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Research article

On positive definite solutions of the matrix equation $X - \sum_{i=1}^{m} A_i^* X^{-p_i} A_i = Q$

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Abstract: In this paper, we used the outstanding properties of the Thompson metric to conclusively demonstrate the existence of a unique positive definite solution for the nonlinear matrix equation $X - \sum_{i=1}^{m} A_i^* X^{-p_i} A_i = Q$ without any additional assumptions. Furthermore, we designed an iterative algorithm to compute this unique positive definite solution, and derive its corresponding error estimate formula. Additionally, we presented three refined existence intervals for positive definite solutions of this equation. Finally, numerical examples were employed to validate the practicability of our iterative algorithm.

Keywords: nonlinear matrix equation; positive definite solution; Thompson metric
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1. Introduction

In this paper, we study the following nonlinear matrix equation:

$$X - \sum_{i=1}^{m} A_i^* X^{-p_i} A_i = Q, \tag{1.1}$$

where $A_i \in \mathbb{C}^{n\times n}$ are given matrices, $0 < p_i \leq 1$ $(i = 1, 2, \ldots, m)$, $Q \in \mathbb{C}^{n\times n}$ is a positive definite matrix, and $X \in \mathbb{C}^{n\times n}$ is an unknown matrix to be solved.

In recent years, researchers have shown great interest in Eq (1.1), with their primary focus being to explore the conditions for the existence of positive definite solutions, perturbation analysis, and developing iterative methods to solve Eq (1.1). Notably, there have been several noteworthy studies on the special case of Eq (1.1) for $m = 1$ and $p_i = 1$ [1, 2], which arises from ladder networks [3], control theory [4], dynamic programming [5], stochastic filtering [6], statistics [7], etc. Equation (1.1) also arises in the computation of the solution for a large sparse linear system given by $M\mathbf{x} = \mathbf{f}$ with $M = \begin{pmatrix} I & A \\ A^* & I \end{pmatrix}$, which arises from a finite difference approximation to an elliptic partial differential equation [8]. Let $M = \tilde{M} +$ diag $[I + X, \mathbf{0}]$, where

$$\tilde{M} = \begin{pmatrix} -X & A \\ A^* & I \end{pmatrix} = \begin{pmatrix} I & \mathbf{0} \\ -A^*X^{-1} & I \end{pmatrix} \begin{pmatrix} -X & A \\ \mathbf{0} & X \end{pmatrix}. \tag{1.2}$$

$\tilde{M}$ can be decomposed into the form of (1.2) if and only if X is a solution of Eq (1.1) for $m = 1$ and $p_i = 1$. Therefore, the linear system $\tilde{M}\mathbf{y} = \mathbf{f}$ can be solved by block LU decomposition. Then $M\mathbf{x} = \mathbf{f}$ can be solved by using the Sherman-Morrison-Woodbury formula.

For $m > 1$, $p_i = 1$, Eq (1.1) originates from the more generalized nonlinear matrix equation

$$X = A^*(\hat{X} - C)^{-1}A + Q, \tag{1.3}$$

where $\hat{X}$ is the $m \times m$ block diagonal matrix with on each diagonal entry the $n \times n$ matrix X, C is an $mn \times mn$ positive semidefinite matrix, A is an arbitrary $mn \times n$ matrix and Q is an $n \times n$ positive definite matrix. Equation (1.3) plays an important role in modeling certain optimal interpolation problems [9]. Let $C = 0$ and $A = (A_1^T, A_2^T, \ldots, A_m^T)^T$, where $A_i, i = 1, 2, \ldots, m$ are $n \times n$ matrices. Thus, Eq (1.3) can be written as $X - \sum_{i=1}^{m} A_i^* X^{-1} A_i = Q$, i.e., the special case of Eq (1.1), where $m > 1$ and $p_i = 1$.

For $m > 1$ and $p_i = 1$, Duan et al. [10] employed the Thompson metric to prove that Eq (1.1) always possesses a unique positive definite solution. In contrast, Weng [11] investigated Eq (1.1) by applying Newton's method and the generalized Smith method, and obtained the maximum positive definite solution of Eq (1.1). Yin and Fang [12] studied the perturbation analysis of the positive definite solution of Eq (1.1), obtained two perturbation bounds, and provided an explicit expression for the condition number of its unique positive definite solution of Eq (1.1). Li et al. [13] proposed a fixed-point accelerated iteration algorithm and proved its convergence using the properties of the Thompson metric.

In the case where $m > 1$ and $p_i > 0$, Li and Zhang [14] successfully proved the existence of a unique positive definite solution for Eq (1.1) under rigorous prerequisites. However, in this paper, we employ the excellent properties of the Thompson metric [15] to Eq (1.1), which differs significantly from the metric employed by [14]. Notably, we establish that Eq (1.1) consistently possesses a unique positive definite solution without any additional assumptions. Furthermore, we design an iterative method to solve Eq (1.1) and present three refined existence intervals for its positive definite solution.

The paper is organized as follows. In Section 2, we prove that Eq (1.1) always has a unique positive definite solution, and propose an iteration method to solve it. In Section 3, we refine the existence intervals for the positive definite solution of Eq (1.1) by providing three more accurate intervals. In Section 4, we provide some numerical examples to verify the feasibility of the iteration method. In Section 5, we draw a conclusion.

Throughout this paper, we use the following notations. Let $\mathbb{C}^{n\times n}, H^{n\times n}$, and $P(n)$ denote the set of all $n \times n$ complex matrices, all $n \times n$ positive semidefinite matrices and all $n \times n$ positive definite matrices, respectively. For $A,\ B \in H^{n\times n}$, $A \leq B$ means that $B - A$ is a positive semidefinite matrix. Let $\lambda_1(A)$ and $\lambda_n(A)$ denote the maximal and minimal eigenvalues of a matrix A. For $A \in \mathbb{C}^{n\times n}$, the symbols $\sigma_1(A)$ and $\sigma_n(A)$ denote the maximal and minimal singular values of A, respectively.

2. Existence of positive definite solutions of Eq (1.1)

In this section, we shall use the Thompson metric to prove Eq (1.1) always has a unique positive definite solution. Before we start, we first introduce the Thompson metric briefly.

Definition 2.1. *[15] Let $P(n)$ be the open convex cone of $n\times n$ positive definite matrices. The Thompson metric on $P(n)$ is defined by*

$$d(A,B) = \log\{\max\{M(A/B), M(B/A)\}\},$$

where $M(A/B) = \inf\{\lambda > 0 : A \le \lambda B\} = \lambda_1(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})$.

Next, we list several excellent properties of the Thompson metric needed in the following proof.

Lemma 2.1. *[16] For any $A, B, C, D \in P(n)$,*

$$d(A + B, C + D) \le \max\{d(A,C), d(B,D)\}.$$

Lemma 2.2. *[17] For any $A, B \in P(n)$ and $n \times n$ nonsingular matrix M, we have*

$$d(A,B) = d(A^{-1}, B^{-1}) = d(M^*AM, M^*BM),$$
$$d(M^*A^rM, M^*B^rM) \le |r|d(A,B),\ r \in [-1,1].$$

Lemma 2.3. *[18] Let A be an $n \times n$ positive semidefinite matrix. Then,*

$$d(A + X, A + Y) \le \frac{\alpha}{\alpha+\beta}d(X,Y),\ \forall X, Y \in P(n),$$

where $\alpha = \max\{\lambda_1(X), \lambda_1(Y)\}$*, and* $\beta = \lambda_n(A)$.

Since the Thompson metric differs from the metric induced by matrix norms, we provide an example to illustrate its properties. Specifically, let us consider the Thompson metric between the following positive definite matrices.

Example 2.1.

$$A = \begin{pmatrix} 4 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 3 \end{pmatrix},\quad B = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{pmatrix},\quad C = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix},\quad D = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 1 \end{pmatrix},$$

$$M = \begin{pmatrix} 1 & 0 & 5 \\ 2 & 1 & 6 \\ 3 & 4 & 0 \end{pmatrix},\quad N = \begin{pmatrix} 4 & 1 & 2 \\ 1 & 5 & -1 \\ 2 & -1 & 6 \end{pmatrix}.$$

A, B, C, and D are positive definite matrices, M is a nonsingular matrix, and N is a positive semidefinite matrix. We use Example 2.1 to verify Lemmas 2.1–2.3. From Definition 2.1, we know that $d(A+B, C+D) = 1.0296$, $d(A,C) = 0.9959$, and $d(B,D) = 1.6561$. Therefore, it follows that $d(A + B, C + D) \le \max\{d(A,C), d(B,D)\}$.

Through simple calculations, we know that $d(A,B) = 0.6109 = d(A^{-1}, B^{-1}) = d(M^*AM, M^*BM)$, and taking $r = \frac{1}{2}$, $d(M^*A^{\frac{1}{2}}M, M^*B^{\frac{1}{2}}M) = 0.2994 < 0.3054 = \frac{1}{2}d(A,B)$, which verifies Lemma 2.2.

According to Lemma 2.3, we obtain $\alpha = 7.6119$, $\beta = 2.1049$, and

$$d(N + A, N + B) = 0.2485 < 0.4786 = \frac{7.6119}{7.6119 + 2.1049} d(A, B),$$

which verifies Lemma 2.3.

Lemma 2.4. *[14] The Eq* (1.1) *always has positive definite solutions.*

Remark 2.1. *Lemma 2.4 just tells us the existence of fixed points, i.e., Eq* (1.1) *always has positive definite solutions, but the uniqueness is still uncertain. The proof of Lemma 2.4 is obtained by the Brouwer fixed point theorem. Therefore, Lemma 2.4 does not provide a method for solving Eq* (1.1)*.*

Lemma 2.5. *[14] If X is a positive definite solution of Eq* (1.1)*, then*

$$Q \le X \le Q + \sum_{i=1}^{m} \frac{A_i^* A_i}{\lambda_n^{p_i}(Q)}.$$

The following theorem shows the existence and uniqueness of positive definite solutions to Eq (1.1), and provides a method to solve it.

Theorem 2.1. *[14] If*

$$q = \sum_{i=1}^{m} p_i \|A_i\|^2 \lambda_n^{-p_i - 1}(Q) < 1, \tag{2.1}$$

then:

(1) The Eq (1.1) *has a unique positive definite solution X^*.*
(2) The iteration

$$X_0 \in \left[Q,\ Q + \sum_{i=1}^{m} \lambda_n^{-p_i}(Q) A_i^* A_i \right],\ X_k = Q + \sum_{i=1}^{m} A_i^* X_{k-1}^{-p_i} A_i,\ k = 1, 2, \ldots \tag{2.2}$$

converges to X. Moreover,

$$\|X_k - X^*\| \le \frac{q^k}{1 - q} \|X_1 - X_0\|.$$

Remark 2.2. *Although Theorem 2.1 tells us of the existence and uniqueness of positive definite solutions to Eq* (1.1)*, it requires the assumption* (2.1)*. This assumption seems a bit strong in some cases.*

Unlike the 2-norm metric used by previous researchers, we first adopt the Thompson metric to prove the existence and uniqueness of Eq (1.1). No additional assumptions need to be made about Eq (1.1).

Theorem 2.2. *Equation* (1.1) *always has a unique positive definite solution X^*, and the iteration*

$$\forall X_0 \in [Q,\ Q + \sum_{i=1}^{m} \frac{A_i^* A_i}{\lambda_n^{p_i}(Q)}],\ X_{k+1} = Q + \sum_{i=1}^{m} A_i^* X_k^{-p_i} A_i,\ k = 0, 1, 2, \ldots \tag{2.3}$$

converges to X^*. Moreover,

$$d(X_k, X^*) \le \frac{L^k}{1 - L} d(X_1, X_0),$$

where

$$L = \frac{\lambda_1(\sum_{i=1}^{m} A_i^* \lambda_n^{-p_i}(Q) A_i)}{\lambda_n(Q) + \lambda_1(\sum_{i=1}^{m} A_i^* \lambda_n^{-p_i}(Q) A_i)}.$$

Proof. Let

$$F(X) = Q + \sum_{i=1}^{m} A_i^* X^{-p_i} A_i,\ X \in \Omega,$$

where $\Omega = \left\{X \mid Q \le X \le Q + \sum_{i=1}^{m} \frac{A_i^* A_i}{\lambda_n^{p_i}(Q)}\right\}$, and it is easy to verify that Ω is a compact convex subset, the map F is continuous on Ω, and $F(\Omega) \subseteq \Omega$. Next, we shall prove that F is a contraction map on Ω.

$\forall X, Y \in \Omega$, let

$$\alpha = \max\left\{\lambda_1(\sum_{i=1}^{m} A_i^* X^{-p_i} A_i), \lambda_1(\sum_{i=1}^{m} A_i^* Y^{-p_i} A_i)\right\},$$

and then we have

$$\begin{aligned}
\alpha &= \max\left\{\lambda_1(\sum_{i=1}^{m} A_i^* X^{-p_i} A_i), \lambda_1(\sum_{i=1}^{m} A_i^* Y^{-p_i} A_i)\right\} \\
&\le \max\left\{\lambda_1(\sum_{i=1}^{m} A_i^* \lambda_n^{-p_i}(Q) A_i), \lambda_1(\sum_{i=1}^{m} A_i^* \lambda_n^{-p_i}(Q) A_i)\right\} \qquad (2.4) \\
&= \lambda_1\left(\sum_{i=1}^{m} A_i^* \lambda_n^{-p_i}(Q) A_i\right).
\end{aligned}$$

We will prove the following inequality holds:

$$d(\sum_{i=1}^{m} A_i^* X^{-p_i} A_i, \sum_{i=1}^{m} A_i^* Y^{-p_i} A_i) \le r d(X, Y), \qquad (2.5)$$

where $r = \max\{p_i \mid i = 1, 2, \ldots, m\}$. We use mathematical induction to prove (2.5). First, we verify that (2.5) holds when $m = 1$. According to Lemma 2.2, we have

$$d(A_1^* X^{-p_1} A_1, A_1^* Y^{-p_1} A_1) \le p_1 d(X, Y) \le r d(X, Y).$$

Thus, (2.5) holds with $m = 1$. Assume that (2.5) holds with $m = k$, i.e.,

$$d(\sum_{i=1}^{k} A_i^* X^{-p_i} A_i, \sum_{i=1}^{k} A_i^* Y^{-p_i} A_i) \le r d(X, Y). \qquad (2.6)$$

Using Lemmas 2.1 and 2.2, we have

$$\begin{aligned}
&d(\sum_{i=1}^{k+1} A_i^* X^{-p_i} A_i, \sum_{i=1}^{k+1} A_i^* Y^{-p_i} A_i) \\
&= d(\sum_{i=1}^{k} A_i^* X^{-p_i} A_i + A_{k+1}^* X^{-p_{k+1}} A_{k+1}, \sum_{i=1}^{k} A_i^* Y^{-p_i} A_i + A_{k+1}^* Y^{-p_{k+1}} A_{k+1}) \\
&\le \max\left\{d(\sum_{i=1}^{k} A_i^* X^{-p_i} A_i, \sum_{i=1}^{k} A_i^* Y^{-p_i} A_i), d(A_{k+1}^* X^{-p_{k+1}} A_{k+1}, A_{k+1}^* Y^{-p_{k+1}} A_{k+1})\right\}
\end{aligned}$$

$$\leq \max\left\{d(\sum_{i=1}^{k} A_i^* X^{-p_i} A_i, \sum_{i=1}^{k} A_i^* Y^{-p_i} A_i), rd(X,Y)\right\}$$
$$\leq rd(X,Y).$$

Thereby (2.5) holds with $m = k + 1$, and thus (2.5) holds.

Using Lemma 2.3 and (2.5), we obtain

$$\begin{aligned} d(F(X),F(Y)) &= d(Q + \sum_{i=1}^{m} A_i^* X^{-p_i} A_i, Q + \sum_{i=1}^{m} A_i^* Y^{-p_i} A_i) \\ &\leq \frac{\alpha}{\beta+\alpha} d(\sum_{i=1}^{m} A_i^* X^{-p_i} A_i, \sum_{i=1}^{m} A_i^* Y^{-p_i} A_i) \\ &\leq \frac{\alpha}{\beta+\alpha} d(X,Y) \\ &\leq \frac{\lambda_1(\sum_{i=1}^{m} A_i^* \lambda_n^{-p_i}(Q) A_i)}{\lambda_n(Q) + \lambda_1(\sum_{i=1}^{m} A_i^* \lambda_n^{-p_i}(Q) A_i)} d(X,Y). \end{aligned} \tag{2.7}$$

Let

$$L = \frac{\lambda_1(\sum_{i=1}^{m} A_i^* \lambda_n^{-p_i}(Q) A_i)}{\lambda_n(Q) + \lambda_1(\sum_{i=1}^{m} A_i^* \lambda_n^{-p_i}(Q) A_i)},$$

and obviously $0 < L < 1$. Thus, F is a contraction map on Ω. Using the Banach fixed point theorem, we know that F has a unique fixed point X^* on Ω, which is the positive definite solution of Eq (1.1). Thereby, the sequences $\{X_k\}_{k\geq 0}$ generated by iteration (2.3) converge to X^*, and the error estimate is as follows:

$$d(X_k, X^*) \leq \frac{L^k}{1-L} d(X_1, X_0).$$

The proof is complete. □

3. The existence interval of positive definite solutions of Eq (1.1)

In this section, we provide three existence intervals for the positive definite solutions to Eq (1.1) that are more precise than those given by Lemma 2.5.

Theorem 3.1. *Let $\hat{X}$ be the positive definite solution of Eq* (1.1)*, and then*

$$\hat{X} \in \left[Q + \sum_{i=1}^{m} A_i^*(Q + \sum_{i=1}^{m} \frac{A_i^* A_i}{\lambda_n^{p_i}(Q)})^{-p_i} A_i,\ Q + \sum_{i=1}^{m} \frac{A_i^* A_i}{\lambda_n^{p_i}(Q)}\right].$$

Proof. Let $\hat{X}$ be the positive definite solution of Eq (1.1), and then we have

$$\hat{X} = Q + \sum_{i=1}^{m} A_i^* \hat{X}^{-p_i} A_i. \tag{3.1}$$

Using Lemma 2.5, we obtain

$$Q \leq \hat{X} \leq Q + \sum_{i=1}^{m} \frac{A_i^* A_i}{\lambda_n^{p_i}(Q)}, \tag{3.2}$$

and because $\lambda_n(Q)I \leq Q$, according to (3.2), this implies

$$(Q + \sum_{i=1}^{m} \frac{A_i^* A_i}{\lambda_n^{p_i}(Q)})^{-p_i} \leq \hat{X}^{-p_i} \leq \lambda_n^{-p_i}(Q)I. \tag{3.3}$$

Thus, the following inequality holds:

$$\sum_{i=1}^{m} A_i^*(Q + \sum_{i=1}^{m} \frac{A_i^* A_i}{\lambda_n^{p_i}(Q)})^{-p_i} A_i \leq \sum_{i=1}^{m} A_i^* \hat{X}^{-p_i} A_i \leq \sum_{i=1}^{m} \frac{A_i^* A_i}{\lambda_n^{p_i}(Q)}. \tag{3.4}$$

According to (3.1) and (3.4), we have

$$\sum_{i=1}^{m} A_i^*(Q + \sum_{i=1}^{m} \frac{A_i^* A_i}{\lambda_n^{p_i}(Q)})^{-p_i} A_i \leq \hat{X} - Q \leq \sum_{i=1}^{m} \frac{A_i^* A_i}{\lambda_n^{p_i}(Q)},$$

and therefore,

$$Q + \sum_{i=1}^{m} A_i^*(Q + \sum_{i=1}^{m} \frac{A_i^* A_i}{\lambda_n^{p_i}(Q)})^{-p_i} A_i \leq \hat{X} \leq Q + \sum_{i=1}^{m} \frac{A_i^* A_i}{\lambda_n^{p_i}(Q)}.$$

The proof is complete. □

Remark 3.1. *For any $n \times n$ positive semidefinite matrices A and B, if $A \leq B$, then $M^*AM \leq M^*BM$ holds, where M is an arbitrary $n \times n$ matrix. Therefore, from (3.3), we can deduce (3.4).*

Theorem 3.2. *Let $\hat{X}$ be the positive definite solution of Eq* (1.1)*, and then*

$$\hat{X} \in [\alpha I, \beta I],$$

where α and β are solutions of the following equations:

$$\begin{aligned} \alpha &= \lambda_n(Q) + \sum_{i=1}^{m} \sigma_n^2(A_i)\beta^{-p_i}, \\ \beta &= \lambda_1(Q) + \sum_{i=1}^{m} \sigma_1^2(A_i)\alpha^{-p_i}. \end{aligned} \tag{3.5}$$

Proof. First, we define two sequences $\{\alpha_s\}_{s\geq 0}$ and $\{\beta_s\}_{s\geq 0}$:

$$\begin{aligned} \alpha_0 &= \lambda_n(Q), \\ \beta_0 &= \lambda_1(Q) + \sum_{i=1}^{m} \sigma_1^2(A_i)\lambda_n^{-p_i}(Q), \\ \alpha_s &= \lambda_n(Q) + \sum_{i=1}^{m} \sigma_n^2(A_i)\beta_{s-1}^{-p_i}, \\ \beta_s &= \lambda_1(Q) + \sum_{i=1}^{m} \sigma_1^2(A_i)\alpha_s^{-p_i}, \ s = 1, 2, \ldots \end{aligned} \tag{3.6}$$

Next, we shall prove that the sequence $\{\alpha_s\}_{s\geq 0}$ is monotonically increasing, and sequence $\{\beta_s\}_{s\geq 0}$ is monotonically decreasing. According to the definition of two sequences, we know that $0 < \alpha_0 < \beta_0$, and

$$\alpha_1 = \lambda_n(Q) + \sum_{i=1}^{m} \sigma_n^2(A_i)\beta_0^{-p_i} = \alpha_0 + \sum_{i=1}^{m} \sigma_n^2(A_i)\beta_0^{-p_i} \geq \alpha_0,$$
$$\beta_1 = \lambda_1(Q) + \sum_{i=1}^{m} \sigma_1^2(A_i)\alpha_1^{-p_i} \leq \lambda_1(Q) + \sum_{i=1}^{m} \sigma_1^2(A_i)\alpha_0^{-p_i} = \beta_0.$$

Assuming $\alpha_{k-1} \leq \alpha_k$ and $\beta_k \leq \beta_{k-1}$ hold, we have

$$\alpha_{k+1} = \lambda_n(Q) + \sum_{i=1}^{m} \sigma_n^2(A_i)\beta_k^{-p_i} \geq \lambda_n(Q) + \sum_{i=1}^{m} \sigma_n^2(A_i)\beta_{k-1}^{-p_i} = \alpha_k,$$
$$\beta_{k+1} = \lambda_1(Q) + \sum_{i=1}^{m} \sigma_1^2(A_i)\alpha_{k+1}^{-p_i} \leq \lambda_1(Q) + \sum_{i=1}^{m} \sigma_1^2(A_i)\alpha_k^{-p_i} = \beta_k.$$

Thus, for all positive integer s, $\alpha_s \leq \alpha_{s+1}$ and $\beta_{s+1} \leq \beta_s$ hold.

Finally, we will prove that $\hat{X} \in [\alpha_s I, \beta_s I]$, $s = 0, 1, 2, \ldots$. According to (3.2) and $\alpha_0 I = \lambda_n(Q)I \leq Q$, we have $\alpha_0 I \leq \hat{X}$. Moreover,

$$\hat{X} \leq Q + \sum_{i=1}^{m} \frac{A_i^* A_i}{\lambda_n^{p_i}(Q)} \leq \left(\lambda_1(Q) + \sum_{i=1}^{m} \sigma_1^2(A_i)\lambda_n^{-p_i}(Q)\right) I = \beta_0 I,$$

and thus, $\hat{X} \in [\alpha_0 I, \beta_0 I]$.

Assuming $\hat{X} \in [\alpha_k I,\ \beta_k I]$ holds, we have

$$\hat{X} = Q + \sum_{i=1}^{m} A_i^* \hat{X}^{-p_i} A_i \geq Q + \sum_{i=1}^{m} A_i^* A_i \beta_k^{-p_i} \geq \left(\lambda_n(Q) + \sum_{i=1}^{m} \sigma_n^2(A_i)\beta_k^{-p_i}\right) I = \alpha_{k+1} I,$$

and

$$\hat{X} = Q + \sum_{i=1}^{m} A_i^* \hat{X}^{-p_i} A_i \leq Q + \sum_{i=1}^{m} A_i^* A_i \alpha_{k+1}^{-p_i} \leq \left(\lambda_1(Q) + \sum_{i=1}^{m} \sigma_1^2(A_i)\alpha_{k+1}^{-p_i}\right) I = \beta_{k+1} I.$$

Thus, $\hat{X} \in [\alpha_s I, \beta_s I]$, $s = 0, 1, 2, \ldots$, and the two sequences $\{\alpha_s\}_{s\geq 0}$, $\{\beta_s\}_{s\geq 0}$ are convergent. Let

$$\alpha = \lim_{s\to\infty} \alpha_s,\ \beta = \lim_{s\to\infty} \beta_s,$$

and therefore, $\hat{X} \in [\alpha I, \beta I]$, and the proof is complete. □

Remark 3.2. *Theorem 3.2 provides a series of existence intervals for the positive definite solutions to Eq* (1.1)*, each of which is more accurate than the previous one, in a manner similar to the theorem of nested intervals. Within these intervals, the parameters α and β can be calculated using an iterative scheme* (3.6).

Corollary 3.1. *Let $\hat{X}$ be the positive definite solution of Eq* (1.1)*, and then*

$$\hat{X} \in \left[Q + \sum_{i=1}^{m} \frac{A_i^* A_i}{\beta^{p_i}}, Q + \sum_{i=1}^{m} \frac{A_i^* A_i}{\alpha^{p_i}} \right],$$

where α and β are solutions of (3.5).

Proof. According to Theorem 3.2, we know that $\hat{X} \in [\alpha I, \beta I]$, and $\beta^{-p_i} I \leq \hat{X}^{-p_i} \leq \alpha^{-p_i} I$. Thus, the following inequality holds:

$$\beta^{-p_i} A_i^* A_i \leq A_i^* \hat{X}^{-p_i} A_i \leq \alpha^{-p_i} A_i^* A_i, \; i = 1, 2, \ldots, m.$$

Since $\hat{X}$ is the positive definite solution of (1.1), we have

$$\hat{X} - Q = \sum_{i=1}^{m} A_i^* \hat{X}^{-p_i} A_i,$$

and

$$\sum_{i=1}^{m} \frac{A_i^* A_i}{\beta^{p_i}} \leq \hat{X} - Q \leq \sum_{i=1}^{m} \frac{A_i^* A_i}{\alpha^{p_i}}.$$

Thus, we obtain

$$Q + \sum_{i=1}^{m} \frac{A_i^* A_i}{\beta^{p_i}} \leq \hat{X} \leq Q + \sum_{i=1}^{m} \frac{A_i^* A_i}{\alpha^{p_i}},$$

and the proof is complete. □

4. Numerical examples

In this section, we demonstrate the feasibility of the iterative algorithm (2.3) for solving Eq (1.1) with some numerical examples. All programs were implemented in MATLAB R2018a on a Windows 10 computer equipped with an Intel i5 Processor (1.6 GHz) and 8 GB of RAM. The residual's infinite norm is defined as follows:

$$\texttt{Res}(X_k) = \|X_k - \sum_{i=1}^{m} A_i^* X_k^{-p_i} A_i - Q\|_\infty,$$

and the stop condition is $\texttt{Res}(X_k) < 1 \times 10^{-8}$.

Example 4.1. We consider Eq (1.1) with $m = 2$, $p_1 = 0.2$, and $p_2 = 0.3$, along with coefficient matrices

$$A_1 = \begin{pmatrix} 0.1190 & 0.3404 & 0.7513 \\ 0.4984 & 0.5853 & 0.2551 \\ 0.9597 & 0.2238 & 0.5060 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0.6991 & 0.5472 & 0.2575 \\ 0.8909 & 0.1386 & 0.8407 \\ 0.9593 & 0.1493 & 0.2543 \end{pmatrix},$$

and Q is the identity matrix. By iterative algorithm (2.3), we take $X_0 = I_3$ and obtain the positive definite solution of Eq (1.1) after 12 steps of iteration,

$$X_{12} = \begin{pmatrix} 3.4973 & 0.8102 & 1.3614 \\ 0.8102 & 1.6359 & 0.5534 \\ 1.3614 & 0.5534 & 2.3172 \end{pmatrix},$$

$\mathtt{Res}(X_{12}) = 5.4431 \times 10^{-9}$.

Example 4.2. We consider Eq (1.1) with $m = 3$ and $p_i = 1$, along with coefficient matrices

$$A_1 = \begin{pmatrix} 1.3 & 0.02 & 0 & 0.02 \\ 0.04 & 1.6 & 0.02 & 0.03 \\ 0.04 & 0.02 & 1.5 & 0.02 \\ 0.05 & 0.02 & 0.01 & 2.5 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 3 & 2 & 1 & 4 \\ 2 & 1 & 1 & 2 \\ 1 & 2 & 3 & 1 \\ 6 & 3 & 5 & 4 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0.2 & 0.03 & 0.02 & 0.01 \\ 0.02 & 0.15 & 0.03 & 0.15 \\ 0.01 & 0.03 & 0.2 & 0.02 \\ 0.01 & 0.02 & 0.01 & 0.03 \end{pmatrix},$$

and Q is the identity matrix. By iterative algorithm (2.3), we take $X_0 = I_4$ and obtain the positive definite solution of Eq (1.1) after 75 steps of iteration,

$$X_{75} = \begin{pmatrix} 7.7116 & 2.6934 & 4.6181 & 3.7489 \\ 2.6934 & 3.6804 & 2.6958 & 1.6474 \\ 4.6181 & 2.6958 & 7.8231 & 2.0123 \\ 3.7489 & 1.6474 & 2.0123 & 6.0860 \end{pmatrix},$$

$\mathtt{Res}(X_{75}) = 9.0619 \times 10^{-9}$.

Example 4.3. We consider Eq (1.1) with $m = 2$, $p_1 = 0.3$, and $p_2 = 0.5$, along with coefficient matrices

$$A_1 = \begin{pmatrix} 0.1622 & 0.2630 & 0.2290 & 0.0782 & 0.8173 & 0.4314 \\ 0.7943 & 0.6541 & 0.9133 & 0.4427 & 0.8687 & 0.9106 \\ 0.3112 & 0.6892 & 0.1524 & 0.1067 & 0.0844 & 0.1818 \\ 0.5285 & 0.7482 & 0.8258 & 0.9619 & 0.3998 & 0.2638 \\ 0.1656 & 0.4505 & 0.5383 & 0.0046 & 0.2599 & 0.1455 \\ 0.6020 & 0.0838 & 0.9961 & 0.7749 & 0.8001 & 0.1361 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0.8693 & 0.3510 & 0.1839 & 0.4909 & 0.7803 & 0.9421 \\ 0.5797 & 0.5132 & 0.2400 & 0.4893 & 0.3897 & 0.9561 \\ 0.5499 & 0.4018 & 0.4173 & 0.3377 & 0.2417 & 0.5752 \\ 0.1450 & 0.0760 & 0.0497 & 0.9001 & 0.4039 & 0.0598 \\ 0.8530 & 0.2399 & 0.9027 & 0.3692 & 0.0965 & 0.2348 \\ 0.6221 & 0.1233 & 0.9448 & 0.1112 & 0.1320 & 0.3532 \end{pmatrix},$$

and Q is the identity matrix. By iterative algorithm (2.3), we take $X_0 = I_6$ and obtain the positive definite solution of Eq (1.1) after 17 steps of iteration,

$$X_{17} = \begin{pmatrix} 2.8756 & 1.1522 & 1.8785 & 1.2880 & 1.3619 & 1.4901 \\ 1.1522 & 2.3073 & 1.1222 & 0.8897 & 0.8918 & 1.1649 \\ 1.8785 & 1.1222 & 3.7261 & 1.4100 & 1.3077 & 1.0731 \\ 1.2880 & 0.8897 & 1.4100 & 2.9204 & 1.2881 & 0.8733 \\ 1.3619 & 0.8918 & 1.3077 & 1.2881 & 2.9089 & 1.4659 \\ 1.4901 & 1.1649 & 1.0731 & 0.8733 & 1.4659 & 2.8979 \end{pmatrix},$$

$\mathtt{Res}(X_{17}) = 2.9125 \times 10^{-9}$.

Example 4.4. Eq (1.1) for $m = p_i = 1$ arises in the computation of the solution for a large sparse linear system $M\mathbf{x} = \mathbf{f}$, where M is a real symmetric matrix of block tridiagonal form $M = \begin{pmatrix} I_n & A \\ A^* & I_n \end{pmatrix}$, and

$$A = \begin{pmatrix} -4 & 1 & & & \\ 1 & -4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -4 \end{pmatrix}_{n\times n},$$

which arises from a finite difference approximation to Poisson's equation on a rectangle, with specified boundary values [8, pp. 9-10], where Q is the identity matrix, taking $n = 100$. By iterative algorithm (2.3), we take $X_0 = I_{100}$ and obtain the positive definite solution of Eq (1.1), after 129 steps of iteration, $\mathtt{Res}(X_{129}) = 8.8759 \times 10^{-9}$. Based on the characteristics of the iterative algorithm (2.3), the computational complexity of the iterative algorithm (2.3) is mainly $O(n^3)$.

5. Conclusions

We have conclusively demonstrated, through the utilization of the Thompson metric, that Eq (1.1) possesses a unique positive definite solution without any additional assumptions. Furthermore, we have designed an iterative method to compute this positive definite solution to Eq (1.1) and have provided three refined existence intervals for the positive definite solution of this equation.

Author contributions

Changzhou Li: Conceptualization, Writing–original draft, Funding acquisition; Chao Yuan: Software, Methodology; Shiliang Chen: Validation. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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