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Research article

Rumor model on social networks contemplating self-awareness and saturated transmission rate

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Abstract: The propagation of rumors indisputably inflicts profound negative impacts on society and individuals. This article introduces a new unaware ignorants-aware ignorants-spreaders-recovereds ($2ISR$) rumor spreading model that combines individual vigilance self-awareness with nonlinear spreading rate. Initially, the positivity of the system solutions and the existence of its positive invariant set are rigorously proved, and the rumor propagation threshold is solved using the next-generation matrix method. Next, a comprehensive analysis is conducted on the existence of equilibrium points of the system and the occurrence of backward bifurcation. Afterward, the stability of the system is validated at both the rumor-free equilibrium and the rumor equilibrium, employing the Jacobian matrix approach as well as the Lyapunov stability theory. To enhance the efficacy of rumor propagation management, a targeted optimal control strategy is formulated, drawing upon the Pontryagin's Maximum principle as a guiding framework. Finally, through sensitivity analyses, numerical simulations, and tests of real cases, we verify the reliability of the theoretical results and further consolidate the solid foundation of the above theoretical arguments.

Keywords: rumor propagation; backward bifurcation; stability; optimal control
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1. Introduction

Rumors are fabricated statements without the existence of facts. Traditionally, rumors are spread orally or interpersonally. In the era of "self-media", where information dissemination is becoming faster and more convenient, rumors have caught the fast train of the internet, expanding unprecedentedly in terms of speed, breadth, and influence. For example, on February 16th, 2024, a self-media account on the internet circulated a video titled "Picking up the Lost Workbook of Primary

School Student Qin Lang in Paris, France." Upon its release, the video garnered immense viewership, generating a significant amount of traffic. Following a thorough investigation and verification conducted by relevant personnel, it was revealed that the video was a malicious fabrication, intended to spread rumors or false information through the internet. Such rumors occupy public resources, disrupt social order, and have a negative impact on society. Therefore, gaining a comprehensive understanding of the mechanisms behind rumor generation and dissemination is crucial for us to better comprehend and respond to such phenomena. By enhancing our information literacy and improving our ability to discern rumors, we can effectively mitigate the harmful effects of rumors.

The original rumor spreading model was developed from the infectious disease model. The famous Daley-Kendall (DK) model was first presented by Daley and Kendall in 1965 [1, 2]. It was the first model to successfully combine rumor spreading and mathematics. Thereafter, based on the DK model, Maki and Thomson further extended the model and proposed the Maki-Thomson (MT) model in [3]. It is noteworthy that as research into the dissemination of rumors intensifies, scholars have further discovered that the process is significantly influenced by the intricate topology of social networks. Watts and Strogatz proposed the small-world network model in 1998 [4] and analyzed the information propagation characteristics on it, which can be regarded as the important milestone in the examination of rumor spreading in small-world networks. As research progressed, a growing number of scholars turned their attention to the intricate dynamics of rumor spreading across complex networks [5–11], and various influencing factors were considered during the spread of rumors [12–17].

The impact of self-awareness on rumor dissemination is primarily manifested in an individual's capacity to discern the authenticity of information and their resistant attitude toward its propagation. When a person possesses a heightened perception, comprehension, and discernment toward specific information, they exhibit a stronger critical faculty toward related content and are less prone to participating in the dissemination of rumors. Therefore, the rumor spreading ability of the aware ignorants can be represented by the nonlinear function $g(S)I$, where $g(S)$ tends to saturation level when S becomes large, i.e.,

$$g(S) = \frac{\beta S}{1 + \delta S}$$

where βS measures the rumor's ability to spread, $\frac{1}{1+\delta S}$ captures the behavioral shifts triggered by an increase in the number of alert and informed ignorants, and δ is the half-saturation constant.

The control of rumors can be better weakened by studying the adverse effects of rumors on the society. Applying optimal control to the rumor spreading model means that the number of rumor spreading individuals can be controlled to the maximum extent at the minimum cost. Jain et al. used Pontryagin's maximum principle and counter news attack mechanism and designed an optimal rumor control system to minimize the density of rumor adopters and control cost [18]. Huo et al. verified that the optimal control strategy can minimize the number of rumor spreaders and effectively control the rumor spreading [19]. It is known that excessive control intensity frequently results in unwarranted resource expenditure, whereas inadequate control intensity fails to effectively contain the dissemination of rumors. Consequently, the implementation of optimized control strategies holds paramount importance in mitigating the rumors spreading.

Based on the above research, this article discusses the dynamic behaviors of rumor spreading in different situations derived from whether the ignorants has the ability to identify. Moreover, we delved into the model's dynamic characteristics and devised an optimal control strategy to mitigate the spread of rumors. Hence, the main contributions in this article are as follows:

(1) A new model of rumor propagation with saturation incidence is proposed, predicated on specific hypotheses, which takes into account the self-awareness of the rumor ignorants and classifies them into two categories according to their degree, thus more closely resembling the phenomenon of rumor spreading in real life.

(2) By calculating the threshold $\mathfrak{R}_0$, the existence of rumor-free equilibrium and rumor equilibrium in the model are analyzed, and the existence of backward bifurcation is discussed. Then, the stability of the equilibrium points is explored by constructing a Lyapunov function.

(3) The optimal control strategy is established for the model using Pontryagin's maximum principle. It is able to find strategies to decrease the impact and spread of rumors by adopting methods such as education or online monitoring of rumor spreaders. The best control effect is achieved with minimum cost.

The rest of the paper is organized as follows. In Section 2, a new model of rumor spreading with saturated transmission rate is proposed. In Section 3, the existence and stability of equilibrium points of the model are considered. An optimal control to reduce rumor spreading is proposed in Section 4. In Section 5, numerical simulations validate the accuracy of the theoretical results. Additionally, Section 6 provides a real-world instance to demonstrate the practical applicability of the model. The paper concludes in Section 7.

2. The basic formulation of the model

In this section, we introduce a rumor propagation model that takes into account users' self-awareness. This self-awareness enables users to be alert to potential risks and hazards associated with rumors, thus maintaining a cautious attitude toward information. Based on the user's status, we categorize them into four different types: Unaware ignorants $I_u(t)$: they refer to individuals who are yet unaware of the rumor, a cohort of unknowns who potentially possess insufficient pertinent knowledge and judgment capabilities. Aware ignorants $I_a(t)$: in the face of rumor spreading, although they will show a more rational and prudent attitude, and be able to think and judge the content of the rumor independently, they actually do not know the specific content and are still in a state of ignorance. Spreaders $S(t)$: these users are individuals who actively spread rumors to others. They further amplify the impact of rumors in social networks. Recovereds $R(t)$: it refers to the process of rumor spreading. More users are initially affected by the rumor and are no longer participating in it.

The rumor's predicted social network spread pattern is based on the following assumptions:

(1) In the social network, the number of users added per unit time is to be W_1 and W_2, respectively. In addition, the eviction rate μ represents the probability of each type of user leaving the system per unit time.

(2) Since I_u gradually becomes alert and aware through improving self-knowledge and actively learning relevant knowledge, they will transform into aware individuals I_a with probability α.

(3) When ignorants come into contact with rumor spreaders, they may be influenced by the rumors and become rumor spreaders. We assume that the probability of $I_u(t)$ and $S(t)$ transforming into $S(t)$ after contact is β_1. The probability of $I_a(t)$ and $S(t)$ becoming $S(t)$ after contact is the saturated propagation rate $\frac{\beta_2 I_a S}{1+\delta S}$.

(4) Those ignorant people who are already alert are able to recognize and resist the misinformation

of rumors directly due to their vigilant and prudent attitude toward information, thus becoming the recovereds with a certain probability of η. Rumor spreaders with probability γ stop spreading rumors and become the recovereds.

Using the above assumptions, W_1, W_2, α, μ, η, γ, β_1, β_2 are all positive constants. The process of $2ISR$ rumor propagation can be characterized as in Figure 1. The model is described by the following system of differential equations:

$$\begin{cases} \dfrac{\mathrm{d}I_u(t)}{\mathrm{d}t} = W_1 - \alpha I_u(t) - \beta_1 I_u(t)S(t) - \mu I_u(t), \\ \dfrac{\mathrm{d}I_a(t)}{\mathrm{d}t} = W_2 + \alpha I_u(t) - \dfrac{\beta_2 I_a(t)S(t)}{1+\delta S(t)} - \mu I_a(t) - \eta I_a(t), \\ \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \beta_1 I_u(t)S(t) + \dfrac{\beta_2 I_a(t)S(t)}{1+\delta S(t)} - \gamma S(t) - \mu S(t), \\ \dfrac{\mathrm{d}R(t)}{\mathrm{d}t} = \eta I_a(t) + \gamma S(t) - \mu R(t), \end{cases} \tag{2.1}$$

with the initial conditions

$$I_u(0) > 0,\ I_a(0) > 0,\ S(0) > 0,\ R(0) > 0.$$

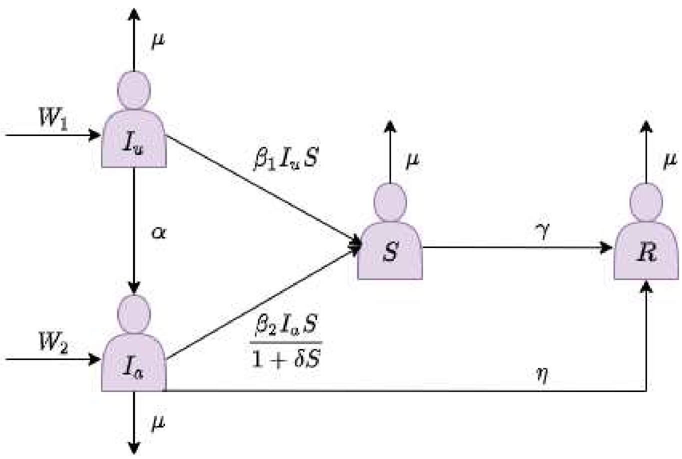


Figure 1. Description diagram of the rumor dynamics.

Based on [20], the following two lemmas will be proved.

Lemma 2.1. *Assume* $(I_u(t), I_a(t), S(t), R(t))$ *is the solution of system* (2.1) *that satisfies the initial conditions, then* $I_u(t), I_a(t), S(t), R(t)$ *are positive for all* $t > 0$.

Proof. First, we prove that $I_u(t)$ and $I_a(t)$ are positive for all $t > 0$. We know that $I_u(0) > 0$ and $I_a(0) > 0$ from the initial conditions of system (2.1). According to the continuity of the solution that satisfies the initial conditions, it is assumed that there is a normal number t_1, which makes $I_u(t) > 0$ and $I_a(t) > 0$ for $t \in [0, t_1)$, and suppose that $I_u(t) = 0$ and $I_a(t) = 0$ when $t = t_1$. $\frac{\mathrm{d}I_u(t_1)}{\mathrm{d}t} = W_1$ and $\frac{\mathrm{d}I_a(t_1)}{\mathrm{d}t} = W_2$ can be obtained from the first equation of system (2.1).

However, $\forall t \in [0, t_1)$, $I_u(t) > I_u(t_1) = 0$ and $I_a(t) > I_a(t_1) = 0$ which cause a contradiction with $\frac{dI_u(t_1)}{dt} > 0$ and $\frac{dI_a(t_1)}{dt} > 0$. Thus, there is a positive number t_1, which makes $I_u(t_1) > 0$ and $I_u(t_1) > 0$.

By the same way, we can prove that the existence of positive $0 < t_1 < t_2 < t_3$ makes $S(t_2) > 0$ and $R(t_3) > 0$, respectively. □

Lemma 2.2. *The feasible region* Ω *is defined as:*

$$\Omega = \left\{(I_u, I_a, S, R) \in R_4^+ : I_u \leq I_u^{\triangle}, I_a \leq I_a^{\triangle}, I_u + I_a + S + R \leq \frac{W_1 + W_2}{\mu}\right\}$$

which is a positive invariant set of system (2.1).

Proof. Let $N(t) = I_u(t) + I_a(t) + S(t) + R(t)$. It is easy to know that

$$\frac{dN(t)}{dt} = W_1 + W_2 - \mu N(t),$$

and then $\limsup_{t\to\infty} N(t) = \frac{W_1+W_2}{\mu}$. Additionally,

$$\frac{dI_u(t)}{dt} = W_1 - \alpha I_u(t) - \beta_1 I_u(t)S(t) - \mu I_u(t) \leq W_1 - (\alpha + \mu)I_u(t),$$
$$\frac{dI_a(t)}{dt} = W_2 + \alpha I_u(t) - \frac{\beta_2 I_a(t)S(t)}{1 + \delta S(t)} - \mu I_a(t) - \eta I_a(t) \leq W_2 + \alpha I_u(t) - (\mu + \eta)I_a(t).$$

It follows that

$$\limsup_{t\to\infty} I_u(t) = \frac{W_1}{\alpha + \mu} := I_u^{\triangle}, \quad \limsup_{t\to\infty} I_a(t) = \frac{W_2}{\mu + \eta} + \frac{\alpha W_1}{(\alpha + \mu)(\mu + \eta)} := I_a^{\triangle}.$$

Therefore, the positive invariant set of system (2.1) is $\Omega = \left\{(I_u, I_a, S, R) \in R_4^+ : I_u \leq I_u^{\triangle}, I_a \leq I_a^{\triangle}, I_u + I_a + S + R \leq \frac{W_1+W_2}{\mu}\right\}$. □

Due to $R(t)$ being independent of the other equations of the system (2.1), we simplify the equations for convenience as follows:

$$\begin{cases} \frac{dI_u(t)}{dt} = W_1 - \alpha I_u(t) - \beta_1 I_u(t)S(t) - \mu I_u(t), \\ \frac{dI_a(t)}{dt} = W_2 + \alpha I_u(t) - \frac{\beta_2 I_a(t)S(t)}{1 + \delta S(t)} - \mu I_a(t) - \eta I_a(t), \\ \frac{dS(t)}{dt} = \beta_1 I_u(t)S(t) + \frac{\beta_2 I_a(t)S(t)}{1 + \delta S(t)} - \gamma S(t) - \mu S(t). \end{cases} \tag{2.2}$$

Remark 1. *A plethora of scholarly works has delved into the dynamics of rumor propagation. However, these studies do not take into account the self-awareness of rumor ignorants. In the context of rumor dissemination, self-awareness of rumors refers to an individual's ability to identify and judge rumors and avoid being misled by them when confronted with information. Consequently, we posit a new rumor propagation model* (2.1)*, which categorizes the ignorants of rumors based on varying degrees of self-awareness.*

3. Stability analysis of system

In this section, we will discuss the stability of the rumor-free equilibrium and rumor equilibrium of the system (2.2). The specific algorithm is as follows.

The rumor-free equilibrium $E_0 = (I_u, I_a, 0)$, where $I_u = \frac{W_1}{\alpha+\mu}$ and $I_a = \frac{W_2}{\mu+\eta} + \frac{\alpha W_1}{(\alpha+\mu)(\mu+\eta)}$, is easy to compute.

Next, the basic regeneration number is calculated by using the next generation matrix method [21]. By writing the third equation in system (2.2) as $\frac{dS(t)}{dt} = \mathcal{F} - \mathcal{V}$ in which $\mathcal{F} = \beta_1 I_u(t)S(t) + \frac{\beta_2 I_a(t)S(t)}{1+\delta S(t)}$ and $\mathcal{V} = \gamma S(t) + \mu S(t)$, let $F = \frac{\partial \mathcal{F}}{\partial S}\Big|_{E_0} = \beta_1 I_u + \beta_2 I_a$ and $V = \frac{\partial \mathcal{V}}{\partial S}\Big|_{E_0} = (\mu + \gamma)$. The spectral radius of the matrix FV^{-1} is the basic reproduction number $\mathfrak{R}_0$ of the system, which can be obtained as follows for the system (2.2):

$$\begin{aligned}\mathfrak{R}_0 &= \rho(FV^{-1}) \\ &= \frac{\beta_1 I_u + \beta_2 I_a}{\mu + \gamma} \\ &= \frac{\alpha\beta_2(W_1 + W_2) + \beta_1 W_1(\mu + \eta) + \mu\beta_2 W_2}{(\mu + \gamma)(\alpha + \mu)(\mu + \eta)}.\end{aligned}$$

Theorem 3.1. *If* $\mathfrak{R}_0 < 1$, *the rumor-free equilibrium* E_0 *of system* (2.2) *is locally asymptotically stable.*

Proof. The Jacobian matrix of system (2.2) at the rumor-free equilibrium E_0 is

$$\mathcal{J}(E_0) = \begin{pmatrix} -\alpha - \mu & 0 & -\beta_1 I_u \\ \alpha & -\mu - \eta & -\beta_2 I_a \\ 0 & 0 & \beta_1 I_u + \beta_2 I_a - \gamma - \mu \end{pmatrix}.$$

By simple calculation, it has

$$(\lambda + \mu + \eta)(\lambda + \alpha + \mu)[\lambda - (\beta_1 I_u + \beta_2 I_a - (\mu + \gamma))] = 0. \tag{3.1}$$

This gives the characteristic roots of (3.1) as

$$\lambda_1 = -\mu - \eta, \;\; \lambda_2 = -\alpha - \mu, \;\; \lambda_3 = \beta_1 I_u + \beta_2 I_a - (\mu + \gamma) = (\mu + \gamma)(\mathfrak{R}_0 - 1).$$

Since $\mathfrak{R}_0 < 1$, λ_1, λ_2, and λ_3 are negative, one can conclude that E_0 is locally asymptotically stable. □

Theorem 3.2. *The rumor-free equilibrium* E_0 *is globally asymptotically stable if* $\mathfrak{R}_0 < 1$.

Proof. To construct the subsequent Lyapunov function

$$V(t) = S(t).$$

By taking the derivative of V, it can be obtained that

$$\frac{\mathrm{dV(t)}}{\mathrm{d}t} = \beta_1 I_u(t)S(t) + \frac{\beta_2 I_a(t)S(t)}{1 + \delta S(t)} - \gamma S(t) - \mu S(t)$$

$$
\begin{aligned}
&\le (\beta_1 I_u^{\triangle} + \beta_2 I_a^{\triangle})S(t) - (\mu+\gamma)S(t) \\
&= (\mu+\gamma)(\Re_0 - 1)S(t).
\end{aligned}
$$

If $\Re_0 < 1$, the $\frac{\mathrm{dV(t)}}{\mathrm{d}t} \le 0$, and $\frac{\mathrm{dV(t)}}{\mathrm{d}t} = 0$ if and only if $S(t) = 0$. In accordance with LaSalle's invariance principle, E_0 is globally asymptotically stable. □

When system (2.2) gets rumor equilibrium $E^* = \{I_u^*, I_a^*, S^*\}$, the system (2.2) satisfies the following equation:

$$
\begin{cases}
W_1 - \alpha I_u^* - \beta_1 I_u^* S^* - \mu I_u^* = 0, \\
W_2 + \alpha I_u^* - \dfrac{\beta_2 I_a^* S^*}{1+\delta S^*} - \mu I_a^* - \eta I_a^* = 0, \\
\beta_1 I_u^* S^* + \dfrac{\beta_2 I_a^* S^*}{1+\delta S^*} - \gamma S^* - \mu S^* = 0.
\end{cases}
\tag{3.2}
$$

We have

$$
\begin{aligned}
I_u^* &= \frac{W_1}{\alpha+\beta_1 S^* + \mu}, \\
I_a^* &= \frac{1+\delta S^*}{(\mu+\eta)(1+\delta S^*) + \beta_2 S^*} \cdot \frac{\alpha(W_1+W_2) + W_2(\beta_1 S^* + \mu)}{\alpha + \beta_1 S^* + \mu}.
\end{aligned}
$$

The result is obtained directly through computation

$$
A(S^*)^2 + BS^* + C = 0, \tag{3.3}
$$

where

$$
\begin{aligned}
A &= \beta_1(\mu+\gamma)((\mu+\eta)\delta + \beta_2), \\
B &= (\mu+\gamma)(\alpha+\mu)((\mu+\eta)\delta+\beta_2) + (\mu+\eta)(\mu+\gamma)\beta_1 \\
&\quad - \beta_1\beta_2(W_1+W_2) - (\mu+\eta)\beta_1 W_1 \delta, \\
C &= (\mu+\eta)(\alpha+\mu)(\mu+\gamma) - \alpha\beta_2(W_1+W_2) - \beta_1 W_1(\mu+\eta) - \mu\beta_2 W_2 \\
&= (\mu+\eta)(\alpha+\mu)(\mu+\gamma)(1-\Re_0).
\end{aligned}
$$

Next, we will solve the positive solution S^* of Eq (3.3). Equation(3.3) has a unique positive root when $C < 0$ ($\Re_0 > 1$); when $C = 0$ ($\Re_0 = 1$), Eq (3.3) has a positive root if and only if $B < 0$; when $C > 0$ ($\Re_0 < 1$), Eq (3.3) has two equal positive roots $S^* = -\frac{B}{2A}$ if $B < 0$ and $\Delta = 0$, then Eq (3.3) has two positive roots if $\Delta > 0$ and $B < 0$.

Furthermore, in accordance with the third, from $\Re_0 < 1$ and $\frac{W_1+W_2}{\mu} = 1$, it follows that $(\mu+\gamma) > \frac{\alpha\beta_2\mu + \beta_1 W_1(\mu+\eta) + \mu\beta_2 W_2}{(\alpha+\mu)(\mu+\eta)}$. Substituting the above equation into B,

$$
\begin{aligned}
B =& (\mu+\gamma)(\alpha+\mu)((\mu+\eta)\delta+\beta_2) + (\mu+\eta)(\mu+\gamma)\beta_1 \\
&- \beta_1\beta_2(W_1+W_2) - (\mu+\eta)\beta_1 W_1\delta, \\
>& [\alpha\beta_2\mu + \beta_1 W_1(\mu+\eta) + \mu\beta_2 W_2]\delta - \beta_1\beta_2\mu - (\mu+\eta)\beta_1 W_1\delta \\
&+ \frac{[\alpha\beta_2\mu + \beta_1 W_1(\mu+\eta) + \mu\beta_2 W_2]\beta_2}{(\mu+\eta)} + \frac{[\alpha\beta_2\mu + \beta_1 W_1(\mu+\eta) + \mu\beta_2 W_2]\beta_1}{(\alpha+\mu)}
\end{aligned}
$$

$$
\begin{aligned}
&>(\alpha\beta_2\mu + \mu\beta_2 W_2)\delta + \beta_1\beta_2 W_1 - \beta_1\beta_2\mu \\
&\quad + \frac{\mu\beta_2^2(\alpha + W_2)}{(\mu+\eta)} + \frac{\beta_1^2 W_1(\mu+\eta) + \mu\beta_1\beta_2(\alpha + W_2)}{(\alpha+\mu)} \\
&>(\alpha\beta_2\mu + \mu\beta_2 W_2)\delta + \beta_1\beta_2 W_1 + \frac{\mu\beta_2^2(\alpha + W_2)}{(\mu+\eta)} + \frac{\beta_1^2 W_1(\mu+\eta)}{(\alpha+\mu)} - \frac{\mu\beta_1\beta_2 W_1}{(\alpha+\mu)} \\
&>(\alpha\beta_2\mu + \mu\beta_2 W_2)\delta + \frac{\mu\beta_2^2(\alpha + W_2)}{(\mu+\eta)} + \frac{\beta_1^2 W_1(\mu+\eta)}{(\alpha+\mu)} > 0.
\end{aligned}
$$

It follows that when $C > 0$, it can be inferred that $B > 0$. Hence, it can be concluded that system (2.2) does not have backward bifurcation.

Denote

$$
\hat{\mathfrak{R}}_0 = 1 - \frac{(\mathcal{P} - Q)^2}{4\beta_1(\mu+\gamma)((\mu+\eta)\delta + \beta_2)[(\mu+\eta)(\alpha+\mu)(\mu+\gamma)(1-\mathfrak{R}_0)]},
$$

where $\mathcal{P} = (\mu+\gamma)(\alpha+\mu)((\mu+\eta)\delta+\beta_2) + (\mu+\eta)(\mu+\gamma)$ and $Q = \beta_1\beta_2(W_1+W_2) + (\mu+\eta)\beta_1 W_1\delta$.

Concerning the pivotal basic reproduction number previously defined, we derive pertinent conclusions regarding the positive equilibrium point in the subsequent theorem.

Theorem 3.3. *For the system* (2.2)*, upon defining* $\mathfrak{R}_0$ *and* $\hat{\mathfrak{R}}_0$ *as aforementioned, we arrive at the following:*

(1). If $\mathfrak{R}_0 > 1$*, there exists a unique rumor equilibrium point denoted as* E^**.*

(2). If $\mathfrak{R}_0 = 1$ *and* $B < 0$*, there is a unique rumor equilibrium point.*

(3). If $\mathfrak{R}_0 \leq 1$ *and* $B \geq 0$*, the system* (2.2) *does not have a rumor equilibrium point.*

(4). If $\mathfrak{R}_0 = \hat{\mathfrak{R}}_0$ *and* $B < 0$*, system* (2.2) *possesses a singular and unique equilibrium point pertaining to rumor spreading.*

(5). If $\mathfrak{R}_0 < \hat{\mathfrak{R}}_0$ *and* $B < 0$*, there is no rumor equilibrium point.*

Proof. Given that $\mathfrak{R}_0 < 1$, system (2.2) admits positive equilibrium points only when the condition $B < 0$ is satisfied. Additionally, letting Δ represent the discriminant of Eq (3.3), the critical value is ascertained by the condition $\Delta = 0$. By solving $\Delta = 0$ in terms of $\mathfrak{R}_0$, we arrive at the previously delineated critical value of $\hat{\mathfrak{R}}_0$.

Moreover, we get the following relationships:

$$
\Delta < 0 \Longleftrightarrow \mathfrak{R}_0 < \hat{\mathfrak{R}}_0,\ \Delta = 0 \Longleftrightarrow \mathfrak{R}_0 = \hat{\mathfrak{R}}_0,\ \Delta > 0 \Longleftrightarrow \mathfrak{R}_0 > \hat{\mathfrak{R}}_0.
$$

When $B < 0$, system (3.3) may possess 0 or 1 positive equilibrium point, depending on whether $\mathfrak{R}_0 < \hat{\mathfrak{R}}_0$ and $\mathfrak{R}_0 = \hat{\mathfrak{R}}_0$, respectively.

In summary, the Theorem 3.4 yields a definitive conclusion regarding the number of positive equilibrium points. □

Next, we assume that

$$
\Psi = \frac{\beta_2 I_a^* S^*}{1 + \delta S^*}
$$

represents the incidence of the model.

Furthermore, we set $\frac{\partial\Psi}{\partial I_a}$ and $\frac{\partial\Psi}{\partial S}$ to be represented by Ψ_{I_a} and Ψ_S, respectively, i.e., $\Psi_{I_a} = \frac{\beta_2 S^*}{1+\delta S^*}$ and $\Psi_S = \frac{\beta_2 I_a^*}{(1+\delta S^*)^2}$; meanwhile, $\Theta_1 = (\alpha+\mu)$, $\Theta_2 = (\mu+\eta)$, and $\Theta_3 = (\mu+\gamma)$.

Theorem 3.4. *If $a_1a_2 - a_3 > 0$, then the rumor equilibrium E^* of system* (2.2) *is locally asymptotically stable, where a_1, a_2, and a_3 are chosen as follows.*

Proof. The Jacobian matrix of system (2.2) at $E^* = (I_u^*, I_a^*, S^*)$ is as follows:

$$\mathcal{J}(E^*) = \begin{pmatrix} -\beta_1 S^* - \Theta_1 & 0 & -\beta_1 I_u^* \\ \alpha & -\Psi_{I_a} - \Theta_2 & -\Psi_S \\ \beta_1 S^* & \Psi_{I_a} & \beta_1 I_u^* + \Psi_S - \Theta_3 \end{pmatrix}.$$

The characteristic equation for matrix $\mathcal{J}(E^*)$ has the following form:

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \tag{3.4}$$

where $a_1 = \Theta_1 + \Theta_2 + \Theta_3 + \beta_1 S^* + \Psi_{I_a} - \beta_1 I_u^* - \Psi_S$, $a_2 = (\Theta_2 + \Theta_3)(\beta_1 S^* + \Theta_1 + \Psi_{I_a}) + \Theta_3(\Theta_2 + \Psi_{I_a}) - (\Theta_1 + \Theta_2)(\beta_1 I_u^* + \Psi_S) - \beta_1 I_u^* \Psi_{I_a} - \beta_1 S^* \Psi_S$, $a_3 = \Theta_3(\Theta_1 + \beta_1 S^*)(\Theta_2 + \Psi_{I_a}) + \alpha\beta_1 I_u^* \Psi_{I_a} - \Theta_1\beta_1 I_u^*(\Theta_2 + \Psi_{I_a}) - \Theta_2\Psi_S(\Theta_1 + \beta_1 S^*)$.

Using the Routh-Hurwitz stability criterion [22], if $a_1, a_2, a_3 > 0$, and $a_1a_2 - a_3 > 0$, then the real parts of all the eigenvalues of Eq (3.4) are negative, which proves that the rumor equilibrium E^* is locally asymptotically stable. □

4. Optimal control model

When rumors continue to spread and cause adverse effects, it is crucial to take targeted measures to contain their spread. To this end, this section details a real-time optimal control approach that aims to achieve effective suppression of rumor spreading within a desired time frame at minimal cost. This control mechanism not only takes into account the dynamics of rumor spreading, but also incorporates a cost control strategy, which provides a powerful tool for practically tackling the rumor spreading problem. The model after adding the control mechanism is given as follows:

$$\begin{cases} \dfrac{\mathrm{d}I_u(t)}{\mathrm{d}t} = W_1 - \alpha I_u(t) - \beta_1 I_u(t)S(t) - \mu I_u(t), \\ \dfrac{\mathrm{d}I_a(t)}{\mathrm{d}t} = W_2 + \alpha I_u(t) - \dfrac{\beta_2 I_a(t)S(t)}{1 + \delta S(t)} - \mu I_a(t) - \eta I_a(t), \\ \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \beta_1 I_u(t)S(t) + \dfrac{\beta_2 I_a(t)S(t)}{1 + \delta S(t)} - \gamma S(t) - \mu S(t) - r(t)S(t), \\ \dfrac{\mathrm{d}R(t)}{\mathrm{d}t} = \eta I_a(t) + \gamma S(t) + r(t)S(t) - \mu R(t), \end{cases} \tag{4.1}$$

in which $r(t)$ represents the intensity of control measures taken to reduce the spread of rumors through education or network supervision. To evaluate the associated costs, we introduce ϕ and κ to signify the mean expenditure required to control and educate S individuals, respectively. Considering that the required time frame is $[0, T]$, the subsequent phase involves commencing the process of formulating the objective function for the optimal control strategy.

$$J(r(t)) = \int_0^T [\phi S(t) + \kappa r^2(t)]\mathrm{d}t. \tag{4.2}$$

The feasible region of $r(t)$ is $U = \{(r(t))|0 \leq r(t) \leq r^{\max}, t \in (0,T]\}$, where $r^{\max}$ denotes the upper bound of $r(t)$. Optimal control r^* satisfy $J(r^*) = \min\{J(r(t)) : (r(t)) \in U\}$.

Next, we construct the Lagrangian function

$$L(S(t), r(t)) = \phi S(t) + \kappa r^2(t).$$

The Hamiltonian function is delineated as

$$\begin{aligned}
&H(I_u(t), I_a(t), S(t), R(t), r(t), \lambda_i(t)) = L(S(t), r(t)) \\
&+\lambda_1(t)\left[W_1 - \alpha I_u(t) - \beta_1 I_u(t)S(t) - \mu I_u(t)\right] \\
&+\lambda_2(t)\left[W_2 + \alpha I_u(t) - \frac{\beta_2 I_a(t)S(t)}{1+\delta S(t)} - \mu I_a(t) - \eta I_a(t)\right] \\
&+\lambda_3(t)\left[\beta_1 I_u(t)S(t) + \frac{\beta_2 I_a(t)S(t)}{1+\delta S(t)} - \gamma S(t) - \mu S(t) - r(t)S(t)\right] \\
&+\lambda_4(t)\left[\eta I_a(t) + \gamma S(t) + r(t)S(t) - \mu R(t)\right],
\end{aligned} \tag{4.3}$$

where $i = 1, 2, 3, 4$. Leveraging Pontryagin's principle of optimality [23], the subsequent theorem is formulated.

Theorem 4.1. *Assuming I_u^ϵ, I_a^ϵ, S^ϵ, and R^ϵ represent the optimal states achieved under the optimal control r^*, there exist adjoint variables $\lambda_i(t)$, $i = 1, \cdots, 4$, that satisfy the following conditions:*

$$\begin{cases}
\dfrac{\mathrm{d}\lambda_1(t)}{\mathrm{d}t} = \lambda_1(t)(\alpha + \beta_1 S^\epsilon + \mu) - \lambda_2(t)\alpha - \lambda_3(t)\beta_1 S^\epsilon, \\
\dfrac{\mathrm{d}\lambda_2(t)}{\mathrm{d}t} = \lambda_2(t)(\dfrac{\beta_2 S^\epsilon}{1+\delta S^\epsilon} + \mu + \eta) - \lambda_3(t)\dfrac{\beta_2 S^\epsilon}{1+\delta S^\epsilon} - \lambda_4(t)\eta, \\
\dfrac{\mathrm{d}\lambda_3(t)}{\mathrm{d}t} = -\phi + \lambda_1\beta_1 I_u^\epsilon + \lambda_2(t)\dfrac{\beta_2 S^\epsilon}{1+\delta S^\epsilon} - \lambda_4(t)(\gamma + r(t)) \\
\qquad\qquad - \lambda_3(t)\left[\beta_1 I_u^\epsilon + \dfrac{\beta_2 I_a^\epsilon}{(1+\delta S^\epsilon)^2} - (\gamma+\mu) - r(t)\right], \\
\dfrac{\mathrm{d}\lambda_4(t)}{\mathrm{d}t} = \lambda_4(t)\mu,
\end{cases} \tag{4.4}$$

with the transversality conditions $\lambda_i(T) = 0$. The optimal control r^* is stipulated as follows:

$$r^* = \max\{\min\{\frac{(\lambda_3 - \lambda_5)S^\epsilon}{2\kappa}, 0\}, r^{\max}\}.$$

Proof. Set $I_u(t) = I_u^\epsilon$, $I_a(t) = I_a^\epsilon$, $S(t) = S^\epsilon$, and $R(t) = R^\epsilon$ using Pontryagin's maximum principle. Upon derivation, the following adjoint equations are obtained.

$$\begin{cases} \frac{d\lambda_1(t)}{dt} = -\frac{\partial H}{\partial I_u(t)} = \lambda_1(t)(\alpha + \beta_1 S^\epsilon + \mu) - \lambda_2(t)\alpha - \lambda_3(t)\beta_1 S^\epsilon, \\ \frac{d\lambda_2(t)}{dt} = -\frac{\partial H}{\partial I_a(t)} = \lambda_2(t)(\frac{\beta_2 S^\epsilon}{1+\delta S^\epsilon} + \mu + \eta) - \lambda_3(t)\frac{\beta_2 S^\epsilon}{1+\delta S^\epsilon} - \lambda_4(t)\eta, \\ \frac{d\lambda_3(t)}{dt} = -\frac{\partial H}{\partial S(t)} = -\phi + \lambda_1\beta_1 I_u^\epsilon + \lambda_2(t)\frac{\beta_2 S^\epsilon}{1+\delta S^\epsilon} - \lambda_4(t)(\gamma + r(t)) \\ \qquad - \lambda_3(t)\left[\beta_1 I_u^\epsilon + \frac{\beta_2 I_a^\epsilon}{(1+\delta S^\epsilon)^2} - (\gamma+\mu) - r(t)\right], \\ \frac{d\lambda_4(t)}{dt} = -\frac{\partial H}{\partial R(t)} = \lambda_4(t)\mu. \end{cases}$$

Adhering to the optimality criteria, the differentiation of Eq (4.3) with regard to $r(t)$ is delineated herein.

$$\frac{\partial H}{\partial r(t)}|_{r(t)=r^*} = 2\kappa r^* - \lambda_3(t)S^\epsilon + \lambda_4(t)S^\epsilon = 0.$$

Then, the optimal control can be obtained

$$r^* = \frac{(\lambda_3(t) - \lambda_4(t))S^\epsilon}{2\kappa}.$$

In other words, the optimal control variables r^* are characterized as

$$r^* = \max\{\min\{\frac{(\lambda_3(t) - \lambda_4(t))S^\epsilon}{2\kappa}, 0\}, r^{\max}\}.$$

□

5. Numerical simulations

In this section, we plan to perform numerical simulation analyses using two different datasets presented in Table 1. Specifically, the Set 1 will be used as a reference to study the trend of rumor dissipation, while the Set 2 will be used as a basis to explore the mechanism of continuous rumor propagation. Choose the following initial value: $I_u(0) = 0.6 - 0.05k$, $I_a(0) = 0.3 - 0.03k$, $S(0) = 0.06 + 0.04k$, and $R(0) = 0.04 + 0.04k$, $k \in [1, 8]$.

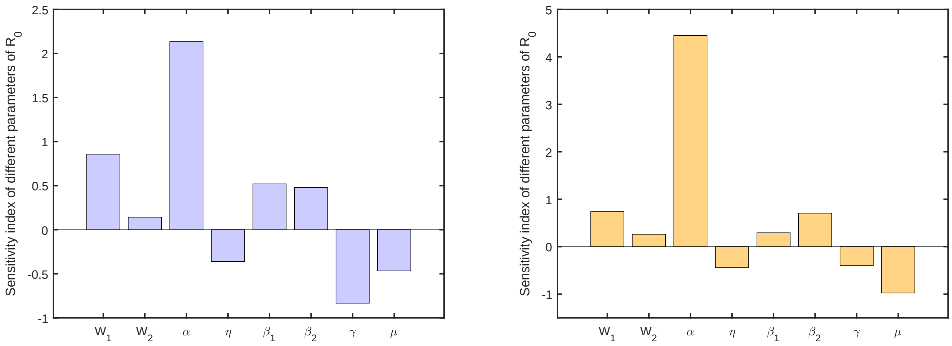
Table 1. The parameter set for simulation.

Parameters	W_1	W_2	α	β_1	β_2	η	γ	μ	δ
Set 1 values	0.03	0.01	0.15	0.26	0.18	0.12	0.2	0.04	0.38
Set 2 values	0.02	0.01	0.17	0.28	0.2	0.05	0.02	0.03	0.38

5.1. Sensitivity analysis

In sensitivity analyses to assess the impact of variable changes on the system as a whole, our primary concern is to determine which parameters have the greatest impact on the basic reproduction number $\Re_0$. For this purpose, we have carried out exhaustive calculations based on the normalized sensitivity index defined in the literature [24] to identify the parameters that are most sensitive to $\Re_0$. The results of the sensitivity assessment of these parameters are presented in Table 1.

If the sensitivity index of a parameter assumes a positive value, it signifies that an increase in the parameter leads to an elevation in the threshold $\Re_0$. Conversely, a negative sensitivity index indicates that an augmentation in the parameter results in a decrease of the threshold $\Re_0$. From Figure 2, it can be seen that regardless of the rumor disappearance or existence, the parameters W_1, W_2, α, β_1, and β_2 have positive sensitivity indices, i.e., $\Re_0$ increases with the increase of these parameters. In contrast, the sensitivity indices of the parameters η, γ, and μ are all less than zero, i.e., $\Re_0$ decreases with an increase in these parameters.



(a) (b)

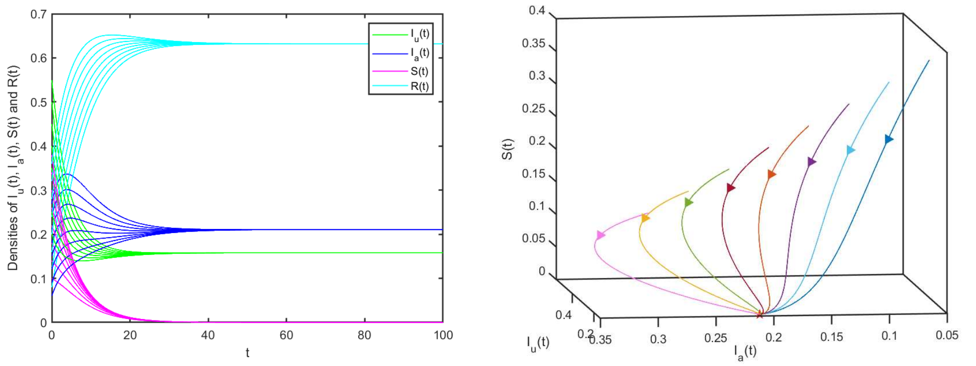
Figure 2. (a) The sensitivity index of $\Re_0$ to different parameters, when $\Re_0 < 1$. (b) The sensitivity index of $\Re_0$ to different parameters, when $\Re_0 > 1$.

5.2. The stability of rumor-free equilibrium and rumor equilibrium

Example 5.1. *Considering the data for parameter Set 1 in Table 1, Figure 3 delves into the scenario where* $\Re_0 = 0.3289 < 1$. *Drawing upon the proof presented in Theorem 3.2, we infer that within the system, the propagation of rumors will progressively attenuate, ultimately leading to their disappearance. Concurrently, the equilibrium* E_0 *of system* (2.1) *is deemed to be globally asymptotically stable. This assertion is visually corroborated in Figure 3(a). Furthermore, Figure 3(b) shows the phase diagram at the rumor-free equilibrium* E_0. *Collectively, Figure 3 demonstrates that regardless of the initial conditions, the impact of rumors will gradually weaken and eventually disappear.*

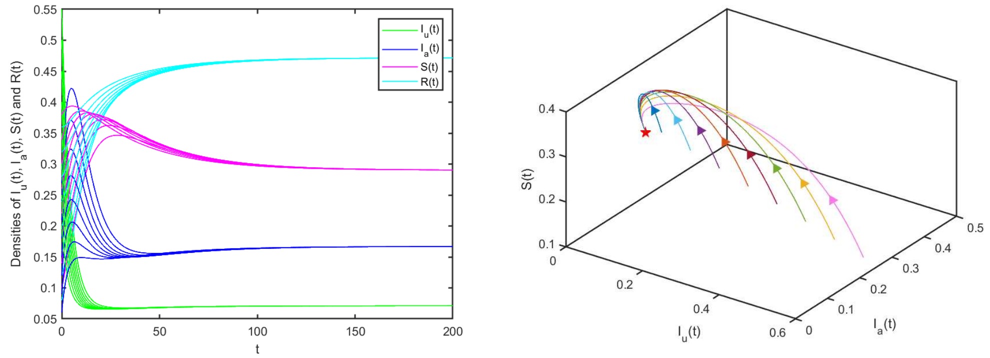
Example 5.2. *By choosing the parameters of Set 2 in Table 1, Figure 4 discusses the case* $\Re_0 = 1.3988 > 1$. *Therefore, according to Theorem 3.5, there exists a rumor equilibrium* E^* *in system* (2.1). *Figure 4(a) clearly shows the dynamic trajectory of all states stable. Figure 4(b) more specifically shows that although the initial values have changed, their final steady-state remains unchanged. This further confirms the stability of system* (2.1) *at the rumor equilibrium* E^*.

Example 5.3. *We use the parameters of Set 2 in Table 1. When other parameters are constant, we select* $\delta = 0$, $\delta = 0.38$, *and* $\delta = 0.8$ *to analyze the effect of ignoramuses with different self-alertness awareness on rumor spreading. Figure 5(a) shows that as the self-awareness of the ignorant increases, the density of spreaders gradually decreases, indicating that they become more cautious about the information they receive after enhancing their self-awareness, thereby reducing the density of spreaders. Similarly,* $\alpha = 0$, $\alpha = 0.38$, *and* $\alpha = 0.8$ *are chosen to discuss the effect of changes in the probability of unaware ignorants switching to aware ignorants. Figure 5(b) shows that as* α *gets larger, the number* *of propagators gets smaller. That is, as the probability of an ignorant person converting to an aware ignorants person rises, the number of rumor spreaders decreases accordingly.*



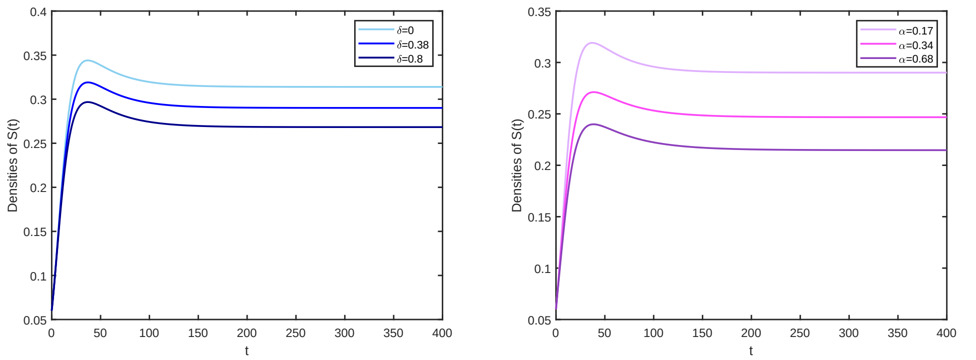
(a) (b)

Figure 3. (a) The evolutions of $I_u(t)$, $I_a(t)$, $S(t)$, and $R(t)$ over time at E_0. (b) Phase diagram with different initial values when $\Re_0 < 1$.



(a) (b)

Figure 4. (a) The evolutions of $I_u(t)$, $I_a(t)$, $S(t)$, and $R(t)$ over time at E^*. (b) Phase diagram with different initial values when $\Re_0 > 1$.



(a) (b)

Figure 5. (a) The trajectories of $S(t)$ with $\Re_0 > 1$, $\delta = 0$, $\delta = 0.38$, and $\delta = 0.8$. (b) The trajectories of $S(t)$ with $\Re_0 > 1$, $\alpha = 0.17$, $\alpha = 0.34$, and $\alpha = 0.68$.

5.3. *Feasibility of optimal control*

This section discusses the effectiveness of optimal control. In the presence of rumors, the initial values are set to be $I_u(0) = 0.6$, $I_a(0) = 0.3$, $S(0) = 0.06$, and $R(0) = 0.04$. The weights of the indicators are $\phi = 2$ and $\kappa = 1$. Figure 6 depicts the variation of $S(t)$ and $R(t)$ over time with and without optimal control. From the figure, it can be concluded that the number of spreaders significantly decreases and the amount count of recovereds increases under optimal control. That is, the government and related departments can increase the investment in public education to improve the public's ability to identify and prevent rumors and control the spread of rumors through various channels, such as school education, community promotion, and media publicity. The trajectory of the optimal control strategy and the corresponding change in control cost over time has been clearly demonstrated in Figure 7.

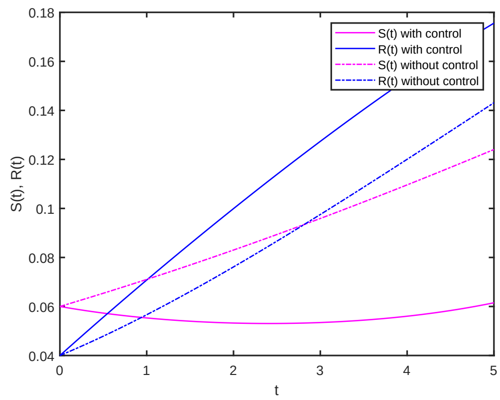


Figure 6. The trajectories of $S(t)$ and $R(t)$ with and without optimal control.

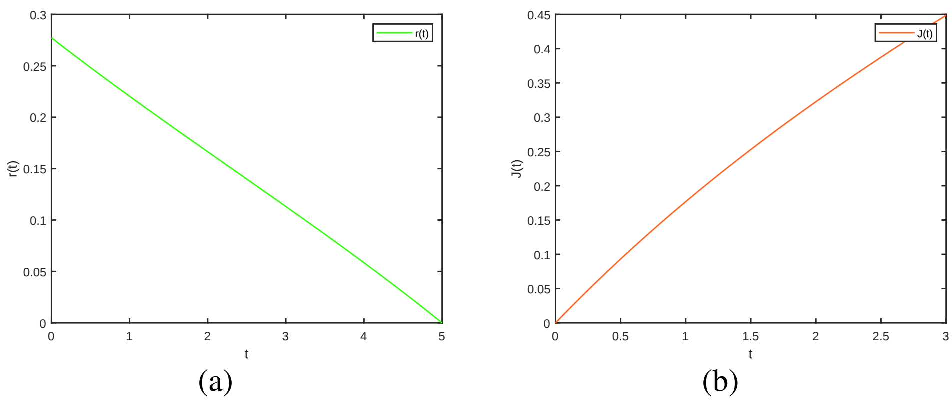
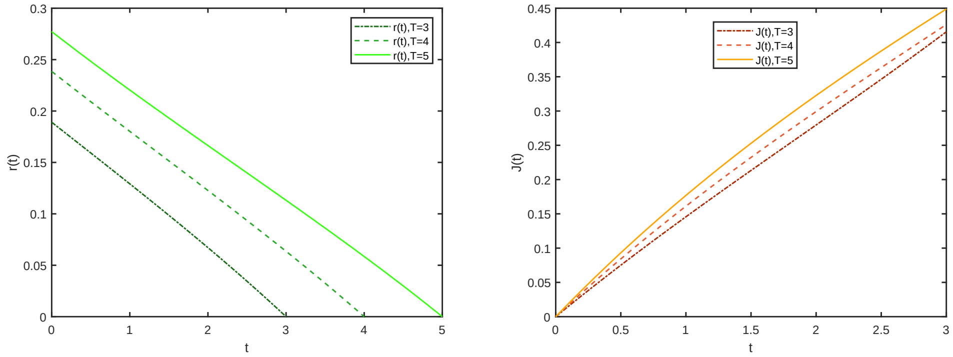


Figure 7. (a) Evolution of optimal control paths $r(t)$ in [0,5]. (b) The trajectories of control cost profiles $J(t)$ between expected intervals [0, 3].

In the context of optimal control, varying the control time T results in distinct trajectories. Figure 8(a) illustrates that the trajectories of $r(t)$ exhibit variations when the control time T is set to 3, 4, and 5, respectively. As depicted in Figure 8(b), the control consumptions $J(t)$ are presented for various control times: $T = 3$, $T = 4$, and $T = 5$. Our observations indicate that as the time T increases, so does the consumption $J(t)$.



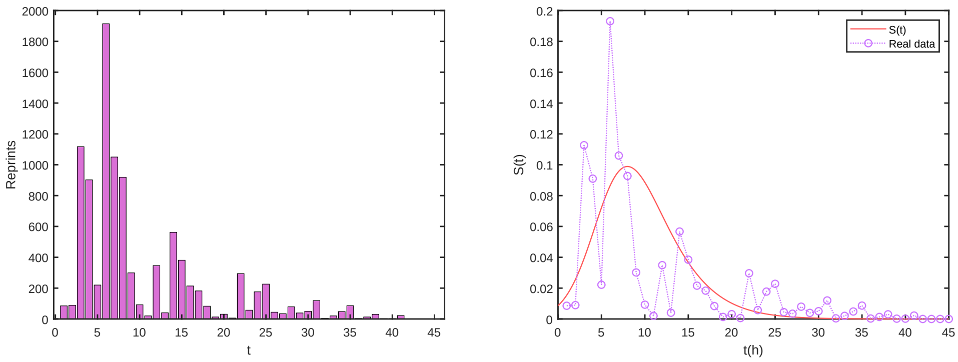
(a) (b)

Figure 8. (a) Evolution of optimal control paths $r(t)$ under varying T. (b) The trajectories of control cost profiles $J(t)$ under varying T.

6. Model application

To validate the pertinent results, a practical example is presented in this section as an application of the model. Utilizing a microblog post regarding "the leader being trampled to death as a result of tourists pulling the tails of elephants" as the data source, simulations are conducted. [25] provides the amount of reprints for the first 44 h of rumor spreading, as shown in Figure 9(a).

Given that users who engage with microblogs possess a specific knowledge base and cultural background, it is postulated that $I_a = 0$. Also inspired by [26], the following parameters are chosen: $\delta = 8 \times 10^{-3}$, $W_2 = 8 \times 10^{-4}$, $\beta_2 = 1 - 1 \times 10^{-15}$, $\mu = 8 \times 10^{-4}$, $\eta = 6.8 \times 10^{-3}$, and $\gamma = 0.37$. By incorporating the aforementioned parameters with the real data, a meticulous fitting process is conducted, yielding Figure 9(b) as the outcome.



(a) (b)

Figure 9. (a) The sensitivity index of $\Re_0$ to different parameters. (b) The progression of rumor spreaders and factual data.

Next, if we adopt an effective control strategy against the spreader, then the rumor is likely to die down quickly. Based on the optimal control presented in Section 5, $\phi = 4$ and $\kappa = 8$ are chosen. The transformation of the rumor spreaders is graphically depicted in Figure 10. The optimal control intensity $r(t)$ is shown in Figure 11(a), while the consumption $J(t)$ is presented in Figure 11(b). It is obvious that under the optimal control strategy (e.g., the government adopts science education and the network regulator bans the spreaders), the number of spreaders will decrease rapidly, while the consumption $J(t)$ will be less than in the other cases.

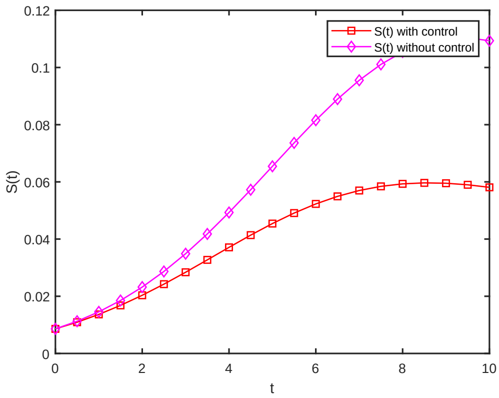
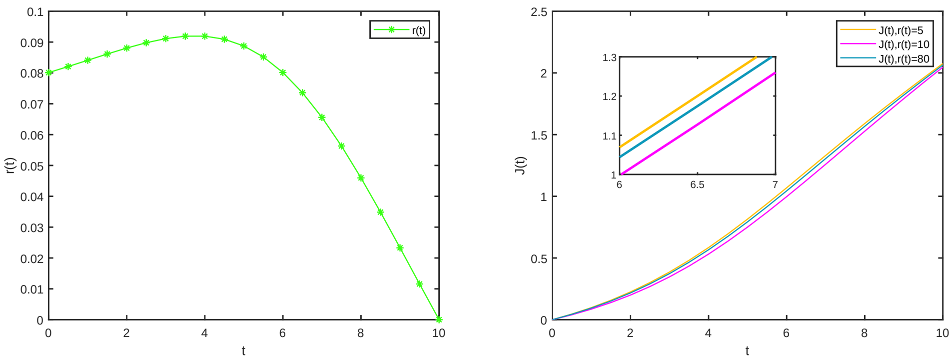


Figure 10. The evolution of $S(t)$ states with and without optimal control.



(a) (b)

Figure 11. (a) The evolution of optimal control strength $r(t)$. (b) The consumption $J(t)$ with different control.

7. Conclusions

This article studies a $2ISR$ rumor propagation model with nonlinear correlation, and the nonlinear occurrence rate of the model can reflect an individual's psychological alertness. To begin, the positivity and positive invariant set of the solution of the established rumor propagation model are proved. Then, the next generation matrix method is used to calculate the rumor propagation threshold $\Re_0$, and the existence of the equilibrium is analyzed, and it is proved that there is no backward bifurcation at the rumor equilibrium. In addition, the main results indicate that when $\Re_0 < 1$, rumors eventually disappear, and the rumor-free equilibrium E_0 is globally asymptotically stable. When $\Re_0 > 1$, rumors continue to spread and the rumor equilibrium E^* is locally asymptotically stable. In order to further control the spread of rumors, we studied the optimal control strategy of increasing education and online regulatory measures. Finally, sensitivity analysis is conducted on various parameters in the model, and the correctness of the theoretical results is verified through numerical simulation. The influence of psychological factor δ on rumor propagation was also simulated. By analyzing real cases, we confirm that the model constructed in this paper not only has theoretical significance, but also has feasibility in practical applications.

Considering the complexity and variability of the internet, the impact of complex network structures on rumor propagation will be considered in the future, as well as the fact that it is not enough to curb rumor spreading by optimizing control strategies. In the future, we will study in depth the effects of other control strategies [27–30] on rumor propagation, with a view to finding a more comprehensive solution. Meanwhile, since the stochastic rumor model can more accurately reflect the complex propagation laws in real life [31–34], this will also be the direction of our subsequent research.

Author contributions

Hui Wang: Conceptualization, Methodology, Validation, Formal analysis, Writing-original draft; Shuzhen Yu: Supervision, Funding acquisition, Validation; Haijun Jiang: Supervision, Writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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