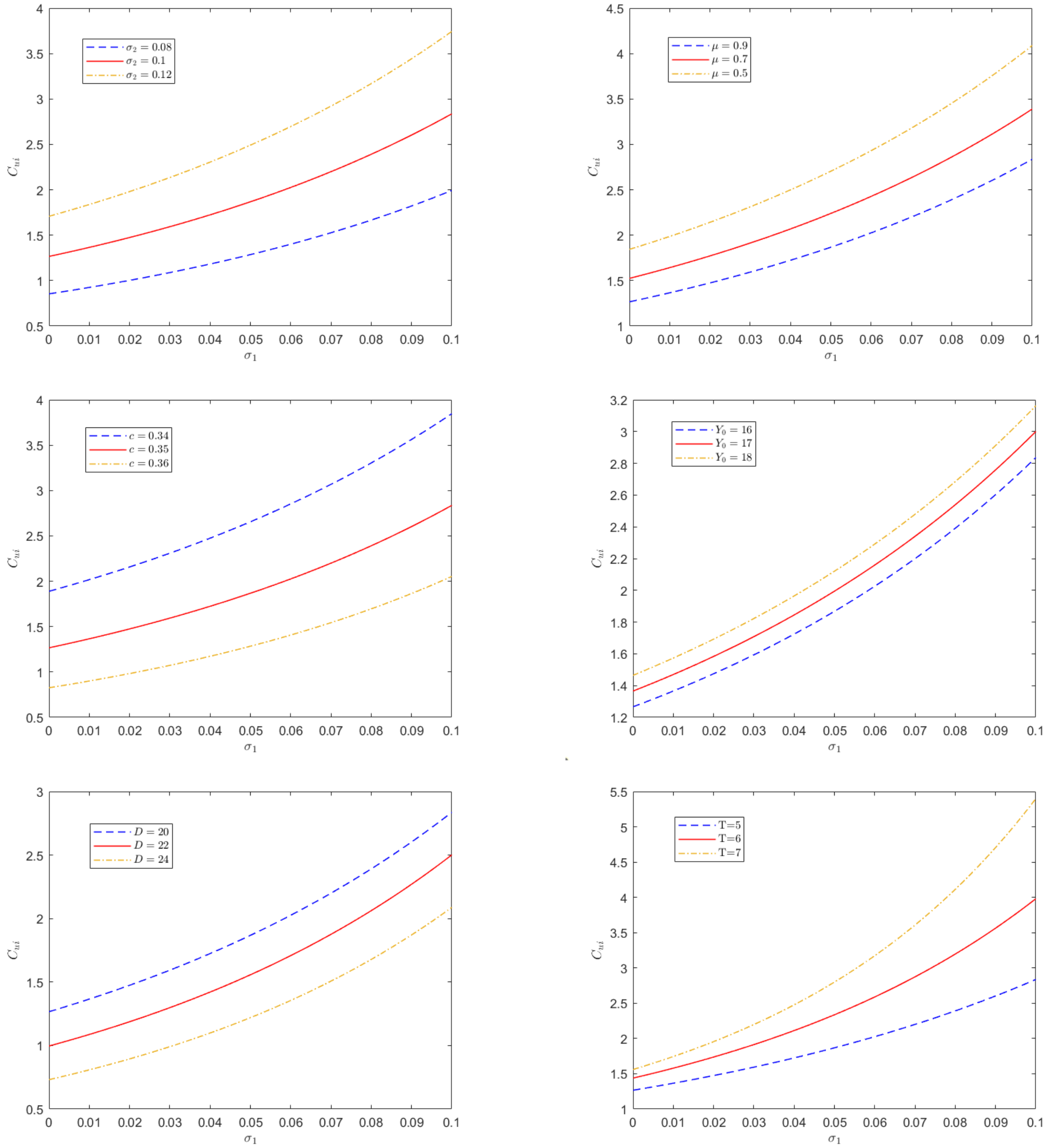

Figure 6. Variation of the price C_{ui} with σ_1.

4.2. *Down-and-in put option*

The algorithm for calculating the price P_{di} is designed according to Corollary 3.1.
Step 0: Set the values of r_0, m, a , σ_1, Y_0, μ, c , σ_2, B, T, and D.
Step 1: Calculate

$$\theta = \Big(1 + \exp\big(\frac{\mu c\pi(\ln Y_0 \exp(-\mu cT) - \ln D)}{\sqrt{3}\sigma_2 - \sqrt{3}\sigma_2 \exp(-\mu cT)} + \frac{\mu\pi}{\sqrt{3}\sigma_2}\big)\Big)^{-1}.$$

Step 2: Set $\alpha_j = j\theta/N$, $j = 1, 2, ..., N-1$, where N is a large positive integer.

Step 3: Set $j = 0$.
Step 4: Set $j \leftarrow j + 1$.
Step 5: Compute the positive deviation

$$\begin{aligned} Z_j =&(B - Y_T^{\alpha_j})^+ \\ =&\max(B - Y_T^{\alpha_j}, 0) \\ =&\max\Big(B - \exp\Big(\Big(\frac{1}{c} + \frac{\sqrt{3}\sigma_2}{\mu c\pi}\ln\frac{\alpha_j}{1-\alpha_j}\Big)\Big(1 - \exp(-\mu cT)\Big) + \ln Y_0 \exp(-\mu cT)\Big), 0\Big). \end{aligned}$$

Step 6: Calculate

$$G_j = \exp\Big(\Big(\frac{r_0}{a} - \frac{m}{a^2} - \frac{\sigma_1\sqrt{3}}{a^2\pi}\ln\frac{\alpha_j}{1-\alpha_j}\Big)\Big(\exp(-aT) - 1\Big) - \Big(\frac{m}{a} + \frac{\sigma_1\sqrt{3}}{a\pi}\ln\frac{\alpha_j}{1-\alpha_j}\Big)T\Big)$$

and $W_j = Z_j \times G_j$. Return to **Step 4** if $j < N - 1$.
Step 7: The option price function is

$$P_{di} = \frac{\theta}{N-1}\sum_{j=1}^{N-1} W_j.$$

Example 4.7. *Assume the initial interest rate* $r_0 = 0.03$*, the initial stock price* $Y_0 = 16$*, and other parameters of model* (3.3) *are* $m = 0.01, a = 0.8, \sigma_1 = 0.01, \mu = 0.9, c = 0.35$*, and* $\sigma_2 = 0.1$*, and the parameters of the option are* $B = 15, T = 5$*, and* $D = 14$*, respectively. Then the price* P_{di} *is* 0.5425.

Considering the price function of the up-and-in call option is similar to that of the down-and-in put option, we only investigate the influence of B and D on the option price P_{di}.

Example 4.8. *Let the strike price B change from 10 to 15 with step 0.01, and the other parameters remain unchanged. Figure 7 displays the results.*

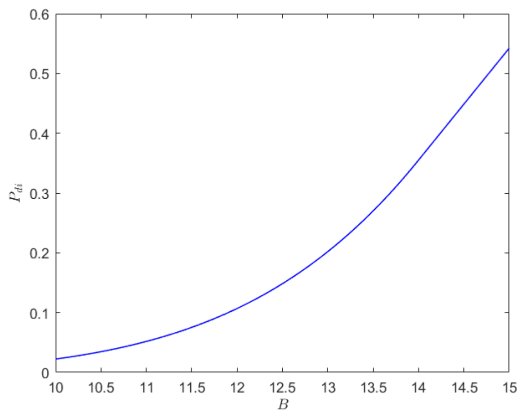

Figure 7. Variation of the price P_{di} with B.

Figure 7 illustrates that the price P_{di} increases with B. The result can be deduced immediately from Theorem 3.1. Since B appears only in the positive deviation, the price P_{di} is a monotonically increasing function of B.

Example 4.9. *Let the barrier level D change from 9 to 14 with step 0.01, and the other parameters remain unchanged. Figure 8 displays the results.*

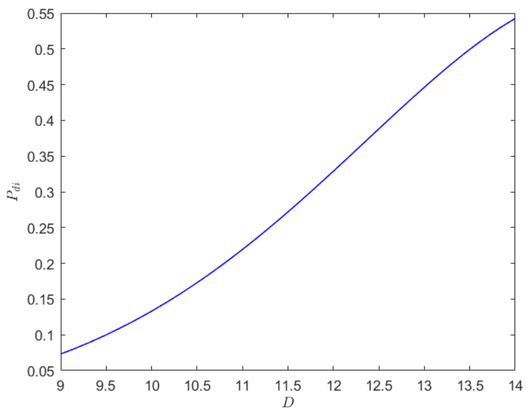

Figure 8. Variation of the price P_{di} with D.

Figure 8 demonstrates that the price P_{di} increases with D. This result may be attributed to the fact that the price of stock is more likely to decrease to the barrier level as D increases and the option is more likely to be activated.

5. Real data analysis

In this section, real financial data are used to illustrate the performances of the four option pricing formulas given in Sections 3.1 and 3.2. Moreover, the method of moments is chosen for estimating the unknown parameters in the model (3.3). The uncertain hypothesis test is utilized in the following example to assess the reasonableness of the estimations.

5.1. Parameter estimation

We choose the Shanghai Interbank Offered Rate (SHIBOR) and the closing price of Haitian food stock for the period of October 20, 2023, to December 27, 2023, which are displayed in Tables 1 and 2.

Table 1. Shanghai Interbank Offered Rate from October 20, 2023 to December 27, 2023.

1.9070	1.9310	1.8770	1.9520	1.6500	1.6260	1.6950	1.7500	1.7890	1.4940
1.6230	1.5790	1.6370	1.7300	1.6470	1.7010	1.7540	1.9130	1.8630	1.9070
1.8822	1.8970	1.8900	1.8990	1.8960	1.8370	1.8090	1.7110	1.6080	1.8580
1.6180	1.7240	1.7140	1.6060	1.6190	1.6300	1.7550	1.7610	1.6340	1.6280
1.5940	1.6230	1.5700	1.5920	1.7110	1.6160	1.5920	1.4940	1.4670	

Table 2. The closing stock prices of Haitian food from October 20, 2023 to December 27, 2023.

35.09	35.07	34.59	35.05	35.26	37.33	37.87	37.65	37.15	36.86	37.53
37.79	38.10	38.02	38.51	38.36	38.25	38.35	39.15	38.35	38.10	38.41
38.23	38.00	37.91	37.98	37.49	37.98	37.81	38.11	37.32	36.75	36.70
36.40	36.39	36.35	36.72	36.57	35.00	34.83	34.69	34.55	34.34	33.96
35.19	35.82	36.30	36.20	36.28						

According to the method of moments for uncertain differential equations [27], the estimations of the parameters in the model (3.3) are $m = 0.0122, a = 0.7139, \sigma_1 = 0.0011$, and $\mu = 0.8669, c = 0.2774, \sigma_2 = 0.0166$. Thus, the model (3.3) can be expressed as

$$\begin{cases} dr_t = (0.0122 - 0.7139 r_t)dt + 0.0011 dC_{1t}, \\ dY_t = 0.8669(1 - 0.2774 \ln Y_t)Y_t dt + 0.0166 Y_t dC_{2t}. \end{cases} \tag{5.1}$$

As we can see from Figures 9 and 10, all the observations of interest rate and stock price fall between the 0.05-path and the 0.95-path, which implies that the estimates are acceptable.

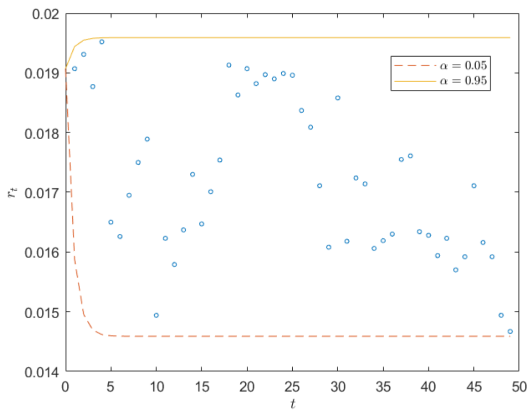

Figure 9. α-paths and observations r_t.

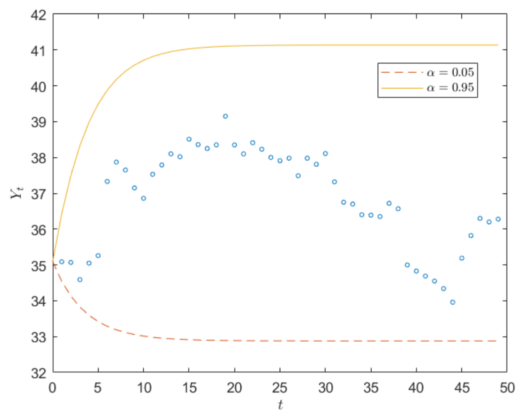

Figure 10. α-paths and observations Y_t.

5.2. Hypothesis test

In this part, we employ the uncertain hypothesis testing proposed by Zhang et al. [28] to assess how well the uncertain model (5.1) fits the observed data.

For the first differential equation in the model (3.3),

$$dr_t = (m - ar_t)dt + \sigma_1 dC_{1t}$$

by using the Euler difference, we obtained

$$\frac{r_{t_{j+1}} - r_{t_j} - (m - ar_{t_j})(t_{j+1} - t_j)}{\sigma_1(t_{j+1} - t_j)} = \frac{C_{t_{j+1}} - C_{t_j}}{t_{j+1} - t_j}.$$

Since

$$\frac{C_{t_{j+1}} - C_{t_j}}{t_{j+1} - t_j} \sim \mathcal{N}(0, 1),$$

it can be obtained that

$$\omega_j = \frac{r_{t_{j+1}} - r_{t_j} - (m - ar_{t_j})(t_{j+1} - t_j)}{\sigma_1(t_{j+1} - t_j)} \sim \mathcal{N}(0, 1).$$

Similarly, for the second differential equation, we have

$$z_j = \frac{Y_{t_{j+1}} - Y_{t_j} - \mu(1 - c \ln Y_{t_j})Y_{t_j}(t_{j+1} - t_j)}{\sigma_2 Y_{t_j}(t_{j+1} - t_j)} \sim \mathcal{N}(0, 1).$$

The sample values of ω_j and z_j can be obtained from the observed data of r_{t_j} and Y_{t_j}, where $j = 1, 2, ..., 48$.

The issue of determining whether the model (5.1) fits the data well is converted into a test to verify whether ω_j and z_j obey the standard normal uncertain distribution $\mathcal{N}(0, 1)$. Let the significance level α take the value of 0.05, and the two rejection domains are

$$W_1 = \{(\omega_1, \omega_2, ..., \omega_{48}) : \text{there are at least 3 index } j\text{'s with } 1 \leq j \leq 48 \text{ such that } \omega_j < -2.0198 \text{ or } \omega_j > 2.0198\}$$

and

$$W_2 = \{(z_1, z_2, ..., z_{48}) : \text{there are at least 3 index } j\text{'s with } 1 \leq j \leq 48 \text{ such that } z_j < -2.0198 \text{ or } z_j > 2.0198\}.$$

We can see that only $\omega_9 = -2.1621 \notin [-2.0198, 2.0198]$ in Figure 11, thus $(\omega_1, \omega_2, ..., \omega_{48}) \notin W_1$. It can also be found that $z_9 > 2.0198$ and $z_{42} < -2.0198$ in Figure 12, so we have $(z_1, z_2, ..., z_{48}) \notin W_2$. In summary, we can conclude that the model (5.1) is able to fit the observed data well.

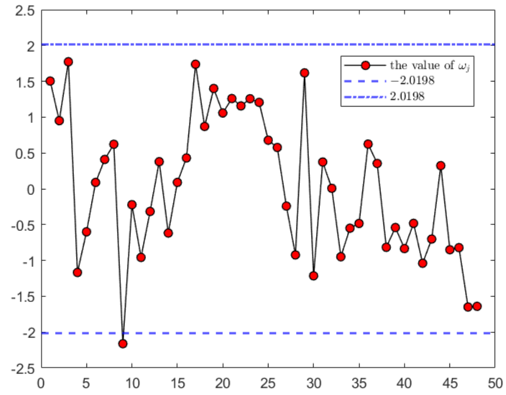

Figure 11. Residual plot of interest rate.

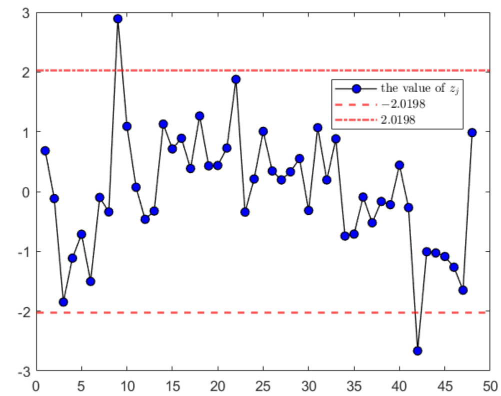

Figure 12. Residual plot of stock data.

5.3. Numerical results

We use the option price formulas given in Sections 3.1 and 3.2 to calculate the option prices with Haitian food stock as the underlying asset. Suppose that Y_0 is 37.33, r_0 is 1.6260%, and the expiration date T is 8. According to these initial conditions, we calculate the option prices under three different models, including the Black–Scholes model

$$\begin{cases} dX_t = rX_t dt, \\ dY_t = \mu Y_t dt + \sigma Y_t dW_t, \end{cases} \tag{5.2}$$

where r is a constant, the stochastic exponential Ornstein–Uhlenbeck model with stochastic interest rates

$$\begin{cases} dr_t = (m - ar_t)dt + \sigma_1 dW_{1t}, \\ dY_t = \mu(1 - c\ln Y_t)Y_t dt + \sigma_2 Y_t dW_{2t}, \end{cases} \tag{5.3}$$

and the model (3.3) used in this paper. The four barrier option prices on the three models are outlined in Table 3. According to Table 3, it can be observed that the price estimates under the model (3.3) are higher than those of the two stochastic models, with the exception of the down-and-out call option.

Table 3. Prices of the four barrier options under different models.

	C_{ui}	P_{di}	P_{uo}	C_{do}
strike price B	38	35	38	35.5
barrier level D	40	34	40	34
$B-S$ model	0.1569	0.0361	0.9716	1.5868
stochastic $O-U$ model	0.0525	0.0033	0.5228	1.6138
uncertain $O-U$ model	0.2242	0.1438	1.3018	1.4960

Finally, based on the three different models, some numerical results for the up-and-in call option are given to visually demonstrate the sensitivity of the up-and-in call option price to the parameters, including barrier level, strike price and expiration date. Tables 4 and 5 illustrate that the price C_{ui} decreases with the barrier level and the strike price, respectively. Table 6 indicates an increasing trend in the price C_{ui} by varying T from 6 to 10. As seen in Tables 4–6, for the comparison of the up-and-in call option prices with different parameters, the uncertain $O-U$ model's price estimates are closer to the stochastic $B-S$ model's price estimates.

Table 4. The price C_{ui} with different barrier levels.

barrier level D	38.5	39	39.5	40
$B-S$ model	0.3088	0.2663	0.2165	0.1569
stochastic $O-U$ model	0.1858	0.1501	0.0938	0.0525
uncertain $O-U$ model	0.3537	0.3178	0.2717	0.2242

Table 5. The price C_{ui} with different strike prices.

strike price B	37	37.5	38	38.5
$B-S$ model	0.2135	0.1752	0.1569	0.1247
stochastic $O-U$ model	0.0816	0.0675	0.0525	0.0344
uncertain $O-U$ model	0.2907	0.2575	0.2242	0.1909

Table 6. The price C_{ui} with different expiration dates.

expiration date T	8	9	10	11
$B-S$ model	0.1569	0.1782	0.2118	0.2353
stochastic $O-U$ model	0.0525	0.0565	0.0633	0.0714
uncertain $O-U$ model	0.2242	0.2408	0.2530	0.2617

6. Conclusions

This paper primarily focused on the pricing issue of the European barrier option in the uncertain exponential Ornstein–Uhlenbeck model with a floating interest rate. The price functions of knock-in and knock-out barrier options were given. Then we developed numerical algorithms to compute the option prices and provided several numerical examples to show the effect of parameters on the option prices. In the end, we chose Haitian food stock as the underlying asset to demonstrate how to obtain the option prices and compare the option prices under different models.

Author contributions

Shaoling Zhou: Responsible for the review and editing of the manuscript, as well as project administration. Huixin Chai: Responsible for visualization and preparation of the original draft of the manuscript. Xiaosheng Wang: Responsible for the supervision of the research project. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors have no relevant financial or non-financial interests to disclose.

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Appendix

Suppose the stock price of Haitian food follows the stochastic differential equation

$$dY_t = \mu(1 - c \ln Y_t)Y_t dt + \sigma_2 Y_t dW_t,$$

where W_t is a Wiener process, μ, c and σ_2 are unknown parameters. Based on the stock price data shown in Table 2, using Maximum Likelihood Estimation (MLE), we obtain the estimates of the parameters in the above stochastic differential equation as

$$\mu^* = 0.4002, c^* = 0.2761, \sigma_2^* = 0.0159.$$

Hence, we get a stochastic stock model

$$dY_t = 0.4002(1 - 0.2761 \ln Y_t)Y_t dt + 0.0159 Y_t dW_t.$$

By using the Euler difference, we obtain

$$\varepsilon_j = \frac{Y_{t_{j+1}} - Y_{t_j} - \mu(1 - c \ln Y_{t_j})Y_{t_j}(t_{j+1} - t_j)}{\sigma_2 Y_{t_j}(t_{j+1} - t_j)} = \frac{W_{t_{j+1}} - W_{t_j}}{t_{j+1} - t_j} \sim N(0, 1).$$

Therefore, the sample value of ε_j can be regarded as a sample of the standard normal distribution $N(0, 1)$.

Next we test whether the stochastic stock model fits the stock price of Haitian-food by the "Shapiro-Wilk" test. That is, we should determine whether the sample values of ε_j are derived from the standard normal distribution $N(0, 1)$. The Shapiro function is used in the test with a significance level of 0.05. The result indicates that the value of P is 0.0172, which implies that the sample values of ε_j don't come from the standard normal distribution $N(0, 1)$. Therefore, the stochastic stock model mentioned in this part is not suitable for the observed stock data well.