AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(9): 25646–25658.
DOI: 10.3934/math.20241253
Received: 20 June 2024
Revised: 06 August 2024
Accepted: 15 August 2024
Published: 04 September 2024

Research article

The property $(\omega\pi)$ as a generalization of the a-Weyl theorem

Wei Xu[1], **Elvis Aponte**[2,*] **and Ponraj Vasanthakumar**[3]

[1] Computer Science Department, New York University, 251 Mercer Street, New York, NY 10012, USA

[2] Departamento de Matemáticas, Facultad de Ciencias Naturales y Matemáticas, Escuela Superior Politécnica del Litoral, ESPOL, Campus Gustavo Galindo, km. 30.5 vía Perimetral, Guayaquil, 090902, Ecuador

[3] Department of Mathematics, Dr. N.G.P. Institute of Technology, Kalapatti Road, Coimbatore-641048, Tamilnadu, India

* **Correspondence:** Email: ecaponte@espol.edu.ec.

Abstract: In this paper, for a bounded linear operator defined on a complex Banach space of infinite dimension, we consider the set of isolated points in its approximate point spectrum, which are eigenvalues of finite multiplicity; this set can be equal to the spectrum of the operator but without its upper semi-Fredholm spectrum, and this relation or equality defines in the literature a new spectral property called the property $(\omega\pi)$ and is a generalization of the classical a-Weyl theorem. We establish some characterizations and consequences about the property $(\omega\pi)$, some with topological aspects. Furthermore, we study this property through the Riesz functional calculus. Part of the spectral structure of a linear operator verifying property $(\omega\pi)$ is described, obtaining some associated properties.

Keywords: property $(\omega\pi)$; upper semi-Fredholm operator; SVEP
Mathematics Subject Classification: Primary 47A10, 47A11; Secondary 47A53, 47A55

1. Introduction

It is known that in the algebra $\mathcal{L}(X)$ of bounded linear operators defined on an infinite-dimensional complex Banach space X, there are several studies on the Weyl-type and Browder-type theorems; see, for example, [2, 4, 11, 13, 22, 29]; and from these studies several spectral properties are generated, including those defined in [27], which are more than forty-four properties. These properties are studied separately in search of new relationships or results between them, and new properties have been created from them, which are versions generalized or modified. However, among these investigations, the upper spectrum of semi-Fredholm has received little attention compared to the more common consideration of Weyl, Browder, or Drazin spectra.

These studies generate a theory that has several applications; for example, in a general context see [18], or particularly in the field of physics, it is applied in electromagnetism, see [16], in the field of artificial intelligence, it is applied from the point of view of statistical analysis of massive data considering the normalized data as the spectrum of an operator, and also to make dimensionality reduction, see [12, 28].

On the other hand, in [8, 24–26], some of the relations involved with the upper semi-Fredholm spectrum are studied, but they are not of Weyl type. Therefore, it is interesting to consider Weyl-type relations to investigate and innovate the theory on this subject and thus explore new applications.

Considering the information provided above, we are motivated to make the following contributions to the literature: In Section 2, we introduce the necessary preliminaries, and then in Section 3 until Section 6, using classical techniques of local spectral theory, we define and study a Weyl-type property that relates to the upper semi-Fredholm spectrum; we call it the property $(\omega\pi)$. This new spectral property for a linear operator state that "the upper semi-Fredholm spectrum subtracted from the spectrum coincides with the eigenvalues of finite multiplicity that are isolated in the approximate point spectrum." From this property, we see different relationships that exist between spectra, including the upper semi-Fredholm spectrum. This property generalizes the a-Weyl theorem in the sense that it implies, but the converse is not true, which we will justify with an example. Also, in Section 3, we see the necessary conditions that a linear operator in $\dot{\mathcal{L}}(X)$ must have to verify the property $(\omega\pi)$, while in Section 4, we see the sufficient conditions. In Section 5, we study the property $(\omega\pi)$ under the Riesz functional calculus, and finally, in Section 6, we characterize it.

Our contributions may be useful for further studies; for example, it remains to study the property $(\omega\pi)$ under perturbations or for the tensor product of two operators verifying it, which is of interest because it could correlate with the results obtained in [6, 10, 14, 20, 21]. The study is also pending for conjugate operators, see [5], or for operators equivalent in norms, see [19].

2. Preliminaries

In this section, we give some terminologies and definitions to the development of results, and refer to [2], for details about notations and terminologies.

For $T \in \dot{\mathcal{L}}(X)$, $\dot{\alpha}(T)$ and $\dot{\beta}(T)$ represent the dimension of the kernel of T, $ker(T)$, and the codimension of the range of T, $T(X)$, respectively.

An operator $T \in \dot{\mathcal{L}}(X)$ is *upper semi-Fredholm*, in notation $T \in \dot{\Phi}_+(X)$, if $\dot{\alpha}(T) < \infty$ and $T(X)$ is closed, or is *lower semi-Fredholm*, in notation $T \in \dot{\Phi}_-(X)$, if $\dot{\beta}(T) < \infty$. The class of *Fredholm* operators is given by $\dot{\Phi}(X) := \dot{\Phi}_+(X) \cap \dot{\Phi}_-(X)$, and the class of semi-Fredholm operators is given by $\dot{\Phi}\pm(X) := \dot{\Phi}_+(X) \cup \dot{\Phi}_-(X)$. Now, if $T \in \dot{\Phi}\pm(X)$, we define the index of T as $\tilde{\text{ind}}\,(T) := \dot{\alpha}(T) - \dot{\beta}(T)$. It is represented by the class of *Weyl operators* by the set

$$\dot{W}(X) := \{T \in \dot{\Phi}(X) : \tilde{\text{ind}}\, T = 0\},$$

and the class of *upper semi-Weyl operators* by the set

$$\dot{W}_+(X) := \{T \in \dot{\Phi}_+(X) : \tilde{\text{ind}}\, T \leq 0\}.$$

The *ascent* of an operator $T \in \dot{\mathcal{L}}(X)$ is defined as the smallest nonnegative integer $p := \hat{p}(T)$ such that $\ker T^p = \ker T^{p+1}$, and if it does not exist, we consider it as $\hat{p}(T) = \infty$. Analogously, the *descent* of T is defined as the smallest nonnegative integer $q := \hat{q}(T)$ such that $T^q(X) = T^{q+1}(X)$, and if it does not exist, we consider it as $\hat{q}(T) = \infty$.

The set represented by the class of *Drazin invertible operators* is

$$D(X) := \{T \in \dot{\mathcal{L}}(X) : \hat{p}(T).\hat{q}(T) < \infty\},$$

and by the class of *Left Drazin invertible operators* is

$$D_+(X) := \{T \in \dot{\mathcal{L}}(X) : \hat{p}(T) < \infty \text{ and the set } T^{p+1}(X) \text{ is closed}\}.$$

The class of *Browder operators* is the set denoted by

$$\dot{B}(X) := \{T \in \dot{\Phi}(X) : \hat{p}(T).\hat{q}(T) < \infty\}$$

and the class of *upper semi-Browder operators* is the set denoted by

$$\dot{B}_+(X) := \{T \in \dot{\Phi}_+(X) : \hat{p}(T) < \infty\}.$$

For $T \in \dot{\mathcal{L}}(X)$ and a nonnegative integer n, we put by $T_{[n]}$, the restriction of T on $T^n(X)$, viewed as a map of space $T^n(X)$ in itself (where $T_{[0]} = T$).

Now, $T \in \dot{\mathcal{L}}(X)$ is said to be *semi-B-Fredholm* (resp., *B-Fredholm, upper semi-B-Fredholm*) if for some integer $n \geq 0$ the range $T^n(X)$ is closed and $T_{[n]}$ is a semi-Fredholm operator (resp., Fredholm, upper semi-Fredholm). If T is a semi-B-Fredholm operator, then its index is $\tilde{\text{ind}}\, T = \tilde{\text{ind}}\, T_{[n]}$. In this way, T is said to be *B-Weyl* (respectively, *upper semi-B-Weyl*) if for some integer $n \geq 0$ the range $T^n(X)$ is closed and $T_{[n]}$ is a Weyl operator (respectively, upper semi-Weyl).

An operator is called bounded below if it is injective with closed range. In addition, we point out the following notations that will be used in this article. For $T \in \dot{\mathcal{L}}(X)$ the:

- Spectrum: $\varrho(T) := \{\psi \in \mathbb{C} : \psi I - T \text{ is not invertible}\}$.
- Approximate point spectrum: $\varrho_a(T) := \{\psi \in \mathbb{C} : \psi I - T \text{ is not bounded below}\}$.
- Upper semi-Fredholm spectrum: $\varrho_{uf}(T) := \{\psi \in \mathbb{C} : \psi I - T \notin \dot{\Phi}_+(X)\}$.
- Weyl spectrum: $\varrho_w(T) := \{\psi \in \mathbb{C} : \psi I - T \notin \dot{W}(X)\}$.
- Upper semi-Weyl spectrum: $\varrho_{uw}(T) := \{\psi \in \mathbb{C} : \psi I - T \notin \dot{W}_+(X)\}$.
- Upper semi B-Weyl spectrum: $\varrho_{ubw}(T) := \{\psi \in \mathbb{C} : \psi I - T \text{ is not upper semi B-Weyl}\}$.
- Browder spectrum: $\varrho_b(T) := \{\psi \in \mathbb{C} : \psi I - T \notin \dot{B}(X)\}$.
- Upper semi-Browder spectrum: $\varrho_{ub}(T) := \{\psi \in \mathbb{C} : \psi I - T \notin \dot{B}_+(X)\}$.
- Drazin invertible spectrum: $\varrho_d(T) := \{\psi \in \mathbb{C} : \psi I - T \notin D(X)\}$.
- Left Drazin invertible spectrum: $\varrho_{ld}(T) := \{\psi \in \mathbb{C} : \psi I - T \notin D_+(X)\}$.
- Resolvent: $\rho(T) = \mathbb{C} \setminus \varrho(T)$.
- Approximate point resolvent: $\rho_a(T) = \mathbb{C} \setminus \varrho_a(T)$.
- Upper semi-Fredholm resolvent: $\rho_{uf}(T) = \mathbb{C} \setminus \varrho_{uf}(T)$.
- Dual operator, it is denoted by T^*.

Now, we give an important property in local spectral theory. The localized version of this property has been introduced by Finch [15], and in the framework of Fredholm theory, this property has been characterized in several ways; see Chapter 3 of [1].

It is said that an operator $T \in \dot{\mathcal{L}}(X)$ verifies the *single-valued extension property* at $\psi_0 \in \mathbb{C}$ (abbreviated, SVEP at ψ_0), if for every open disc $\mathbb{D}_{\psi_0} \subseteq \mathbb{C}$ centered at ψ_0 the only analytic function $f : \mathbb{D}_{\psi_0} \to X$ which satisfies the equation

$$(\psi I - T)f(\psi) = 0 \quad \text{for all } \psi \in \mathbb{D}_{\psi_0},$$

is the function $f \equiv 0$ on $\mathbb{D}_{\psi_0}$. An operator T verifies the SVEP if T verifies the SVEP at every point $\psi \in \mathbb{C}$. It is clear that T verifies the SVEP at every point of resolvent $\rho(T) := \mathbb{C} \setminus \varrho(T)$. Moreover, from the identity theorem for analytic functions, it is easily seen that T verifies the SVEP at every point of the boundary $\partial\varrho(T)$ of the spectrum. In particular, T and T^* verify the SVEP at every isolated point of the spectrum. We have

$$\hat{p}(\psi I - T) < \infty \Rightarrow T \text{ verifies the SVEP at } \psi, \tag{2.1}$$

and dually,

$$\hat{q}(\psi I - T) < \infty \Rightarrow T^* \text{ verifies the SVEP at } \psi, \tag{2.2}$$

see [1, Theorem 3.8]. Furthermore, from the definition of localized SVEP is easily seen that

$$\varrho_a(T) \text{ does not cluster at } \psi \Rightarrow T \text{ verifies the SVEP at } \psi. \tag{2.3}$$

Remark 2.1. All the implications (2.1)–(2.3) are equivalences whenever $\psi I - T$ is quasi-Fredholm, see [3]. In particular, these equivalences hold whenever $\psi I - T$ is a semi-Fredholm operator.

Remark 2.2. [1] It is well known that for $T \in \dot{\mathcal{L}}(X)$ are given:

- $\hat{p}(T) < +\infty \Rightarrow \dot{\alpha}(T) \leq \dot{\beta}(T)$.
- $\hat{q}(T) < +\infty \Rightarrow \dot{\alpha}(T) \geq \dot{\beta}(T)$.
- $\hat{p}(T).\hat{q}(T) < +\infty \Rightarrow \hat{p}(T) = \hat{q}(T) \Rightarrow \dot{\alpha}(T) = \dot{\beta}(T)$.
- $\dot{\alpha}(T) = \dot{\beta}(T)$ and $\hat{p}(T) < +\infty$, or $\hat{q}(T) < +\infty \Rightarrow \hat{p}(T) = \hat{q}(T)$.

For the set of all isolated (resp., interior) points of $K \subseteq \mathbb{C}$, we denote it as $\operatorname{iso} K$ (resp., $Int(K)$).

Now, for $T \in \dot{\mathcal{L}}(X)$, we take into account the following sets:

$$\pi^0(T) := \varrho(T) \setminus \varrho_b(T), \quad \pi_a^0(T) := \varrho_a(T) \setminus \varrho_{ub}(T),$$

$$\pi(T) := \varrho(T) \setminus \varrho_d(T), \quad \pi_a(T) := \varrho_a(T) \setminus \varrho_{ld}(T).$$

$$\mathcal{E}_a^0(T) := \{\psi \in \operatorname{iso} \varrho_a(T) : 0 < \dot{\alpha}(\psi I - T) < \infty\}.$$

The *quasi-nilpotent part* of $T \in \dot{\mathcal{L}}(X)$ is defined to be the set

$$H_0(T) := \{x \in X : \lim_{n\to\infty} \|T^n x\|^{1/n} = 0\}.$$

The analytical core of $T \in \dot{\mathcal{L}}(X)$ is the set $K(T)$ of all $x \in X$ such that there exists a sequence $(u_n) \subset X$ and a constant $\delta > 0$ verifying:

(1) $x = u_0$, and $Tu_{n+1} = u_n$ for every $n \in \mathbb{Z}_+$;

(2) $\|u_n\| \leq \delta^n \|x\|$ for every $n \in \mathbb{Z}_+$.

Definition 2.3. An operator $T \in \dot{\mathcal{L}}(X)$ is said to satisfy:

(i) a-Weyl's theorem if $\varrho_a(T) \setminus \varrho_{uw}(T) = \mathcal{E}_a^0(T)$;
(ii) a-Browder's theorem if $\varrho_a(T) \setminus \varrho_{uw}(T) = \pi_a^0(T)$.

3. The property $(\omega\pi)$ and its effects

In this section, we introduce a new spectral property of Weyl type, denoted by $(\omega\pi)$. In the study of spectral properties, it is always interesting to check whether the spectrum of an operator coincides with its approximate point spectrum. Since such equality allows us to obtain many results, we show that we achieve it if the operator verifies the property $(\omega\pi)$. Moreover, the upper semi-Fredholm spectrum, upper semi-Weyl spectrum, upper semi-Browder spectrum, the Browder spectrum and the Weyl spectrum of an operator are equal if the operator verifies the property $(\omega\pi)$. The property $(\omega\pi)$ is related to the theorems of a-Weyl and a-Browder, but they are not equivalent. On the other hand, the necessary conditions have been taken to demonstrate that the spectrum of an operator is equal to its upper semi-Fredholm spectrum. On the other hand, we have taken the necessary conditions to prove that the spectrum of an operator is equal to its upper semi-Fredholm spectrum, since such equality does not always occur, as shown in the following example.

Example 3.1. Let $X = \ell^2(\mathbb{N}))$ and $T \in \dot{\mathcal{L}}(X)$, given by

$$T(x_1, x_2, \ldots) = (0, x_2, x_3, \ldots).$$

Note that $\varrho(T) = \{0, 1\}$ and $\varrho_{uf}(T) = \{1\}$, so that $\varrho(T) \neq \varrho_{uf}(T)$. Since, $\mathcal{E}_a^0(T) = \{0\}$ we have

$$\varrho(T) \setminus \varrho_{uf}(T) = \mathcal{E}_a^0(T).$$

We have already given before reference what the property $(\omega\pi)$, follows using some sets we present a more formal definition.

Definition 3.2. An operator $T \in \dot{\mathcal{L}}(X)$ verifies the property $(\omega\pi)$ if

$$\varrho(T) \setminus \varrho_{uf}(T) = \mathcal{E}_a^0(T).$$

In what follows, we will say that $T \in \dot{\mathcal{L}}(X)$ has $(\omega\pi)$ to indicate that the operator T verifies the property $(\omega\pi)$.

Note that the operator in the previous example has $(\omega\pi)$. We will see in Corollary 4.3 that several classes of operators verify the property $(\omega\pi)$, in particular some operators of the class $H(p)$, which is defined as follows: $T \in \dot{\mathcal{L}}(X)$ is said to have the property $H(p)$ if

$$H_0(\psi I - T) = \ker(\psi I - T)^p \quad \text{for all } \psi \in \mathbb{C},$$

for some $p = \hat{p}(\psi) \in \mathbb{N}$, see [23].

Several classes of linear operators maintain the equality $\varrho(T) = \varrho_a(T)$, while others do not. Under the effects of the property $(\omega\pi)$, the approximate point spectrum is equal to the spectrum.

Theorem 3.3. *If $T \in \dot{\mathcal{L}}(X)$ has $(\omega\pi)$, then $\varrho(T) = \varrho_a(T)$.*

Proof. Let $\psi \in \varrho(T)$. Suppose that $\psi \notin \varrho_a(T)$, then $\psi \in \varrho(T)\backslash\varrho_{uf}(T) = \mathcal{E}_a^0(T)$. Hence $\psi \in \text{iso}\,\varrho_a(T)$, a contradiction, because $\psi I - T$ is injective. Thus, $\psi \in \varrho_a(T)$. Hence, $\varrho(T) \subseteq \varrho_a(T)$. Reverse inclusion always occurs. Therefore, $\varrho(T) = \varrho_a(T)$. □.

The following theorem establishes the relationship of the property $(\omega\pi)$ with a-Weyl's theorem and a-Browder's theorem.

Theorem 3.4. *Let $T \in \dot{\mathcal{L}}(X)$ such that T has $(\omega\pi)$, then:*

(i) T satisfies a-Weyl's theorem.

(ii) T satisfies a-Browder's theorem.

Proof. (i) Assume that T has $(\omega\pi)$. Since $\varrho_a(T) \subseteq \varrho(T)$ and $\varrho_{uf}(T) \subseteq \varrho_{uw}(T)$, we have that $\varrho_a(T)\backslash\varrho_{uw}(T) \subseteq \mathcal{E}_a^0(T)$. On the other hand, let $\psi \in \mathcal{E}_a^0(T)$, so ψ is an isolated point in the approximate point spectrum. But $\psi I - T$ is upper semi-Fredholm with finite ascent. By [1, Theorem 3.4], $\tilde{\text{ind}}(\psi I - T) \leq 0$, so that $\psi I - T$ is upper semi-Weyl, then $\psi \notin \varrho_{uw}(T)$. Hence, $\psi \in \varrho_a(T)\backslash\varrho_{uw}(T)$. Thus, $\mathcal{E}_a^0(T) \subseteq \varrho_a(T)\backslash\varrho_{uw}(T)$. Therefore, T satisfies a-Weyl's theorem.
(ii) It follows from [11, Corollary 3.5] and part (i). □.

The following example shows that the converse of the above theorem is not true.

Example 3.5. Let the operator $T = 0 \oplus R_s$ on $\ell^2(\mathbf{N}) \oplus \ell^2(\mathbf{N})$, where 0 is zero operator, and R_s is the right shift operator that verifies the SVEP, and so $\varrho_{ub}(R_s) = \varrho_{uf}(R_s)$ are defined by $0(x_1, x_2, x_3, \ldots) = (0, 0, 0, \ldots)$ and $R_s(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots)$, respectively. Since $\varrho(T) = D(0, 1)$ it is the closed unit disc, $\varrho_a(T) = \varrho_{uw}(T) = \varrho_{ub}(T) = \varrho_{uf}(T) = C(0, 1) \cup \{0\}$, (where $C(0, 1)$ is the unit circle), and $\mathcal{E}_a^0(T) = \emptyset$, so T verifies both the a-Weyl and the a-Browder theorem but does not verify the property $(\omega\pi)$.

One of the main tasks of spectral theory is to investigate the equality between different spectra of a linear operator. But little attention has been paid to including the upper semi-Fredholm spectrum. However, the property $(\omega\pi)$ helps that the upper semi-Fredholm spectrum can be related to other spectra, and the following theorem points out the effects that derive from this property.

Theorem 3.6. *Let $T \in \dot{\mathcal{L}}(X)$. If T has $(\omega\pi)$, then $\varrho_{uf}(T) = \varrho_{uw}(T) = \varrho_{ub}(T) = \varrho_b(T) = \varrho_w(T)$.*

Proof. Let $\psi \notin \varrho_{uf}(T)$, we consider $\psi \in \varrho(T)\backslash\varrho_{uf}(T) = \mathcal{E}_a^0(T)$, so $\psi \in \text{iso}\,\varrho_a(T)$, whereby T verifies the SVEP at ψ. Since $\psi I - T$ is upper semi Fredholm, $\hat{p}(\psi I - T) < \infty$, so by [1, Theorem 3.4], $\psi \notin \varrho_{uw}(T)$, hence, $\varrho_{uf}(T) = \varrho_{uw}(T)$; also by Theorem 3.4 part (ii), $\varrho_{uw}(T) = \varrho_{ub}(T)$. Thus, $\varrho_{uf}(T) = \varrho_{uw}(T) = \varrho_{ub}(T)$. On the other hand, by Theorem 3.3 and [9, Lemma 2.1], we obtain that $\varrho_{ub}(T) = \varrho_b(T)$. Finally, by Theorem 3.4 part (ii) and [29, Theorem 3.9], it turns out that $\varrho_w(T) = \varrho_b(T)$. □.

Corollary 3.7. *Let $T \in \dot{\mathcal{L}}(X)$. If T has $(\omega\pi)$, then for $\psi \notin \varrho_{uf}(T)$, T and T^* verify the SVEP at ψ.*

Under the effects of the property $(\omega\pi)$, poles of T, poles of T of finite rank, left poles of T, and left poles of T of finite rank, coincide with eigenvalues of T of finite multiplicity that are isolated points in the approximate point spectrum.

Theorem 3.8. *If $T \in \dot{\mathcal{L}}(X)$ has $(\omega\pi)$, then $\pi(T) = \pi^0(T) = \pi_a(T) = \pi_a^0(T) = \mathcal{E}_a^0(T)$.*

Proof. By Theorem 3.3, it is clear that $\varrho(T) = \varrho_a(T)$. Then, by [9, Lemma 2.1], we have that $\varrho_d(T) = \varrho_{ld}(T)$ and $\varrho_b(T) = \varrho_{ub}(T)$. Note that $\varrho_d(T) \subseteq \varrho_b(T)$. But if $\psi \notin \varrho_d(T)$, so by [17, Proposition 50.2], it turns out that $\psi \in \mathcal{E}_a^0(T)$, whereby $\psi \notin \varrho_{uf}(T)$, thus, by the Theorem 3.6, we obtain that $\psi \notin \varrho_b(T)$, then $\varrho_b(T) = \varrho_d(T)$. Therefore, $\pi(T) = \pi^0(T) = \pi_a(T) = \pi_a^0(T) = \mathcal{E}_a^0(T)$. □.

4. Sufficient conditions implying the property $(\omega\pi)$

We have seen in the previous section the nature of an operator having $(\omega\pi)$, so in this section, we investigate some conditions to ensure that an operator has $(\omega\pi)$.

Remark 4.1. Let $T \in \dot{\mathcal{L}}(X)$ and $\psi \in \varrho_a(T)$; if $\psi I - T$ is a semi-Fredholm operator and T verifies the SVEP at ψ, then by Remark 2.1, we have $\psi \in \operatorname{iso}\varrho_a(T)$.

If the polaroid operator $T \in \dot{\mathcal{L}}(X)$ verifies the SVEP and its spectrum is equal to the approximate point spectrum, then T has $(\omega\pi)$. In fact, if $\psi \in \mathcal{E}_a^0(T)$, then $\psi \in \operatorname{iso}\varrho_a(T)$, but T is polaroid, whereby $\hat{p}(\psi I - T) = \hat{q}(\psi I - T) < \infty$, as $\dot{\alpha}(\psi I - T) < \infty$, we deduce that $\mathcal{E}_a^0(T) \subseteq \varrho(T) \setminus \varrho_{uf}(T)$. The inverse inclusion is followed by Remark 4.1 and the fact that T is a polaroid operator. Note by Theorem 3.6, that $\varrho_{uw}(T) = \varrho_w(T)$. Hence, a-Weyl's and Weyl's theorems are equivalent to T.

Thus, we have obtained the following theorem.

Theorem 4.2. *If the polaroid operator $T \in \dot{\mathcal{L}}(X)$ verifies the SVEP and $\varrho(T) = \varrho_a(T)$, then T has $(\omega\pi)$.*

Corollary 4.3. *The following class of operators have the property $(\omega\pi)$.*

- *(i) The bounded operators belonging to class $H(p)$ such that $\varrho(T) = \varrho_a(T)$.*
- *(ii) Operators defined in a Hilbert space such that $\varrho(T) = \varrho_a(T)$ and which are algebraically paranormal, algebraically quasi-class A, or analytically paranormal operators.*

Proof. All these classes of operators in (i) and (ii) are polaroid and verify the SVEP, see [4]. Since $\varrho(T) = \varrho_a(T)$, it follows that by Theorem 4.2, the property $(\omega\pi)$ is verified. □.

Example 4.4. Each convolution operator T_μ on $L_1(G)$ is $H(1)$, where $L_1(G)$ is the group algebra of a locally compact abelian group G. Also, we have that $\varrho(T_\mu) = \varrho_a(T_\mu)$; see [1, Corollary 5.88]. Therefore, by Corollary 4.3, it turns out that T_μ has $(\omega\pi)$.

The Fredholm theory, also known as local spectral theory, has been developed in relation to localized SVEP. This theory illustrates how localized SVEP helps establish conditions for an operator to have $(\omega\pi)$.

Theorem 4.5. *If $T \in \dot{\mathcal{L}}(X)$ verifies the SVEP at each $\psi \notin \varrho_{uf}(T)$, $\varrho(T) = \varrho_a(T)$, and $\operatorname{iso}\varrho(T) = \emptyset$, then $\varrho(T) = \varrho_{uf}(T)$. In particular, T has $(\omega\pi)$.*

Proof. Clearly $\varrho_{uf}(T) \subseteq \varrho(T)$. On the other hand, let $\psi \in \varrho(T)$, and suppose $\psi \notin \varrho_{uf}(T)$, then T verifies the SVEP ψ, but $\psi I - T$ is upper semi-Fredholm, so by the Remark 4.1, $\psi \in \operatorname{iso}\varrho(T)$, this is a contradiction, for which $\psi \in \varrho_{uf}(T)$. Hence $\varrho(T) = \varrho_{uf}(T)$.

Note that $\mathcal{E}_a^0(T) = \emptyset$ whenever $\operatorname{iso}\varrho_a(T) = \emptyset$. Therefore, T has $(\omega\pi)$. □.

Corollary 4.6. *Let $T \in \dot{\mathcal{L}}(X)$. If T^* verifies the SVEP at each $\psi \notin \varrho_{lf}(T)$, $\varrho(T) = \varrho_s(T)$, and $\operatorname{iso}\varrho(T^*) = \emptyset$, then T^* has $(\omega\pi)$.*

Proof. We have that $\varrho_{uf}(T^*) = \varrho_{lf}(T)$, $\varrho(T^*) = \varrho_s(T) = \varrho_a(T^*)$, and $\operatorname{iso}\varrho(T^*) = \emptyset$. So, as in Theorem 4.5, it turns out that T^* has $(\omega\pi)$. □.

5. The property $(\omega\pi)$ and the Riesz calculus

In this section, with the help of the Riesz functional calculus and through the property $(\omega\pi)$, we focus on enforcing that $\varrho_{uw}(f(T)) = f(\varrho_{uw}(T))$ with $f \in \mathcal{H}(\varrho(T))$, where $\mathcal{H}(\varrho(T))$ is the set of all analytic functions defined in an open neighborhood of $\varrho(T)$. For $f \in \mathcal{H}(\varrho(T))$, we define $f(T)$ as in the Riesz functional calculus. $\mathcal{H}_i(\varrho(T))$ denotes the set of functions belonging to $\mathcal{H}(\varrho(T))$ that are injective.

Note that by Theorem 2.40 of [1], if $T \in \dot{\mathcal{L}}(X)$ verifies the SVEP, then $f(T)$ verifies the SVEP, analogous for T^* and $f(T^*)$. Note that by continuity of f it turns out that iso $\varrho(f(T)) = \emptyset$ if iso $\varrho(T) = \emptyset$. Therefore, by Theorem 4.5 and Corollary 4.6, we have the next corollary.

Corollary 5.1. *Let $T \in \dot{\mathcal{L}}(X)$ and $f \in \mathcal{H}(\varrho(T))$.*

(i) If T verifies the SVEP, $\varrho(T) = \varrho_a(T)$, and iso $\varrho(T) = \emptyset$*, then $f(T)$ has $(\omega\pi)$.*
(ii) If T^* verifies the SVEP, $\varrho(T) = \varrho_s(T)$, and iso $\varrho(T) = \emptyset$*, then $f(T^*)$ has $(\omega\pi)$.*

The class of upper semi-Fredholm operators is a regularity, while the class of upper semi-Weyl operators is not, i.e., it is not always true that $\varrho_{uw}(f(T)) = f(\varrho_{uw}(T))$, for $f \in \mathcal{H}(\varrho(T))$ and $T \in \dot{\mathcal{L}}(X)$. With $f \in \mathcal{H}_i(\varrho(T))$, we find two conditions for $f(T)$ has $(\omega\pi)$.

Theorem 5.2. *Let $T \in \dot{\mathcal{L}}(X)$ that has $(\omega\pi)$ and $f \in \mathcal{H}_i(\varrho(T))$. Then, $f(T)$ has $(\omega\pi)$ if and only if $\varrho_{uw}(f(T)) = f(\varrho_{uw}(T))$.*

Proof. ($\Rightarrow$) Since T and $f(T)$ have $(\omega\pi)$, it follows that $\varrho_{uf}(T) = \varrho_{uw}(T)$ and $\varrho_{uf}(f(T)) = \varrho_{uw}(f(T))$. Hence, by the regularity of the class of the upper semi-Fredholm operators, it is obtained that $\varrho_{uw}(f(T)) = f(\varrho_{uw}(T))$.
($\Leftarrow$) If $\psi \in \mathcal{E}_a^0(f(T))$, then $\psi \in \text{iso}\,\varrho_a(f(T)) = \text{iso}\, f(\varrho_a(T))$ and therefore $\exists\, \theta \in \varrho_a(T)$ such that $\psi = f(\theta)$, and by the continuity of f, $\theta \in \text{iso}\,\varrho_a(T)$.

Note that $h(\mu) := f(\theta) - f(\mu)$ is zero in $\mu = \theta$, whereby $h(\mu) := (\theta - \mu)g(\mu)$, where $g(\mu)$ is an analytic function that does not vanish in $\varrho(T)$. Then, $h(T) = (\theta I - T)g(T) = f(\theta)I - f(T) = \psi I - f(T)$. But, as $g(T)$ is invertible, it follows that $\theta \in \mathcal{E}_a^0(T)$. Since T has $(\omega\pi)$, $\theta \notin \varrho_{uf}(T)$ and so $\psi \notin f(\varrho_{uf}(T)) = \varrho_{uf}(f(T))$, because f is one to one. Hence $\psi \in \varrho(f(T)) \setminus \varrho_{uf}(f(T))$. Thus, we deduce that

$$\mathcal{E}_a^0(f(T)) \subseteq \varrho(f(T)) \setminus \varrho_{uf}(f(T)).$$

On the other hand, by hypothesis, Theorem 3.6, and regularity of the class of the semi-Browder operators, we get that

$$\varrho_{uw}(f(T)) = f(\varrho_{uw}(T)) = f(\varrho_{uf}(T)) = f(\varrho_b(T)) = \varrho_{uf}(f(T)) = \varrho_b(f(T)).$$

Hence $\varrho(f(T)) \setminus \varrho_{uf}(f(T)) \subseteq \mathcal{E}_a^0(f(T))$. We conclude that $f(T)$ has $(\omega\pi)$. $\square$.

A variant of Browder's theorem is the property (bz), studied depth in [8], and is given for $T \in \dot{\mathcal{L}}(X)$ if

$$\varrho_a(T) \setminus \varrho_{uf}(T) = \varrho_a(T) \setminus \varrho_{ub}(T).$$

Note that by Theorem 3.6 it follows that property $(\omega\pi)$ implies property (bz). Among other things, for all $f \in \mathcal{H}(\varrho(T))$, it is obtained that if T has $(\omega\pi)$, so $\varrho_{uw}(f(T)) = f(\varrho_{uw}(T))$, see [8, Corollary 3.8].

Corollary 5.3. *If $T \in \dot{\mathcal{L}}(X)$ has $(\omega\pi)$ and $f \in \mathcal{H}(\varrho(T))$, then $\varrho_{uw}(f(T)) = f(\varrho_{uw}(T))$.*

By Theorem 5.2, we have the following corollary. Particularly, this applies to all the operators mentioned in Corollary 4.3.

Corollary 5.4. *Let $T \in \dot{\mathcal{L}}(X)$ which has $(\omega\pi)$, and $f \in \mathcal{H}_i(\varrho(T))$. Then, $f(T)$ has $(\omega\pi)$.*

6. Characterizations of the property $(\omega\pi)$

In the following, we provide characterizations for the operators that satisfy the property $(\omega\pi)$. We begin by defining for $T \in \dot{\mathcal{L}}(X)$ the following set:

$$\dot{\Theta}(T) := (\varrho(T) \setminus \varrho_{uf}(T)) \cup \mathcal{E}_a^0(T).$$

Then T has $(\omega\pi)$ is equivalent to the elements of $\dot{\Theta}(T)$ being poles of finite rank for T, also for T^*; additionally, there are other equivalences with topological aspects. This is confirmed by the following results.

Theorem 6.1. *Let $T \in \dot{\mathcal{L}}(X)$. Then the following statements are equivalent:*

(i) T has $(\omega\pi)$.
(ii) T and T^* verifies the SVEP for $\psi \in \dot{\Theta}(T)$.
(iii) If $\psi \in \dot{\Theta}(T)$, then $\psi \notin \varrho_b(T)$.

Proof. $(i) \Rightarrow (ii)$. By Theorem 3.3 and hypothesis, we have $\mathcal{E}_a^0(T) = \dot{\Theta}(T)$. If $\psi \in \dot{\Theta}(T)$, then $\psi \in$ iso $\varrho(T)$, so by [1, Corollary 3.21] it turns out that T and T^* verify the SVEP at $\psi \in \dot{\Theta}(T)$.

$(ii) \Rightarrow (iii)$. If $\psi \in \mathcal{E}_a^0(T)$, then $0 < \dot{\alpha}(\psi I - T) < \infty$, so adjoint is a semi-Fredholm operator, and by hypothesis we have that $\hat{q}(\psi I - T) < \infty$, whereby $\dot{\beta}(\psi I - T) \le \dot{\alpha}(\psi I - T) < \infty$. Hence, if $\psi \in \dot{\Theta}(T)$, then $\psi I - T$ is semi-Fredholm. So that by Remark 2.1, $\hat{p}(\psi I - T) = \hat{q}(\psi I - T) < \infty$, this implies that $\dot{\beta}(\psi I - T) = \dot{\alpha}(\psi I - T) < \infty$. Therefore, $\psi I - T$ is Browder. i.e., $\psi \notin \varrho_b(T)$.

$(iii) \Rightarrow (i)$ Let $\psi \in \varrho(T) \setminus \varrho_{uf}(T)$, so $\psi \in \dot{\Theta}(T)$, by hypothesis $\psi \notin \varrho_b(T)$, this implies that $\psi \in \mathcal{E}_a^0(T)$, we deduce that $\varrho(T) \setminus \varrho_{uf}(T) \subseteq \mathcal{E}_a^0(T)$. The reverse inclusion is clear. Therefore, T has $(\omega\pi)$. □.

Corollary 6.2. *If $T \in \dot{\mathcal{L}}(X)$ has $(\omega\pi)$, then $\forall\psi \in \dot{\Theta}(T)$, the codimension of $K(\psi I - T)$ is finite.*

Proof. Let $\psi \in \dot{\Theta}(T)$, so by Theorem 6.1, $p := \hat{p}(\psi I - T) = \hat{q}(\psi I - T) < +\infty$. This implies that $X = H_0(\psi I-T) \oplus K(\psi I-T) = N(\psi I-T)^p \oplus (\psi I-T)^p(X)$, see [2, Theorem 6.62]. Also, $\dot{\alpha}(\psi I-T) < +\infty$, and so $\dot{\beta}(\psi I - T) < +\infty$, hence $\dot{\beta}(\psi I - T)^p < +\infty$. Since $K(\psi I - T) = (\psi I - T)^p(X)$, we obtain that $K(\psi I - T)$ has a codimension that is finite. □.

For $T \in \dot{\mathcal{L}}(X)$, define the set $\Delta^{uf}(T) = \varrho(T) \setminus \varrho_{uf}(T)$ and $\Delta^{uw}(T) = \varrho(T) \setminus \varrho_{uw}(T)$. Since $\varrho_{uf}(T) \subseteq \varrho_{uw}(T)$, $\Delta^{uw}(T) \subseteq \Delta^{uf}(T)$. With this terminology, we have the following characterization with topological aspects:

Theorem 6.3. *For $T \in \dot{\mathcal{L}}(X)$, the following statements are equivalent:*

(i) T has $(\omega\pi)$.
(ii) $Int(\Delta^{uf}(T)) = \emptyset$ and $\mathcal{E}_a^0(T) = \pi^0(T)$.

(iii) $\Delta^{uf}(T) \subseteq \partial\varrho(T)$ *and* $\mathcal{E}_a^0(T) \subseteq Cl(\Delta^{uf}(T))$.

Proof. $(i) \Rightarrow (ii)$. Clearly $Int(\mathcal{E}_a^0(T)) = \emptyset$, thus by hypothesis $Int(\Delta^{uf}(T)) = \emptyset$. Also, by Theorem 3.8, $\mathcal{E}_a^0(T) = \pi^0(T)$.

$(ii) \Rightarrow (i)$. Assume that $Int(\Delta^{uf}(T)) = \emptyset$. By [7, Theorem 6] and [8, Theorem 3.5], we have $\varrho(T) = \varrho_a(T)$ and $\varrho_{uf}(T) = \varrho_{ub}(T)$. Note that by [9, Lemma 2.1], it turns out that $\varrho_{ub}(T) = \varrho_b(T)$. Thus, $\varrho(T)\backslash\varrho_{uf}(T) = \varrho_a(T)\backslash\varrho_{ub}(T) = \pi_a^0(T) = \pi^0(T) = \mathcal{E}_a^0(T)$. Therefore, T has $(\omega\pi)$.

$(iii) \Rightarrow (i)$. Let $\psi \in \Delta^{uf}(T)$, so by hypothesis $\psi \in \partial\varrho(T)$, whereby T and T^* verify the SVEP at ψ, then by the Remark 2.1 $\psi \notin \varrho_b(T)$. Hence, $\psi \in \mathcal{E}_a^0(T)$. On the other hand, by hypothesis $\mathcal{E}_a^0(T) \subseteq Cl(\Delta^{uf}(T)) = \Delta^{uf}(T) \cup \text{acc}\,(\Delta^{uf}(T))$, it turns out that $\mathcal{E}_a^0(T) \subseteq \Delta^{uf}(T)$. Therefore, T has $(\omega\pi)$.

$(i) \Rightarrow (iii)$. By the Theorem 3.3, $\varrho(T) = \varrho_a(T)$, and each isolated point is a boundary point, we get that $\Delta^{uf}(T) \subseteq \partial\varrho(T)$. Clearly, $\mathcal{E}_a^0(T) = \Delta^{uf}(T) \subseteq Cl(\Delta^{uf}(T))$. □.

Corollary 6.4. *For* $T \in \dot{\mathcal{L}}(X)$*, the following statements are equivalent:*

(i) T *has* $(\omega\pi)$.
(ii) $Int(\dot{\Theta}(T)) = \emptyset$ *and* $\mathcal{E}_a^0(T) = \pi^0(T)$.
(iii) $\dot{\Theta}(T) \subseteq \partial\varrho(T)$ *and* $\mathcal{E}_a^0(T) \subseteq Cl(\Delta^{uf}(T))$.
(vi) $Int(\Delta^{uf}(T)) = \emptyset$ *and* $\mathcal{E}_a^0(T) = \pi^0(T)$.
(v) $\Delta^{uf}(T) \subseteq \partial\varrho(T)$ *and* $\mathcal{E}_a^0(T) \subseteq Cl(\Delta^{uf}(T))$.

On the other hand, the fact that $\rho_a(T)$ is connected implies that $\varrho_a(T) = \varrho(T)$. Indeed, if $\rho_a(T)$ is connected, then it has a unique component, says Ω; since $\rho(T) \subseteq \rho_a(T)$, we have that $\rho(T) \subseteq \Omega$. Since both T and T^* verify the SVEP at every point of $\rho(T)$, by [2, Corollary 2.125], it follows that both T and T^* verify the SVEP at every point of Ω. If $\psi \notin \varrho_a(T)$, then $\psi \in \Omega$, so both T and T^* verify the SVEP at ψ, and as $\psi I - T$ is a semi-Fredholm operator, by Remark 2.1 it follows that $\hat{p}(\psi I - T) = \hat{q}(\psi I - T) = 0$, and so $\psi \notin \varrho(T)$. Therefore, $\varrho_a(T) = \varrho(T)$. Also, if $\rho_{uf}(T)$ is connected, then T verifies the SVEP for all $\psi \in \rho_{uf}(T)$, since $\rho(T) \subseteq \rho_{uf}(T)$. In this way, we conclude that $\varrho_{uf}(T) = \varrho_{ub}(T) = \varrho_b(T)$. Hence, $\varrho(T) \setminus \varrho_{uf}(T) \subseteq \mathcal{E}_a^0(T)$. But if additionally $\mathcal{E}_a^0(T) \subseteq \pi(T)$, then T has $(\omega\pi)$.

Example 6.5. The Césaro operator is given by:

$$(C_p f)(\psi) := \frac{1}{\psi}\int_0^{\psi} \frac{f(\mu)}{(1-\mu)}d\mu, \quad \text{for all } f \in \mathbb{H}_{\hat{p}}(\mathbb{D}) \text{ and } \psi \in \mathbb{D}.$$

Where, $\mathbb{H}_{\hat{p}}(\mathbb{D})$ is the Hardy space, for $1 < p < \infty$. Note that C_p verifies the property (bz), see [8, Example 3.10]. Also, $\emptyset \neq \Delta^{uf}(C_p) \subseteq Int(\varrho(C_p))$. Thus, (v) in Corollary 6.3 is not given. Therefore, the operator C_p does not have $(\omega\pi)$.

Thus, it is clear that the converse of the following result does not hold.

Corollary 6.6. *The property* $(\omega\pi)$ *implies the property* (bz).

Another strong variant of classical Browder's theorem and their generalized versions, which was studied in [9], is the property (V_Π), given for $T \in \dot{\mathcal{L}}(X)$ if

$$\varrho(T) \setminus \varrho_{uw}(T) = \pi(T).$$

Clearly, by Theorem 3.8, if T has $(\omega\pi)$, then T satisfies the property (V_Π). In this case, for all $f \in \mathcal{H}(\varrho(T))$, we have that $f(T)$ verifies the property (V_Π), because T satisfies property (bz) and it is always true that $\varrho_d(f(T)) = f(\varrho_d(T))$. Note that the property (V_Π) does not imply the property (bz).

Example 6.7. The operators discussed in Example 4.4 verify the property $(\omega\pi)$ and then verify Corollary 6.3. Also, they verify the property (V_Π), and are polaroid, then by [9, Theorem 4.5], these operators verify the property (V_E) defined in [27]. Then, every property mentioned in [27, Table 1], which are more than forty-four properties, is verified equivalently for the operators discussed in Example 4.4; see [27, Theorem 2.27].

7. Conclusions

- From Section 3, we obtain that if; $\varrho(T) \neq \varrho_a(T)$, or $\varrho_w(T) \neq \varrho_{uw}(T)$, or $\varrho_b(T) \neq \varrho_{ub}(T)$, or $\varrho_d(T) \neq \varrho_{ld}(T)$, or T does not verify a-Weyl's theorem, then T does not have $(\omega\pi)$.
- From Section 4, we obtain that the class of operators that are polaroid with SVEP and that their spectrum coincides with their approximate point spectrum is numerous, and their elements verify the property $(\omega\pi)$.
- From Section 5, we see that $f(T)$ has $(\omega\pi)$, if $f \in \mathcal{H}_i(\varrho(T))$ and T has $(\omega\pi)$. Also, the class of upper semi-Weyl operators that verify the property $(\omega\pi)$ is a regularity, i.e., for $f \in \mathcal{H}(\varrho(T))$, it turns out that $\varrho_{uw}(f(T)) = f(\varrho_{uw}(T))$.
- From Section 6, if $Int(\Delta^{uf}(T)) \neq \emptyset$, or $Int(\dot{\Theta}(T)) \neq \emptyset$, then T does not have $(\omega\pi)$.
- The study of property $(\omega\pi)$ under perturbations and tensor product is still pending.

Author contributions

Elvis Aponte, Wei Xu and Ponraj Vasanthakumar: Conceptualization; Elvis Aponte and Wei Xu: Methodology; Wei Xu: Supervision of language; Elvis Aponte and Ponraj Vasanthakumar: Writing original. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors express their gratitude to the referees for their contributions to the final version of this manuscript.

Conflict of interest

The authors have no conflict of interest to declare.

References

1. P. Aiena, *Fredholm and local spectral theory*, Springer, 2004.
2. P. Aiena, *Fredholm and local spectral theory II: With application to weyl-type theorems*, Springer, **2235** (2018).

3. P. Aiena, *Quasi-Fredholm operators and localized SVEP*, 2007.
4. P. Aiena, E. Aponte, E. Balzan, Weyl type theorems for left and right polaroid operators, *Integr. Equat. Oper. Th.*, **66** (2010), 1–20. https://doi.org/10.1007/s00020-009-1738-2
5. P. Aiena, F. Burderi, S. Triolo, Local spectral properties under conjugations, *Mediterr. J. Math.*, **18** (2021), 1–20. https://doi.org/10.1007/s00009-021-01731-7
6. E. Aponte, J. Sanabria, L. Vásquez, Perturbation theory for property (VE) and tensor product, *Mathematics*, **9** (2021), 2275. https://doi.org/10.3390/math9212775
7. E. Aponte, Property (*az*) through topological notions and some applications, *T. A Razmadze Math. In.*, **176** (2022), 417–425.
8. E. Aponte, J. Soto, E. Rosas, Study of the property (bz) using local spectral theory methods, *Arab J. Basic Appl. Sci.*, **30** (2023), 665–674. https://doi.org/10.1080/25765299.2023.2278217
9. E. Aponte, J. Macías, J. Sanabria, J. Soto, Further characterizations of property (VΠ) and some applications, *Proyecciones (Antofagasta)*, **39** (2020), 1435–1456. https://doi.org/10.22199/issn.0717-6279-2020-06-0088
10. E. Aponte, N. Jayanthi, D. Quiroz, P. Vasanthakumar, Tensor product of operators satisfying Zariouh's property (gaz), and stability under perturbations, *Axioms*, **11** (2022), 225. https://doi.org/10.3390/axioms11050225
11. M. Berkani, J. Koliha, Weyl type theorems for bounded linear operators, *Acta Sci. Math.(Szeged)*, **69** (2003), 359–376.
12. Y. X. Chen, Y. J. Chi, J. Q. Fan, C. Ma, Spectral methods for data science: A statistical perspective, *Found. Tre. Mach. Learn.*, **14** (2021), 566–806. https://doi.org/10.1561/2200000079
13. L. A. Coburn, Weyl's theorem for nonnormal operators, *Mich. Math. J.*, **13** (1966), 285–288. https://doi.org/10.1307/mmj/1031732778
14. B. P. Duggal, Tensor products and property (w), *Rend. Circ. Mat. Palerm.*, **60** (2011), 23–30. https://doi.org/10.1007/s12215-011-0023-9
15. J. Finch, The single valued extension property on a Banach space, *Pac. J. Math.*, **58** (1975), 61–69. https://doi.org/10.1016/0009-8981(75)90485-4
16. G. W. Hanson, A. B. Yakovlev, *Operator theory for electromagnetics: An introduction*, Springer Science & Business Media, 2013.
17. H. Heuser, *Functional analysis*, A Wiley-Interscience publication, Wiley, 1982.
18. A. Jeribi, *Spectral theory and applications of linear operators and block operator matrices*, Springer, 2015. https://doi.org/10.1007/978-3-319-17566-9
19. P. S. Johnson, S. Balaji, On linear operators with closed range, *J. Appl. Math. Bioinform.*, **1** (2011), 175.
20. C. S. Kubrusly, B. P. Duggal, On Weyl's theorem for tensor products, *Glasgow Math. J.*, **55** (2013), 139–144. https://doi.org/10.1017/S0017089512000407
21. C. S. Kubrusly, B. P. Duggal, On Weyl and Browder spectra of tensor products, *Glasgow Math. J.*, **50** (2008), 289–302. https://doi.org/10.1017/S0017089508004205

22. V. Müller, *Spectral theory of linear operators and spectral systems in Banach algebras*, Springer Science & Business Media, **139** (2007).

23. M. Oudghiri, Weyl's and Browder's theorems for operators satisfying the SVEP, *Stud. Math.*, **1** (2004), 85–101. https://doi.org/10.4064/sm163-1-5

24. K. B. Ouidren, A. Ouahab, H. Zariouh, On a class of (bz)-operators, *Rend. Circ. Mat. Palerm. Ser. 2*, **72** (2023), 4169–4177. https://doi.org/10.1007/s12215-023-00884-6

25. B. Ouidren, H. Zariouh, New approach to a-Weyl's theorem through localized SVEP and Riesz-type perturbations, *Linear Multilinear A.*, **70** (2022), 3231–3247. https://doi.org/10.1080/03081087.2020.1833823

26. B. Ouidren, H. Zariouh, New approach to a-Weyl's theorem and some preservation results, *Rend. Circ. Mat. Palerm. Ser. 2*, **70** (2021), 819–833. https://doi.org/10.1007/s12215-020-00525-2

27. J. Sanabria, L. Vasquez, C. Carpintero, E. Rosas, O. Garcia, On strong variations of Weyl type theorems, *Acta Math. Univ. Comen.*, **86** (2017), 345–356.

28. L. K. Saul, K. Q. Weinberger, F. Sha, J. Ham, D. D. Lee, *Spectral methods for dimensionality reduction*, The MIT Press, 2006. https://doi.org/10.7551/mitpress/6173.003.0022

29. S. V. Djordjević, Y. M. Han, Browder's theorem and spectral continuity, *Glasgow Math. J.*, **42** (2000), 479–486. https://doi.org/10.1017/S0017089500030147