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Research article

Novel operators in the frame of primal topological spaces

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Abstract: Our aim in this paper is to define more concepts that are related to primal topological space. We introduce new operators called γ-diamond and γ^*-diamond and explore their main characterizations. We provide results and examples regarding to these operators. Using these new operators, we create a weaker version of the original topology. Additionally, we present some results related to compatibility.

Keywords: primal; primal topological space; γ-diamond and γ^*-diamond
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1. Introduction

Many topological structures were an active area in the study of various spheres of mathematics, such as natural and social sciences to solve numerous natural problems. Choquet [1] developed the theory of grills, which he introduced in 1947. Subsequently, in 1966, Kuratowski [2] investigated and studied ideals concepts where the concept of ideal is the inverse of the filter. Many researchers utilized grill structures, including general topology [3] and fuzzy topology [4], etc. It is worth noting that R. Vaidyanathaswamy introduced the concept of localization theory in set-topology in [5] and [6]. Moreover, this topic was highly discussed in [7] by D. Janković et al. Furthermore, D. Sarkar dicussed fuzzy ideals in fuzzy set theory and how to generate new fuzzy topologies from old using fuzzy ideals. Additionally, he studied the concept of fuzzy local functions and the notion of compatibility of fuzzy ideals with fuzzy topologies in [8]. On the other hand, A. Kandil et al. introduced the notion of soft local functions in [9]. Z. Amee et al. represented cluster soft closed sets in terms of several forms of soft sets, which was a development of the concept of soft local functions [10].

Recently, the notions of primal structure were discussed in [11] where primals are the dual of the notion of grills. Additionally, they studied the relationship between primal topological spaces and topological spaces. Promoting the fast development of primal topological space, Al-Shami et al. [12] defined the soft primal soft topology and investigated its basic properties. Moreover, Al-Omari et al. [13] presented a novel type of primal soft operator. Also, Ameen et al. [14] introduced the concept of fuzzy primal. The work of Al-Omari et al. [15] studing proximity spaces inspired by primal and others [16, 17] had a significant impact on the development of operators in primal topological spaces.

In this work, we investigate and introduce a new operator named γ-diamond and study the relationships between it and other primal operators. Moreover, we introduce a new topology via the γ^*-diamond operator and study several fundamental properties. The principal characteristics of these notions are were defined and examined by the researchers of [18–20]. In Section 3, we present new results related on to the γ-diamond operator. We also use the concept of γ-diamond to provide a weaker topology than the one presented in Section 4. Moreover, we present some basic results regarding to compatibility in Section 5.

2. Preliminaries

For the duration of this document, (M, σ) and (S, σ) (briefly, M and S) denote topological spaces unless specified otherwise. For any $G \subset M$, we denote the closure of G by $cl(G)$ and the interior of G by $Int(G)$. We will use 2^M to refer to the power set of M. We use the symbol $\sigma(x)$ to denote the family of open sets that contains x. If F is any subset of M such that $F \in \sigma^c$, then F is a closed subset of M. We use the symbol σ_θ to mention the class of θ-open [21] sets in M; that is, $\sigma_\theta = \{W \in \sigma \mid \forall w \in W\ \exists\, G \in \sigma(w)$ such that $w \in G \subseteq cl(G) \subseteq W\}$. Moreover, $cl_\theta(G) = \{g \in M \mid cl(W) \cap G \neq \emptyset\ \ \forall\ W \in \sigma(g)\}$ and $Int_\theta(G) = \{\bigcup_{\alpha \in \Lambda} U_\alpha \mid U_\alpha \subseteq G,\ U_\alpha \in \sigma_\theta\ \ \forall\ \alpha \in \Lambda\}$. We now obtain the following notions and findings, which are necessary for the following section.

Definition 2.1. *[1] The family $\mathcal{G}$ of 2^M is a grill on M if $\mathcal{G}$ meets the following requirements:*

(a) $\emptyset \notin \mathcal{G}$,
(b) if $m_1 \cup m_2 \in \mathcal{G}$, we have that $m_1 \in \mathcal{G}$ or $m_2 \in \mathcal{G}$,
(c) if $m_1 \in \mathcal{G}$ and $m_1 \subseteq m_2$, we get that $m_2 \in \mathcal{G}$.

Definition 2.2. *[11] The family $\mathcal{P} \subseteq 2^M$ is called a primal on M, where M is a nonempty set if and only if the following circumstances are met:*

(a) $M \notin \mathcal{P}$,
(b) if $m_1 \cap m_2 \in \mathcal{P}$, then $m_2 \in \mathcal{P}$ or $m_1 \in \mathcal{P}$,
(c) if $m_2 \in \mathcal{P}$ and $m_1 \subseteq m_2$, then $m_1 \in \mathcal{P}$.

Corollary 2.1. *[11] The family $\mathcal{P} \subseteq 2^M$ is a primal on M if and only if the following circumstances are met:*

(a) $M \notin \mathcal{P}$,
(b) if $m_2 \notin \mathcal{P}$ and $m_1 \notin \mathcal{P}$, then $m_1 \cap m_2 \notin \mathcal{P}$,

(c) if $m_2 \notin \mathcal{P}$ and $m_2 \subseteq m_1$, then $m_1 \notin \mathcal{P}$.

A primal $\mathcal{P}$ [11] on M with a topological space (M, σ) is a primal topological space $(M, \sigma, \mathcal{P})$ indicated by $\mathcal{PTS}$.

Definition 2.3. *[11] Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$. We define a function $(\cdot)^\diamond : 2^M \to 2^M$ as $A^\diamond(M, \sigma, \mathcal{P}) = \{x \in M : A^c \cup U^c \in \mathcal{P}$ for all $U \in \sigma(x)\}$ for any set $A \subseteq M$. We will use the symbol $A^\diamond_{\mathcal{P}}$ to denote $A^\diamond(M, \sigma, \mathcal{P})$.*

Definition 2.4. *[11] Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$. We define a function $cl^\diamond : 2^M \to 2^M$ as $cl^\diamond(T) = T \cup T^\diamond$, where $T \subseteq M$.*

Definition 2.5. *[11] Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$. We define $\sigma^\diamond$ as $\sigma^\diamond = \{T \subseteq M : cl^\diamond(T^c) = T^c\}$.*

Definition 2.6. *[17] Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$. For $T \subseteq M$, we define the function $\Pi : 2^M \to 2^M$ as follows: $\Pi(T)(\mathcal{P}, \sigma) = \Pi(T) = \{x \in M : T^c \cup (cl(V))^c \in \mathcal{P}$ for all $V \in \sigma(x)\}$, where $\sigma(x) = \{V \in \sigma : x \in V\}$.*

Lemma 2.1. *[17] Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$. Then, for any $T \subseteq M$ we have $T^\diamond_{\mathcal{P}} \subseteq \Pi(T)$.*

Definition 2.7. *Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$ and let S be any subset of M. An operator γ is called idempotent if and only if $\gamma(\gamma(S)) = \gamma(S)$.*

Definition 2.8. *[17] Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$. Then, we define the operator $\overrightarrow{\Pi}(S)$ for the set $S \subseteq M$ as $\overrightarrow{\Pi}(S) = \{s \in M \mid \exists\, W \in \sigma(s)$ such that $(cl(W) - S)^c \notin \mathcal{P}\}$.*

Theorem 2.1. *[17] Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$. Consider the set $\beta = \{K \subseteq M : K \subseteq \overrightarrow{\Pi}(K)\}$. Then, β is a topological space on M and $K \in \beta$ is called β-open.*

3. On γ-diamond operator

This section introduces a new a primal structure called a γ-diamond operator. The fundamental properties of this structure are presented.

Definition 3.1. *Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$. For $T \subseteq M$, define a function $\gamma : 2^M \to 2^M$ as: $\gamma(T)(\mathcal{P}, \sigma) = \{x \in M : T^c \cup (W^\diamond)^c \in \mathcal{P}$ for all $W \in \sigma(x)\}$. For the avoidance of uncertainty, $\gamma(T)(\mathcal{P}, \sigma)$ is succinctly described by $\gamma(T)$ and is known as the primal γ-diamond operator of A pertaining to σ and $\mathcal{P}$.*

Theorem 3.1. *([11]) Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$. Then, the following claims are true for $T, S \subseteq M$.*

(1) $\emptyset^\diamond = \emptyset$.
(2) $cl(T^\diamond) = T^\diamond$.
(3) $(T^\diamond)^\diamond \subseteq T^\diamond$.
(4) if $T \subseteq S$, then $T^\diamond \subseteq S^\diamond$.
(5) $T^\diamond \cup S^\diamond = (T \cup S)^\diamond$.
(6) $(T \cap S)^\diamond \subseteq T^\diamond \cap S^\diamond$.

Lemma 3.1. *Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$ and $T \subseteq M$. Then, $T^\diamond \subseteq cl(T)$.*

Proof. Let $t \in T^\diamond$. Then, $U^c \cup T^c \in \mathcal{P}$ for all $U \in \sigma(t)$. Thus, $U^c \cup T^c \neq M$ for all $U \in \sigma(t)$. Hence, $U \cap T \neq \emptyset$ for all $U \in \sigma(t)$, implying that $t \in cl(T)$. Hence, $T^\diamond \subseteq cl(T)$. □

Theorem 3.2. *Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$ and $T \subseteq M$. Then, $\gamma(T) \subseteq \Pi(T)$.*

Proof. Let $t \in \gamma(T)$. Then, $T^c \cup (W^\diamond)^c \in \mathcal{P}$ for every $W \in \sigma(t)$. From Lemma 3.1, $T^c \cup (cl(W))^c \subseteq T^c \cup (W^\diamond)^c$ for every $W \in \sigma(t)$, which implies that $T^c \cup (cl(W))^c \in \mathcal{P}$. Hence, $t \in \Pi(A)$. □

Hence, we know from Theorem 3.2 that $\gamma(T) \subseteq \Pi(T)$. The following examples show that $\Pi(T) \not\subseteq \gamma(T)$ in general.

Example 3.1. *Let $T = \{1, 2, 3, 4\}$, $\sigma = \{\emptyset, T, \{4\}, \{1, 3\}, \{1, 3, 4\}\}$, and $\mathcal{P} = \{\emptyset, \{3\}, \{4\}, \{3, 4\}\}$. For a subset $S = \{1, 2, 3\}$, we have $\gamma(S) = \emptyset$ and $\Pi(S) = S$*

Example 3.2. *Consider the set of natural numbers $\mathbb{N}$. Define the topological space ν on $\mathbb{N}$ such that $U \in \nu$ if and only if $U = \mathbb{N}$ or $1 \notin U$. Let $\mathcal{P}$ be defined on $\mathbb{N}$ as $T \in \mathcal{P}$ if and only if $1 \notin T$. Then, $(\mathbb{N}, \nu, \mathcal{P})$ is primal topological space. Let $S \subseteq \mathbb{N}$. Then, there are two options:*
Case 1. $1 \in S$. Let $n \in \mathbb{N}$ and let $V \in \nu(n)$ be arbitrary. From the definition of ν, we know that $1 \in cl(V)$. Then, $1 \notin S^c \cup (cl(V))^c$, which implies that $S^c \cup (cl(V))^c \in \mathcal{P}$ and then $n \in \Pi(S)$. Hence, $\Pi(S) = \mathbb{N}$.
Case 2. $1 \notin S$. Then, $1 \in S^c \cup (cl(V))^c$ for every $V \in \nu$, which implies that $S^c \cup (cl(V))^c \notin \mathcal{P}$. Hence, $\Pi(S) = \emptyset$.

$$\therefore \Pi(S) = \begin{cases} \mathbb{N}, & \text{if } 1 \in S \\ \emptyset, & \text{if } 1 \notin S \end{cases}$$

Now, we want to find $\gamma(S)$.
Case 1. $1 \in S$. Let $n \in \mathbb{N}$ and let $V \in \nu(n)$ be arbitrary. Then, we have two subcases:

Subcase 1.1. $n = 1$. Then, if $V \in \nu(1)$, $V = \mathbb{N}$. As $V^\diamond = \{n \in \mathbb{N} \mid U^c \cup V^c \in \mathcal{P}\ \forall\ U \in \nu(1)\}$, $1 \in V^\diamond$, which implies that $1 \in \gamma(S)$.

Subcase 1.2. $n \neq 1$. Set $V = \{n\}$. Then, $V^\diamond = \emptyset$, which implies that $n \notin \gamma(S)$ since $(V^\diamond)^c = \mathbb{N}$.
Thus, in this case $\gamma(S) = \{1\}$.
Case 2. $1 \notin S$. Then, $1 \in S^c \cup (V^\diamond)^c$ for every $V \in \nu$, which implies that $S^c \cup (V^\diamond)^c \notin \mathcal{P}$. Hence, $\gamma(S) = \emptyset$.

$$\therefore \gamma(S) = \begin{cases} \{1\}, & \text{if } 1 \in S \\ \emptyset, & \text{if } 1 \notin S \end{cases}$$

Lemma 3.2. *[11] Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$. Then, the following holds:*

(1) If $\sigma^c - \{M\} \subseteq \mathcal{P}$, then $S \subseteq S^\diamond$ for all $S \in \sigma$.
(2) If $S^c \in \sigma$, then $S^\diamond \subseteq S$.

Lemma 3.3. *[17] Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$. A subset $F \subseteq M$ is closed in β iff $\Pi(F) \subseteq F$.*

Lemma 3.4. *Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$ and $\sigma^c - \{M\} \subseteq \mathcal{P}$. Then, for all $T \subseteq M$, $T^\diamond \subseteq \gamma(T) \subseteq \Pi(T)$.*

Proof. Let $t \in T^\diamond$. Then, $T^c \cup W^c \in \mathcal{P}$ for every $W \in \sigma(t)$. By Lemma 3.2, we have that $T^c \cup (W^\diamond)^c \subseteq T^c \cup W^c \in \mathcal{P}$ for every $W \in \sigma(t)$. Thus, $T^c \cup (W^\diamond)^c \in \mathcal{P}$ for all $W \in \sigma(t)$, which implies that $t \in \gamma(T)$. Hence, $T^\diamond \subseteq \gamma(T) \subseteq \Pi(T)$. □

Lemma 3.5. *[17] Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$ and $T \subseteq M$. Then,*

(1) $cl(T) = cl_\theta(T)$ if T is open.
(2) $\Pi(T) = cl(\Pi(T)) \subseteq cl_\theta(T)$.

Theorem 3.3. *Let $(M, \sigma, \mathcal{P})$ and $(M, \sigma, \mathfrak{J})$ be two $\mathcal{PTS}$ and let $T, S \subseteq M$. Thus, the following properties hold:*

(1) If $T \subseteq S$, then $\gamma(T) \subseteq \gamma(S)$.
(2) If $\mathfrak{J} \subseteq \mathcal{P}$, then $\gamma(T)(\mathfrak{J}) \subseteq \gamma(T)(\mathcal{P})$.
(3) $\gamma(T)$ is closed.
(4) $\gamma(T) \subseteq \Pi(T) \subseteq cl_\theta(T)$.
(5) If $T \subseteq \gamma(T)$ and $\gamma(T)$ is open, then $\gamma(T) = \Pi(T) = cl_\theta(T)$.
(6) If $T^c \notin \mathcal{P}$, then $\gamma(T) = \emptyset$ and $\gamma(\emptyset) = \emptyset$.
(7) $\gamma(T \cup S) = \gamma(T) \cup \gamma(S)$.

Proof. (1) Let $s \notin \gamma(S)$. Then, there is $W \in \sigma(s)$ such that $S^c \cup (W^\diamond)^c \notin \mathcal{P}$. Since $S^c \cup (W^\diamond)^c \subseteq T^c \cup (W^\diamond)^c$, then $T^c \cup (W^\diamond)^c \notin \mathcal{P}$. Hence, $s \notin \gamma(T)$. Therefore, $M - \gamma(S) \subseteq M - \gamma(T)$ or $\gamma(T) \subseteq \gamma(S)$.

(2) Let $t \notin \gamma(T)(\mathcal{P})$. Then, there is $W \in \sigma(t)$ such that $T^c \cup (W^\diamond)^c \notin \mathcal{P}$. Since $\mathcal{J} \subseteq \mathcal{P}$, then $T^c \cup (W^\diamond)^c \notin \mathfrak{J}$ and $t \notin \gamma(T)(\mathfrak{J})$. Therefore, $\gamma(T)(\mathfrak{J}) \subseteq \gamma(T)(\mathcal{P})$.

(3) Since $\gamma(T) \subseteq cl(\gamma(T))$ in general, let $t_1 \in cl(\gamma(T))$. Then, $\gamma(T) \cap W \neq \emptyset$ for every $W \in \sigma(t_1)$. Thus, there is $t_2 \in \gamma(T) \cap W$, and hence $W \in \sigma(t_2)$. Since $t_2 \in \gamma(T)$, then $T^c \cup (W^\diamond)^c \in \mathcal{P}$ which implies that $t_1 \in \gamma(T)$. Hence, $cl(\gamma(T)) \subseteq \gamma(T)$, and so $cl(\gamma(T)) = \gamma(T)$, which is equivalent to that $\gamma(T)$ is closed.

(4) By Theorem 3.2, we know that $\gamma(A) \subseteq \Pi(A)$. Then, it remains to show that $\Pi(T) \subseteq cl_\theta(T)$. Let $t \in \Pi(T)$. Then, $T^c \cup (cl(G))^c \in \mathcal{P}$ for every $G \in \sigma(t)$. As $M \notin \mathcal{P}$, then $T^c \cup (cl(G))^c \neq M$, which implies that $(T^c \cup (cl(G))^c)^c \neq M^c$ for every $G \in \sigma(t)$. Then, $T \cap cl(G) \neq \emptyset$ for every $G \in \sigma(t)$. Therefore, $t \in cl_\theta(T)$.

(5) Let $T \subseteq M$. By (4) we have $\gamma(T) \subseteq \Pi(T) \subseteq cl_\theta(T)$. Since $T \subseteq \gamma(T)$, then $cl_\theta(T) \subseteq cl_\theta(\gamma(T))$. By (1) in Lemma 3.5, we get that $\Pi(T) \subseteq cl_\theta(T) \subseteq cl_\theta(\gamma(T))$ since $\gamma(T)$ is open. Then, $cl(\gamma(T)) = \gamma(T) \subseteq \Pi(T) \subseteq cl_\theta(T)$. Therefore, $\gamma(T) = \Pi(T) = cl_\theta(T)$.

(6) Suppose that $T^c \notin \mathcal{P}$ and let $t \in T$. Since $T^c \subseteq T^c \cup (W^\diamond)^c$ for every $W \in \sigma(t)$, then $T^c \cup (W^\diamond)^c \notin \mathcal{P}$ for all $W \in \sigma(t)$. Hence, $\gamma(T) = \emptyset$.

(7) Since $T \subseteq T \cup S$ and $S \subseteq T \cup S$, then $\gamma(T) \subseteq \gamma(T \cup S)$ and $\gamma(S) \subseteq \gamma(T \cup S)$ by (1); hence, $\gamma(T) \cup \gamma(S) \subseteq \gamma(T \cup S)$. Let $r \notin \gamma(T) \cup \gamma(S)$. Then, $r \notin \gamma(T)$ and $r \notin \gamma(S)$. Therefore, there exist $W_1, W_2 \in \sigma(r)$ such that $T^c \cup (W_1^\diamond)^c \notin \mathcal{P}$ and $S^c \cup (W_2^\diamond)^c \notin \mathcal{P}$. Hence, $[T^c \cup (W_1^\diamond)^c] \cup (W_2^\diamond)^c \notin \mathcal{P}$ and $[S^c \cup (W_2^\diamond)^c] \cup (W_1^\diamond)^c \notin \mathcal{P}$. Moreover,

$$\begin{aligned}
&[T^c \cup (W_1^\diamond)^c] \cup (W_2^\diamond)^c \ \cap \ [S^c \cup (W_2^\diamond)^c] \cup (W_1^\diamond)^c \notin \mathcal{P} \\
&= [T^c \cap S^c] \ \cup \ [(W_1^\diamond)^c \cup (W_2^\diamond)^c] \notin \mathcal{P} \\
&= [T \cup S]^c \ \cup \ [W_1^\diamond \cap W_2^\diamond]^c \notin \mathcal{P}.
\end{aligned}$$

Since $(W_1 \cap W_2)^\diamond \subseteq W_1^\diamond \cap W_2^\diamond$ and $W_1 \cap W_2 \in \sigma(r)$, then $[T \cup S]^c \cup [W_1^\diamond \cap W_2^\diamond]^c \subseteq [T \cup S]^c \cup [(W_1 \cap W_2)^\diamond]^c \notin \mathcal{P}$, which implies that $r \notin \gamma(T \cup S)$. Hence, $\gamma(T \cup S) = \gamma(T) \cup \gamma(S)$. □

Lemma 3.6. *Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$. If $G \in \sigma_\theta$, then $G \cap \gamma(K) = G \cap \gamma(G \cap K) \subseteq \gamma(G \cap K)$ for any $K \subseteq M$.*

Proof. Let $r \in G \cap \gamma(K)$. Since $G \in \sigma_\theta$, then there exists $W_1 \in \sigma$ such that $r \in W_1 \subseteq cl(W_1) \subseteq G$. Let W_2 be any open set such that $r \in W_2$. Then, $W_2 \cap W_1 \in \sigma(r)$ and since $r \in \gamma(K)$, we have

$[(W_2 \cap W_1)^\diamond]^c \cup K^c \in \mathcal{P}$. Now, $(W_2^\diamond)^c \cup (G \cap K)^c = (W_2^\diamond)^c \cup G^c \cup K^c \subseteq (W_2^\diamond)^c \cup (cl(W_1))^c \cup K^c$ by using the result from Lemma 3.1. Hence, $(W_2^\diamond)^c \cup (G \cap K)^c \subseteq (W_2^\diamond)^c \cup (cl(W_1))^c \cup K^c \subseteq (W_2^\diamond)^c \cup (W_1^\diamond)^c \cup K^c \subseteq [(W_2 \cap W_1)^\diamond]^c \cup K^c \in \mathcal{P}$. Therefore, $(W_2^\diamond)^c \cup (G \cap K)^c \in \mathcal{P}$. Then, $r \in \gamma(G \cap K)$ which implies that $G \cap \gamma(K) \subseteq \gamma(G \cap K)$. Moreover, $G \cap \gamma(K) \subseteq G \cap \gamma(G \cap K)$, and by Theorem 3.3, $\gamma(G \cap K) \subseteq \gamma(G)$ and $\gamma(G \cap K) \cap G \subseteq \gamma(K) \cap G$. Thus, $G \cap \gamma(K) = G \cap \gamma(G \cap K)$. □

Lemma 3.7. *Let $T, S \subseteq M$ and $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$. Then,*

$$\gamma(S) - \gamma(T) = \gamma(S - T) - \gamma(T).$$

Proof. By (7) in Theorem 3.3, $\gamma(S) = \gamma[(S - T) \cup (T \cap S)] = \gamma(S - T) \cup \gamma(S \cap T) \subseteq \gamma(S - T) \cup \gamma(S)$. Thus, $\gamma(S) - \gamma(T) \subseteq \gamma(S - T) - \gamma(T)$. By (1) in Theorem 3.3, $\gamma(S - T) \subseteq \gamma(S)$; hence, $\gamma(S - T) - \gamma(T) \subseteq \gamma(S) - \gamma(T)$. Therefore, $\gamma(S) - \gamma(T) = \gamma(S - T) - \gamma(T)$. □

Corollary 3.1. *Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$ and $T, S \subseteq M$ such that $S^c \notin \mathcal{P}$. Then, $\gamma(T \cup S) = \gamma(T) = \gamma(S - T)$.*

Proof. Since $S^c \notin \mathcal{P}$, then $\gamma(S) = \emptyset$ by using (6) in Theorem 3.3. By Lemma 3.7, we have $\gamma(T) = \gamma(S - T)$, and by (7) in Theorem 3.3, we obtain $\gamma(T \cup S) = \gamma(T) \cup \gamma(S) = \gamma(T)$. □

Theorem 3.4. *Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$. The following statements are equivalent:*

(a) $\sigma - \{M, \triangle\} \subseteq \mathcal{P}$, where $\triangle = \{U \in \sigma \mid U^\diamond = \emptyset\}$.
(b) If $Y^c \notin \mathcal{P}$, then $Int_\theta(Y) = \emptyset$.
(c) If $\sigma^c - \{M\} \subseteq \mathcal{P}$, then $T \subseteq \gamma(T)$ for every clopen set T.
(d) $M = \gamma(M)$.

Proof. (a) $\Longrightarrow$ (b): Suppose that $Y^c \notin \mathcal{P}$ and $(\sigma - \{M, \triangle\}) \subseteq \mathcal{P}$ and let $r \in Int_\theta(Y)$. Thus, we can find $W \in \sigma$ such that $r \in W \subseteq cl(W) \subseteq Y$. Then, $Y^c \subseteq (cl(W))^c \subseteq (W^\diamond)^c$. Since $Y^c \notin \mathcal{P}$, then $(cl(W))^c \notin \mathcal{P}$ and $(W^\diamond)^c \notin \mathcal{P}$, which contradicts that $\sigma - \{M, \triangle\} \subseteq \mathcal{P}$. Hence, $Int_\theta(Y) = \emptyset$.

(b) $\Longrightarrow$ (c): Let $t \in T$ and suppose $t \notin \gamma(T)$. Then, there is $W_t \in \sigma(t)$ such that $T^c \cup (W_t^\diamond)^c \notin \mathcal{P}$ which implies that $(T \cap W_t^\diamond)^c \notin \mathcal{P}$. Since T is a clopen set, then by (b) and Lemma 3.2 we have $T \cap W_t = Int(W_t \cap T) \subseteq Int(W_t^\diamond \cap T) \subseteq Int_\theta(W_t^\diamond \cap T) = \emptyset$, which is a contradiction since $t \in T \cap W_t$. Then, $t \in \gamma(t)$, and hence $T \subseteq \gamma(T)$.

(c) $\Longrightarrow$ (d): Since M is a clopen set, we get that $M = \gamma(M)$.

(d) $\Longrightarrow$ (a): $M = \gamma(M) = \{a \in M : (W^\diamond)^c \cup M^c = (W^\diamond)^c \in \mathcal{P}$ for each $a \in W \in \sigma\}$. Hence, $(\sigma - \{M, \triangle\}) \subseteq \mathcal{P}$. □

4. New topology via γ^*-diamond operator

In this section, we define a new operator called γ^*-diamond operator. We present some results regarding to this operator including generating a new topology.

Definition 4.1. *Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$. We define the operator $\gamma^* : 2^M \to 2^M$ as:*

$$\gamma^*(S) = \{s \in M : \exists\ W \in \sigma(s) \text{ and } (W^\diamond - S)^c \notin \mathcal{P}\}$$

for every $S \subseteq M$.

The theorem below establishes some essential aspects about the behavior of the γ^*-diamond operator.

Theorem 4.1. *Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$ and let $S, H \subseteq M$. The following statements hold:*

(1) $\gamma^*(S) = [\gamma(S^c)]^c$,
(2) $\gamma^*(S)$ *is open,*
(3) $\gamma^*(S) \subseteq \gamma^*(H)$, *if* $S \subseteq H$,
(4) $\gamma^*(S \cap H) = \gamma^*(S) \cap \gamma^*(H)$,
(5) $\gamma^*(S) = \gamma^*(\gamma^*(S))$ *iff* $\gamma(S^c) = \gamma(\gamma(S^c))$,
(6) $\gamma^*(S) = M - \gamma(M)$, *if* $S^c \notin \mathcal{P}$,
(7) $\gamma^*(S - I) = \gamma^*(S)$, *if* $I^c \notin \mathcal{P}$,
(8) $\gamma^*(S \cup I) = \gamma^*(S)$, *if* $I^c \notin \mathcal{P}$,
(9) $\gamma^*(S) = \gamma^*(H)$, *if* $[(S - H) \cup (H - S)]^c \notin \mathcal{P}$.

Proof. (1) Suppose that $s \in \gamma^*(S)$. Then, there exists $W \in \sigma(s)$ such that $(W^\diamond - S)^c \notin \mathcal{P}$. Since $(W^\diamond - S)^c = (W^\diamond \cap S^c)^c = (W^\diamond)^c \cup S$, $(W^\diamond)^c \cup S \notin \mathcal{P}$, which implies that $s \notin \gamma(S^c)$. Hence, $s \in [\gamma(S^c)]^c$.

Conversely, suppose that $s \in [\gamma(S^c)]^c$. Then, $s \notin \gamma(S^c)$, which implies that there exists $W \in \sigma(s)$ such that $(W^\diamond)^c \cup (S^c)^c \notin \mathcal{P}$. Now, as $(W^\diamond)^c \cup (S^c)^c = (W^\diamond \cap S^c)^c = (W^\diamond - S)^c$, then $(W^\diamond - S)^c \notin \mathcal{P}$. Hence, $s \in \gamma^*(S)$.

(2) By (3) in Theorem 3.3, we know that $\gamma(S^c)$ is closed. Hence, $\gamma^*(S) = [\gamma(S^c)]^c$ is open.

(3) By (1) in Theorem 3.3, we know that if $S \subseteq H$, then $\gamma(S) \subseteq \gamma(H)$. Now, since $S \subseteq H$, then $H^c \subseteq S^c$; hence, $\gamma(H^c) \subseteq \gamma(S^c)$, which implies that $[\gamma(S^c)]^c \subseteq [\gamma(H^c)]^c$. Then, $\gamma^*(S) \subseteq \gamma^*(H)$.

(4) By (3) we have $\gamma^*(S \cap H) \subseteq \gamma^*(S)$ and $\gamma^*(S \cap H) \subseteq \gamma^*(H)$. Hence, $\gamma^*(S \cap H) \subseteq \gamma^*(S) \cap \gamma^*(H)$. Now, let $r \in \gamma^*(S) \cap \gamma^*(H)$. Then, there exist $W_1, W_2 \in \sigma(r)$ such that $(W_1^\diamond - S)^c \notin \mathcal{P}$ and $(W_2^\diamond - H)^c \notin \mathcal{P}$. Let $G = W_1 \cap W_2 \in \sigma(r)$. Since $(W_1^\diamond - S)^c \notin \mathcal{P}$ and $(W_1^\diamond - S)^c \subseteq (G^\diamond - S)^c$, we get that $(G^\diamond - S)^c \notin \mathcal{P}$ and similarly $(G^\diamond - H)^c \notin \mathcal{P}$. Therefore, $[G^\diamond - (S \cap H)]^c = (G^\diamond - S)^c \cap (G^\diamond - H)^c \notin \mathcal{P}$ by Corollary 2.1. Then, $r \in \gamma^*(S \cap H)$. Hence, $\gamma^*(S \cap H) = \gamma^*(S) \cap \gamma^*(H)$.

(5) It follows from the facts:

(a) $\gamma^*(S) = [\gamma(S^c)]^c$.
(b) $\gamma^*(\gamma^*(S)) = M - \gamma[M - (M - \gamma(S^c))] = [\gamma(\gamma(S^c))]^c$.

(6) By Corollary 3.1, we acquire that $\gamma(S^c) = \gamma(M)$ if $S^c \notin \mathcal{P}$. Then, $\gamma^*(S) = [\gamma(S^c)]^c = M - \gamma(M)$.

(7) This is inferred from Corollary 3.1 and $\gamma^*(S - I) = M - \gamma[M - (S - I)] = M - \gamma[(M - S) \cup I] = M - \gamma(M - S) = \gamma^*(S)$.

(8) This is inferred from Corollary 3.1 and $\gamma^*(S \cup I) = M - \gamma[M - (S \cup I)] = M - \gamma[(M - S) - I] = M - \gamma(M - S) = \gamma^*(S)$.

(9) Assume $[(S - H) \cup (H - S)]^c \notin \mathcal{P}$. Let $S - H = I$ and $H - S = J$. Observe that $I^c, J^c \notin \mathcal{P}$ by heredity. Furthermore, we see that $H = (S - I) \cup J$. Thus, $\gamma^*(S) = \gamma^*(S - I) = \gamma^*[(S - I) \cup J] = \gamma^*(H)$ by (7) and (8). □

Remark 4.1. *Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$. Then, by (1) in Theorem 4.1 we have $\gamma^*(M) = [\gamma(M^c)]^c = [\gamma(\emptyset)]^c = \emptyset^c = M$.*

Definition 4.2. *Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$ and let $S \subseteq M$. Then, S is called diamond-open if $S \subseteq \gamma^*(S)$.*

Lemma 4.1. *Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$. Then, every θ-open set is diamond-open.*

Proof. Let $S \subseteq M$ be θ-open. By (1) in Theorem 4.1, we have $\gamma^*(S) = [\gamma(S^c)]^c$. Then, $\gamma(M - S) \subseteq cl_\theta(M-S) = M-S$ since $M-S$ is θ-closed. Thus, $S = M-(M-S) \subseteq M-\gamma(M-S) = \gamma^*(S)$. Hence, S is a diamond-open. □

Theorem 4.2. *Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$. The collection $\sigma_{\gamma^*} = \{S \subseteq M \mid S \subseteq \gamma^*(S)\}$ is a topology on M.*

Proof. By Remark 4.1, $\emptyset, M \in \sigma_{\gamma^*}$. Let $S, T \in \sigma_{\gamma^*}$. Then, $S \subseteq \gamma^*(S)$ and $T \subseteq \gamma^*(T)$. Thus, $S \cap T \subseteq \gamma^*(S) \cap \gamma^*(T) = \gamma^*(S \cap T)$ by (4) in Theorem 4.1. Therefore, $S \cap T \in \sigma_{\gamma^*}$. Let $\{S_\alpha | \alpha \in \Delta\}$ be a family of diamond-open sets. Since $\{S_\alpha \subseteq \gamma^*(S_\alpha) \mid \forall\, \alpha \in \Delta\}$, $S_\alpha \subseteq \gamma^*(S_\alpha) \subseteq \gamma^*(\bigcup_{\alpha\in\Delta} S_\alpha)$ for each $\alpha \in \Delta$. Hence, $\bigcup_{\alpha\in\Delta} S_\alpha \subseteq \gamma^*(\bigcup_{\alpha\in\Delta} S_\alpha)$. Therefore, σ_{γ^*} is topology. □

Lemma 4.2. *Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$. $\overrightarrow{\Pi}(S) \subseteq \gamma^*(S)$ for every subset S of M.*

Proof. From Definition 2.8, we know that $\overrightarrow{\Pi}(S) = \{s \in M \mid \exists\, W \in \sigma(s)$ such that $(cl(W) - S)^c \notin \mathcal{P}\}$. Hence, by Theorem 3.2, we have $\gamma(M-S) \subseteq \Pi(M-S)$. Then, $\overrightarrow{\Pi}(S) = M - \Pi(M-S) \subseteq M-\gamma(M-S) = \gamma^*(S)$. □

Lemma 4.3. *Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$. Then, every β-open subset is diamond-open.*

Proof. Recall that a set S is called β-open if $S \subseteq \overrightarrow{\Pi}(S)$, see Definition 2.1. Let S be β-open. Then, $S \subseteq \overrightarrow{\Pi}(S)$. By Lemma 4.2, $S \subseteq \overrightarrow{\Pi}(S) \subseteq \gamma^*(S)$. Hence, S is diamond-open. □

Lemma 4.4. *Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$ and let $S \subseteq M$. Then,*

$$\gamma(\gamma(S)) \subseteq \gamma(S) \Longleftrightarrow \gamma^*(M-S) \subseteq \gamma^*[\gamma^*(M-S)].$$

Proof. Let $S \subseteq M$. Then,

$$\begin{aligned}
\gamma(\gamma(S)) \subseteq \gamma(S) &\Longleftrightarrow [\gamma(S)]^c \subseteq [\gamma(\gamma(S))]^c \\
&\Longleftrightarrow [\gamma((S^c)^c)]^c \subseteq [\gamma([\gamma((S^c)^c)]^c)^c]^c \\
&\Longleftrightarrow \gamma^*(S^c) \subseteq [\gamma(\gamma^*(S^c))^c]^c \\
&\Longleftrightarrow \gamma^*(M-S) \subseteq \gamma^*[\gamma^*(M-S)].
\end{aligned}$$

□

Corollary 4.1. *Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$ and let $S \subseteq M$. Then,*

$$\gamma(\gamma(S)) \subseteq \gamma(S) \Longleftrightarrow \gamma^*(S) \subseteq \gamma^*(\gamma^*(S)).$$

Proposition 4.1. *Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$. If γ is idempotent, then $\gamma^*(S) - K$ and $\gamma^*(S-K) \in \sigma_{\gamma^*}$ for $S \subseteq M$ and $K^c \in \mathcal{P}$.*

Proof. By (7) in Theorem 4.1 and γ is idempotent, we have

(1) $(\gamma^*(S) - K) \subseteq \gamma^*(S) \subseteq \gamma^*(\gamma^*(S)) = \gamma^*(\gamma^*(S) - K) \Longrightarrow \gamma^*(S) - K \in \sigma_{\gamma^*}$.
(2) $\gamma^*(S-K) = \gamma^*(S) \subseteq \gamma^*(\gamma^*(S)) = \gamma^*(\gamma^*(S-K)) \Longrightarrow \gamma^*(S-K) \in \sigma_{\gamma^*}$.

□

Proposition 4.2. *Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$. The following hold for $S \subseteq M$:*

(1) A subset S is closed in σ_{γ^*} if and only if $\gamma(S) \subseteq S$.
(2) $\sigma_\theta \subseteq \beta \subseteq \sigma_{\gamma^*}$.
(3) If $\Pi(T) = \gamma(T)$ for every $T \subseteq M$, then $\sigma_{\gamma^*} = \beta$.
(4) If $\gamma(\gamma(S)) \nsubseteq \gamma(S)$ and $\Pi(\gamma(S)) \subseteq \gamma(S)$, then $\sigma_{\gamma^*} \nsubseteq \beta$.

Proof. (1): Let S be closed in σ_{γ^*}. Then, $M - S$ is open in σ_{γ^*}, and hence $M - S \subseteq \gamma^*(M - S) = M - [\gamma(S)]$. Thus, $\gamma(S) \subseteq S$.

(2): Let $S \in \sigma_\theta$. We know that $\overrightarrow{\Pi}(S) = M - \Pi(M - S)$. Now, $\Pi(M - S) \subseteq cl_\theta(M - S) = M - S$. Since $M - S$ is θ-closed, then $T = M - (M - S) \subseteq M - \Pi(M - S) = \overrightarrow{\Pi}(S)$ and $\sigma_\theta \subseteq \beta$. Also, if $S \in \beta$, by Lemma 4.2 we get $S \subseteq \overrightarrow{\Pi}(S) \subseteq \gamma^*(S)$. So, $\sigma_\theta \subseteq \beta \subseteq \sigma_{\gamma^*}$.

(3): Let $S \in \sigma_{\gamma^*}$. Then, $S \subseteq \gamma^*(B) = M - \gamma(M - S) = M - \Pi(M - S) = \overrightarrow{\Pi}(S)$ and $\sigma_{\gamma^*} = \beta$.

(4): Since $\Pi(\gamma(S)) \subseteq \gamma(S)$, then $\gamma(S)$ is closed in β by Lemma 3.3, but $\gamma(\gamma(S)) \nsubseteq \gamma(S)$, then by (1) $\gamma(S)$ is not closed in σ_{γ^*}, indicating that $\sigma_{\gamma^*} \nsubseteq \beta$. □

The following examples demonstrate that β and σ are independent.

Example 4.1. *Let $M = \{1, 2, 3\}$ with topology $\sigma = \{\emptyset, M, \{1\}, \{2\}, \{1, 2\}\}$ and a primal $\mathcal{P} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. Then, $\beta = \{\emptyset, M, \{3\}, \{1, 3\}, \{1, 3\}\}$.*

Example 4.2. *Let $\sigma = \{W \subseteq \mathbb{N}$ such that $W = \mathbb{N}$ or $1 \notin W\}$ and $\mathcal{P} = \{W \subseteq \mathbb{N}$ such that $1 \notin W\}$. Let $S \subseteq \mathbb{N}$. Then,*
Case 1. $1 \in S$. As $1 \in (cl(W) - S)^c$, then $(cl(W) - S)^c \notin \mathcal{P}$ for every $W \in \sigma$.
Hence, $\overrightarrow{\Pi}(S) = \mathbb{N}$, which implies that $S \subseteq \overrightarrow{\Pi}(S)$.
Case 2. $1 \notin S$. As $1 \notin (cl(W) - S)^c$, then $(cl(W) - S)^c \in \mathcal{P}$, which implies that $\overrightarrow{\Pi}(S) = \emptyset$, and then $S \subseteq \overrightarrow{\Pi}(S) \Longleftrightarrow S = \emptyset$.

Hence, $S \subseteq \overrightarrow{\Pi}(S)$ if and only if $S = \emptyset$ or $1 \in S$. Therefore, $\beta = \{S, \emptyset \mid 1 \in S\}$.

Theorem 4.3. *Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$. If for each $S \subseteq M$ we have $\gamma(\gamma(S)) \subseteq \gamma(S)$, then $cl_{\gamma^*}(S) = S \cup \gamma(S)$.*

Proof. Since $\gamma(A \cup \gamma(S)) = \gamma(S) \cup \gamma(\gamma(S)) = \gamma(S) \subseteq S \cup \gamma(S)$, we know that $S \cup \gamma(S)$ is a closed set in σ_{γ^*} containing A by Proposition 4.2. Let us demonstrate that $S \cup \gamma(S)$ is the smallest closed set in σ_{γ^*} containing S. Let $s \in \gamma(S) \cup S$. If $s \in S$, then $s \in cl_{\gamma^*}(S)$. If $s \in \gamma(S)$, then $S^c \cup (W^\diamond)^c \in \mathcal{P}$ for every open set $W \in \sigma(s)$. We have $(W^\diamond)^c \cup [cl_{\gamma^*}(S)]^c \in \mathcal{P}$ because $[cl_{\gamma^*}(S)]^c \subseteq S^c$ Therefore, $s \in \gamma[cl_{\gamma^*}(S)]$ and since $cl_{\gamma^*}(S)$ is closed in σ_{γ^*}, then $\gamma[cl_{\gamma^*}(S)] \subseteq cl_{\gamma^*}(S)$. Now, by (1) in Proposition 4.2, we have $s \in cl_{\gamma^*}(S)$. Hence, $cl_{\gamma^*}(S) = S \cup \gamma(S)$ for all $S \subseteq M$. □

Lemma 4.5. *[22] Let (M, σ) be a $\mathcal{TS}$. If either $S \in \sigma$ or $T \in \sigma$, then $Int((cl(S \cap T))) = Int(cl(S)) \cap Int(cl(T))$.*

Theorem 4.4. *Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$ and let $\sigma_{\gamma^{**}} = \{S \subseteq M : S \subseteq Int(cl(\gamma^*(S)))\}$. Then, $\sigma_{\gamma^{**}}$ forms a topology on M.*

Proof. By item (2) in Theorem 4.1, $\gamma^\star(S)$ is an open set for any $S \subseteq M$ and $\sigma_{\gamma^*} \subset \sigma_{\gamma^{**}}$. Thus, $\emptyset, M \in \sigma_{\gamma^{**}}$. Let $A, B \in \sigma_{\gamma^{**}}$. Then, using Theorem 4.1 and Lemma 4.5, we obtain that $S \cap$

$T \subset Int\,(cl(\gamma^\star(S))) \cap Int\,(cl(\gamma^\star(T))) = Int\,(cl(\gamma^\star(S) \cap \gamma^\star(T))) = Int\,(cl(\gamma^\star(S \cap T)))$. Therefore, $S \cap T \in \sigma_{\gamma^{**}}$. Let $S_\alpha \in \sigma_{\gamma^{**}}$ for each $\alpha \in I$. Then, $S_\alpha \subseteq Int\,[cl(\gamma^\star(S_\alpha))]$ for each $\alpha \in I$. Now, by (3) in Theorem 4.1, we get that $Int\,[cl(\gamma^\star(S_\alpha))] \subseteq Int\,[cl(\gamma^\star(\cup S_\alpha))]$ for all $\alpha \in I$ and $\cup S_\alpha \subset Int\,[cl(\gamma^\star(\cup S_\alpha))]$. Therefore, $\cup S_\alpha \in \sigma_{\gamma^{**}}$. Thus, $\sigma_{\gamma^{**}}$ is a topology on M. □

Proposition 4.3. *Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$. We have the following:*

(1) $\varphi \subseteq \sigma_{\gamma^{**}}$.
(2) $\sigma_{\gamma^*} \subseteq \sigma_{\gamma^{**}}$.
(3) T is closed subset in $\sigma_{\gamma^{**}} \Longleftrightarrow cl(Int(\gamma(T))) \subseteq T$.

Proof. Recall that $\varphi = \{S \subseteq M \mid S \subseteq Int(cl(\overrightarrow{\Pi}(S)))\}$, see [17].
(1) Let S be any subset of M. We know that $\overrightarrow{\Pi}(S) \subseteq \gamma^*(S)$ by Lemma 4.2. Then, $Int(cl(\overrightarrow{\Pi}(S))) \subseteq Int(cl(\gamma^*(S)))$. Hence, $\varphi \subseteq \sigma_{\gamma^{**}}$.
(2) Let S be diamond-open. Then, $S \subseteq \gamma^*(S)$. Since $\gamma^*(S)$ is open, we get $S \subseteq \gamma^*(S) \subseteq Int(cl(\gamma^*(S)))$. Thus, $\sigma_{\gamma^*} \subseteq \sigma_{\gamma^{**}}$.
(3) Let T be closed in $\sigma_{\gamma^{**}}$. Then, $M - T$ is open in $\sigma_{\gamma^{**}} \Longleftrightarrow M - T \subseteq Int(cl(\gamma^*(M - T))) = Int(cl(M - [\gamma(T)])) \subseteq [M - cl(Int(\gamma(T)))]$. Hence, $cl(Int(\gamma(T))) \subseteq T$. □

A necessary condition for the tight inequality between these two topologies is given by the lemma that follows.

Lemma 4.6. *Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$ and let $S \subseteq M$. If $\sigma_{\gamma^*} \subsetneq \sigma_{\gamma^{**}}$, then there exists $s \in S$ such that*

(a) $[T^\diamond - S]^c \in \mathcal{P}$ for each $T \in \sigma(s)$;
(b) There exist $W \in \sigma(s)$ and an open set $K \subseteq W$ such that, $[K^\diamond - S]^c \notin \mathcal{P}$.

Proof. If $\sigma_{\gamma^*} \subsetneq \sigma_{\gamma^{**}}$, then there exists $S \in \sigma_{\gamma^{**}} - \sigma_{\gamma^*}$. Since $S \notin \sigma_{\gamma^*}$, there exists $s \in S$ such that

$$\begin{aligned} s \notin \gamma^*(S) &\Longleftrightarrow s \notin M - \gamma[M - S] \\ &\Longleftrightarrow s \in \gamma[M - S] \\ &\Longleftrightarrow \forall\ T \in \sigma(s),\ (T^\diamond)^c \cup S \in \mathcal{P} \\ &\Longleftrightarrow \forall\ T \in \sigma(s),\ [T^\diamond \cap S^c]^c \in \mathcal{P} \\ &\Longleftrightarrow \forall\ T \in \sigma(s),\ [T^\diamond - S]^c \in \mathcal{P}. \end{aligned}$$

Since $S \in \sigma_{\gamma^{**}}$, then for all $r \in S$, we have

$$\begin{aligned} r \in Int\,(cl(\gamma^*(S))) &\Longleftrightarrow \exists\, W \in \sigma(r),\ W \subseteq cl(\gamma^*(S)) \\ &\Longleftrightarrow \exists\, W \in \sigma(r),\ \forall\ z \in W, \forall\ H \in \sigma(z),\ H \cap \gamma^*(S) \neq \emptyset \\ &\Longleftrightarrow \exists\, W \in \sigma(r),\ \forall\ H \subseteq W,\ [H \in \sigma \Rightarrow\ H \cap \gamma^*(S) \neq \emptyset] \\ &\Longleftrightarrow \exists\, W \in \sigma(r),\ \forall\ H \subseteq W,\ [H \in \sigma \Rightarrow\ H \cap [M - \gamma(M - S)] \neq \emptyset] \\ &\Longleftrightarrow \exists\, W \in \sigma(r),\ \forall\ H \subseteq W,\ [H \in \sigma \Rightarrow\ H - \gamma(M - S) \neq \emptyset] \\ &\Longleftrightarrow \exists\, W \in \sigma(r),\ \forall H \subseteq W,\ [H \in \sigma \Rightarrow [\exists\ K \subseteq W\ (K \in \sigma \Longleftrightarrow [K^\diamond - S]^c \notin \mathcal{P})]. \end{aligned}$$

□

Theorem 4.5. *Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$ and let $S \subseteq M$. If $\gamma(cl(Int(\gamma(S)))) \subseteq cl(Int(\gamma(S)))$, then $cl_{\gamma^{\star\star}}(S) = S \cup cl(Int(\gamma(S)))$.*

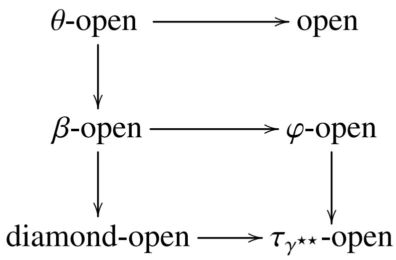
Proof. By Proposition 4.3 and since $\gamma(S)$ is a closed set, we have

$$\begin{aligned} cl(Int(\gamma[S \cup cl(Int(\gamma(S)))])) &= cl(Int(\gamma(S) \cup \gamma(cl(Int(\gamma(S)))))) \\ &\subseteq cl(Int(\gamma(S) \cup cl(Int(\gamma(S))))) \\ &= cl(Int(\gamma(S))) \subseteq S \cup cl(Int(\gamma(S))) \end{aligned}$$

by Proposition 4.3, and we that have $S \cup cl(Int(\gamma(S)))$ is a closed subset in $\sigma_{\gamma^{**}}$ containing S. Now, we want to show that $S \cup cl(Int(\gamma(S)))$ is the smallest closed set in $\sigma_{\gamma^{**}}$ containing S. Let $r \in S \cup cl(Int(\gamma(S)))$. If $r \in S$, then $r \in cl_{\gamma^{**}}(S)$. Suppose that $r \in cl(Int(\gamma(S)))$. Since $S \subseteq cl_{\gamma^{\star\star}}(S)$, then $r \in cl(Int(\gamma(S))) \subseteq cl(Int(\gamma(cl_{\gamma^{\star\star}}(S))))$. As $cl_{\gamma^{\star\star}}(S)$ is closed in $\sigma_{\gamma^{\star\star}}$, then by Proposition 4.3 we have $r \in cl_{\gamma^{\star\star}}(S)$. Thus, $S \cup cl(Int(\gamma(S))) \subseteq cl_{\gamma^{**}}(S)$. Since $cl_{\gamma^{**}}(S)$ is the smallest closed set in $\sigma_{\gamma^{\star\star}}$ containing S, then $cl_{\gamma^{\star\star}}(S) = S \cup cl(Int(\gamma(S)))$.

□

The following diagram and examples show the link between the results such as the concept of topologies β, $\tau_{\gamma^\star}$, φ and $\tau_{\gamma^{\star\star}}$.



(Diagram I)

The following example illustrates the relations between the concepts.

Example 4.3. *Let $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, and the primal $\mathcal{P} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. It is clear that $\beta = \{\emptyset, X, \{c\}, \{b, c\}, \{a, c\}\}$, $\tau_\theta = \{\emptyset, X\}$, and $\tau_{\gamma^\star} = \varphi = \tau_{\gamma^{\star\star}} = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$, as shown by the following table. If $A \subseteq X$:*

Table 1. Details on illustrates the relations between the concepts I.

A	$cl(A)$	$\Pi(X-A)$	$\overrightarrow{\Pi}(A)$	$A^\diamond$	$\gamma(A)$	$\gamma^\star(A)$	$Int(Cl(\overrightarrow{\Pi}(A)))$	$Int(Cl(\gamma^\star(A)))$
$\emptyset$	$\emptyset$	X	$\emptyset$	$\emptyset$	$\emptyset$	$\{a,b\}$	$\emptyset$	X
X	X	$\emptyset$	X	$\{c\}$	$\{c\}$	X	X	X
$\{a\}$	$\{a,c\}$	X	$\emptyset$	$\emptyset$	$\emptyset$	$\{a,b\}$	$\emptyset$	X
$\{b\}$	$\{b,c\}$	X	$\emptyset$	$\emptyset$	$\emptyset$	$\{a,b\}$	$\emptyset$	X
$\{c\}$	$\{c\}$	$\emptyset$	X	$\{c\}$	$\{c\}$	X	X	X
$\{a,b\}$	X	X	$\emptyset$	$\emptyset$	$\emptyset$	$\{a,b\}$	$\emptyset$	X
$\{a,c\}$	$\{a,c\}$	$\emptyset$	X	$\{c\}$	$\{c\}$	X	X	X
$\{b,c\}$	$\{b,c\}$	$\emptyset$	X	$\{c\}$	$\{c\}$	X	X	X

Example 4.4. *Let $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, X, \{a\}, \{c\}, \{b, c\}, \{a, c\}\}$ and the primal $\mathcal{P} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. It is clear that $\beta = \varphi = \{\emptyset, X, \{a\}, \{c\}, \{b, c\}, \{a, c\}\}$, $\tau_\theta = \{\emptyset, X, \{a\}, \{b, c\}\}$, and $\tau_{\gamma^\star} = \tau_{\gamma^{\star\star}} = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$, as shown by the following table. If $A \subseteq X$:*

Table 2. Details on illustrates the relations between the concepts II.

A	$cl(A)$	$\Pi(X-A)$	$\overrightarrow{\Pi}(A)$	$A^\diamond$	$\gamma(A)$	$\gamma^\star(A)$	$Int(Cl(\overrightarrow{\Pi}(A)))$
$\emptyset$	$\emptyset$	$\{b, c\}$	$\{a\}$	$\emptyset$	$\emptyset$	X	$\{a\}$
X	X	$\emptyset$	X	$\{b\}$	$\emptyset$	X	X
$\{a\}$	$\{a\}$	$\{b, c\}$	$\{a\}$	$\emptyset$	$\emptyset$	X	$\{a\}$
$\{b\}$	$\{b\}$	$\{b, c\}$	$\{a\}$	$\emptyset$	$\emptyset$	X	$\{a\}$
$\{c\}$	$\{b, c\}$	$\emptyset$	X	$\emptyset$	$\emptyset$	X	X
$\{a, b\}$	$\{a, b\}$	$\{b, c\}$	$\{a\}$	$\emptyset$	$\emptyset$	X	$\{a\}$
$\{a, c\}$	X	$\emptyset$	X	$\{b\}$	$\emptyset$	X	X
$\{b, c\}$	$\{b, c\}$	$\emptyset$	X	$\{b\}$	$\emptyset$	X	X

5. Compatibility via primal topological spaces

This section introduces a new a primal structure, namely a compatible space. The fundamental properties of this structure are presented.

Definition 5.1. *Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$ and let $S \subseteq M$. Then:*

(1) If $S^c \cup S^\diamond \notin \mathcal{P}$, then σ is suitable for $\mathcal{P}$, [16].
(2) If $[cl(W)]^c \cup S^c \notin \mathcal{P}$ for $W \in \sigma(s)$ where $s \in S$, then σ is Π-suitable for $\mathcal{P}$, [17].

If σ is suitable for $\mathcal{P}$, then σ is Π-suitable for $\mathcal{P}$.

Definition 5.2. *Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$ and let $S \subseteq M$. σ is said to be compatible with $\mathcal{P}$ if the following condition holds:*
If for every $s \in S$ there exists $W \in \sigma(s)$ such that $(W^\diamond)^c \cup S^c \notin \mathcal{P}$, then $S^c \notin \mathcal{P}$.

Proposition 5.1. *Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$ such that σ is Π-suitable for $\mathcal{P}$. Then, σ is compatible.*

Proof. Let σ be Π-suitable for $\mathcal{P}$ and $S \subseteq M$. Assume that for each $s \in S$ there exists $W \in \sigma(s)$ such that $[cl(W)]^c \cup S^c \notin \mathcal{P}$. Since $W^\diamond \subseteq cl(W)$, then $[cl(W)]^c \cup S^c \subseteq [W^\diamond]^c \cup S^c \notin \mathcal{P}$. Therefore, $S^c \notin \mathcal{P}$; hence, σ is compatible with $\mathcal{P}$. □

Theorem 5.1. *Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$ and let $S \subseteq M$, then the following statements are equivalent:*

(1) σ is compatible for $\mathcal{P}$.
(2) If there exists an open cover $\mathcal{W}$ for the set S such that $\forall\ W \in \mathcal{W}$, then $[(W_s)^\diamond]^c \cup S^c \notin \mathcal{P}$, and then $S^c \notin \mathcal{P}$.
(3) If $S \cap \gamma(S) = \emptyset$, then $S^c \notin \mathcal{P}$.
(4) $(S - \gamma(S))^c \notin \mathcal{P}$.
(5) If there is no nonempty subset $R \subseteq S$ such that $R \subseteq \gamma(R)$, then $S^c \notin \mathcal{P}$.

Proof. (1) $\Longrightarrow$ (2): The evidence is clear.

(2) $\Longrightarrow$ (3): Let $s \in S \subseteq M$. As $S \cap \gamma(S) = \emptyset$, then $s \notin \gamma(S)$ which implies that $\exists W_s \in \sigma(s)$ with $[(W_s)^\diamond]^c \cup S^c \notin \mathcal{P}$. Consequently, we have $S \subseteq \cup\{W_s : s \in S\}$ and $W_s \in \sigma(s)$. Hence, by (2) $S^c \notin \mathcal{P}$.

(3) $\Longrightarrow$ (4): Suppose that $S \cap \gamma(S) = \emptyset$. Then, $S - \gamma(S) \subseteq S$ and $(S - \gamma(S)) \cap \gamma(S - \gamma(S)) \subseteq (S - \gamma(S)) \cap \gamma(S) = \emptyset$. Hence, by using (3) we get $(S - \gamma(S))^c \notin \mathcal{P}$.

(4) $\Longrightarrow$ (5): Assume that $(S - \gamma(S))^c \notin \mathcal{P}$. Set $J = S - \gamma(S)$. Then, $S = J \cup (S \cap \gamma(S))$. By Theorem 3.3, we get that $\gamma(S) = \gamma(J) \cup \gamma(S \cap \gamma(S)) = \gamma(S \cap \gamma(S))$. Now, if $R = S \cap \gamma(S) \subseteq S$, then $R = S \cap \gamma(S \cap \gamma(S)) \subseteq \gamma(S \cap \gamma(S)) = \gamma(R) = \emptyset$ by item (6) of Theorem 3.3. Therefore, $S \cap \gamma(S) = \emptyset$ and $S - \gamma(S) = S$, we have $(S - \gamma(S))^c = S^c \notin \mathcal{P}$.

(5) $\Longrightarrow$ (1): Let $s \in S$ and let $W \in \sigma(s)$ such that $(W^\diamond)^c \cup S^c \notin \mathcal{P}$. Then, $S \cap \gamma(S) = \emptyset$ because if there is $r \in S \cap \gamma(S)$, and then for every $H \in \sigma(r)$ we have $(H^\diamond)^c \cup S^c \in \mathcal{P}$, which is a contradiction. Suppose that $R \subseteq S$ such that $R \subseteq \gamma(R)$. Then, $R = R \cap \gamma(R) \subseteq S \cap \gamma(S) = \emptyset$. Thus, S does not contains a nonempty set R with $R \subseteq \gamma(R)$, which implies that, by (5), $S^c \notin \mathcal{P}$. Thus, σ is compatible for the primal $\mathcal{P}$. □

Theorem 5.2. *Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$ and let $S \subseteq M$. If σ is compatible for the primal $\mathcal{P}$, then the following statements are equivalent:*

(1) If $S \cap \gamma(S) = \emptyset$, then $\gamma(S) = \emptyset$.
(2) $\gamma(S - \gamma(S)) = \emptyset$.
(3) $\gamma(S \cap \gamma(S)) = \gamma(S)$.

Proof. We want first to show that if σ is compatible for $\mathcal{P}$ and if $S \cap \gamma(S) = \emptyset$, then $\gamma(S) = \emptyset$. Since $S \cap \gamma(S) = \emptyset$, then by using (3) in Thoerem 5.1 we get that $S^c \notin \mathcal{P}$. Hence, by (6) in Theorem 3.3 we have $\gamma(S) = \emptyset$.

(1) $\Longrightarrow$(2): Suppose that if $S \cap \gamma(S) = \emptyset$, then $\gamma(S) = \emptyset$. We want to show that $\gamma(K) = \emptyset$ where $K = S - \gamma(S)$. Then,

$$\begin{aligned} K \cap \gamma(K) &= (S - \gamma(S)) \cap \gamma(S - \gamma(S)) \\ &= (S \cap (M - \gamma(S))) \cap \gamma(S \cap (M - \gamma(S))) \\ &\subseteq [S \cap (M - \gamma(S))] \cap [\gamma(S) \cap (\gamma(M - \gamma(S)))] = \emptyset. \end{aligned}$$

By (1), we get that $\gamma(K) = \emptyset$.

(2) $\Longrightarrow$ (3): Assume that $\gamma(S - \gamma(S)) = \emptyset$.

$$\begin{aligned} S &= (S - \gamma(S)) \cup (S \cap \gamma(S)) \\ \gamma(S) &= \gamma[(S - \gamma(S)) \cup (S \cap \gamma(S))] \\ &= \gamma(S - \gamma(S)) \cup \gamma(S \cap \gamma(S)) \\ &= \gamma(S \cap \gamma(S)). \end{aligned}$$

(3) $\Longrightarrow$ (1): Suppose that $\gamma(S) = \gamma(S \cap \gamma(S))$ and $\gamma(S) \cap S = \emptyset$. Then,

$$\gamma(\gamma(S) \cap S) = \gamma(S) \Longrightarrow \gamma(\emptyset) = \gamma(S).$$

By Theorem 3.3, $\gamma(\emptyset) = \emptyset$. Hence, $\gamma(S) = \emptyset$. □

Theorem 5.3. *Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$. Then, σ is compatible for $\mathcal{P}$ if and only if $[\gamma^*(S) - S]^c \notin \mathcal{P}$ for every $S \subseteq M$.*

Proof. First, let σ be compatible for $\mathcal{P}$ and let $S \subseteq M$. We want to show that $[\gamma^*(S) - S]^c \notin \mathcal{P}$. Let $s \in \gamma^*(S) - S$. Then, $s \in \gamma^*(S) = [\gamma(S)^c]^c$ and $s \notin S$, which implies that $s \notin \gamma(M - S)$. Hence, $\exists W \in \sigma(s)$ such that $(W^\diamond)^c \cup S \notin \mathcal{P}$. Since σ is compatible, then $S \notin \mathcal{P}$. As $S \subseteq [\gamma^*(S) - S]^c$ and $S \notin \mathcal{P}$, $[\gamma^*(S) - S]^c \notin \mathcal{P}$.

Second, Let, $S \subseteq M$ such that $[\gamma^*(S) - S]^c \notin \mathcal{P}$. We want to show that σ is compatible. Suppose that for every $s \in S$ there exists $W \in \sigma(s)$ such that $[(W^\diamond)^c \cup S^c] \notin \mathcal{P}$. Note that for every $S \subseteq M$, $\gamma^*(S^c) - (S^c) = S - \gamma(S) = \{s \in S \mid \exists W \in \sigma(s) \text{ such that } s \in (W^\diamond)^c \cup S^c \notin \mathcal{P}\}$. As a result, we get that $[S - \gamma(S)]^c = [\gamma^*(S^c) - (S^c)]^c \notin \mathcal{P}$; hence, by item (4) of Theorem 5.1, σ is compatible for $\mathcal{P}$. □

Theorem 5.4. *Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$ such that σ is compatible for $\mathcal{P}$ and the primal diamond operator is idempotent. Then, $\sigma_{\gamma^*} = \{\gamma^*(S) - T : S \subseteq M, \ T^c \notin \mathcal{P}\}$.*

Proof. Let $S, T \subseteq M$. We want first to prove that all the sets are of the form $\gamma^*(S) - T$ in σ_{γ^*}. By using the results from Theorem 4.1 and Corollary 4.1, we have $\gamma^*(S) - T \subseteq \gamma^*(S) \subseteq \gamma^*[\gamma^*(S)] \subseteq \gamma^*[\gamma^*(S) - T]$. By Theorem 4.2, we get that $\gamma^*(S) - T \in \sigma_{\gamma^*}$.

Conversely, let $S \in \sigma_{\gamma^*}$. Therefore, $S \subseteq \gamma^*(S)$. Since σ is compatible for $\mathcal{P}$, then by Theorem 5.3, we have $[\gamma^*(S) - S]^c \notin \mathcal{P}$. Let $T = \gamma^*(S) - S$. Therefore, $S = \gamma^*(S) - T$ and $T^c \notin \mathcal{P}$. Thus, $S \in \{\gamma^*(S) - T : S \subseteq M, \ T^c \notin \mathcal{P}\} = \sigma_{\gamma^*}$. □

Theorem 5.5. *Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$ and σ is be compatible for $\mathcal{P}$. Then, for every $T \in \sigma_\theta$ and $S \subseteq M$, $[\gamma(T \cap S)]^\diamond \subseteq \gamma(T \cap S) \subseteq \gamma(T \cap \gamma(S)) \subseteq cl_\theta(T \cap \gamma(S))$.*

Proof. By (3) in Theorem 5.2 , we have $\gamma(S \cap T) = \gamma((S \cap T) \cap \gamma(S \cap T))$ By (1) in Theorem 3.3, we get that $\gamma((S \cap T) \cap \gamma(S \cap T)) \subseteq \gamma(T \cap \gamma(S))$. Additionally, by Theorem 3.3 and Lemma 3.1, $[\gamma(T \cap S)]^\diamond \subseteq cl(\gamma(T \cap S)) = \gamma(T \cap S) \subseteq \gamma(T \cap \gamma(S)) \subseteq cl_\theta(T \cap \gamma(S))$. □

We now examine some of a compatible structure's primal qualities and investigate some of its attributes via primal topological spaces.

Proposition 5.2. *Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$ and σ be compatible for $\mathcal{P}$. If $T \subseteq \gamma(S) \cap \gamma^*(S)$ and $T \neq \emptyset$ is open, then $[T - S]^c \notin \mathcal{P}$ and $(T^\diamond)^c \cup S^c \in \mathcal{P}$ for $T, S \subseteq M$.*

Proof. Since σ is compatible for $\mathcal{P}$, then by Theorem 5.3, we have that $[\gamma^*(S) - S]^c \notin \mathcal{P}$. Given that $T \subseteq \gamma(S) \cap \gamma^*(S)$ such that T is a nonempty open set, as $[\gamma^*(S) - S]^c \subseteq [T - S]^c$, then $[T - S]^c \notin \mathcal{P}$ by heredity. Since T is an open nonempty set and $T \subseteq \gamma(S)$, then $(T^\diamond)^c \cup S^c \in \mathcal{P}$ by the definition of $\gamma(S)$. □

We say that $S = T$ [mod $\mathcal{P}$] if $[(S - T) \cup (T - S)]^c \notin \mathcal{P}$, where [mod $\mathcal{P}$] is an equivalence relation. By (9) in Theorem 4.1, we have, if $S = T$ [mod $\mathcal{P}$], then $\gamma^*(S) = \gamma^*(T)$.

Lemma 5.1. *Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$ and let σ be compatible for $\mathcal{P}$. If S, $T \in \sigma_\theta$, and $\gamma^*(S) = \gamma^*(T)$, then $S = T$ [mod $\mathcal{P}$].*

Proof. Let $S \in \sigma_\theta$. Then, by Lemma 4.1 we have $S \subseteq \gamma^*(S)$; hence, $S - T \subseteq \gamma^*(S) - T = \gamma^*(T) - T$ and $[\gamma^*(T) - T]^c \notin \mathcal{P}$ by Theorem 5.3. Consequently, $[S - T]^c \notin \mathcal{P}$ and $[T - S]^c \notin \mathcal{P}$. Now, $(S - T)^c \cap (T - S)^c = [(S - T) \cup (T - S)]^c \notin \mathcal{P}$ by additivity. Hence, $S = T$ [mod $\mathcal{P}$]. □

Definition 5.3. *Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$ and let $S \subseteq M$. We say that S is a Baire set pertaining to σ and $\mathcal{P}$, and we write $S \in \mathcal{B}_\theta$ if there exists $W \in \sigma_\theta$ such that $S = W$ [mod $\mathcal{P}$].*

Theorem 5.6. *Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$ such that σ is compatible for $\mathcal{P}$. If $S, T \in \mathcal{B}_\theta$ and $\gamma^*(S) = \gamma^*(T)$, then $S = T$ [mod $\mathcal{P}$].*

Proof. Let $W_1, W_2 \in \sigma_\theta$ such that $S = W_1$ [mod $\mathcal{P}$] and $T = W_2$ [mod $\mathcal{P}$]. Then, by using the result (9) in Theorem 4.1, we have $\gamma^*(S) = \gamma^*(W_1)$ and $\gamma^*(T) = \gamma^*(W_2)$. Since $\gamma^*(S) = \gamma^*(T)$, then $\gamma^*(W_1) = \gamma^*(W_2)$, which implies that $W_1 = W_2$ [mod $\mathcal{P}$] by Lemma 5.1. Thus, $S = T$ [mod $\mathcal{P}$] by transitivity. □

Theorem 5.7. *Let $(M, \sigma, \mathcal{P})$ be a $\mathcal{PTS}$. If $\sigma - \{M\} \subseteq \mathcal{P}$ such that σ is compatible for the primal $\mathcal{P}$, then $\gamma^*(S) \subseteq \gamma(S)$ for any set $S \subseteq M$.*

Proof. Let $s \in \gamma^*(S)$. Suppose that $s \notin \gamma(S)$. Then, there exists $W_s \in \sigma(s)$ such that $[W_s^\diamond \cap S]^c \notin \mathcal{P}$. Since $s \in \gamma^*(A)$, then there exists $H \in \sigma(s)$ and $[H^\diamond - S]^c \notin \mathcal{P}$. Thus, $[(W_s \cap H)^\diamond \cap S]^c \notin \mathcal{P}$ and $[(W_s \cap H)^\diamond - S]^c \notin \mathcal{P}$ by heredity. Consequently, $[(W_s \cap H)^\diamond]^c = [(W_s \cap H)^\diamond \cap S]^c \cap [(W_s \cap H)^\diamond - S]^c \notin \mathcal{P}$. Since $[(W_s \cap H)^\diamond]^c \in \sigma(s)$, which is a contradiction to $\sigma - \{M\} \subseteq \mathcal{P}$, $s \in \gamma(S)$. Hence, $\gamma^*(S) \subseteq \gamma(S)$. □

6. Conclusions

Acharjee et al. introduced a new mathematical structure called primal in [11], which is the inverse of the concept of grills. They also provided results that connect topological spaces with primal topological spaces. Since the concept of primal topological spaces has been quickly developed Al-Shami et al. in [12] defined soft primal soft topology. Moreover, Al-Omari et al. has introduced a new structure, called the soft primal, in [13], and investigated its properties and applications. Also, Ameen et al. presented results regarded to the concept of fuzzy primal in [14]. Furthermore, Al-Omari found out a new class of proximity spaces called primal-proximity spaces, which are derived from the notion of primal in [15]. This study had an important role in the improvement of operators in primal topological spaces. The aim of this study is to introduce and examine some novel operators based on primal structures, which enrich the field of primal topological spaces by generating new frameworks that enable us to formulate new notions and properties. Moreover, by using these operators, we were able to construct a new topological space that will help to discover new notations and applications in this area. This also contributes significantly in the improvement of other topological notions such as fuzzy and soft primal topological spaces. In future work, we will explore more results regarding to the primal topological spaces.

Author contributions

O. Alghamdi: Visualization, Writing-original draft, Writing–review & editing; A. Al-Omari: Conceptualization, Methodology, Investigation, Writing-review & editing; M. H. Alqahtani: Writing-review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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