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Research article

Flag-transitive non-symmetric 2-designs with λ prime and exceptional groups of Lie type

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Abstract: This paper contributes to the classification of flag-transitive 2-(v, k, λ) designs. Let $\mathcal{D}$ be a non-trivial and non-symmetric 2-(v, k, λ) design with λ prime and G be a flag-transitive point-primitive automorphism group of $\mathcal{D}$. A recent work by the first author and Chen has proven that the socle of G is either a nonabelian simple group or an elementary abelian p-group for some prime p. In this paper, we focus on the case where the socle of G is an exceptional group of Lie type and give all possible parameters of such 2-designs.

Keywords: 2-design; flag-transitive; exceptional group of Lie type; primitive group
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1. Introduction

A 2-(v, k, λ) design $\mathcal{D}$ is a pair $(\mathcal{P}, \mathcal{B})$, where $\mathcal{P}$ is a set of v points, and $\mathcal{B}$ is a set of k-subsets of $\mathcal{P}$ called blocks, such that any 2 points are contained in exactly λ blocks. A flag is a point-block pair (α, B) with $\alpha \in B$. The Fisher's inequality in [8, 1.3.8] shows that the number of blocks is at least v. Design $\mathcal{D}$ is said to be non-symmetric if $v < b$ and non-trivial if $2 < k < v - 1$. We always assume $\mathcal{D}$ to be non-trivial and non-symmetric in this paper. An automorphism of $\mathcal{D}$ is a permutation of $\mathcal{P}$ that leaves $\mathcal{B}$ invariant. All automorphisms of the design $\mathcal{D}$ form a group called the full automorphism group of $\mathcal{D}$, denoted by $Aut(\mathcal{D})$. Let $G \leq Aut(\mathcal{D})$. The design $\mathcal{D}$ is called point (block, flag)-transitive if G acts transitively on the set of points (blocks, flags) and point-primitive if G acts primitively on $\mathcal{P}$, that is, G does not preserve a partition of $\mathcal{P}$ in classes of size c with $1 < c < v$.

For decades, works have been done on the classification of 2-designs admitting a transitive automorphism group. In 1988, Buekenhout, Delandtsheer, and Doyen first proved in [5] that the flag-transitive automorphism group of a 2-$(v, k, 1)$ design must be of affine or almost simple type.

Then, the classification of flag-transitive 2-$(v, k, 1)$ designs was given in [6] by a six-person team, except for the case of the one-dimensional affine type. In recent years, some researchers have focused on into classifying 2-(v, k, λ) designs with general λ admitting flag-transitive automorphism group, such as [1, 3, 12, 16, 25–27]. Moreover, some of the works also considered classification of such designs admitting automorphism groups in a weaker condition, namely, block-transitive rather than flag-transitive [21–24].

The current paper tackles the 2-(v, k, λ) designs where λ is a prime. In [25], Zhang and Chen reduced the flag-transitive, point-primitive automorphism groups of such 2-designs either to the affine type (with an elementary abelian p-group as socle) or to the almost simple type (with a nonabelian simple socle). Hence, it is possible to classify such 2-designs based on the classification of simple groups. The aim of this paper is to consider the case when the socle of the automorphism group G is an exceptional simple group of Lie type. Note that groups $G_2(2)$, ${}^2G_2(3)$, ${}^2B_2(2)$, and ${}^2F_4(2)$ are not simple, so they are not under consideration in this work. It is also worth noting that the symmetric 2-designs with exceptional simple socle have been studied in [1, 2, 20]. The main result of the current paper is the following:

Theorem 1.1. *Let $\mathcal{D}$ be a non-symmetric 2-(v, k, λ) design with λ prime and G a flag-transitive automorphism group of $\mathcal{D}$. If the socle T of G is an exceptional Lie type simple group in characteristic p, then one of the following holds:*

(1) *T is ${}^2B_2(q)$ with $q = 2^{2n+1} > 2$ and $(v, k, \lambda) = (q^2 + 1, q, q - 1)$, where $q - 1$ is a Mersenne prime;*
(2) *T is $G_2(q)$, and $(v, k, \lambda) = (\frac{q^3(q^3-1)}{2}, \frac{q^3}{2}, q + 1)$ where $q > 2$ is even and $q + 1$ is a Fermat prime.*

Remark 1.1. For the parameters in Theorem 1.1(1), the design $\mathcal{D}$ is described in [26]. For the parameters in Theorem 1.1(2), the existence of such a design remains uncertain at this time.

2. Preliminaries

We begin with some well-established results about the parameters of 2-(v, k, λ) designs and the automorphism groups of them. For any point α, we denote by r the number of blocks that contain α, as it is a constant.

Lemma 2.1. ([8]) *For a 2-(v, k, λ) design $\mathcal{D}$, it is well known that*

(1) $bk = vr$;
(2) $\lambda(v - 1) = r(k - 1)$;
(3) $\lambda v < r^2$.

Lemma 2.2. ([8, Section 1.2]) *Assume that G is an automorphism group of $\mathcal{D}$. Then the flag-transitivity of G is equivalent to one of the following:*

(1) *G is point-transitive, and the point stabilizer G_α is transitive on all blocks that contain α;*
(2) *G is block-transitive, and the block stabilizer G_B is transitive on the k points in block B.*

Lemma 2.3. ([7]) *Assume that G is a flag-transitive automorphism group of $\mathcal{D}$, and T is the socle of G. Then, we have*

(1) *$r \mid |G_\alpha|$, where G_α is the point-stabilizer of G;*
(2) *$r \mid \lambda d_i$, where d_i is any nontrivial subdegree of G.*

3. Proof of Theorem 1.1

Assume that λ is a prime. Then either $(\lambda, r) = 1$ or $\lambda \mid r$. For the former case, by the results of [26], we immediately obtain the following Lemma:

Lemma 3.1. *Assume that G and $\mathcal{D}$ satisfy the hypothesis of Theorem 1.1. If $(\lambda, r) = 1$, then $T = {}^2B_2(q)$ with $q = 2^{2n+1} \geq 8$, and $\mathcal{D}$ is a 2-$(q^2+1, q, q-1)$ design with $q-1$ a Mersenne prime. In particular, $2n+1$ is prime.*

Therefore, we always assume $\lambda \mid r$ in the remaining content. Let $r_0 = \frac{r}{\lambda}$. We get the following from Lemmas 2.1 and 2.3.

Lemma 3.2. *Assume that $\mathcal{D}$ is a 2-(v, k, λ) design where λ is a prime divisor of r, and G is a flag-transitive automorphism group of $\mathcal{D}$. Then we have*

(1) $v < \lambda r_0^2$;
(2) *r_0 divides the greatest common divisor of $|G_\alpha|$, $v-1$ and all nontrivial subdegrees of G.*

3.1. Parabolic maximal subgroups

Since G is point-primitive, the point stabilizer G_α is a maximal subgroup of G. In this section, we first deal with the case when G_α is a maximal parabolic subgroup of G.

Lemma 3.3. *Assume that $T = {}^2B_2(q)$ with $q = 2^{2n+1} > 2$. Then G_α cannot be the maximal parabolic subgroup of G.*

Proof. If G_α is a maximal parabolic subgroup of G, we know that $|G_\alpha| = fq^2(q-1)$ with $f \mid (2n+1)$ from [19], and hence $v = q^2 + 1$. Then, according to (1) and (2) in Lemma 2.1 and the fact $\lambda \mid r$, we further get $k - 1 \mid q^2$ and $b = \frac{\lambda q^2(q^2+1)}{k(k-1)}$. Since G is flag-transitive, Lemma 2.2 implies that $|G_B| = \frac{|G|}{b} = \frac{fk(k-1)(q-1)}{\lambda}$. All maximal subgroups of G can be read off from [19], and let M be any one of them with $G_B \leq M$. The fact that $|G_B|$ divides $|M|$ implies that M is the maximal parabolic subgroup of G, and $k(k-1)$ divides λq^2. This forces $k = \lambda$, for otherwise $k(k-1) \mid q^2$, which is a contradiction. It follows that G_B is primitive on B, and so T_B is transitive on B. Namely, $|T_B : T_{\gamma,B}| = k$ for any point $\gamma \in B$. On the other hand, since M is parabolic, there exists a point α such that $M = G_\alpha$. That is to say, $T_B \leq T_\alpha$ and therefore $T_{\gamma,B} \leq T_{\gamma,\alpha}$ for $\gamma \in B$. Since the stabilizer of any two points in ${}^2B_2(q)$ is a cyclic group of order $q-1$ by [9, p.187], $|T_{\gamma,B}|$ divides $(q-1)$. Also, $|T : T_{\gamma,\alpha}|$ divides bk by the flag-transitivity of G. It follows that $(k-1) \mid \lambda$, which holds only when $\lambda = k = 2$, for it has been proved that $k = \lambda$ above. This is impossible as $\mathcal{D}$ is nontrivial. □

Lemma 3.4. *Assume that $T = {}^2G_2(q)$ with $q = 3^{2n+1} > 3$. Then G_α cannot be the maximal parabolic subgroup of G.*

Proof. If G_α is the maximal parabolic subgroup of G, then we know that $|G_\alpha| = fq^3(q-1)$ with $f \mid (2n+1)$ from [11], and so $v = q^3 + 1$. Similar as to Lemma 3.3, we have

$$b = \frac{\lambda v(v-1)}{k(k-1)} = \frac{\lambda q^3(q^3+1)}{k(k-1)}$$

and $k-1 \mid q^3$. Let f_1 be a divisor of f such that $|G_B : T_B| = f_1$. Then by the flag-transitivity of G, we get

$$|T_B| = \frac{f(q-1)k(k-1)}{f_1\lambda}.$$

Here, we also consider the maximal subgroups M of ${}^2G_2(q)$ such that $T_B \leq M$. From [11], either M is parabolic, or $M \cong \mathbb{Z}_2 \times PSL_2(q)$.

If M is a parabolic subgroup, then $k(k-1) \mid \lambda q^3$. Since $k-1 \mid q^3$, we have $k \mid \lambda$ and therefore $\lambda = k$. It follows that $\lambda - 1 \mid q^3$ and $\lambda = 3^{n_1} + 1$, which forces $\lambda = k = 2$, for λ is prime. However, now we get $b = q^3(q^3+1) > \binom{v}{2}$, which is obviously impossible. Hence, in the remaining part of the proof, we assume that $T_B \leq \mathbb{Z}_2 \times PSL_2(q)$.

According to the list of the maximal subgroups of $PSL_2(q)$ in [4, Tables 8.1 and 8.2], T_B is isomorphic to a subgroup of $\mathbb{Z}_2 \times A_4$, $\mathbb{Z}_2 \times D_{q\pm1}$, $\mathbb{Z}_2 \times ([q] : \mathbb{Z}_{\frac{q-1}{2}})$ or $\mathbb{Z}_2 \times PSL_2(q_0)$ with $q_0^\ell = q = 3^{2n+1}$. Obviously, the former two cases are impossible as $k-1 \mid q^3$. Then, if $T_B \lesssim \mathbb{Z}_2 \times ([q] : \mathbb{Z}_{\frac{q-1}{2}})$, we also have $\lambda = k$, a contradiction again. For the last case, the condition that $|T_B|$ divides $|\mathbb{Z}_2 \times PSL_2(q_0)|$ forces $q_0 = q$, which implies that T_B is isomorphic to $\mathbb{Z}_2 \times PSL_2(q)$ or $PSL_2(q)$. Then, by $|T : T_B| \mid b$, we have $k(k-1) \mid q(q+1)\lambda$. This, together with $k-1 \mid q^2$, implies that $k-1 \mid q$ when $\lambda \neq 3$, and $k-1 \mid 3q$ when $\lambda = 3$. Furthermore, the facts that $q+1$ is the smallest degree of non-trivial action of $PSL_2(q)$ since q is an odd power of 3 and that $|T_B : T_{\alpha,B}|$ divides k imply $k = q+1$. Hence, $|T_B| = \frac{fk(k-1)(q-1)}{f_1\lambda} = \frac{q(q^2-1)}{a}$, with $a = 1$ or 2 when T_B is $\mathbb{Z}_2 \times PSL_2(q)$ or $PSL_2(q)$, respectively. It follows that $\lambda \mid f$ when T_B is $\mathbb{Z}_2 \times PSL_2(q)$, or $\lambda = 2$ when T_B is $PSL_2(q)$.

Let $\mathcal{R}$ be the Ree unital of order q (which is a 2-$(q^3+1, q+1, 1)$ design). For the former case, let σ be the central involution of $\mathbb{Z}_2 \times PSL_2(q)$. It can be deduced from [15] that σ fixes a block ℓ of $\mathcal{R}$ pointwise and preserves a point-partition $\mathcal{S}_\sigma$ of $\mathcal{R} \setminus \ell$ into $q^2 - q$ blocks, each of them invariant by σ. Now, $\mathbb{Z}_2 \times PSL_2(q)$ induces $PSL_2(q)$ on $\mathcal{S}_\sigma \cup \{\ell\}$, and $PSL_2(q)$ preserves ℓ acting on this one in its natural 2-transitive action of degree $q+1$. Further, $PSL_2(q)$ partitions $\mathcal{S}_\sigma$ into two orbits each of length $\frac{q^2-q}{2}$. Thus, ℓ is the unique $\mathbb{Z}_2 \times PSL_2(q)$-orbit of points of $\mathcal{R}$ of length $q+1$. Note that $k = q+1$, which means $B = \ell$. This means that $|B^G| = |\ell^G| = q^2(q^2-q+1)$ by [6], and so $\lambda = 1$, which contradicts with λ being prime. For the latter case, the block stabilizer T_ℓ for the Ree unital is $\mathbb{Z}_2 \times PSL_2(q)$, and $\mathbb{Z}_2$ fixed all points in ℓ. However, since $\alpha^{T_B} \subseteq \alpha^{T_\ell}$ and $|\alpha^{T_B}| = |\alpha^{T_\ell}| = q+1$, we have $\alpha^{T_B} = \alpha^{T_\ell}$. This means that $\mathbb{Z}_2$ fixed all points in B, and so $\mathbb{Z}_2 \in T_B$, an obvious contradiction. □

For the remaining possibility of T in $\mathcal{T}$, where

$$\mathcal{T} = \{{}^2F_4(q), {}^3D_4(q), G_2(q), F_4(q), E_6^\epsilon(q), E_7(q), E_8(q)\},$$

we use the following Lemma from [14] to prove that G_α cannot be the maximal parabolic subgroup. Note that in the following we denote by n_p the p-part of n and $n_{p'}$ the p'-part of n, i.e., $n_p = p^t$ where $p^t \mid n$ but $p^{t+1} \nmid n$, and $n_{p'} = n/n_p$.

Lemma 3.5. ([14]) *Assume that T is a simple group of Lie type in characteristic p and acts on the set of cosets of a maximal parabolic subgroup. Then T has a unique subdegree which is a power of p except when T is $L_d(q)$, $\Omega_{2m}^+(q)$ (m is odd) or $E_6(q)$.*

Lemma 3.6. *If $T \in \mathcal{T}$, then G_α cannot be a parabolic subgroup of G.*

Proof. By Lemma 3.5, for all cases where $T \in \mathcal{T} \setminus \{E_6(q)\}$, there is a unique subdegree which is a power of p. Then, Lemma 3.2 implies that r_0 divides $|v-1|_p$. Since we also have λ divides $|G_\alpha|$, we can easily check that r_0 is too small to satisfy the condition $v < \lambda r_0^2$. Therefore, we assume that $T = E_6(q)$. If G contains a graph automorphism, or $G_\alpha \cap T$ is P_2 or P_4, then there is also a unique subdegree that is a power of p. However, r_0 is too small again. If $G_\alpha \cap T$ is P_3 with type A_1A_4, we have $\lambda \leq \frac{q^5-1}{q-1}$ by $\lambda \mid |G_\alpha|$ and

$$v = \frac{(q^3+1)(q^4+1)(q^9-1)(q^6+1)(q^4+q^2+1)}{(q-1)}.$$

Moreover, from [1, Proposition 6.3], we know that there exist two nontrivial subdegrees: $q^{13}\frac{q^5-1}{q-1}$ and $\frac{q(q^5-1)(q^4-1)}{(q-1)^2}$. Lemma 3.2 then implies that r divides $\lambda q\frac{q^5-1}{q-1}$. However, the condition $v < \lambda r_0^2$ cannot be satisfied again. If $G_\alpha \cap T$ is P_1 with type D_5, then

$$v = \frac{(q^8+q^4+1)(q^9-1)}{q-1},$$

and there exist two nontrivial subdegrees (see [13]): $\frac{q(q^3+1)(q^8-1)}{(q-1)}$ and $\frac{q^8(q^4+1)(q^5-1)}{(q-1)}$. It follows that $r \mid \lambda q(q^4+1)$. This, together with $\lambda \mid |G_\alpha|$, implies that $r^2 < \lambda^2 q^2 (q^4+1)^2 < \lambda v$, which is contradictive with Lemma 2.1. □

3.2. *Non-parabolic maximal subgroups*

In this section, we assume that G_α is a non-parabolic maximal subgroup of G.

Lemma 3.7. *Assume that G and $\mathcal{D}$ satisfy the hypothesis of Theorem 1.1. Then, $|G| < |G_\alpha|^3$.*

Proof. From Lemma 2.3, we know that r divides every nontrivial subdegree of G, and so r divides $|G_\alpha|$. Since $v < r^2$ by (3) of Lemma 2.1, it follows that $|G| < |G_\alpha|^3$. □

Lemma 3.7 implies that G_α is a large maximal non-parabolic subgroup of G. The type of G_α can be read from [1, Table 2]. Note that Theorem 1.1(2) just corresponds to the non-parabolic case here, with $T = G_2(q)$ and the type of G_α being $SL_3^\epsilon(q).2$.

Lemma 3.8. *Assume that G and $\mathcal{D}$ satisfy the hypothesis of Theorem 1.1. If $T = G_2(q)$ and the type of G_α is $SL_3^\epsilon(q).2$ with $\epsilon = \pm$, then $\epsilon = -$, T is flag-transitive on $\mathcal{D}$, and the parameters of $\mathcal{D}$ are $(v,b,r,k,\lambda) = (\frac{q^3(q^3-1)}{2}, (q+1)(q^6-1), (q+1)(q^3+1), \frac{q^3}{2}, q+1)$, where q is even, and $\lambda = q+1$ is a Fermat prime.*

Proof. It is obvious that $|T_\alpha| = 2q^3(q^2-1)(q^3-\epsilon 1)$, and hence $v = \frac{1}{2}q^3(q^3+\epsilon 1)$. We first deal with the case when q is even. Since $G_2(2)$ is not simple ($G_2(2) \cong PSU_3(3):2$), we assume that $q > 2$. From [17, Section 3, Case 8], we know that r divides $\lambda(q^3-\epsilon 1)$. Then, the equality $\lambda(v-1) = r(k-1)$ from Lemma 2.1 implies that there exists an odd integer t dividing $(q^3-\epsilon 1)$ such that

$$k = t\frac{(q^3+\epsilon 2)}{2} + 1 \quad \text{and} \quad r = \frac{\lambda(q^3-\epsilon 1)}{t}.$$

Obviously, the fact that $k < r$ implies $t < \lambda$. Moreover, by Lemma 2.1 we have

$$b = \frac{\lambda v(v-1)}{k(k-1)} = \frac{\lambda q^3(q^6-1)(q^3+\epsilon 2)}{4k(k-1)} = \frac{\lambda q^3(q^6-1)}{2kt}. \tag{3.1}$$

Note also that $(2k, q^3 - \epsilon 1) \mid 3t + \epsilon 2$, $(2k, q^3 + 1) \mid t + \epsilon 2$, $(k, \frac{q^3}{2}) \mid t + \epsilon 1$, and therefore $(2k, q^2 - 1) \mid (t + \epsilon 2)(3t + \epsilon 2)$. Since b is an integer, it follows from (3.1) that $k \mid \lambda \frac{q^3}{2}(q^3 - \epsilon 1)(q^3 + \epsilon 1)$. Hence, we have

$$\frac{t(q^3 + \epsilon 2)}{2} + 1 \mid \lambda(t + \epsilon 2)(t + \epsilon 1)(3t + \epsilon 2). \tag{3.2}$$

Since $3t + \epsilon 2 \leq 5t$, it follows that $q^3 + \epsilon 2 < 10\lambda(t + \epsilon 2)(t + \epsilon 1)$ except when $t = 1$ and $\epsilon = -$. When $t \neq 1$, the above together with $t < \lambda$ further implies that λ cannot be a prime divisor of $|Out(T)|$, and hence λ divides $|SL_3^\epsilon(q).2|$.

In the following, we prove that $t = 1$. Obviously, $t \neq 2$, for t is odd. When $t \geq 3$, we have $\frac{r}{\lambda} \leq \frac{1}{3}(q^3 - \epsilon 1)$ and $\frac{3t}{2} < \lambda$ by $t\frac{(q^3+\epsilon 2)}{2} < k \leq r \leq \lambda \frac{(q^3-\epsilon 1)}{3}$. Now, assume that $\lambda \mid k$. Then λ divides $(2q^3(q^2 - 1)(q^3 - \epsilon 1), 2k)$, and it follows that $\lambda \mid 4(t + \epsilon 1)(t + \epsilon 2)(3t + \epsilon 2)^2$. Since $\frac{3}{2}t < \lambda$, we have $\lambda = 3t + \epsilon 2$, or $\epsilon = +$ and $\lambda = \frac{3t+2}{2}$. If $\lambda = 3t + 2\epsilon$, then $k < r$ forces $(t, \lambda, \epsilon) = (5, 17, +), (5, 13, -), (3, 11, +), (3, 7, -)$, or $(1, 5, +)$. Note that $k \mid \lambda(t + \epsilon 2)(t + \epsilon 1)(3t + \epsilon 2)$, and we check each case and know that it is impossible. If $\lambda = \frac{3t+2}{2}$, then we get $(t, \lambda) = (4, 7)$, which can be ruled out similarly. Hence, $\lambda \nmid k$, and it follows (3.2) that $t > q$. On the other hand, since $|T : T_B| \mid b$, there exists an integer f_1 dividing f such that $f_1|T : T_B| = b$ and

$$|T_B| = \frac{2f_1 q^3(q^2 - 1)k}{\lambda t}.$$

Since $\lambda \nmid k$ and $\lambda > t > q \geq 2$, λ is a divisor of f_1, $(q - 1)$, $q + 1$, or q, and so $\lambda \leq q + 1$. Since $q < t < \frac{2}{3}\lambda$, we get a contradiction. Therefore, $t = 1$ as we claim.

Let $t = 1$. Then, $\frac{r}{\lambda} = (q^3 - \epsilon 1)$, and $k = \frac{(q^3+2\epsilon)}{2} + 1$ with q even. If $\epsilon = +$, then $r = \lambda(q^3 - 1)$, and $k = \frac{q^3+4}{2}$. Since b is an integer, we get that $q^3 + 4$ divides $\lambda q^3(q^6 - 1)$. It follows that $q^3 + 4 \mid 60\lambda$, and so λ divides $q^3 + 4$, which is impossible as λ is a prime divisor of $2q^3(q^2 - 1)(q^3 - 1)$. We now assume that $\epsilon = -$. Then, $k = \frac{q^3}{2}$ and $b = \lambda(q^6 - 1)$, and $r = \lambda(q^3 + 1)$ for $q \geq 4$. Moreover, in this case $|T_B| = \frac{f_1 q^6(q^2-1)}{\lambda}$ and we further find that T_B is contained in a maximal parabolic subgroup $M = q^5 : GL_2(q)$ of $G_2(q)$. Since G is flag-transitive, Lemma 2.2 implies that $|SU_3(q).2 : T_{\alpha,B}|$ divides $\lambda(q^3 + 1)$. Using the maximal subgroup list for $SU_3(q)$ provided in [4, Tables 8.5 and 8.6], we get that $T_{\alpha,B}$ is isomorphic to a subgroup of $M_1 = q^3 : C_{q^2-1}.2$. If $T_{\alpha,B} = M_1$ or $\lambda \nmid q^2 - 1$, then $T_{\alpha,B}$ contains a cyclic group of order $q^2 - 1$, which contradicts $T_{\alpha,B} \leq T_B \leq q^5 : GL_2(q)$. Hence, $|M_1 : T_{\alpha,B}| = \lambda$ divides $q^2 - 1$. This also implies that T is flag-transitive, and so $|T : M||M : T_B| = \lambda(q^6 - 1)$. It follows that $|M : T_B| = |GL_2(q) : T_B \cap GL_2(q)| = \lambda(q - 1)$, which gives $|T_B \cap GL_2(q)| = q\frac{(q^2-1)}{\lambda}$. Then, using the list of maximal subgroups of $SL(2, q)$ provided in [4, Tables 8.1 and 8.2], we get that $\lambda \nmid q - 1$, and so $\lambda \mid q + 1$, which further implies that $\lambda = q + 1$. This is to say, if such design exists, then the design parameters tuple is $(v, b, r, k, \lambda) = (\frac{q^3(q^3-1)}{2}, (q + 1)(q^6 - 1), (q + 1)(q^3 + 1), \frac{q^3}{2}, q + 1)$, where $\lambda = q + 1$ is a Fermat prime.

Now, we assume that q is odd. Then, we conclude that r divides $\lambda \frac{(q^3-\epsilon 1)}{2}$ from [17, Section 4, Case 1, $i = 1$]. Let $rt = \lambda \frac{(q^3-\epsilon 1)}{2}$. Similar as in the even case, we also have $t = 1$. That is to say, $k = q^3 + \epsilon 2 + 1$ and $r = \lambda \frac{(q^3-\epsilon 1)}{2}$. When $\epsilon = +$, the fact of k dividing $\lambda q^3(q^6 - 1)$ $q^3 + 3$ implies that $q^3 + 3$ divides 24λ, and so λ divides $q^3 + 3$, which is impossible as λ is a prime divisor of $2q^3(q^2 - 1)(q^3 - 1)$. If $\epsilon = -$, we have $k = q^3 - 1$, and so $b = \frac{\lambda q^3(q^3+1)}{4}$. We consider a maximal subgroup M containing T_B. It is proven later that $M \cong T_B \cong SL_3(q).2$ and hence that is unique. The fact that $|T : M| \mid b$ implies that M is

$SL_3(q).2$ by [4, Tables 8.41 and 8.42] and that $|T : M| = \frac{q^3(q^3+1)}{2}$. It follows that $2|M : T_B| \mid \lambda$, which forces $\lambda = 2$ and $M = T_B \cong SL_3(q).2$. Since $T_\alpha \cong SU_3(q).2$ and $r = q^3 + 1$, we have $T_{\alpha,B} \cong q^3.C_{q^2-1}.2$ or $q^3.C_{q^2-1}$. According to the maximal subgroups of $SL_3(q)$ in [4, Tables 8.3 and 8.4], we know that $T_{\alpha,B}$ is isomorphic to a subgroup of $q^2.GL_2(q).2$, which is impossible. □

All other types of G_α in [1, Table 2], except two cases which we will discuss in Lemma 3.10, can be ruled out using the method stated below. First, for each possibility of G_α, the order of G_α and the value of v can be determined. We can hence get an upper bound of λ according to $\lambda \mid |G_\alpha|$. Then, to get an upper bound of r_0, we consider the divisors of $|G_\alpha|$ in two parts: $\prod_{i=1}^{i_1} \Phi_i$ for which Φ_i divides v, and $\prod_{j=1}^{i_2} \Psi_j = |G_\alpha|/\prod_{i=1}^{i_1} \Phi_i$. Obviously, all Φ_i are coprime with $v-1$. For each Ψ_j, we calculate the remainder $\bar{\Psi}_j$ of Ψ_j divided by $v-1$. This implies that $(|G_\alpha|, v-1)$ divides $|Out(T)| \prod_{j=1}^{i_2} \bar{\Psi}_j$, which implies that $r_0 \leq |Out(T)| \prod_{j=1}^{i_2} \bar{\Psi}_j$. Finally, one can check that the values of r_0 for all these cases are too small to satisfy the condition that $v < \lambda r_0^2$. That is, no new designs arise in these cases. To be more explicit, we take $T = E_8(q)$ as an example.

Lemma 3.9. *Assume that G and $\mathcal{D}$ satisfy the hypothesis of Theorem 1.1. If $T = E_8(q)$ with $q = p^e$, then G_α cannot be a non-parabolic maximal subgroup of G.*

Proof. Let $T = E_8(q)$. Then, it follows from [1, Table 2] that the type of G_α is one of the following:

$$\{A_1(q)E_7(q),\ D_8(q),\ E_8(q^{\frac{1}{2}}),\ E_8(q^{\frac{1}{3}}),\ A_2^\epsilon(q)E_6^\epsilon(q)\ \}.$$

For the case that G_α is of type $A_1(q)E_7(q)$, we have $\lambda < q^8$ since $\lambda \mid |G_\alpha|$ and $v = q^{56}(q^6+1)(q^{10}+1)(q^{12}+1)\frac{q^{30}-1}{q^2-1}$ by $v = |G : G_\alpha|$. Obviously, $q(q^6+1) \mid v$ and $\frac{q^{30}-1}{q^2-1} \mid v$, which also implies $\frac{q^6-1}{q^2-1} \mid v$ and $\frac{q^{10}-1}{q^2-1} \mid v$. This means $(|G_\alpha|, v-1)$ divides $|Out(T)|(q^2-1)^5(q^8-1)(q^{14}-1)(q^{18}-1)$. Since $r_0 \mid (|G_\alpha|, v-1)$, we have $r_0 < q^{51}$. However, Lemma 3.2 shows $q^{112} < v < \lambda r_0^2 < q^{110}$, a contradiction.

For the case that G_α is of type $D_8(q)$, we have $\lambda < q^7$ and

$$v = q^{64}(q^{12}+q^6+1)(q^{16}+q^8+1)\frac{(q^{10}+1)(q^{30}-1)}{q^4-1}.$$

Since $v-1 \equiv 2(\bmod\ q^4+1)$, $(v-1, q^4+1) = 2$ or 1. This, together with $q \mid v$ and $\frac{q^{30}-1}{q^2-1} \mid v$, implies that $(|G_\alpha|, v-1)$ divides $4|Out(T)|(q^2-1)^3(q^4-1)^3(q^{12}-1)(q^{14}-1)$. It follows that $r_0 \leq 4|Out(T)|q^{44} < 4q^{45}$, and $q^{128} < v < \lambda r_0^2 < 4q^{97}$, which is a contradiction.

Assume that G_α is of type $E_8(q^{\frac{1}{2}})$. Then, $\lambda \leq q^{15}$ and $v = q^{60}(q+1)(q^4+1)(q^6+1)(q^7+1)(q^9+1)(q^{10}+1)(q^{12}+1)(q^{15}+1)$. Since q, q^3+1, q^4+1, q^5+1, and q^6+1 are divisors of v, we get that $(|G_\alpha|, v-1)$ divides $|Out(T)|(q-1)^2(q^3-1)^2(q^5-1)(q^7-1)(q^9-1)(q^{15}-1)$. It follows that $r_0 < q^{45}$, and so $q^{124} < v < \lambda r_0^2 < q^{105}$, a contradiction again.

Assume that G_α is of type $A_2^\epsilon(q)E_6^\epsilon(q)$ or $E_8(q^{\frac{1}{3}})$. Then, since G_α is non-parabolic, the Tits lemma in [18, 1.6] implies that p divides $v = |G : G_\alpha|$, and so $(|G_\alpha|, v-1)$ is coprime with p. It follows that $r_0 \leq |G_\alpha|_{p'}$ as r_0 divides $(|G_\alpha|, v-1)$. This implies that $v < \lambda |Out(T)|^2 |T_\alpha|_{p'}^2$ by Lemma 3.2, which cannot be satisfied when G_α is of type $A_2^\epsilon(q)E_6^\epsilon(q)$ or $E_8(q^{\frac{1}{3}})$. □

Lemma 3.10. *Assume that G and $\mathcal{D}$ satisfy the hypothesis of Theorem 1.1. Then the type of G_α cannot be either $(q-\epsilon 1)D_5^\epsilon(q)$ when $T = E_6^\epsilon(q)$ or $(q-\epsilon 1)E_6^\epsilon(q)$ when $T = E_7(q)$.*

Proof. Assume that T is $E_6^\epsilon(q)$ and G_α is of type $(q-\epsilon 1)D_5^\epsilon(q)$. Then, $\lambda < 2q^4$ as λ divides $|G_\alpha|$ and $v = \frac{q^{16}(q^9-\epsilon 1)(q^{12}-1)}{(3,q-1)(q-\epsilon 1)(q^4-1)}$. In addition, we know from [1, Theorem 4.1] that there exist two subdegrees: $q^8(q^5-\epsilon)(q^4+1)$ and $q^{10}(q^3+\epsilon)(q^8-1)$. Since r_0 divides the greatest common divisors of every non-trivial subdegree and $v-1$ (Lemma 2.3), we have $(r_0, p) = 1$, and so $r_0 \mid 2(q-\epsilon 1)(q^4+1)$, which implies that r_0 is too small to satisfy $v < \lambda r_0^2$ again.

If T is $E_7(q)$ and G_α is of type $(q-\epsilon 1)E_6^\epsilon(q)$, we have $\lambda \leq 2q^6$ and $v = q^{27}(q^5+\epsilon 1)(q^9+\epsilon 1)\frac{(q^{14}-1)}{q-\epsilon 1}$. [1, Theorem 4.1] shows that there exist two subdegrees, which divide $q^{12}(q^5-\epsilon)(q^9-\epsilon)$ and$(4, q^m - 1\epsilon)q^{16}(q^5-\epsilon)(\frac{q^{12}-1}{q^4-1})$, respectively. However, by Lemma 2.3 we know that r_0 is too small again. □

Proof of Theorem 1.1. It follows immediately from Lemmas 3.1–3.10. □

4. Conclusions

In this paper, we figure out all possible parameters of 2-(v, k, λ) designs $\mathcal{D}$ (with λ prime) that admit flag-transitive point-primitive automorphism groups with an exceptional Lie type socle. Our work contributes to the classification of flag-transitive 2-(v, k, λ) designs. In addition, the cases that the automorphism groups of such designs with classical socle will be the main focus in our future work.

Author contributions

Y. Zhang: Data curation, writing-review and editing; J. Shen: Writing-original draft. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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