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Research article

Global well-posedness and scattering of the four dimensional cubic focusing nonlinear Schrödinger system

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Abstract: In this paper, the Cauchy problem for a class of coupled system of the four-dimensional cubic focusing nonlinear Schrödinger equations was investigated. By exploiting the double Duhamel method and the long-time Strichartz estimate, the global well-posedness and scattering were proven for the system below the ground state. In our proof, we first established the variational characterization of the ground state, and obtained the threshold of the global well-posedness and scattering. Second, we showed that the non-scattering is equivalent to the existence of an almost periodic solution by following the concentration-compactness/rigidity arguments of Kenig and Merle [17] (*Invent. Math.*, **166** (2006), 645–675). Then, we obtained the global well-posedness and scattering below the threshold by excluding the almost periodic solution.

Keywords: nonlinear Schrödinger system; global well-posedness; ground state; scattering; almost periodic solution
Mathematics Subject Classification: 35A15, 35B15, 35P25, 35Q55

1. Introduction

In this paper, we consider the following cubic-focusing energy-critical nonlinear Schrödinger system on $\mathbb{R}^4$

$$\begin{cases} i\partial_t u_j(t,x) + \Delta u_j(t,x) = -F_j(\mathbf{u}), \\ \mathbf{u}(0,x) = \mathbf{u}_0 = (u_{1,0}, u_{2,0}) \in (\dot{H}^1_x(\mathbb{R}^4))^2, \end{cases} \tag{1.1}$$

where $j = 1,2, \mathbf{u} = \mathbf{u}(t,x) = \big(u_1(t,x), u_2(t,x)\big) : \mathbb{R}\times\mathbb{R}^4 \to \mathbb{C}^2$ is the unknown function and the nonlinear term $F_j(\mathbf{u})$ is given by the relation

$$F_j(\mathbf{u}) := \Big(|u_j(t,x)|^2 u_j(t,x) + |u_k(t,x)|^2 u_j(t,x)\Big),$$

where $k = 1, 2$, and $k \neq j$.

The nonlinear Schrödinger system (1.1) enjoys the following conservation laws.

1) Mass conservation

$$M\Big(u_1(t), u_2(t)\Big) = \int_{\mathbb{R}^4} \Big(|u_1(t,x)|^2 + |u_2(t,x)|^2\Big)dx = M\Big(u_1(0), u_2(0)\Big). \tag{1.2}$$

2) Energy conservation

$$\begin{aligned} E(u_1(t), u_2(t)) &= \frac{1}{2}\int_{\mathbb{R}^4} (|\nabla u_1(t,x)|^2 + |\nabla u_2(t,x)|^2)dx - \frac{1}{4}\int_{\mathbb{R}^4} \Big(|u_1(t,x)|^2 + |u_2(t,x)|^2\Big)^2 dx \\ &= E(u_1(0), u_2(0)). \end{aligned} \tag{1.3}$$

System (1.1) is called energy-critical since a solution (u_1, u_2) to system (1.1) is invariant under the scaling

$$\Big(u_1(t,x), u_2(t,x)\Big) \mapsto \lambda\Big(u_1(\lambda^2 t, \lambda x), u_2(\lambda^2 t, \lambda x)\Big).$$

We first briefly recall the results of the Cauchy problem for the single nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = \mu f(u), \\ u(0,x) = u_0 \in \dot{H}^1(\mathbb{R}^d), \end{cases} \tag{1.4}$$

where dimension $d \geq 3$, $\mu = \pm 1$, $f(u) = |u|^{\frac{4}{d-2}}u$, and $u(t,x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$. By rescaling the values of u, it is possible to restrict attention to the case $\mu = -1$ or $\mu = 1$; these are known as the focusing and defocusing equations, respectively.

There is a large number of works on problem (1.4). In the defocusing case, Bourgain [3] first proved the global well-posedness and scattering for radial initial data in dimensions ($d = 3, 4$) by introducing the induction on energy method. Grillakis [13] demonstrated global regularity for the three-dimensional energy-critical defocusing NLS with spherically symmetric initial data, that is, he proved that smooth spherically symmetric initial data lead to a globally smooth solution. This result can be deduced a posteriori from [3]; however, the argument in [13] is rather different. Further advancements in the spherically symmetric case were made by Tao [32], who extended the result to higher dimensions. For non-radial initial data, Colliander et al. [5] achieved a significant breakthrough by proving the global well-posedness and scattering for $d = 3$, introducing a wealth of new ideas and tools to the problem. Specifically, the authors utilized an interaction Morawetz inequality (introduced in [4]), which is more suitable for the non-radial case than the Morawetz inequality used in previous works. Subsequently, Ryckman and Visan [28] extended the argument from [5] to $d = 4$, and Visan [34] obtained global well-posedness and scattering for $d \geq 5$. Interested readers can refer to [8, 19, 35] for an easier understanding of the global well-posedness and scattering for $d = 3, 4, 5$ using the long-time Strichartz estimate of Dodson [10].

In the focusing case, Kenig and Merle [17] established global well-posedness and scattering when $d = 3, 4, 5$ for radial initial data, where they first put forward the concentration compactness/rigidity method. Killip and Visan [18] extended this to global well-posedness and scattering for non-radial initial data when $d \geq 5$. Later, Dodson [11] obtained global well-posedness and scattering for non-radial initial data in $d = 4$. However, it seems difficult to solve the problem for $d = 3$ due to the fact that the stationary solution $W(x)$ of problem (1.4) does not belong to $L^2(\mathbb{R}^d)$ when $d = 3, 4$. Here, $W(x)$ denotes the ground state. Therefore, the case ($d = 3$) remains an open question.

We can summarize the above research results in the following theorem.

Theorem 1.1. *Let $u_0 \in \dot{H}^1(\mathbb{R}^d)$, when $\mu = 1$, the corresponding solution u to problem* (1.4) *is global for $d \geq 3$.*

When $\mu = -1$ and $E(u_0) < E(W)$, the following results hold.

1) If $\|\nabla u_0\|_{L^2(\mathbb{R}^d)} < \|\nabla W\|_{L^2(\mathbb{R}^d)}$, then the solution u of problem (1.4) is global for $d \geq 4$, and this is also true for $d = 3$ in the radial case.

2) If $\|\nabla u_0\|_{L^2(\mathbb{R}^d)} > \|\nabla W\|_{L^2(\mathbb{R}^d)}$, and if either $u_0 \in L^2(\mathbb{R}^d)$ or $u_0 \in H^1(\mathbb{R}^d)$ is radial, then the solution u of problem (1.4) blows up in finite time for $d \geq 3$.

Furthermore, the global solution scatters in $\dot{H}^1(\mathbb{R}^d)$.

When $\mu = -1$, there exists global solution to problem (1.4) that does not scatter, that is,

$$W(t, x) = W(x) := \frac{1}{\left(1 + \frac{|x|^2}{d(d-2)}\right)^{\frac{d-2}{2}}}, \tag{1.5}$$

which solves the nonlinear elliptic equation

$$\Delta W + |W|^{\frac{4}{d-2}} W = 0, \tag{1.6}$$

then $W(t, x) = W(x)$ is a stationary solution to problem (1.4). Moreover, the energy of ground state W is

$$E(W) = \frac{1}{2}\|\nabla W\|^2_{L^2(\mathbb{R}^d)} - \frac{d-2}{2d}\|W\|^{\frac{2d}{d-2}}_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} = d^{-1} C_d^{-d}.$$

Here, the sharp constant

$$C_d^{-d} = \|\nabla W\|^2_{L^2(\mathbb{R}^d)} = \|W\|^{\frac{2d}{d-2}}_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}, \tag{1.7}$$

comes from the Sobolev embedding inequality

$$\|u\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} \leq C_d \|\nabla u\|_{L^2(\mathbb{R}^d)}. \tag{1.8}$$

By following the arguments in [1, 31], we obtain W is a maximizer in the sharp Sobolev embedding inequality.

There are also some results on the nonlinear Schrödinger equation of fourth order. Miao et al. [26] considered the focusing energy-critical nonlinear Schrödinger equation of fourth order $iu_t + \Delta^2 u = |u|^{\frac{8}{d-4}} u$ with $d \geq 5$. They proved that if a maximal-lifespan radial solution $u : I \times \mathbb{R}^d \to \mathbb{C}$ obeys $\sup_{t \in I} \|\Delta u(t)\|_2 \leq \|\Delta W\|_2$, then it is global and scatters both forward and backward in time. In addition, if a solution has both energy and kinetic energy lower than those of the ground state W at some point in time, then the solution is global and scatters. After that, they considered the defocusing energy-critical nonlinear Schrödinger equation of fourth order $iu_t + \Delta^2 u = -|u|^{\frac{8}{d-4}} u$ in [25]. They proved that any finite energy solution is global and scatters both forward and backward in time in dimensions $d \geq 9$.

Some results on nonlinear Schrödinger system were also obtained. For example, Cheng et al. [9] obtained the global well-posedness and scattering of the cubic focusing infinite coupled nonlinear Schrödinger system on $\mathbb{R}^2$ below the threshold in $L_x^2 h^1(\mathbb{R}^2 \times \mathbb{Z})$. Gao and Wang [14] proved a concentration result for blow-up solutions of the coupled Schrödinger equations with non-spherically symmetric initial data in $H^1(\mathbb{R}^2)$. There are also some papers on other types of nonlinear Schrödinger systems, for instance, [21, 36, 37]. Compared to the single nonlinear Schrödinger equation, results on the nonlinear Schrödinger system is relatively short.

The main motivation of this article is to provide a satisfactory answer to the following Theorem 1.2 and to extend the results of single nonlinear Schrödinger (NLS) equation to a class of coupled systems, which contributes to a better understanding of the long-time evolution between systems. Such finite coupled nonlinear Schrödinger system has already been applied in nonlinear optics; see [2] and the references therein. It provides a useful approximation for describing the propagation of self-trapped, mutually incoherent wave packets in nonlinear optics. In order to prove the following Theorem 1.2, we first establish the variational characterization of the ground state and determine the threshold for global well-posedness and scattering, which is a pivotal initial step. Subsequently, we employ the approach developed by Kenig and Merle [17], utilizing concentration-compactness/rigidity method to reduce the problem of global well-posedness and scattering to the exclusion of almost periodic solutions. In other words, we need to exclude the existence of almost periodic solutions to system (1.1) that satisfy $K = \int_{\mathbb{R}} N(t)^{-2} dt < \infty$ and $K = \int_{\mathbb{R}} N(t)^{-2} dt = \infty$. We prove the global well-posedness and scattering of the four-dimensional cubic focusing energy-critical nonlinear Schrödinger system below the ground state. Our main results are listed as follows.

Theorem 1.2. *If the initial data* $\boldsymbol{u}_0 \in (\dot{H}_x^1(\mathbb{R}^4))^2$ *satisfies*

$$E(\mathbf{u}_0) < E(\mathbf{W}), \quad \|\mathbf{u}_0\|_{(\dot{H}^1(\mathbb{R}^4))^2} < \|\mathbf{W}\|_{(\dot{H}^1(\mathbb{R}^4))^2},$$

where $\mathbf{W} = (\frac{1}{\sqrt{2}} W, \frac{1}{\sqrt{2}} W)$, W is shown in (1.5), then the corresponding solution $\mathbf{u}$ to system (1.1) is globally well-posed and scatters for both time directions.

Remark 1.3. If not otherwise specified, the following $\mathbf{W} = (\frac{1}{\sqrt{2}} W, \frac{1}{\sqrt{2}} W)$, W denotes ground state and is the stationary solution to problem (1.4), when $\mu = -1$. The specific characterizations of ground state $\mathbf{W}$ can be attained in Section 3.

The proof of Theorem 1.2 is based on the following three theorems.

Theorem 1.4. *(Reduction to almost periodic solution, [17, 18]) If system* (1.1) *is not globally well-posed and scattering for all data satisfying* $E(\boldsymbol{u}_0) < E(\boldsymbol{W}), \|\boldsymbol{u}_0\|_{(\dot{H}^1(\mathbb{R}^4))^2} < \|\boldsymbol{W}\|_{(\dot{H}^1(\mathbb{R}^4))^2}$, *then there exists a non-zero solution* $\boldsymbol{u}$ *to system* (1.1) *that is almost periodic for the entire time of its existence.*

So far, reducing almost periodic solutions has become a standard technique in the analysis of dispersive equations at critical regularity. Their existence was first established by Keraani [15] in the context of the mass-critical nonlinear Schrödinger equation, and was initially employed by Kenig, Merle [17] as a tool to prove global well-posedness. As mentioned above, Theorem 1.4 has been proven in [17, 18].

Next, is the definition of almost periodicity in Theorem 1.4.

Definition 1.5. If there exists $N(t) : I \to (0, \infty)$ and $x(t) : I \to \mathbb{R}^4$ such that for all $t \in I$, $\frac{1}{N(t)}\mathbf{u}(\frac{x-x(t)}{N(t)})$ lies in a compact set $K \subset \dot{H}^1(\mathbb{R}^4)$, then $\mathbf{u}$ is almost periodic for all $t \in I$, where I is the maximal interval of its existence.

Theorem 1.6. *The only almost periodic solution on the maximal interval of its existence I satisfying* $\|\nabla \boldsymbol{u}(t)\|_{(L_t^\infty L_x^2(I\times\mathbb{R}^4))^2} < \|\nabla \boldsymbol{W}\|_{(L^2(\mathbb{R}^4))^2}$ *is* $\boldsymbol{u} \equiv 0$.

Theorem 1.7. *To prove Theorem* 1.6*, it suffices to show that the only global, almost periodic solution to system* (1.1) *on* $\mathbb{R}$ *satisfying*

$$N(t) \geq 1, N(0) = 1,$$

is $\mathbf{u} \equiv 0$.

The outline of the paper is as follows. In Section 2, we present the local well-posedness of system (1.1) and some analytic tools. In Section 3, we establish variational characterizations of the ground state. In Section 4, we give that the non-scattering is equivalent to the existence of almost periodic solutions. In Section 5, we prove the long time Strichartz estimate, which plays a pivotal role in proving the main results. In Section 6, we first preclude the almost periodic solution to system (1.1) satisfying $K = \int_{\mathbb{R}} N(t)^{-2}dt < \infty$ (see Theorem 6.1). Second, we exclude the almost periodic solution to system (1.1) with $K = \int_{\mathbb{R}} N(t)^{-2}dt = \infty$ (see Theorem 6.2). Our study excursion comes to an end with the provided conclusion in Section 7.

2. Notation and preliminaries

In this section, we will briefly recall the local well-posedness theory of system (1.1) and some basic results.

2.1. Some notations

We use the following notations throughout this paper. We will use $X \lesssim Y$ whenever there exists some constant $C > 0$ such that $X \leq CY$. Similarly, we use $X \sim Y$ if $X \lesssim Y \lesssim X$. For any space-time slab $I \times \mathbb{R}^4$, we use $(L_t^q L_x^r(I \times \mathbb{R}^4))^2$ to denote the Banach space time of function $\mathbf{u} : I \times \mathbb{R}^4 \to \mathbb{C}^2$ whose norm is

$$\|\mathbf{u}\|_{(L_t^q L_x^r(I\times\mathbb{R}^4))^2} := (\int_I \|\mathbf{u}\|^q_{(L_x^r(\mathbb{R}^4))^2} dt)^{\frac{1}{q}}$$

with the usual modifications when q or r are equal to infinity. When $q = r$, we abbreviate $L_t^q L_x^r$ as $L_{t,x}^q$.

We define the Fourier transform on $\mathbb{R}^4$ to be

$$\mathscr{F}\mathbf{f}(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^4} e^{-ix\xi}\mathbf{f}(x)dx.$$

For $s \in \mathbb{R}$, we define the fractional differentiation/integral operator

$$\mathscr{F}(|\nabla|^s \mathbf{f}(\xi)) := |\xi|^s \mathscr{F}\mathbf{f}(\xi),$$

which in turn define the homogeneous Sobolev norm

$$\|\mathbf{f}\|_{(\dot{H}^s(\mathbb{R}^4))^2} := \||\nabla|^s \mathbf{f}\|_{(L^2(\mathbb{R}^4))^2}.$$

2.2. Local theory and analytic tools

In this subsection, we will review local well-posedness and some important estimates. We first give the precise definition of solution in the energy spaces.

Definition 2.1. A function $\mathbf{u} : I \times \mathbb{R}^4 \to \mathbb{C}^2$ on a nonempty time interval $t_0 \in I \subset \mathbb{R}$ is a strong $\dot{H}^1$ solution of system (1.1) if it lies in the class $C_t^0 \dot{H}_x^1(K \times \mathbb{R}^4) \bigcap L_{t,x}^6(K \times \mathbb{R}^4)$ for all compact $K \subset I$, and obeys the Duhamel formula

$$u_j(t_1) = e^{i(t_1-t_0)\Delta} u_{j,0} + i \int_{t_0}^{t_1} e^{i(t_1-t)\Delta} F_j(\mathbf{u}(t))dt$$

for all $t_1 \in I$ and $j = 1, 2$. We refer to the interval I as the lifespan of $\mathbf{u}$. It is said that $\mathbf{u}$ is a maximal-lifespan solution if the solution cannot be extended to any strictly larger interval, and that $\mathbf{u}$ is a global solution if $I = \mathbb{R}$.

Next, we will give the definition of scattering size and blow up.

Definition 2.2. The scattering size of a solution to system (1.1) on a time interval I by

$$S_I(\mathbf{u}) := \int_I \int_{\mathbb{R}^4} |\mathbf{u}(t, x)|^6 dxdt.$$

Definition 2.3. A solution $\mathbf{u}$ to system (1.1) blows up forward in time if there exists a time $t_1 \in I$ such that

$$S_{[t_1, \sup I)}(\mathbf{u}) = \infty,$$

and that $\mathbf{u}$ blows up backward in time if there exists a time $t_1 \in I$ such that

$$S_{(\inf I, t_1]}(\mathbf{u}) = \infty.$$

Let $e^{it\Delta}$ be the free Schrödinger evolution. From the explicit formula

$$e^{it\Delta}\mathbf{f}(x) = \frac{1}{(4\pi it)^2} \int_{\mathbb{R}^4} e^{i|x-y|^2/4t} \mathbf{f}(y)dy,$$

one easily obtains the standard dispersive inequality

$$\|e^{it\Delta}\mathbf{f}\|_{(L^\infty(\mathbb{R}^4))^2} \lesssim |t|^{-2} \|\mathbf{f}\|_{L^1(\mathbb{R}^4))^2},$$

for all $t \neq 0$.

A different way to express the dispersive effect of the operator $e^{it\Delta}$ is in terms of space-time integrability. To state the estimates, we first need the following definition.

Definition 2.4. (Admissible pairs) For $d \geq 3$, a pair of exponents (q, r) is an admissible pair if

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, 2 \leq q, r \leq \infty,$$

and $(d, q, r) \neq (2, 2, \infty)$.

For a fixed space-time slab $I \times \mathbb{R}^d$, we define the Strichartz norm

$$\|\mathbf{u}\|_{(S^0(I))^2} := \sup \|\mathbf{u}\|_{(L_t^q L_x^r(I\times\mathbb{R}^d))^2}, \quad \|\mathbf{u}\|_{(\dot{S}^s(I))^2} := \||\nabla|^s \mathbf{u}\|_{(S^0(I))^2},$$

where (q, r) is an admissible pair and $s \in \mathbb{R}$. We write $S^0(I)$ for the closure of all test functions under this norm and denote by $N^0(I)$ the dual of $S^0(I)$.

Now, we are ready to state the standard Strichartz estimates.

Theorem 2.5. *(Strichartz estimates) Let I be a compact time interval, and let $\boldsymbol{u} : I \times \mathbb{R}^4 \to \mathbb{C}^2$ be a solution to the forced Schrödinger system*

$$i\partial_t u_j(t, x) + \Delta u_j(t, x) = -F_j(\mathbf{u}),$$

then for any $t_0 \in I$

$$\|\nabla u_j\|_{S^0(I)} \lesssim \|u_j(t_0)\|_{\dot{H}^1(\mathbb{R}^d)} + \|\nabla F_j\|_{N^0(I)}.$$

Proof. We treat the non-endpoint case following [12, 29]. For the endpoint $(q, r) = (2, \frac{2d}{d-2})$ in dimensions $d \geq 3$, see [16]. For failure of the $d = 2$ endpoint, see [27]. □

Theorem 2.6. *(Littlewood-Paley) For any $1 < p < \infty$,*

$$\left\| \left(\Sigma_N |P_N \boldsymbol{f}|^2 \right)^{1/2} \right\|_{(L^p(\mathbb{R}^4))^2} \sim_{p,d} \|\boldsymbol{f}\|_{(L^p(\mathbb{R}^4))^2}.$$

Theorem 2.7. *(Sobolev embedding, [33]) For $1 \leq p \leq q \leq \infty, N \in \mathbb{Z}$,*

$$\|P_N \boldsymbol{f}\|_{(L^q(\mathbb{R}^4))^2} \lesssim 2^{4N(\frac{1}{p} - \frac{1}{q})} \|P_N \boldsymbol{f}\|_{(L^p(\mathbb{R}^4))^2}.$$

Theorem 2.8. *(Bernstein's estimate) For any $s \in \mathbb{R}, N \in \mathbb{Z}, 1 < p < \infty$,*

$$2^{Ns} \|P_N \boldsymbol{f}\|_{(L^p(\mathbb{R}^4))^2} \sim \||\nabla|^s P_N \boldsymbol{f}\|_{(L^p(\mathbb{R}^4))^2}.$$

Theorem 2.9. *(Maximal Strichartz estimate) Suppose $t, t_0 \in I$, and*

$$v_j(t) = \int_{t_0}^{t} e^{i(t-s)\Delta} F_j(s) ds,$$

then for $d = 4, q > 4, j = 1, 2$,

$$\left\| \sup 2^{N(\frac{4}{q} - 2)} \|P_N v_j(t)\|_{L_x^q(\mathbb{R}^4)} \right\|_{L_t^2(I)} \lesssim_q \|F_j\|_{L_t^2 L_x^1(I\times\mathbb{R}^4)}.$$

With the aid of Theorem 2.5, the following local well-posedness theory is obtained. Because the proof is relatively standard, we do not review it here and refer interested readers to [6, 7].

Theorem 2.10. *(Local well-posedness) Assume $\boldsymbol{u}_0 \in (\dot{H}^1(\mathbb{R}^4))^2$ and $t_0 \in \mathbb{R}$, there exists a unique maximal-lifespan $\boldsymbol{u} : I \times \mathbb{R}^4 \to \mathbb{C}^2$ to system (1.1) with initial data $\boldsymbol{u}(t_0) = \boldsymbol{u}_0$. This solution has the following properties*

1) Local existence: I is an open neighborhood of t_0.

2) Blow-up criterion: If $\sup I < \infty$, then $S_{[t_1, \sup I)}(\mathbf{u}) = \infty$. Similarly, if $\inf I < \infty$, then $S_{(\inf I, t_1]}(\mathbf{u}) = \infty$.

3) Scattering: If $\sup I = \infty$, and $\mathbf{u}$ does not blow up forward in time, then $\mathbf{u}$ scatters forward in time, that is, there exists a unique $\mathbf{u}_+ \in (\dot{H}^1(\mathbb{R}^4))^2$ such that

$$\lim_{t\to\infty} \|\mathbf{u}(t) - e^{it\Delta}\mathbf{u}_+\|_{(\dot{H}^1(\mathbb{R}^4))^2} = 0.$$

4) Small data global existence: If $\|\nabla \mathbf{u}_0\|_{(L^2(\mathbb{R}^4))^2}$ is sufficiently small, then $\mathbf{u}$ is a global solution, which does not blow up either forward or backward in time. Indeed, in this case, $S_{\mathbb{R}}(\mathbf{u}) \leq \|\nabla \mathbf{u}_0\|^6_{(L^2(\mathbb{R}^4))^2}$.

3. Variational characterization of the ground state

In this section, we mainly study the variational characterizations of the ground state. Let us first define the ground state to the system (1.1).

Set ground state $\mathbf{W} = (W_1, W_2)$ of the system (1.1), that is $\mathbf{W}$, is stationary solution of system (1.1), and $\mathbf{W}$ solves the following elliptic system

$$\begin{cases} \Delta W_1 = -|W_1|^2 W_1 - |W_2|^2 W_1, \\ \Delta W_2 = -|W_2|^2 W_2 - |W_1|^2 W_2, \end{cases} \tag{3.1}$$

then by subtracting the second equation from the first equation of (3.1), we obtain

$$\Delta(W_1 - W_2) = \left(|W_1|^2 + |W_2|^2\right)\left(W_2 - W_1\right). \tag{3.2}$$

Multiplying the Eq (3.2) by $\left(W_1 - W_2\right)$, it follows that

$$(W_1 - W_2)\Delta(W_1 - W_2) = -\left(|W_1|^2 + |W_2|^2\right)\left(W_1 - W_2\right)^2. \tag{3.3}$$

Then, integrating by parts over $\mathbb{R}^4$, we have

$$\int_{\mathbb{R}^4} \left|\nabla(W_1 - W_2)\right|^2 dx = -\int_{\mathbb{R}^4} \left(|W_1|^2 + |W_2|^2\right)\left(W_1 - W_2\right)^2 dx, \tag{3.4}$$

which implies $W_1 = W_2$ in $\dot{H}^1(\mathbb{R}^4)$. Hence, the system (3.1) can be reduced to the single nonlinear elliptic equation

$$\Delta\widetilde{W} = -2|\widetilde{W}|^2\widetilde{W}. \tag{3.5}$$

The positive solution of (3.5) is known as radially symmetric and unique up to translation and dilation (see [22]) and is identified by

$$W(t,x) = W(x) := \frac{1}{1 + \frac{|x|^2}{8}}, \tag{3.6}$$

which uniquely solves (1.6) (see [31]). As we all know, the solution of (3.5) minimizes the corresponding energy functional (see [30]), and W obtains the best constant of the Sobolev embedding equality (1.8), that is (1.7), is shown.

The ground state is associated with the best constant in the vector-valued sharp Gagliardo-Nirenberg inequality:

$$2\int_{\mathbb{R}^4}(|u_1|^2+|u_2|^2)^2dx-\int_{\mathbb{R}^4}(|u_1|^4+|u_2|^4)dx\le C_{res}\Big(\int_{\mathbb{R}^4}(|u_1|^2+|u_2|^2)dx\Big)\Big(\int_{\mathbb{R}^4}(|\nabla u_1|^2+|\nabla u_2|^2)dx\Big),$$

we define the Weinstein functional

$$F(\mathbf{u}):=\frac{2\int_{\mathbb{R}^4}(|u_1|^2+|u_2|^2)^2dx-\int_{\mathbb{R}^4}(|u_1|^4+|u_2|^4)dx}{\Big(\int_{\mathbb{R}^4}(|u_1|^2+|u_2|^2)dx\Big)\Big(\int_{\mathbb{R}^4}(|\nabla u_1|^2+|\nabla u_2|^2)dx\Big)}.$$

By standard variational argument, a maximizer $\mathbf{W}=(W_1,W_2)$ of the Weinstein functional $F(\mathbf{u})$ weakly solves the system (3.1), if it exists.

If a maximizer $\mathbf{W}=(W_1,W_2)$ exists and we assume it is non-negative, then by a standard argument using the maximum principle, each component W_j of such a non-negative maximizer is indeed strictly positive. In addition, due to strict positivity of each W_j and (3.4), we can derive $W_1=W_2$.

Therefore, under spatial translation and dilation, $\mathbf{W}=(W_1,W_2)=(\frac{1}{\sqrt{2}}W,\frac{1}{\sqrt{2}}W)$ is the unique positive solution of (3.1) that minimizes the corresponding energy functional.

4. Reduction to the almost periodic solution

In this section, we will show that the non-scattering is equivalent to the existence of almost periodic solutions. For any $0\le E\le\|\nabla\mathbf{W}\|_{(L^2(\mathbb{R}^4))^2}$, we define

$$L(E):=\sup\Big\{S_R(\mathbf{u})\Big|\|\mathbf{u}\|_{(L_t^\infty\dot{H}_x^1)^2}\le E\Big\},$$

thus, $L:\Big[0,\|\nabla\mathbf{W}\|_{(L^2(\mathbb{R}^4))^2}\Big]\to\Big[0,\infty\Big]$ is a nondecreasing function with $L\Big(\|\nabla\mathbf{W}\|_{(L^2(\mathbb{R}^4))^2}\Big)=\infty$, and by [18, Lemma 1.4], we see that L is continuous.

To prove system (1.1) is globally well-posed, and scatters satisfying $E(\mathbf{u}_0)<E(\mathbf{W})$ and $\|\mathbf{u}_0\|_{(\dot{H}^1(\mathbb{R}^4))^2}<\|\mathbf{W}\|_{(\dot{H}^1(\mathbb{R}^4))^2}$, it suffices to prove that $L(E)<\infty$ for $E<\|\nabla\mathbf{W}\|_{(L^2(\mathbb{R}^4))^2}$. Therefore, if Theorem 1.2 does not hold, then by the continuity of $L(E)$, there exists $E_c<\|\nabla\mathbf{W}\|_{(L^2(\mathbb{R}^4))^2}$ such that $L(E_c)=\infty$, and according to L is nondecreasing function, $L(E)<\infty$ for all $E<E_c$. E_c is called minimal energy. If we can prove $E_c=\|\nabla\mathbf{W}\|_{(L^2(\mathbb{R}^4))^2}$, then the global well-posedness, and scattering are established. Suppose $E_c<\|\nabla\mathbf{W}\|_{(L^2(\mathbb{R}^4))^2}$, by following the concentration-compactness/rigidity arguments in [17, 18], we obtain the following theorem.

Theorem 4.1. *Assume $E_c<\|\nabla\boldsymbol{W}\|_{(L^2(\mathbb{R}^4))^2}$, there exists a solution $\boldsymbol{u}\in C_t^0\dot{H}_x^1(K\times\mathbb{R}^4)\bigcap L_{t,x}^6(K\times\mathbb{R}^4)$ of system (1.1) with $\sup_{t\in I}\|\nabla\boldsymbol{u}(t)\|^2_{(L^2(\mathbb{R}^4))^2}=E_c$, which is almost periodic in the sense that exists $\Big(N(t),x(t)\Big)\in\mathbb{R}^+\times\mathbb{R}^4$ such that for any $\eta>0$, there exists $C(\eta)<\infty$ satisfying for any $t\in I$,*

$$\int_{|x-x(t)>\frac{C(\eta)}{N(t)}}|\nabla\boldsymbol{u}(t,x)|^2dx+\int_{|\xi|>C(\eta)N(t)}|\xi|^2|\hat{\boldsymbol{u}}(t,\xi)|^2d\xi<\eta. \tag{4.1}$$

Remark 4.2. Note that we have the freedom to modify $N(t)$ by any bounded function of t, provided that we also modify compactness modulus function C accordingly. In particular, one could restrict $N(t)$ to be a constant locally if one wishes to. Thus, recall that [20, Lemma 5.21], one can choose $N(t)$ such that

$$|N'(t)| \lesssim N(t)^3, \tag{4.2}$$

and

$$\int_I N(t)^2 dt \lesssim_{\mathbf{u}} \int_I \int_{\mathbb{R}^4} |\mathbf{u}(t,x)|^6 dx dt \lesssim_{\mathbf{u}} 1 + \int_I N(t)^2 dt. \tag{4.3}$$

Sketch of proof of Theorem 1.7. Suppose $\mathbf{u}$(t) is an almost periodic solution to system (1.1), one can take a limit of $\mathbf{u}(t_n)$ in $\dot{H}^1/G$ (G is a symmetry group) and deduce a solution to system (1.1) satisfying either

$$N(t) \geq 1, N(0) = 1,$$

or that $\mathbf{u}$(t) blows up in finite time.

First, at $t = 0$ and by time reversal symmetry, suppose $\mathbf{u}$(t) blows up as $t \to 0$, then by (4.2) and (4.3), $N(t) \to \infty$ as $t \to 0$.

The next step is to prove that $\int_{\mathbb{R}^4} |\mathbf{u}(t,x)|^2 dx = 0$ for any $t > 0$, which implies the solution $\mathbf{u}$ is identically zero, thus contradicting that $\mathbf{u}$ blows up in finite time.

For any $R > 0$, we define

$$M_R(t) := \int_{\mathbb{R}^4} \phi\Big(\frac{x}{R}\Big)^2 |\mathbf{u}(t,x)|^2 dx,$$

where ϕ is a smooth, radial function, such that

$$\phi(r) = \begin{cases} 1 & |r| \leq 1, \\ 0 & |r| \geq 2. \end{cases}$$

By (4.1) and Hölder's inequality, we obtain

$$\lim_{t \to 0} M_R(t) = 0. \tag{4.4}$$

Moreover, by integration by parts, one has

$$\partial_t M_R(t) \leq \frac{1}{R} \int_{\mathbb{R}^4} \phi'\Big(\frac{x}{R}\Big)\phi\Big(\frac{x}{R}\Big) |\nabla \mathbf{u}(t,x)||\mathbf{u}(t,x)| dx \leq \frac{1}{R}\Big(M_R(t)\Big)^{1/2} \|\nabla \mathbf{u}(t,x)\|^2_{(L^2(\mathbb{R}^4))^2}. \tag{4.5}$$

Therefore, (4.4) combined with the fundamental theorem of calculus and (4.5) implies

$$\int_{\mathbb{R}^4} |\mathbf{u}(t,x)|^2 dx = 0$$

for any $t > 0$, then according to conservation of mass (1.2), it implies $\mathbf{u} \equiv 0$, which contradicts $\mathbf{u}$ blowing up in finite time. Hence, Theorem 1.2 has been reduced to Theorem 1.7.

5. Long-time Strichartz estimate

The main result of this section is a long-time Strichartz estimate; since the usual interaction Morawetz estimate is not positive definite in the focusing case, we will rely on long-time Strichartz estimate based on $K = \int_I N(t)^{-2} dt$.

Theorem 5.1. *(Long-time Strichartz estimate) Suppose I is an interval and given above K, then for any $l \in \mathbb{Z}$,*

$$\Big(\sum_{k\le l}\|\boldsymbol{u}_k\|^2_{(\dot S^1(I\times\mathbb{R}^4))^2}\Big)^{1/2}+2^{2l}\Big\|\sup_{k\ge l}2^{-2k}\|\boldsymbol{u}_k\|_{(L^\infty_x(\mathbb{R}^4))^2}\Big\|_{(L^2_t(I))^2}\lesssim(1+2^{4l}K)^{1/2}. \tag{5.1}$$

Remark: To simplify notation, it is convenient to write $\mathbf{u}_k$ instead of $P_k\mathbf{u}$.

Proof. It follows from Theorem 2.7, Strichartz estimate, and Theorem 2.8 that

$$\begin{aligned}2^{4l}\sum_{k\ge l}2^{-4k}\|e^{i(t-t_0)\Delta}\mathbf{u}_k(t_0)\|^2_{(L^2_tL^\infty_x(I\times\mathbb{R}^4))^2}&\lesssim 2^{4l}\sum_{k\ge l}2^{-2k}\|e^{i(t-t_0)\Delta}\mathbf{u}_k(t_0)\|^2_{(L^2_tL^4_x(I\times\mathbb{R}^4))^2}\\ &\lesssim 2^{2l}\|\mathbf{u}_{\ge l}(t_0)\|^2_{(L^2(\mathbb{R}^4))^2}\lesssim\|\nabla\mathbf{u}(t_0)\|^2_{(L^2(\mathbb{R}^4))^2}\lesssim 1,\end{aligned} \tag{5.2}$$

and

$$\sum_{k\le l}\|\nabla e^{i(t-t_0)\Delta}\mathbf{u}_k(t_0)\|^2_{(L^2_tL^4_x(I\times\mathbb{R}^4))^2}\lesssim\|\nabla\mathbf{u}(t_0)\|^2_{(L^2(\mathbb{R}^4))^2}\lesssim 1. \tag{5.3}$$

Let

$$\begin{aligned}\|\mathbf{u}\|_{(Y(I\times\mathbb{R}^4))^2}=&\sup_l 2^{2l}(1+2^{4l}K)^{-1/2}\Big\|\sup_{k\ge l}2^{-2k}\|\mathbf{u}_l(t)\|_{(L^\infty_x(\mathbb{R}^4))^2}\Big\|_{(L^2_t(I))^2}\\ &+\sup_l(1+2^{4l}K)^{-1/2}\Big(\sum_{k\le l}2^{2k}\|\mathbf{u}_k(t)\|^2_{(L^2_tL^4_x(I\times\mathbb{R}^4))^2}\Big)^{1/2}.\end{aligned} \tag{5.4}$$

Our goal is to use (5.2), (5.3) and the smallness of $\mathbf{u}$ away from the scale $N(t)$ to prove an estimate of the form

$$\|\mathbf{u}\|_{(Y(I\times\mathbb{R}^4))^2}\lesssim 1+\eta\|\mathbf{u}\|_{(Y(I\times\mathbb{R}^4))^2}. \tag{5.5}$$

To this end, we decompose

$$F(\mathbf{u})=F(\mathbf{u}_{\ge l})+O(\mathbf{u}^2_{\ge l}\mathbf{u}_{\le l})+O(\mathbf{u}^2_{\le l}\mathbf{u}_{\ge l})+F(\mathbf{u}_{\le l}). \tag{5.6}$$

By (4.1), it is possible to choose $c(\eta)>0$ such that

$$\|\mathbf{u}_{\le c(\eta)N(t)}\|_{(L^\infty_t\dot H^1_x(\mathbb{R}\times\mathbb{R}^4))^2}\le\eta. \tag{5.7}$$

Step 1. Estimation for $F(\mathbf{u}_{\ge l})$.

By Bernstein's inequality, we obtain

$$\begin{aligned}&\|P_{\le cN(t)}\mathbf{u}_{\ge l}\|^3_{(L^6_tL^3_x(I\times\mathbb{R}^4))^2}\\ &\lesssim\Big\|\sum_{l\le k_1\le k_2\le k_3}\|P_{\le cN(t)}\mathbf{u}_{k_1}\|_{(L^\infty_x(\mathbb{R}^4))^2}\|P_{\le cN(t)}\mathbf{u}_{k_2}\|_{(L^2_x(\mathbb{R}^4))^2}\|P_{\le cN(t)}\mathbf{u}_{k_3}\|_{(L^2_x(\mathbb{R}^4))^2}\Big\|_{(L^2_t(I))^2}\\ &\lesssim\|P_{\le cN(t)}\mathbf{u}\|^2_{(L^\infty_t\dot H^1_x(\mathbb{R}\times\mathbb{R}^4))^2}\Big\|\sup_{k\ge l}2^{-2k}\|\mathbf{u}_k\|_{(L^\infty_x(\mathbb{R}^4))^2}\Big\|_{(L^2_t(I))^2}\\ &\lesssim\eta^2 2^{-2l}(1+2^{4l}K)^{1/2}\|\mathbf{u}\|_{(Y(I\times\mathbb{R}^4))^2},\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}\|P_{\ge cN(t)}\mathbf{u}\|^3_{(L^6_tL^3_x(I\times\mathbb{R}^4))^2}&\lesssim\Big(\int_I\|\mathbf{u}_{\ge cN(t)}\|^2_{(L^2_x(\mathbb{R}^4))^2}\|\mathbf{u}\|^4_{(L^4_x(\mathbb{R}^4))^2}dt\Big)^{1/2}\\ &\lesssim\|\mathbf{u}\|^4_{(L^\infty_t\dot H^1_x(\mathbb{R}\times\mathbb{R}^4))^2}\Big(\int_I c^{-2}N(t)^{-2}dt\Big)^{1/2}\\ &\lesssim c^{-1}K^{1/2}.\end{aligned} \tag{5.9}$$

Then, combining (5.8) with (5.9), we see that

$$\|\mathbf{u}_{\geq l}^3\|_{(L_t^2L_x^1(I\times\mathbb{R}^4))^2} \lesssim \eta^2 2^{-2l}(1+2^{4l}K)^{1/2}\|\mathbf{u}\|_{(Y(I\times\mathbb{R}^4))^2} + c^{-1}K^{1/2}. \tag{5.10}$$

It follows from Theorem 2.9 and (5.10) that

$$\begin{aligned}\Big\|\sup_{k\geq l} 2^{-2k}\|P_k\int_{t_0}^t e^{i(t-\tau)\Delta}F(\mathbf{u}_{\geq l})d\tau\|_{(L_x^\infty(\mathbb{R}^4))^2}\Big\|_{(L_t^2(I))^2} &\lesssim \|\mathbf{u}_{\geq l}^3\|_{(L_t^2L_x^1(I\times\mathbb{R}^4))^2}\\ \lesssim \eta^2 2^{-2l}(1+2^{4l}K)^{1/2}\|\mathbf{u}\|_{(Y(I\times\mathbb{R}^4))^2} + c^{-1}K^{1/2}.\end{aligned} \tag{5.11}$$

Therefore,

$$\begin{aligned}&\Big(\sum_{k\leq l} 2^{2k}\|P_k\int_{t_0}^t e^{i(t-\tau)\Delta}F(\mathbf{u}_{\geq l})d\tau\|^2_{(L_t^2L_x^4(I\times\mathbb{R}^4))^2}\Big)^{1/2}\\ &\lesssim \Big(\sum_{k\leq l} 2^{4k}\Big)^{1/2}\|\mathbf{u}_{\geq l}^3\|_{(L_t^2L_x^1(I\times\mathbb{R}^4))^2}\\ &\lesssim \eta^2(1+2^{4l}K)^{1/2}\|\mathbf{u}\|_{(Y(I\times\mathbb{R}^4))^2} + c^{-1}K^{1/2}2^{2l}.\end{aligned} \tag{5.12}$$

Step 2. Estimation for $O(\mathbf{u}_{\geq l}^2\mathbf{u}_{\leq l})$.

By Sobolev embedding theorem, Littlewood-Paley theorem, and interpolation, then,

$$\begin{aligned}\|\mathbf{u}_{\leq l}\|_{(L_{t,x}^6(I\times\mathbb{R}^4))^2} &\lesssim \|\nabla\mathbf{u}_{\leq l}\|_{(L_t^6L_x^{12/5}(I\times\mathbb{R}^4))^2}\\ &\lesssim \|\nabla\mathbf{u}_{\leq l}\|^{2/3}_{(L_t^\infty L_x^2(I\times\mathbb{R}^4))^2}\Big(\sum_{k\leq l}2^{2k}\|\mathbf{u}_k\|^2_{(L_t^2L_x^4(I\times\mathbb{R}^4))^2}\Big)^{1/6}\\ &\lesssim (1+2^{4l}K)^{1/6}\|\mathbf{u}\|^{1/3}_{(Y(I\times\mathbb{R}^4))^2}.\end{aligned} \tag{5.13}$$

By Hölder's inequality, Theorem 2.7, (5.10), and (5.13), one has

$$\begin{aligned}&\|\mathbf{u}_{\geq l}^2\mathbf{u}_{\leq l}\|_{(L_t^2L_x^{4/3}(I\times\mathbb{R}^4))^2} \lesssim \|\mathbf{u}_{\geq l}\|^2_{(L_t^6L_x^3(I\times\mathbb{R}^4))^2}\|\mathbf{u}_{\leq l}\|_{(L_t^6L_x^{12}(I\times\mathbb{R}^4))^2}\\ &\lesssim 2^{l/3}\|\mathbf{u}_{\geq l}\|^2_{(L_t^6L_x^3(I\times\mathbb{R}^4))^2}\|\mathbf{u}_{\leq l}\|_{(L_{t,x}^6(I\times\mathbb{R}^4))^2}\\ &\lesssim 2^{l/3}\Big(\eta^2 2^{-2l}(1+2^{4l}K)^{1/2}\|\mathbf{u}\|_{(Y(I\times\mathbb{R}^4))^2} + c^{-1}K^{1/2}\Big)^{2/3}(1+2^{4l}K)^{1/6}\|\mathbf{u}\|^{1/3}_{(Y(I\times\mathbb{R}^4))^2}.\end{aligned} \tag{5.14}$$

Theorem 2.7, Strichartz estimate, and (5.14) imply

$$\begin{aligned}&\Big\|\sup_{k\geq l}2^{-2k}\|P_k\int_{t_0}^t e^{i(t-\tau)\Delta}O(\mathbf{u}_{\geq l}^2\mathbf{u}_{\leq l})d\tau\|_{(L_x^\infty(\mathbb{R}^4))^2}\Big\|_{(L_t^2(I))^2}\\ &\lesssim 2^{-l}\Big\|\int_{t_0}^t e^{i(t-\tau)\Delta}O(\mathbf{u}_{\geq l}^2\mathbf{u}_{\leq l})d\tau\Big\|_{(L_t^2L_x^4(I\times\mathbb{R}^4))^2}\\ &\lesssim 2^{-l}\|\mathbf{u}_{\geq l}^2\mathbf{u}_{\leq l}\|_{(L_t^2L_x^{4/3}(I\times\mathbb{R}^4))^2}\\ &\lesssim 2^{-2l/3}\Big(\eta^2 2^{-2l}(1+2^{4l}K)^{1/2}\|\mathbf{u}\|_{(Y(I\times\mathbb{R}^4))^2} + c^{-1}K^{1/2}\Big)^{2/3}(1+2^{4l}K)^{1/6}\|\mathbf{u}\|^{1/3}_{(Y(I\times\mathbb{R}^4))^2}\\ &\lesssim 2^{-2l}\eta^{4/3}(1+2^{4l}K)^{1/2}\|\mathbf{u}\|_{(Y(I\times\mathbb{R}^4))^2} + 2^{-2l/3}c^{-2/3}K^{1/3}(1+2^{4l}K)^{1/6}\|\mathbf{u}\|^{1/3}_{(Y(I\times\mathbb{R}^4))^2}.\end{aligned} \tag{5.15}$$

It follows from Strichartz estimate and (5.15) that

$$\begin{aligned}&\Big(\sum_{k\le l}2^{2k}\|P_k\int_{t_0}^{t}e^{i(t-\tau)\Delta}O(\mathbf{u}_{\ge l}^2\mathbf{u}_{\le l})d\tau\|^2_{(L_t^2L_x^4(I\times\mathbb{R}^4))^2}\Big)^{1/2}\lesssim 2^l\|\mathbf{u}_{\ge l}^2\mathbf{u}_{\le l}\|_{(L_t^2L_x^{4/3}(I\times\mathbb{R}^4))^2}\\&\lesssim \eta^{4/3}(1+2^{4l}K)^{1/2}\|\mathbf{u}\|_{(Y(I\times\mathbb{R}^4))^2}+2^{4l/3}c^{-2/3}K^{1/3}(1+2^{4l}K)^{1/6}\|\mathbf{u}\|^{1/3}_{(Y(I\times\mathbb{R}^4))^2}.\end{aligned}\tag{5.16}$$

Step 3. Estimation for $F(\mathbf{u}_{\le l})$.

By Hölder's inequality, Sobolev embedding theorem, and (4.1), one has

$$\begin{aligned}\Big\|(P_{\le cN(t)}\mathbf{u}_{\le l})\Big\|^2_{(L_t^2L_x^4(I\times\mathbb{R}^4))^2}&\lesssim \|\mathbf{u}_{\le l}\|_{(L_t^2L_x^\infty(I\times\mathbb{R}^4))^2}\|\mathbf{u}_{\le cN(t)}\|_{(L_t^\infty L_x^4(I\times\mathbb{R}^4))^2}\\&\lesssim \|\nabla\mathbf{u}_{\le l}\|_{(L_t^2L_x^4(I\times\mathbb{R}^4))^2}\|\mathbf{u}_{\le cN(t)}\|_{(L_t^\infty L_x^4(I\times\mathbb{R}^4))^2}\\&\lesssim \eta(1+2^{4l}K)^{1/2}\|\mathbf{u}\|_{(Y(I\times\mathbb{R}^4))^2}.\end{aligned}\tag{5.17}$$

Bernstein's inequality and Sobolev embedding theorem yield

$$\begin{aligned}\Big\|(P_{\ge cN(t)}\mathbf{u}_{\le l})\Big\|^2_{(L_t^2L_x^4(I\times\mathbb{R}^4))^2}&\lesssim 2^l\Big(\int_I\|\mathbf{u}_{>cN(t)}\|^2_{(L_x^2(\mathbb{R}^4))^2}\|\mathbf{u}_{\le l}\|^2_{(L_x^\infty(\mathbb{R}^4))^2}dt\Big)^{1/2}\\&\lesssim c^{-1}K^{1/2}2^{2l}.\end{aligned}\tag{5.18}$$

Then, combining (5.17) with (5.18), we obtain

$$\|\mathbf{u}_{\le l}^2\|_{(L_t^1L_x^2(I\times\mathbb{R}^4))^2}\lesssim \eta(1+2^{4l}K)^{1/2}\|\mathbf{u}\|_{(Y(I\times\mathbb{R}^4))^2}+c^{-1}K^{1/2}2^{2l}.\tag{5.19}$$

It follows from Theorem 2.9 and (5.19) that

$$\begin{aligned}&\Big\|\sup_{k\ge l}2^{-2k}\|P_k\int_{t_0}^{t}e^{i(t-\tau)\Delta}F(\mathbf{u}_{\le l})d\tau\|_{(L_x^\infty(\mathbb{R}^4))^2}\Big\|_{(L_t^2(I))^2}\lesssim \|\mathbf{u}_{\le l}^2\|_{(L_t^1L_x^2(I\times\mathbb{R}^4))^2}\\&\lesssim \eta(1+2^{4l}K)^{1/2}\|\mathbf{u}\|_{(Y(I\times\mathbb{R}^4))^2}+c^{-1}K^{1/2}2^{2l}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\Big(\sum_{k\le l}2^{2k}\|P_k\int_{t_0}^{t}e^{i(t-\tau)\Delta}F(\mathbf{u}_{\le l})d\tau\|^2_{(L_t^2L_x^4(I\times\mathbb{R}^4))^2}\Big)^{1/2}\\&\lesssim\Big(\sum_{k\le l}2^{4k}\Big)^{1/2}\|\mathbf{u}_{\le l}^2\|_{(L_t^1L_x^2(I\times\mathbb{R}^4))^2}\\&\lesssim 2^{2l}\eta(1+2^{4l}K)^{1/2}\|\mathbf{u}\|_{(Y(I\times\mathbb{R}^4))^2}+c^{-1}K^{1/2}2^{4l}.\end{aligned}\tag{5.20}$$

Step 4. Estimation for $O(\mathbf{u}_{\le l}^2\mathbf{u}_{\ge l})$.

Using (5.17) and (5.18), then,

$$\begin{aligned}&\Big\|\nabla O(\mathbf{u}_{\le l}^2\mathbf{u}_{>l})\Big\|_{(L_t^2L_x^{4/3}(I\times\mathbb{R}^4))^2}+\|\nabla\mathbf{u}_{\le l}^3\|_{(L_t^2L_x^{4/3}(I\times\mathbb{R}^4))^2}\\&\lesssim \|\nabla\mathbf{u}\|_{(L_t^\infty L_x^2(I\times\mathbb{R}^4))^2}\|\mathbf{u}_{\le l}^2\|_{(L_t^1L_x^2(I\times\mathbb{R}^4))^2}\\&\lesssim c^{-1}K^{1/2}2^{2l}+\eta(1+2^{4l}K)^{1/2}\|\mathbf{u}\|_{(Y(I\times\mathbb{R}^4))^2}.\end{aligned}\tag{5.21}$$

Therefore, we see that

$$\Big(\sum_{k\le l}2^{2k}\|P_k\int_{t_0}^{t}e^{i(t-\tau)\Delta}O(\mathbf{u}_{\le l}^2\mathbf{u}_{>l})d\tau\|_{(L_t^2L_x^4(I\times\mathbb{R}^4))^2}^2\Big)^{1/2} \lesssim c^{-1}K^{1/2}2^{2l}+\eta(1+2^{4l}K)^{1/2}\|\mathbf{u}\|_{(Y(I\times\mathbb{R}^4))^2}, \tag{5.22}$$

Using Sobolev embedding theorem, Bernstein's inequality, Strichartz estimate, and (5.22), we get

$$\begin{aligned}&\Big\|\sup_{k\ge l}2^{-2k}\|P_k\int_{t_0}^{t}e^{i(t-\tau)\Delta}O(\mathbf{u}_{\le l}^2\mathbf{u}_{>l})d\tau\|_{(L_x^\infty(\mathbb{R}^4))^2}\Big\|_{(L_t^2(I))^2}\\ &\lesssim 2^{-2l}\Big\|\nabla\int_{t_0}^{t}e^{i(t-\tau)\Delta}O(\mathbf{u}_{\le l}^2\mathbf{u}_{>l})d\tau\Big\|_{(L_t^2L_x^4(I\times\mathbb{R}^4))^2}\\ &\lesssim c^{-1}K^{1/2}+2^{-2l}\eta(1+2^{4l}K)^{1/2}\|\mathbf{u}\|_{(Y(I\times\mathbb{R}^4))^2}.\end{aligned} \tag{5.23}$$

Combining (5.2), (5.3), (5.11), (5.15), (5.16), (5.20), (5.22), and (5.23), it is obvious to get

$$\|\mathbf{u}\|_{(Y(I\times\mathbb{R}^4))^2}\lesssim c(\eta)^{-1}+\eta\|\mathbf{u}\|_{(Y(I\times\mathbb{R}^4))^2}.$$

By choosing $\eta>0$ sufficiently small, Theorem 5.1 is proved. □

Remark 5.2. By Theorem 2.9 and above analysis, we have also proved

$$2^{2l}\Big\|\sup_{k\ge l}2^{-4k/3}\|\mathbf{u}_k\|_{(L_x^6(\mathbb{R}^4))^2}\Big\|_{(L_t^2(I))^2}\lesssim(1+2^{4l}K)^{1/2}, \tag{5.24}$$

and this result will be applied later.

6. Exclusion of the almost periodic solution

In this section, we prove the main results by considering the following two cases.
Case 1: $K=\int_{\mathbb{R}}N(t)^{-2}dt<\infty$.
Case 2: $K=\int_{\mathbb{R}}N(t)^{-2}dt=\infty$.

First, let us prove the nonexistence of almost periodic solution for system (1.1) under Case 1. The following theorem is obtained.

Theorem 6.1. *There is no almost periodic solution for system* (1.1) *satisfying* $K=\int_{\mathbb{R}}N(t)^{-2}dt<\infty$ *in Theorem* 4.1.

Proof. By (4.1), for any $\eta>0$, there exists $l_0(\eta)$ such that

$$\|P_{\le l_0}\mathbf{u}(t)\|_{(L_t^\infty\dot{H}_x^1(\mathbb{R}\times\mathbb{R}^4))^2}\le\eta.$$

Let k_0 be the integer such that $2^{k_0}\le K^{-1/4}\le 2^{k_0+1}$, by Duhamel formula, for $l\le k_0$ and $t\in[-T,T]$, then,

$$\nabla P_{\le l}\mathbf{u}(t)=\nabla P_{\le l}\mathbf{u}(-T)-i\nabla P_{\le l}\int_{-T}^{t}e^{i(t-\tau)\Delta}F(\mathbf{u}(\tau))d\tau.$$

For $l\le l_0(\eta)$ and k_0,

$$\Big\|\nabla F(\mathbf{u}_{\le l})\Big\|_{(L_t^2L_x^{4/3}([-T,T]\times\mathbb{R}^4))^2}\lesssim\eta^2\|\nabla\mathbf{u}_{\le l}\|_{(L_t^2L_x^4([-T,T]\times\mathbb{R}^4))^2}. \tag{6.1}$$

Exploiting Theorem 2.7 and Hölder's inequality, we derive

$$\begin{aligned}
&\Big\|\nabla P_{\le l}O(\mathbf{u}_{\le l}^2\mathbf{u}_{l\le\cdot\le k_0})\Big\|_{(L_t^2L_x^{4/3}([-T,T]\times\mathbb{R}^4))^2}\lesssim 2^l\Big\|\nabla P_{\le l}O(\mathbf{u}_{\le l}^2\mathbf{u}_{l\le\cdot\le k_0})\Big\|_{(L_t^2L_x^1([-T,T]\times\mathbb{R}^4))^2}\\
&\lesssim 2^l\|\mathbf{u}_{l\le\cdot\le k_0}\|_{(L_t^\infty L_x^2(\mathbb{R}\times\mathbb{R}^4))^2}\|\nabla\mathbf{u}_{\le l}\|_{(L_t^2L_x^4([-T,T]\times\mathbb{R}^4))^2}\|\nabla\mathbf{u}_{\le l}\|_{(L_t^\infty L_x^2(\mathbb{R}\times\mathbb{R}^4))^2}\\
&\lesssim \eta\|\nabla\mathbf{u}_{\le l}\|_{(L_t^2L_x^4(I\times\mathbb{R}^4))^2}.
\end{aligned}\tag{6.2}$$

It follows from Bernstein's inequality and Theorem 2.7 that

$$\begin{aligned}
&\Big\|\nabla P_{\le l}O(\mathbf{u}_{l\le\cdot\le k_0}^2\mathbf{u})\Big\|_{(L_t^2L_x^{4/3}([-T,T]\times\mathbb{R}^4))^2}\lesssim 2^{2l}\Big\|\nabla P_{\le l}O(\mathbf{u}_{l\le\cdot\le k_0}^2\mathbf{u})\Big\|_{(L_t^2L_x^{4/5}([-T,T]\times\mathbb{R}^4))^2}\\
&\lesssim 2^{2l}\sum_{l\le k_1\le k_2\le k_3}\|\mathbf{u}_{k_1}\|_{(L_t^2L_x^4([-T,T]\times\mathbb{R}^4))^2}\|\mathbf{u}_{k_2}\|_{(L_t^\infty L_x^2([-T,T]\times\mathbb{R}^4))^2}\|\mathbf{u}_{k_3}\|_{(L_t^\infty L_x^4([-T,T]\times\mathbb{R}^4))^2}\\
&\lesssim\Big(\sum_{l\le m\le k_0}2^{l-m}\|\mathbf{u}_m\|_{(\dot S^1([-T,T]\times\mathbb{R}^4))^2}\Big)\Big(\sum_{l\le m\le k_0}2^{l-m}\|\mathbf{u}_m\|_{(L_t^\infty\dot H^1([-T,T]\times\mathbb{R}^4))^2}\Big).
\end{aligned}\tag{6.3}$$

Combining (4.1) with (6.1)–(6.3), obviously,

$$\begin{aligned}
&\Big\|\nabla P_{\le l}F(\mathbf{u}_{\le k_0})\Big\|_{(L_t^2L_x^{4/3}([-T,T]\times\mathbb{R}^4))^2}\lesssim\eta\|\mathbf{u}_{\le l}\|_{(\dot S^1([-T,T]\times\mathbb{R}^4))^2}\\
&\quad+\Big(\sum_{l\le m\le k_0}2^{l-m}\|\mathbf{u}_m\|_{(\dot S^1([-T,T]\times\mathbb{R}^4))^2}\Big)\Big(\sum_{l\le m\le k_0}2^{l-m}\|\mathbf{u}_m\|_{(L_t^\infty\dot H^1([-T,T]\times\mathbb{R}^4))^2}\Big).
\end{aligned}\tag{6.4}$$

Using Sobolev embedding theorem, Bernstein's inequality, Strichartz estimate, (4.1), and Theorem 5.1, then,

$$\begin{aligned}
&\Big\|\nabla P_{\le l}\Big[F(\mathbf{u})-F(\mathbf{u}_{\le k_0})\Big]\Big\|_{(L_t^2L_x^{4/3}([-T,T]\times\mathbb{R}^4))^2}\lesssim 2^l\|\mathbf{u}_{\ge k_0}^3\|_{(L_t^2L_x^1([-T,T]\times\mathbb{R}^4))^2}\\
&+2^l\|\mathbf{u}_{l\le\cdot\le k_0}\|_{(L_t^4L_x^8([-T,T]\times\mathbb{R}^4))^2}^2\|\mathbf{u}_{>k_0}\|_{(L_t^\infty L_x^2([-T,T]\times\mathbb{R}^4))^2}\\
&+2^l\|\nabla\mathbf{u}_{\le l}\|_{(L_t^2L_x^4([-T,T]\times\mathbb{R}^4))^2}\|\mathbf{u}_{\le l}\|_{(L_t^\infty\dot H_x^1([-T,T]\times\mathbb{R}^4))^2}\|\mathbf{u}_{>k_0}\|_{(L_t^\infty L_x^2([-T,T]\times\mathbb{R}^4))^2}\\
&\lesssim 2^{2l}K^{1/2}+2^{l-k_0}\Big(\sum_{l\le m\le k_0}2^{l-m}\|\nabla\mathbf{u}_m\|_{(L_t^2L_x^4([-T,T]\times\mathbb{R}^4))^2}\Big)+2^{l-k_0}\eta\|\nabla\mathbf{u}_{\le l}\|_{(L_t^2L_x^4([-T,T]\times\mathbb{R}^4))^2}.
\end{aligned}\tag{6.5}$$

Therefore, we see that

$$\begin{aligned}
&\|\mathbf{u}_{\le l}\|_{(\dot S^1([-T,T]\times\mathbb{R}^4))^2}\lesssim\Big\|\nabla P_{\le l}\mathbf{u}(-T)\Big\|_{(L_x^2(\mathbb{R}^4))^2}+\eta\|\mathbf{u}_{\le l}\|_{(\dot S^1([-T,T]\times\mathbb{R}^4))^2}\\
&+\Big(\sum_{l\le m\le k_0}2^{l-m}\|\mathbf{u}_{\le m}\|_{(\dot S^1([-T,T]\times\mathbb{R}^4))^2}\Big)\Big(\sum_{m\ge l}2^{l-m}\|\nabla\mathbf{u}_m\|_{(L_t^\infty L_x^2([-T,T]\times\mathbb{R}^4))^2}\Big)+2^{2l}K^{1/2}.
\end{aligned}\tag{6.6}$$

It follows from (4.1) that

$$\sum_{l\le m}\|\mathbf{u}_l\|_{(L_t^\infty\dot H_x^1([-T,T]\times\mathbb{R}^4))^2}\lesssim\eta+2^{l-l_0(\eta)}.$$

Since $K=\int_{\mathbb{R}}N(t)^{-2}dt<\infty, N(-T)\to+\infty$ as $T\to+\infty$, for any l,

$$\inf_T\Big\|\nabla P_{\le l}\mathbf{u}(-T)\Big\|_{(L_x^2(\mathbb{R}^4))^2}=0.\tag{6.7}$$

Let $\alpha_l = \|\mathbf{u}_{\leq l}\|_{(\dot{S}^1([-T,T]\times\mathbb{R}^4))^2}$, by Theorem 5.1, we have

$$\|\mathbf{u}_{\leq k_0}\|_{(\dot{S}^1([-T,T]\times\mathbb{R}^4))^2} \lesssim 1$$

uniformly in T, then by (6.6) and (6.7),

$$\alpha_l \lesssim \eta \sum_{l\leq m\leq k_0} 2^{l-m}\alpha_l + 2^{2l}K^{1/2} + 2^{l-l_0}. \tag{6.8}$$

Let $\beta_m = \sum_{m\leq l\leq k_0} 2^{3(m-l)/4}\alpha_l$, clearly $\beta_m \leq \alpha_m$ for any m, then by (6.8), we can deduce $\beta_m \lesssim K^{1/2}2^{3m/4}$. Plugging $\|\mathbf{u}_{\leq l}\|_{(\dot{S}^1(\mathbb{R}\times\mathbb{R}^4))^2} \lesssim 2^{3l/4}$ for $l \leq k_0$ back into (6.6),

$$\|\mathbf{u}_{\leq l}\|_{(\dot{S}^1(\mathbb{R}\times\mathbb{R}^4))^2} \lesssim 2^{3l/2}K.$$

In particular, this means that

$$\|\mathbf{u}\|_{(H^{-1/4}(\mathbb{R}^4))^2} \lesssim K. \tag{6.9}$$

By Bernstein's inequality, interpolation, (4.1), and (6.9), for any $\eta > 0$, then

$$\begin{aligned}\|\mathbf{u}\|_{(L^2(\mathbb{R}^4))^2} &\lesssim \|P_{\leq N(t)/C(\eta)}\mathbf{u}\|^{4/5}_{(H^{-1/4}(\mathbb{R}^4))^2}\|P_{\leq N(t)/C(\eta)}\mathbf{u}\|^{1/5}_{(\dot{H}^1(\mathbb{R}^4))^2} + \|P_{\geq N(t)/C(\eta)}\mathbf{u}\|_{(L^2(\mathbb{R}^4))^2}\\ &\lesssim K^{2/3}\eta^{1/5} + \frac{C(\eta)}{N(t)}.\end{aligned} \tag{6.10}$$

Since $N(t) \to +\infty$ as $t \to +\infty$, there exists $\eta(t) \to 0$ such that (6.10) implies

$$\|\mathbf{u}\|_{(L^2_x(\mathbb{R}^4))^2} \to 0.$$

Therefore, conservation of mass (1.2) implies $\mathbf{u} \equiv 0$. □

In what follows, let us illustrate the nonexistence of almost periodic solution for system (1.1) under Case 2. The following theorem is obtained.

Theorem 6.2. *If $\boldsymbol{u}$ is an almost periodic solution to system* (1.1) *with $K = \int_{\mathbb{R}} N(t)^{-2}dt = \infty$ in Theorem* 4.1, *then $\boldsymbol{u} \equiv 0$.*

To prove the Theorem 6.2, we first need the following theorems and lemmas.

Theorem 6.3. *If $\boldsymbol{u}$ is an almost periodic solution to system* (1.1) *satisfying $N(t) \geq 1$ on $\mathbb{R}$, then*

$$\|\boldsymbol{u}(t)\|_{(L^\infty_t L^3_x(\mathbb{R}\times\mathbb{R}^4))^2} < \infty.$$

Proof. See Dodson [11]. It will turn out that the proof for the single case works with a slight modification, so we omit the proof. □

Next, we prove that in a general sense, the L^2-norm of an almost periodic solution satisfy $\int_{\mathbb{R}} N(t)^{-2}dt = \infty$ logarithmic divergence.

Lemma 6.4. *Suppose $\psi \in C_0^\infty(\mathbb{R}^4)$ is a positive, radial, decreasing function,*

$$\psi(x)=\begin{cases}1 & |x|\le 1,\\ 0 & |x|\ge 2.\end{cases}\tag{6.11}$$

If $K=\int_I N(t)^{-2}dt$, then for any $1\le R\le K^{1/5}$,

$$\int_I\iint_{\mathbb{R}^4\times\mathbb{R}^4}|\mathbf{u}(t,y)|^2\psi\Big(\frac{N(t)(x-y)}{R}\Big)\Big[|\nabla\mathbf{u}(t,x)|^2+|\mathbf{u}(t,x)|^4\Big]dxdydt\lesssim K\Big(1+\ln R\Big).\tag{6.12}$$

Proof. The proof is similar to a single case, so we omit its proof; interested readers should consult Colliander et al. [5] or Killip, Visan [19]. □

Now, we consider the case when $N(t)\equiv 1$.

Theorem 6.5. *If $\boldsymbol{u}$ is an almost periodic solution for system* (1.1) *satisfying $N(t)\equiv 1$ on $\mathbb{R}$, and $\|\boldsymbol{u}\|_{(L_t^\infty\dot H^1(\mathbb{R}\times\mathbb{R}^4))^2}<\|\nabla\boldsymbol{W}\|_{(L_x^2(\mathbb{R}^4))^2}$, then $\boldsymbol{u}\equiv 0$.*

Proof. Set $\psi\in C_0^\infty(\mathbb{R}^4)$ as a radial function satisfying (6.11) and J a large number such that $e^J\le K^{1/10}$, then let

$$\phi(x-y)=\frac{1}{J}\int_1^{e^J}\frac{1}{R}\int_{\mathbb{R}^4}\psi^2\Big(\frac{x}{R}-s\Big)\psi^2\Big(\frac{y}{R}-s\Big)dsdR=\frac{1}{J}\int_1^{e^J}\frac{1}{R}\int_{\mathbb{R}^4}\psi^2\Big(\frac{x-y}{R}-s\Big)\psi^2(s)dsdR.$$

We notice that $\psi(s)=0$ for $|s|\ge 2$, so $\phi(x-y)$ is supported on $|x-y|\le 4e^J$ and that $\|\phi\|_{L^\infty}$ is uniformly bounded.

Next, we need to estimate the derivatives of ϕ. Let us illustrate for $k=1,2,3$,

$$|\nabla^k\phi(x)|\lesssim\frac{1}{J}\frac{1}{|x|^k}.\tag{6.13}$$

In fact, since $\psi(s)=0$ for $|s|\ge 2$,

$$\begin{aligned}\nabla\phi(x)&=\frac{2}{J}\iint_1^{e^J}\frac{1}{R^2}\psi\Big(\frac{x}{R}-s\Big)\psi'\Big(\frac{x}{R}-s\Big)\psi^2(s)\frac{(\frac{x}{R}-s)}{|\frac{x}{R}-s|}dsdR\\&=\frac{2}{J}\int_{\frac{|x|}{4}}^{e^J}\int\frac{1}{R^2}\psi\Big(\frac{x}{R}-s\Big)\psi'\Big(\frac{x}{R}-s\Big)\psi^2(s)\frac{(\frac{x}{R}-s)}{|\frac{x}{R}-s|}dsdR\lesssim\frac{1}{J}\frac{1}{|x|}.\end{aligned}$$

For $k=2,3$, we can obtain by similar computation that $\nabla^2\phi(x)\lesssim\frac{1}{J}\frac{1}{|x|^2}$, $\nabla^3\phi(x)\lesssim\frac{1}{J}\frac{1}{|x|^3}$.

Define the interaction Morawetz action

$$M(t):=\iint_{\mathbb{R}^4\times\mathbb{R}^4}|\mathbf{u}(t,y)|^2\phi(x-y)(x-y)\cdot\mathrm{Im}[\bar{\mathbf{u}}\nabla\mathbf{u}](t,x)dxdy.$$

By Hölder's inequality, Sobolev embedding theorem, and Young's inequality, then

$$\sup|M(t)|\lesssim\|\mathbf{u}\|^3_{(L_t^\infty L_x^4(I\times\mathbb{R}^4))^2}\|\nabla\mathbf{u}\|_{(L_t^\infty L_x^2(I\times\mathbb{R}^4))^2}\|(x-y)\phi(x-y)\|_{L^{4/3}(\mathbb{R}^4)}\lesssim e^{4J}.\tag{6.14}$$

Integrating by parts, we obtain

$$M'(t)=2\iint_{\mathbb{R}^4\times\mathbb{R}^4}|\mathbf{u}(t,y)|^2\phi(x-y)\Big[|\nabla\mathbf{u}(t,x)|^2-|\mathbf{u}(t,x)|^4\Big]dxdy\tag{6.15}$$

$$
\begin{aligned}
&-2\iint_{\mathbb{R}^4\times\mathbb{R}^4}\text{Im}[\bar{\mathbf{u}}\partial_l\mathbf{u}](t,y)\phi(x-y)\text{Im}[\bar{\mathbf{u}}\partial_l\mathbf{u}](t,x)dxdy && (6.16)\\
&+2\iint_{\mathbb{R}^4\times\mathbb{R}^4}|\mathbf{u}(t,y)|^2(\partial_k\phi(x-y))(x-y)_l\Big[\text{Re}(\partial_l\bar{\mathbf{u}}\partial_k\mathbf{u})(t,x)-\frac{1}{4}\delta_{lk}|\mathbf{u}(t,x)|^4\Big]dxdy && (6.17)\\
&-2\iint_{\mathbb{R}^4\times\mathbb{R}^4}\text{Im}[\bar{\mathbf{u}}\partial_k\mathbf{u}](t,y)(\partial_k\phi(x-y))(x-y)_l\text{Im}[\bar{\mathbf{u}}\partial_l\mathbf{u}](t,x)dxdy && (6.18)\\
&-\frac{1}{2}\iint_{\mathbb{R}^4\times\mathbb{R}^4}|\mathbf{u}(t,x)|^2\Big(\partial_l\Delta\phi(x-y)(x-y)_l\Big)|\mathbf{u}(t,y)|^2dxdy. && (6.19)
\end{aligned}
$$

Recall (6.12) and (6.13), then

$$
\begin{aligned}
\int_I (6.18)+(6.19)dt &\lesssim \frac{1}{J}\int_I\int_{|x-y|\le 4e^J}\int_{|x-x(t)|\ge C(\eta)}|\mathbf{u}(t,y)|^2\Big[|\nabla\mathbf{u}(t,x)|^2+|\mathbf{u}(t,x)|^4\Big]dxdydt\\
&\quad+\frac{1}{J}\int_I\int_{|x-y|\le 4e^J}\int_{|x-x(t)|\le C(\eta)}|\mathbf{u}(t,y)|^2\Big[|\nabla\mathbf{u}(t,x)|^2+|\mathbf{u}(t,x)|^4\Big]dxdydt\\
&\lesssim \frac{\eta}{J}\Big(\int_I\sup\int_{|x-y|\le 4e^J}|\mathbf{u}(t,y)|^2dydt\Big)+\frac{1}{J}\int_I\int_{|x-x(t)|\le 8e^J}|\mathbf{u}(t,x)|^2dxdt\\
&\lesssim \eta K+\frac{1}{J}\int_I\int_{|x-x(t)|\le 8e^J}|\mathbf{u}(t,x)|^2dxdt.
\end{aligned} \tag{6.20}
$$

If (6.20) provides a bound on $\int_I\int_{|x-x(t)|\le 4e^J}|\mathbf{u}(t,x)|^2dxdt$, then by (6.12), we see that

$$
\int_I\int_{|x-x(t)|\le e^{J/2}}|\mathbf{u}(t,x)|^2dxdt\lesssim K=\int_I N(t)^{-2}dt. \tag{6.21}
$$

Substituting (6.21) into (6.20), this implies that the left-hand side of (6.21)$\ll K$, which by (4.1) and Bernstein's inequality forces $\mathbf{u}\equiv 0$.

Next, we will utilize (6.13) to estimate (6.19), that is

$$
\begin{aligned}
\int_I (6.19)dt &\lesssim \frac{1}{J}\int_I\iint_{|x-y|\le 4e^J}|\mathbf{u}(t,x)|^2\frac{1}{|x-y|^2}|\mathbf{u}(t,y)|^2dxdydt\\
&\lesssim \frac{1}{J}\int_I\iint_{|x-y|\le 4e^J}|P_h\mathbf{u}_{\ge c(\eta)}(t,x)|^2\frac{1}{|x-y|^2}|\mathbf{u}_h(t,y)|^2dxdydt\\
&\quad+\frac{1}{J}\int_I\iint_{|x-y|\le 4e^J}|P_h\mathbf{u}_{\le c(\eta)}(t,x)|^2\frac{1}{|x-y|^2}|\mathbf{u}_h(t,y)|^2dxdydt\\
&\quad+\frac{1}{J}\int_I\iint_{|x-y|\le 4e^J}|\mathbf{u}_m(t,x)|^2\frac{1}{|x-y|^2}|\mathbf{u}_m(t,y)|^2dxdydt,
\end{aligned}
$$

where $P_h=P_{\ge K^{-1/4}}$ and $P_m=1-P_h$.

By Bernstein's inequality and Hardy's inequality, since $N(t) \equiv 1$,

$$\begin{aligned}
&\frac{1}{J}\int_I \iint_{|x-y|\leq 4e^J} |P_h \mathbf{u}_{\geq c(\eta)}(t,x)|^2 \frac{1}{|x-y|^2} |\mathbf{u}_h(t,y)|^2 dxdydt \\
&\lesssim \frac{1}{J}\int_I \|\mathbf{u}_{\geq c(\eta)}(t)\|^2_{(L^2(\mathbb{R}^4))^2} \Big(\sup \int \frac{1}{|x-y|^2} |\mathbf{u}_h(t,y)|^2 dy\Big) dt \\
&\lesssim \frac{1}{J}\frac{1}{c(\eta)^2} \|\nabla \mathbf{u}\|^2_{(L_t^\infty L_x^2(I\times\mathbb{R}^4))^2} \int_I N(t)^{-2} dt \\
&\lesssim \frac{K}{J}\frac{1}{c(\eta)^2}.
\end{aligned} \tag{6.22}$$

It follows from Hölder's inequality and Young's inequality that

$$\begin{aligned}
&\frac{1}{J}\int_I \iint_{|x-y|\leq 4e^J} |P_h \mathbf{u}_{\leq c(\eta)}(t,x)|^2 \frac{1}{|x-y|^2} |\mathbf{u}_h(t,y)|^2 dxdydt \\
&\lesssim \frac{1}{J}\int_I \sum_{2^l\leq 4e^J} 2^{-2l} \iint_{2^l\leq |x-y|\leq 2^{l+1}} |P_{>-l} P_h \mathbf{u}_{\leq c(\eta)}(t,x)|^2 |\mathbf{u}_h(t,y)|^2 dxdydt \\
&\quad + \frac{1}{J}\int_I \sum_{2^l\leq 4e^J} 2^{-2l} \iint_{2^l\leq |x-y|\leq 2^{l+1}} |P_{\leq -l} P_h \mathbf{u}_{\leq c(\eta)}(t,x)|^2 |\mathbf{u}_h(t,y)|^2 dxdydt \\
&\lesssim \frac{1}{J}\int_I \Big(\sum_l 2^{-2l} \|P_{>-l} P_h \mathbf{u}_{\leq c(\eta)}(t,x)\|^2_{(L^2(\mathbb{R}^4))^2}\Big)\Big(\sup \int_{|x-y|\leq 4e^J} |\mathbf{u}_h(t,x)|^2 dx\Big) dt \\
&\quad + \frac{1}{J}\sum_{1\leq 2^l\leq 4e^J} 2^{-2l} 2^{10l/3} \|P_{\leq -l} P_h \mathbf{u}_{\leq c(\eta)}(t,x)\|^2_{(L_t^2 L_x^4(I\times\mathbb{R}^4))^2} \|\mathbf{u}_h\|^2_{(L_t^\infty L_x^3(I\times\mathbb{R}^4))^2} \\
&\quad + \frac{1}{J}\sum_{l\leq 0} 2^{2l} \|P_{\leq c(\eta)} \mathbf{u}_h\|^2_{(L_t^2 L_x^4(I\times\mathbb{R}^4))^2} \|\mathbf{u}_h\|^2_{(L_t^\infty L_x^4(I\times\mathbb{R}^4))^2}.
\end{aligned}$$

For any fixed t, by Bernstein's inequality and rearranging the order of summation, then

$$\begin{aligned}
&\sum_l 2^{-2l} \|P_{>-l} \mathbf{u}_{\leq c(\eta)}(t)\|^2_{(L^2(\mathbb{R}^4))^2} \\
&\lesssim \sum_l \sum_{-l<k_1\leq k_2} 2^{-2l} \|P_{k_1} \mathbf{u}_{\leq c(\eta)}(t)\|_{(L^2(\mathbb{R}^4))^2} \|P_{k_2} \mathbf{u}_{\leq c(\eta)}(t)\|_{(L^2(\mathbb{R}^4))^2} \\
&\lesssim \sum_l \sum_{-l<k_1\leq k_2} 2^{-2l-k_1-k_2} \Big(2^{k_1} 2^{k_2} \|P_{k_1} \mathbf{u}_{\leq c(\eta)}(t)\|_{(L^2(\mathbb{R}^4))^2} \|P_{k_2} \mathbf{u}_{\leq c(\eta)}(t)\|_{(L^2(\mathbb{R}^4))^2}\Big) \\
&\lesssim \sum_{k_1\leq k_2} 2^{k_1-k_2} \|\nabla P_{k_1} \mathbf{u}_{\leq c(\eta)}(t)\|_{(L^2(\mathbb{R}^4))^2} \|\nabla P_{k_2} \mathbf{u}_{\leq c(\eta)}(t)\|_{(L^2(\mathbb{R}^4))^2} \lesssim \eta^2.
\end{aligned}$$

By (6.12), we obtain

$$\frac{1}{J}\int_I \Big(\sum_l 2^{-2l} \|P_{>-l} \mathbf{u}_{\leq c(\eta)}(t)\|^2_{(L^2(\mathbb{R}^4))^2}\Big)\Big(\sup \int_{|x-y|\leq 4e^J} |\mathbf{u}_h(t)|^2 dx\Big) dt \lesssim \eta^2 K. \tag{6.23}$$

Applying Theorem 6.3, Bernstein's inequality, (6.12), Theorem 5.1, $P_h = P_{\geq K^{1/4}}$, Hölder's inequality, and Young's inequality, then

$$\begin{aligned}&\frac{1}{J}\sum_{1\le 2^l\le 4e^J}2^{-2l}2^{10l/3}\|P_{\le -l}P_h\mathbf{u}_{\le c(\eta)}\|^2_{(L_t^2L_x^4(I\times\mathbb{R}^4))^2}\|\mathbf{u}_h\|^2_{(L_t^\infty L_x^3(I\times\mathbb{R}^4))^2}\\&\lesssim \frac{1}{J}\sum_{1\le 2^l\le 4e^J}2^{-2l/3}K\lesssim \frac{K}{J}.\end{aligned}\tag{6.24}$$

Since $N(t)\equiv 1$ and $\mathbf{u}\in(L_t^\infty L_x^4(I\times\mathbb{R}^4))^2$,

$$\frac{1}{J}\sum_{l\le 0}2^{2l}\|P_{\le c(\eta)}\mathbf{u}_h\|^2_{(L_t^2L_x^4(I\times\mathbb{R}^4))^2}\|\mathbf{u}_h\|^2_{(L_t^\infty L_x^4(I\times\mathbb{R}^4))^2}\lesssim\frac{K}{J}.\tag{6.25}$$

Finally, Theorem 6.3, Theorem 5.1, and Sobolev embedding theorem imply

$$\|\mathbf{u}_m^2\|_{(L_t^2L_x^3(I\times\mathbb{R}^4))^2}\lesssim\|\nabla\mathbf{u}_m\|_{(L_t^2L_x^4(I\times\mathbb{R}^4))^2}\|\mathbf{u}_m\|_{(L_t^\infty L_x^3(I\times\mathbb{R}^4))^2}\lesssim 1.$$

Hölder's inequality indicates

$$\begin{aligned}&\int_I\iint_{|x-y|\le 4e^J}|\mathbf{u}_m(t,x)|^2\frac{1}{|x-y|^2}|\mathbf{u}_m(t,y)|^2dxdydt\\&\lesssim K^{1/2}e^{2J}\|\mathbf{u}_m^2\|_{(L_t^2L_x^3(I\times\mathbb{R}^4))^2}\|\mathbf{u}\|^2_{(L_t^\infty L_x^3(I\times\mathbb{R}^4))^2}\\&\lesssim K^{1/2}e^{2J}.\end{aligned}\tag{6.26}$$

Then, combining (6.22)–(6.26), we obtain

$$\int_I(6.19)dt\lesssim\frac{K}{J}\frac{1}{c(\eta)^2}+\eta^2K+K^{1/2}e^{2J}.\tag{6.27}$$

Decompose

$$\phi(x-y)=\frac{1}{J}\int_1^{e^J}\frac{1}{R}\int_{\mathbb{R}^4}\psi^2\Big(\frac{x}{R}-s\Big)\psi^2\Big(\frac{y}{R}-s\Big)dsdR.$$

For each $R,\ s,\ t$, there exists a $\xi(R,s,t)$ such that

$$\begin{aligned}&\int_{\mathbb{R}^4}\psi^2\Big(\frac{x}{R}-s\Big)\mathrm{Im}\Big(\overline{e^{ix\cdot\xi(R,s,t)}\mathbf{u}}\nabla e^{ix\cdot\xi(R,s,t)}\mathbf{u}\Big)(t,x)dx\\&=\int_{\mathbb{R}^4}\xi(R,s,t)\Big|\psi(\frac{x}{R}-s)\mathbf{u}(t,x)\Big|^2dx+\int_{\mathbb{R}^4}\mathrm{Im}(\overline{\mathbf{u}}\nabla\mathbf{u})(t,x)dx=0.\end{aligned}$$

Moreover, for any fixed s,t,

$$\iint_{\mathbb{R}^4\times\mathbb{R}^4}\psi^2\Big(\frac{x}{R}-s\Big)\psi^2\Big(\frac{y}{R}-s\Big)|\nabla\mathbf{u}(t,x)|^2|\mathbf{u}(t,y)|^2-\mathrm{Im}(\overline{\mathbf{u}}\nabla\mathbf{u})(t,x)\mathrm{Im}(\overline{\mathbf{u}}\nabla\mathbf{u})(t,y)dxdy$$

is invariant under the Galilean transformation $\mathbf{u}\mapsto e^{-ix\cdot\xi(R,s,t)}\mathbf{u}$. Therefore, for any R,s,t, it is possible to choose $\xi(R,s,t)$ that removes the momentum squared term.

Integrating by parts, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^4}\psi^2\Big(\frac{x}{R}-s\Big)\Big[|\nabla(e^{-ix\cdot\xi(R,s,t)}\mathbf{u}(t,x))|^2-|\mathbf{u}(t,x)|^4\Big]dx\\&=\int_{\mathbb{R}^4}\Big|\nabla\Big(\psi(\frac{x}{R}-s)e^{-ix\cdot\xi(R,s,t)}\mathbf{u}(t,x)\Big)\Big|^2dx-\int_{\mathbb{R}^4}\Big|\psi(\frac{x}{R}-s)\mathbf{u}(t,x)\Big|^2|\mathbf{u}(t,x)|^2dx\\&\quad+\int_{\mathbb{R}^4}|\mathbf{u}(t,x)|^2\Big(\psi(\frac{x}{R}-s)\Delta\psi(\frac{x}{R}-s)\Big)dx.\end{aligned}$$

By (1.8) and $\|\mathbf{u}\|_{(L_t^\infty\dot{H}^1(I\times\mathbb{R}^4))^2}\le(1-\overline{\delta})\|\mathbf{W}\|_{(\dot{H}^1(\mathbb{R}^4))^2}$, we have

$$\|\mathbf{u}\|_{(L_x^4(\mathbb{R}^4))^2}\le(1-\overline{\delta})\|\mathbf{W}\|_{(L_x^4(\mathbb{R}^4))^2}.$$

Thus,

$$\begin{aligned}&\int_{\mathbb{R}^4}\Big|\nabla\Big(\psi(\frac{x}{R}-s)e^{-ix\cdot\xi(R,s,t)}\mathbf{u}(t,x)\Big)\Big|^2dx-\int_{\mathbb{R}^4}\Big|\psi(\frac{x}{R}-s)e^{-ix\cdot\xi(R,s,t)}\mathbf{u}(t,x)\Big|^2|\mathbf{u}(t,x)|^2dx\\&\ge\Big\|\nabla\Big(\psi(\frac{x}{R}-s)e^{-ix\cdot\xi(R,s,t)}\mathbf{u}(t,x)\Big)\Big\|^2_{(L^2(\mathbb{R}^4))^2}-(1+\frac{\overline{\delta}}{2})\Big\|\psi(\frac{x}{R}-s)\mathbf{u}\Big\|^2_{(L^4(\mathbb{R}^4))^2}\Big\|\mathbf{u}\Big\|^2_{(L^4(\mathbb{R}^4))^2}\\&\quad+\frac{\overline{\delta}}{2}\Big\|\psi(\frac{x}{R}-s)\mathbf{u}\Big\|^4_{(L^4(\mathbb{R}^4))^2}\\&\ge\frac{\overline{\delta}}{2}\Big\|\psi(\frac{x}{R}-s)\mathbf{u}\Big\|^4_{(L^4(\mathbb{R}^4))^2}+\frac{\overline{\delta}}{2}\Big\|\nabla\Big(\psi(\frac{x}{R}-s)e^{-ix\cdot\xi(R,s,t)}\mathbf{u}(t,x)\Big)\Big\|^2_{(L^2(\mathbb{R}^4))^2}\end{aligned}$$

Finally, if $|\frac{x}{R}-s|\le 2$ and $|\frac{y}{R}-s|\le 2$, $|\frac{x-y}{R}|\le 4$, then

$$\int_{\mathbb{R}^4}\Big|\psi(\frac{x}{R}-s)\Big|\Big|\Delta\psi(\frac{x}{R}-s)\Big|\Big|\psi(\frac{y}{R}-s)\Big|^2ds\lesssim\frac{1}{R^2}\psi(\frac{x-y}{4R}).$$

Therefore, it follows from (6.27) that

$$\begin{aligned}&\int_1^{e^J}\frac{1}{R^3}\iint_{\mathbb{R}^4\times\mathbb{R}^4}\psi(\frac{x-y}{R})|\mathbf{u}(t,x)|^2|\mathbf{u}(t,x)|^2dxdydt\\&\lesssim\int_I\iint_{|x-y|\le 8e^J}|\mathbf{u}(t,x)|^2\frac{1}{|x-y|^2}|\mathbf{u}(t,y)|^2dxdydt\\&\lesssim\frac{K}{J}\frac{1}{c(\eta)^2}+\eta^2K+K^{1/2}e^{2J}.\end{aligned}\tag{6.28}$$

Now, by (6.11), for $|x-y|\le\frac{R}{2}$.

$$\int_{\mathbb{R}^4}\psi^4(\frac{x}{R}-s)\psi^2(\frac{y}{R}-s)ds\gtrsim 1,$$

then,

$$\int_{\mathbb{R}^4}\psi^4(\frac{x}{R}-s)\psi^2(\frac{y}{R}-s)ds\gtrsim\psi\Big[\frac{4(x-y)}{R}\Big],$$

and

$$\frac{1}{J}\int_1^{e^J}\frac{1}{R}\psi\Big[\frac{4(x-y)}{R}\Big]dR \gtrsim \psi\Big(\frac{x-y}{e^{J/2}}\Big). \tag{6.29}$$

In fact, for any $c>0$,

$$\frac{1}{J}\int_1^{e^J}\frac{1}{R}\psi\Big[\frac{4(x-y)}{R}\Big]dR \gtrsim_c \psi\Big(\frac{x-y}{e^{J(1-c)}}\Big).$$

Therefore, by (6.14), (6.20), (6.27)–(6.29), we have

$$\begin{aligned} e^{4J} \gtrsim \int_I M'(t)dt \gtrsim \bar{\delta}\int_I \iint_{|x-y|\le e^{J/2}} |\mathbf{u}(t,x)|^4|\mathbf{u}(t,y)|^2 dxdydt \\ -\frac{K}{J}\frac{1}{c(\eta)^2} - \eta^2 K - e^{2J}K^{1/2} - \frac{1}{J}\int_I\int_{|x-x(t)|\le 8e^J} |\mathbf{u}(t,x)|^2 dxdt. \end{aligned} \tag{6.30}$$

By (4.1), if $\mathbf{u}$ is a non-zero almost periodic solution to system (1.1), then $\|\mathbf{u}(t)\|_{(L_x^4(\mathbb{R}^4))^2}$ is uniformly bounded for all $t\in I$, similar to $\int_{|x-x(t)|\le C(\eta)}|\mathbf{u}(t,x)|^4dx$. Therefore, for J large,

$$\begin{aligned} &\iint_{|x-y|\le e^{J/2}} |\mathbf{u}(t,y)|^4|\mathbf{u}(t,x)|^2 dxdy \\ &\ge \int_{|x-x(t)|\le\frac{1}{2}e^{J/2}}\int_{|y-x(t)|\le\frac{1}{2}e^{J/2}} |\mathbf{u}(t,y)|^4|\mathbf{u}(t,x)|^2 dxdy \\ &\ge \int_{|x-x(t)|\le\frac{1}{2}e^{J/2}} |\mathbf{u}(t,x)|^2 dx. \end{aligned}$$

Substituting this into (6.30), then

$$\begin{aligned} &\bar{\delta}\int_I\int_{|x-x(t)|\le\frac{1}{2}e^{J/2}} |\mathbf{u}(t,x)|^2 dxdt \\ &\lesssim e^{4J} + \frac{K}{J}\frac{1}{c(\eta)^2} + \eta^2K + K^{1/2}e^{2J} + \frac{1}{J}\int_I\int_{|x-x(t)|\le 8e^J} |\mathbf{u}(t,x)|^2 dxdt. \end{aligned}$$

Repeating this argument, it is easily obtained that

$$\begin{aligned} &\bar{\delta}^2\int_I\int_{|x-x(t)|\le\frac{1}{2}e^{J/2}} |\mathbf{u}(t,x)|^2 dxdt \\ &\lesssim e^{8J} + \frac{K}{J}\frac{1}{c(\eta)^2} + \eta^2K + K^{1/2}e^{4J} + \frac{1}{J^2}\int_I\int_{|x-x(t)|\le 128e^{2J}} |\mathbf{u}(t,x)|^2 dxdt. \end{aligned}$$

We can choose $e^J = K^{1/10}$, and combine (6.12), then

$$\int_I\int_{|x-x(t)|\le\frac{1}{2}e^{J/2}} |\mathbf{u}(t,x)|^2 dxdt \lesssim \eta^2 K + \frac{K}{\ln K}\frac{1}{c(\eta)^2}. \tag{6.31}$$

Since $\eta>0$ is arbitrary, we can deduce by (6.31) that there exists a sequence $t_n\in\mathbb{R}$ such that $R_n\to\infty$ and

$$\int_{|x-x(t_n)|\le R_n^{1/4}} |\mathbf{u}(t_n,x)|^2 dx \to 0. \tag{6.32}$$

Therefore, combining (4.1) with (6.32), we can deduce that $\mathbf{u}\equiv 0$. □

It is possible to generalize to any satisfying $\int_{\mathbb{R}} N(t)^{-2}dt = \infty, N(t) \geq 1$ by using the argument in the case that $N(t) \equiv 1$. $N(t)$ is replaced with a $\tilde{N}(t)$ that satisfies the following conditions

$$(a)\ N(t) \gtrsim 1, \quad (b)\ |N'(t)| \lesssim N(t)^3, \quad (c)\ \int_I N(t)^{-2}dt \lesssim K, \quad (d)\ \int_I \frac{|N'(t)|}{N(t)^5}dt \ll K.$$

To simplify notation, let $N_m(t)$ denote $\tilde{N}_m(t)$.

Definition 6.6. Let

$$\frac{1}{N_0(t)} = \|\mathbf{u}_h(t)\|^3_{(L^3_x(\mathbb{R}^4))^2},$$

where $N_0(t)$ satisfies the above conditions.

We refer to [11] possibly after modifying $N_0(t)$ by some function $\varepsilon < \alpha(t) < \frac{1}{\varepsilon}, N_0(t) \mapsto \alpha(t)N_0(t)$, such that

$$(a)\ N_0(t) \gtrsim 1, \quad (b)\ |N_0'(t)| \lesssim N_0(t)^3, \quad (c)\ \int_I N_0(t)^{-2}dt \lesssim K.$$

The following argument is similar to $N(t) \equiv 1$, and we define

$$M(t) := \iint_{\mathbb{R}^4\times\mathbb{R}^4} |\mathbf{u}(t,y)|^2\phi\Big((x-y)N_m(t)\Big)(x-y)_j \cdot \mathrm{Im}[\bar{\mathbf{u}}\partial_j\mathbf{u}](t,x)dxdy.$$

Since $N_m(t) \gtrsim 1$, by Hölder inequality and Young's inequality, $|M(t)| \lesssim \frac{e^{4J}}{N_m(t)^4} \lesssim e^{4J}$, we have

$$\begin{aligned}
M'(t) = {} & 2\iint_{\mathbb{R}^4\times\mathbb{R}^4} |\mathbf{u}(t,y)|^2\phi\Big((x-y)N_m(t)\Big)\Big[|\nabla\mathbf{u}(t,x)|^2 - |\mathbf{u}(t,x)|^4\Big]dxdy && (6.33)\\
& - 2\iint_{\mathbb{R}^4\times\mathbb{R}^4} \mathrm{Im}[\bar{\mathbf{u}}\partial_l\mathbf{u}](t,y)\phi\Big((x-y)N_m(t)\Big)\mathrm{Im}[\bar{\mathbf{u}}\partial_l\mathbf{u}](t,x)dxdy && (6.34)\\
& + 2\iint_{\mathbb{R}^4\times\mathbb{R}^4} |\mathbf{u}(t,y)|^2\Big(\partial_k\phi((x-y)N_m(t))\Big)(x-y)_l\Big[\mathrm{Re}(\partial_l\bar{\mathbf{u}}\partial_k\mathbf{u})(t,x) - \frac{1}{4}\delta_{lk}|\mathbf{u}(t,x)|^4\Big]dxdy && (6.35)\\
& - 2\iint_{\mathbb{R}^4\times\mathbb{R}^4} \mathrm{Im}[\bar{\mathbf{u}}\partial_k\mathbf{u}](t,y)\Big(\partial_k\phi((x-y)N_m(t))\Big)(x-y)_l\mathrm{Im}[\bar{\mathbf{u}}\partial_l\mathbf{u}](t,x)dxdy && (6.36)\\
& - \frac{1}{2}\iint_{\mathbb{R}^4\times\mathbb{R}^4} |\mathbf{u}(t,x)|^2\Big(\partial_l\Delta(\phi(x-y)N_m(t))(x-y)_l\Big)|\mathbf{u}(t,y)|^2dxdy && (6.37)\\
& + \iint_{\mathbb{R}^4\times\mathbb{R}^4} \phi'\Big((x-y)N_m(t)\Big)(x-y)_l|x-y||\mathbf{u}(t,y)|^2N_m'(t)\mathrm{Im}[\bar{\mathbf{u}}\partial_l\mathbf{u}](t,x)dxdy. && (6.38)
\end{aligned}$$

By (4.2), Theorem 5.1, Hölder's inequality, Young's inequality, and $N_m(t) \gtrsim 1$,

$$\begin{aligned}
\int_I (6.38)dt \lesssim {} & \frac{1}{J}\int_I \iint_{|x-y|\leq\frac{4e^J}{N_m(t)}} |\mathbf{u}(t,y)|^2|x-y|\frac{|N_m'(t)|}{N_m(t)}|\nabla\mathbf{u}(t,x)||\mathbf{u}(t,x)|dxdydt\\
\lesssim {} & \frac{e^{3J}}{J}\int_I \frac{|N_m'(t)|}{N_m(t)^4}\|\mathbf{u}_h(t)\|^3_{(L^3(\mathbb{R}^4))^2}\|\nabla\mathbf{u}(t)\|_{(L^2(\mathbb{R}^4))^2}dt\\
& + \frac{e^{5J}}{J}\int_I \frac{|N_m'(t)|}{N_m(t)^6}\|\mathbf{u}_l(t)\|^3_{(L^6(\mathbb{R}^4))^2}\|\nabla\mathbf{u}(t)\|_{(L^2(\mathbb{R}^4))^2}dt && (6.39)\\
\lesssim {} & \frac{e^{3J}}{J}\int_I \frac{|N_m'(t)|}{N_m(t)^5}dt + \frac{e^{5J}}{J}\|\mathbf{u}_l(t)\|^3_{(L^6_{t,x}(I\times\mathbb{R}^4))^2}\|\nabla\mathbf{u}(t)\|_{(L^\infty_t L^2_x(I\times\mathbb{R}^4))^2}\Big(\int_I N_m(t)^{-6}dt\Big)^{1/2}\\
\lesssim {} & 2^{-4m+4}K\frac{e^{3J}}{J} + 4\frac{e^{3J}}{J} + K^{1/2}\frac{e^{5J}}{J}.
\end{aligned}$$

Next, by using the argument in the case that $N(t) \equiv 1$ to estimate the other terms,

$$\begin{aligned}(6.33) + (6.34) \gtrsim \frac{\overline{\delta}}{2} \iint_{\mathbb{R}^4\times\mathbb{R}^4} \psi\Big(\frac{4(x-y)N_m(t)}{e^{11J/12}}\Big)|\mathbf{u}(t,x)|^2|\mathbf{u}(t,y)|^4 dxdy \\ - \frac{1}{J}\iint_{|x-y|\le \frac{4e^J}{N_m(t)}} \frac{1}{|x-y|^2}|\mathbf{u}(t,x)|^2|\mathbf{u}(t,y)|^2 dxdy.\end{aligned} \tag{6.40}$$

$$\begin{aligned}\int_I (6.35) + (6.36) dt &\lesssim \frac{1}{J}\int_I \int_{|x-y|\le \frac{4e^J}{N_m(t)}} \int_{|x-x(t)|\ge \frac{C(\eta)}{N(t)}} |\mathbf{u}(t,y)|^2\Big[|\nabla \mathbf{u}(t,x)|^2 + |\mathbf{u}(t,x)|^4\Big] dxdydt \\ &\quad + \frac{1}{J}\int_I \int_{|x-y|\le \frac{4e^J}{N_m(t)}} \int_{|x-x(t)|\le \frac{C(\eta)}{N(t)}} |\mathbf{u}(t,y)|^2\Big[|\nabla \mathbf{u}(t,x)|^2 + |\mathbf{u}(t,x)|^4\Big] dxdydt \\ &\lesssim \frac{\eta}{J}\Big(\int_I \sup \int_{|x-y|\le \frac{4e^J}{N_m(t)}} |\mathbf{u}(t,y)|^2 dydt\Big) + \frac{1}{J}\int_I \int_{|x-x(t)|\le \frac{8e^J}{N_m(t)}} |\mathbf{u}(t,x)|^2 dxdt \\ &\lesssim \eta K + \frac{1}{J}\int_I \int_{|x-x(t)|\le \frac{8e^J}{N_m(t)}} |\mathbf{u}(t,x)|^2 dxdt.\end{aligned} \tag{6.41}$$

Moreover,

$$\begin{aligned}&\frac{1}{J}\int_I \iint_{|x-y|\le 4e^J} |\mathbf{u}(t,x)|^2 \frac{1}{|x-y|^2}|\mathbf{u}(t,y)|^2 dxdydt \\ &\lesssim \frac{1}{J}\int_I \iint_{|x-y|\le \frac{4e^J}{N_m(t)}} \Big|P_h \mathbf{u}_{\ge c(\eta)N(t)}(t,x)\Big|^2 \frac{1}{|x-y|^2}|\mathbf{u}_h(t,y)|^2 dxdydt \\ &\quad + \frac{1}{J}\int_I \iint_{|x-y|\le \frac{4e^J}{N_m(t)}} \Big|P_h \mathbf{u}_{\le c(\eta)N(t)}(t,x)\Big|^2 \frac{1}{|x-y|^2}|\mathbf{u}_h(t,y)|^2 dxdydt \\ &\quad + \frac{1}{J}\int_I \iint_{|x-y|\le \frac{4e^J}{N_m(t)}} |\mathbf{u}_m(t,x)|^2 \frac{1}{|x-y|^2}|\mathbf{u}_m(t,y)|^2 dxdydt.\end{aligned}$$

By Bernstein's inequality and Hardy's inequality,

$$\begin{aligned}&\frac{1}{J}\int_I \iint_{|x-y|\le \frac{4e^J}{N_m(t)}} \Big|P_h \mathbf{u}_{\ge c(\eta)N(t)}(t,x)\Big|^2 \frac{1}{|x-y|^2}|\mathbf{u}_h(t,y)|^2 dxdydt \\ &\lesssim \frac{1}{J}\int_I \|\mathbf{u}_{\ge c(\eta)N(t)}(t)\|^2_{(L^2(\mathbb{R}^4))^2}\Big(\sup \int_{\mathbb{R}^4} \frac{1}{|x-y|^2}|\mathbf{u}_h(t,y)|^2 dy\Big) dt \\ &\lesssim \frac{1}{J}\frac{1}{c(\eta)^2}\int_I N(t)^{-2} dt = \frac{K}{J}\frac{1}{c(\eta)^2}.\end{aligned}$$

It follows from Hölder's inequality and Young's inequality that

$$
\begin{aligned}
&\frac{1}{J}\int_I\iint_{|x-y|\le\frac{4e^J}{N_m(t)}}\left|P_h\mathbf{u}_{\le c(\eta)N(t)}(t,x)\right|^2\frac{1}{|x-y|^2}|\mathbf{u}_h(t,y)|^2dxdydt\\
&\lesssim\frac{1}{J}\int_I\sum_{2^l\le\frac{4e^J}{N_m(t)}}2^{-2l}\iint_{2^l\le|x-y|\le2^{l+1}}\left|P_{>-l}P_h\mathbf{u}_{\le c(\eta)N(t)}(t,x)\right|^2|\mathbf{u}_h(t,y)|^2dxdydt\\
&\quad+\frac{1}{J}\int_I\sum_{2^l\le\frac{4e^J}{N_m(t)}}2^{-2l}\iint_{2^l\le|x-y|\le2^{l+1}}\left|P_{\le-l}P_h\mathbf{u}_{\le c(\eta)N(t)}(t,x)\right|^2|\mathbf{u}_h(t,y)|^2dxdydt\\
&\lesssim\frac{1}{J}\int_I\Big(\sum_l 2^{-2l}\|P_{>-l}P_h\mathbf{u}_{\le c(\eta)N(t)}(t,x)\|^2_{(L^2_x(\mathbb{R}^4))^2}\Big)\Big(\sup\int_{|x-y|\le4e^J}|\mathbf{u}_h(t,x)|^2dx\Big)dt\\
&\quad+\frac{1}{J}\int_I\sum_{\frac{1}{N_0(t)}\le2^l\le\frac{4e^J}{N_m(t)}}2^{2l}\|P_{\le-l}P_h\mathbf{u}_{\le c(\eta)N(t)}(t,x)\|^2_{(L^6_x(\mathbb{R}^4))^2}\|\mathbf{u}_h\|^2_{(L^3_x(\mathbb{R}^4))^2}dt\\
&\quad+\frac{1}{J}\int_I\sum_{2^l\le\frac{1}{N_0(t)}}2^{2l}\|P_{\le c(\eta)N(t)}\mathbf{u}_h\|^2_{(L^4_x(\mathbb{R}^4))^2}\|\mathbf{u}_h\|^2_{(L^4_x(\mathbb{R}^4))^2}dt.
\end{aligned}
$$

Now, for any fixed t, by Bernstein's inequality and rearranging the order of summation,

$$
\begin{aligned}
&\sum_l 2^{-2l}\|P_{>-l}\mathbf{u}_{\le c(\eta)N(t)}(t)\|^2_{(L^2(\mathbb{R}^4))^2}\\
&\lesssim\sum_l\sum_{-l<k_1\le k_2}\|P_{k_1}\mathbf{u}_{\le c(\eta)N(t)}(t)\|_{(L^2(\mathbb{R}^4))^2}\|P_{k_2}\mathbf{u}_{\le c(\eta)N(t)}(t)\|_{(L^2(\mathbb{R}^4))^2}\\
&\lesssim\sum_l\sum_{-l<k_1\le k_2}2^{-2l-k_1-k_2}\Big(2^{k_1}2^{k_2}\|P_{k_1}\mathbf{u}_{\le c(\eta)N(t)}(t)\|_{(L^2(\mathbb{R}^4))^2}\|P_{k_2}\mathbf{u}_{\le c(\eta)N(t)}(t)\|_{(L^2(\mathbb{R}^4))^2}\Big)\\
&\lesssim\sum_{k_1\le k_2}2^{k_1-k_2}\|\nabla P_{k_1}\mathbf{u}_{\le c(\eta)N(t)}(t)\|_{(L^2(\mathbb{R}^4))^2}\|\nabla P_{k_2}\mathbf{u}_{\le c(\eta)N(t)}(t)\|_{(L^2(\mathbb{R}^4))^2}\lesssim\eta^2.
\end{aligned}
$$

By (6.12), we see that

$$
\frac{1}{J}\int_I\Big(\sum_l 2^{-2l}\|P_{>-l}\mathbf{u}_{\le c(\eta)N(t)}(t)\|^2_{(L^2(\mathbb{R}^4))^2}\Big)\Big(\sup\int_{|x-y|\le4e^J}|\mathbf{u}_h(t)|^2dx\Big)dt\lesssim\eta^2K
$$

Since $N(t)$ is variable, we have

$$
\begin{aligned}
&\frac{1}{J}\int_I\sum_{\frac{1}{N_0(t)}\le2^l\le\frac{4e^J}{N_m(t)}}2^{2l}\|P_{\le-l}P_h\mathbf{u}_{\le c(\eta)N(t)}(t,x)\|^2_{(L^6_x(\mathbb{R}^4))^2}\|\mathbf{u}_h\|^2_{(L^3_x(\mathbb{R}^4))^2}dt\\
&\lesssim\frac{1}{J}\int_I\sum_{\frac{1}{N_0(t)}\le2^l\le\frac{4e^J}{N_m(t)}}2^{-2l/3}\Big(2^{4l/3}\|P_{\le-l}P_h\mathbf{u}_{\le c(\eta)N(t)}(t,x)\|_{(L^6_x(\mathbb{R}^4))^2}\Big)^2\|\mathbf{u}_h\|^2_{(L^3_x(\mathbb{R}^4))^2}dt\\
&\lesssim\int_I\Big(\sup_{2^l\ge K^{-1/4}}2^{4l/3}\|P_l\mathbf{u}\|_{(L^6_x(\mathbb{R}^4))^2}\Big)^2\Big(\sup\sum_{2^l\ge\frac{1}{N_0(t)}}2^{-2l/3}\|\mathbf{u}_h\|^2_{(L^3_x(\mathbb{R}^4))^2}\Big)dt\lesssim\frac{K}{J}.
\end{aligned}
$$

By (4.1) and $\mathbf{u} \in (L_t^\infty L_x^4(I \times \mathbb{R}^4))^2$,

$$\frac{1}{J}\int_I \sum_{2^l \leq \frac{1}{N_0(t)}} 2^{2l} \|P_{\leq c(\eta)N(t)}\mathbf{u}_h\|^2_{(L_x^4(\mathbb{R}^4))^2}\|\mathbf{u}_h\|^2_{(L_x^4(\mathbb{R}^4))^2} dt \lesssim \frac{\eta}{J}\int_I N_0(t)^{-2}dt \lesssim \frac{\eta}{J}K.$$

Finally, by (6.26) and $N_m(t) \gtrsim 1$,

$$\int_I \iint_{|x-y|\leq \frac{4e^J}{N_m(t)}} |\mathbf{u}_m(t,x)|^2 \frac{1}{|x-y|^2}|\mathbf{u}_m(t,y)|^2 dxdydt \lesssim e^{2J}K^{1/2}.$$

Therefore,

$$\frac{1}{J}\int_I \iint_{|x-y|\leq 4e^J} |\mathbf{u}(t,x)|^2 \frac{1}{|x-y|^2}|\mathbf{u}(t,y)|^2 dxdydt \lesssim \frac{K}{J}\frac{1}{c(\eta)^2} + \eta^2 K + e^{2J}K^{1/2}. \tag{6.42}$$

We choose m such that $2^{4m} = e^{10J/3}$, since e^J is large and $N(t) \gtrsim N_0(t) \sim N_1(t)$,

$$\frac{\overline{\delta}}{2}\int_I \iint_{\mathbb{R}^4\times\mathbb{R}^4} \psi\Big(\frac{4(x-y)N_m(t)}{e^{11J/12}}\Big)|\mathbf{u}(t,y)|^2|\mathbf{u}(t,x)|^4 dxdydt \gtrsim \frac{\overline{\delta}}{2}\int_I \int_{|x-x(t)|\geq \frac{e^{11J/12}}{8N_m(t)}} |\mathbf{u}(t,x)|^2 dxdt. \tag{6.43}$$

Combining (6.43) with (6.39)–(6.42), $\sup_{t\in I}|M(t)| \lesssim e^{4J}$, we see that

$$\begin{aligned}
&\overline{\delta}\int_I \int_{|x-x(t)|\leq \frac{e^{11J/12}}{8N_m(t)}} |\mathbf{u}(t,x)|^2 dxdt \\
&\lesssim \frac{1}{J}\int_I \int_{|x-x(t)|\leq \frac{8e^J}{N_m(t)}} |\mathbf{u}(t,x)|^2 dxdt + \eta K + 2^{-4m}K\frac{e^{3J}}{J} + \frac{e^{3J}}{J} + K^{1/2}\frac{e^{5J}}{J} + e^{4J} + \frac{K}{J}\frac{1}{c(\eta)^2},
\end{aligned}$$

therefore,

$$\begin{aligned}
&\overline{\delta}^2 \int_I \int_{|x-x(t)|\leq \frac{e^{11J/12}}{8N_m(t)}} |\mathbf{u}(t,x)|^2 dxdt \\
&\lesssim \frac{\overline{\delta}}{J}\int_I \int_{|x-x(t)|\leq \frac{8e^J}{N_m(t)}} |\mathbf{u}(t,x)|^2 dxdt + \overline{\delta}\Big(\eta K + 2^{-4m}K\frac{e^{3J}}{J} + \frac{e^{3J}}{J} + K^{1/2}\frac{e^{5J}}{J} + e^{4J} + \frac{K}{J}\frac{1}{c(\eta)^2}\Big) \\
&\lesssim \frac{1}{J^2}\int_I \int_{|x-x(t)|\leq \frac{512e^{12J/11}}{N_m(t)}} |\mathbf{u}(t,x)|^2 dxdt + \eta K + 2^{-4m}K\frac{e^{36J/11}}{J} + \frac{e^{36J/11}}{J} \\
&\quad + K^{1/2}\frac{e^{60J/11}}{J} + e^{48J/11} + \frac{K}{J}\frac{1}{c(\eta)^2}.
\end{aligned}$$

Let us choose J and m such that $2^{4m} = e^{10J/3}$ and $K = e^{12J}$,

$$\begin{aligned}
&\overline{\delta}^2 \int_I \int_{|x-x(t)|\leq \frac{e^{11J/12}}{8N_m(t)}} |\mathbf{u}(t,x)|^2 dxdt \\
&\lesssim \frac{1}{J^2}\int_I \int_{|x-x(t)|\leq \frac{512e^{12J/11}}{N_m(t)}} |\mathbf{u}(t,x)|^2 dxdt + \eta K + K\frac{e^{-2J/33}}{J} + K^{21/22} + \frac{K}{J}\frac{1}{c(\eta)^2}.
\end{aligned} \tag{6.44}$$

Now, we are able to complete the proof of Theorem 6.2.

Proof of Theorem 6.2. Let us prove by contradiction. Assume $\mathbf{u}$ is a non-zero, almost periodic solution for system (1.1). Set I be an interval satisfying

$$K = \int_I N(t)^{-2} dt.$$

Combining (6.44) with Lemma 6.4, we deduce that

$$\overline{\delta}^2 \int_I \int_{|x-x(t)| \le \frac{e^{11J/12}}{8N_m(t)}} |\mathbf{u}(t,x)|^2 dx dt \lesssim \eta K + \frac{1}{c(\eta)^2} \frac{K}{\ln K}.$$

Since any $\eta > 0$ and $\int_{\mathbb{R}} N(t)^{-2} dt = \infty$, let us choose an increasing sequence of interval I whose union makes up $\mathbb{R}$, combining $N_m(t) \lesssim 2^m N(t)$ with $2^{4m} = e^{10J/3}$, $K = e^{12J}$, there exists a sequence $t_n \in \mathbb{R}$ and $R_n \to \infty$ such that

$$N(t_n)^2 \int_{|x-x(t)| \le \frac{R_n}{N(t_n)}} |\mathbf{u}(t_n, x)|^2 dx \to 0.$$

However, by (4.1) we see that $\|\mathbf{u}(t_n)\|_{(\dot{H}^1(\mathbb{R}^4))^2} \to 0$, then the conservation of energy (1.3) implies $\mathbf{u} \equiv 0$.

7. Conclusions

In summary, we prove the global well-posedness and scattering of the four-dimensional cubic focusing energy-critical nonlinear Schrödinger (NLS) system below threshold in the non-radial case. Despite $\mathbf{W}$ being a stationary solution of system (1.1), and $\mathbf{W}$ giving an example of an almost periodic solution that does not lie in $(L^2(\mathbb{R}^4))^2$, we are able to combine this logarithmically divergent result with the long-time Strichartz estimate to establish an interaction Morawetz estimate, proving Theorem 1.7. First, we establish the variational characterization of the ground state and derive the threshold of the global well-posedness and scattering, which is a crucial step. Then, we adapt the strategy of Kenig and Merle [17], using a concentration-compactness/rigidity method to reduce the global well-posedness and scattering to the exclusion of almost periodic solution, that is, we need to preclude the almost periodic solution to system (1.1) satisfying $K = \int_{\mathbb{R}} N(t)^{-2} dt < \infty$ and $K = \int_{\mathbb{R}} N(t)^{-2} dt = \infty$. In the future, we plan to study the Schrödinger-Hirota equation, see [23, 24].

Author contributions

Yonghang Chang: writing–original draft, writing–review & editing, Menglan Liao: supervision, writing–original draft, writing–review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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