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Research article

Local well-posedness results for the nonlinear fractional diffusion equation involving a Erdélyi-Kober operator

Wei Fan and Kangqun Zhang*

School of Mathematics and Physics, Nanjing Institute of Technology, Nanjing, 211167, China

* **Correspondence:** Email: chkq@njit.edu.cn, chkqnju@hotmail.com.

Abstract: In this paper, we study an initial boundary value problem of a nonlinear fractional diffusion equation with the Caputo-type modification of the Erdélyi-Kober fractional derivative. The main tools are the Picard-iteration method, fixed point principle, Mittag-Leffler function, and the embedding theorem between Hilbert scales spaces and Lebesgue spaces. Through careful analysis and precise calculations, the priori estimates of the solution and the smooth effects of the Erdélyi-Kober operator are demonstrated, and then the local existence, uniqueness, and stability of the solution of the nonlinear fractional diffusion equation are established, where the nonlinear source function satisfies the Lipschitz condition or has a gradient nonlinearity.

Keywords: fractional diffusion equation; Erdélyi-Kober fractional derivative; Caputo fractional derivative; nonlinearity; well-posedness
Mathematics Subject Classification: 35D30, 35K58, 35R11

1. Introduction

In this paper, we study an initial boundary value problem of the nonlinear diffusion equation with space-time fractional derivative

$$\begin{cases} t^{-\beta\gamma}{}_{*}D_{\beta}^{\alpha,\gamma}u(t,x) + (-\Delta)^{\mu}u(t,x) = f(u,\nabla u), & x\in\Omega, t\in\mathbb{R}^+,\\ u(t,x)=0, & x\in\partial\Omega, t\in\mathbb{R}^+,\\ \lim_{t\to 0^+} t^{\beta(\alpha+1)}u(t,x)=u_0(x), & x\in\Omega \end{cases} \tag{1.1}$$

with a modified initial datum condition, where ∇ is the gradient operator, $\Delta = \sum_{i=1}^{n} \partial_{x_i}^2$ is the Laplace operator, ${}_{*}D_{\beta}^{\alpha,\gamma}$ is the Caputo-type modification of the Erdélyi-Kober fractional differential operator with γ-th order, parameters $\alpha \in (-1,+\infty)$, $\beta \in (0,\infty)$, $\gamma \in (0,1]$, $\mu \in (0,1]$, and $\Omega \subseteq \mathbb{R}^n$ is a bounded domain.

Fractional calculus is an important subject in mathematics, physics, biology, economics, and many other different fields since it is usually used to describe the property of memory and heredity of many materials [1, 2]. Riemann-Liouville derivatives, Caputo derivatives, and Erdélyi-Kober derivatives are the well-known ones. The Riemann-Liouville fractional derivative is always employed in mathematical texts and not frequently used in applications. The Caputo definition of a fractional derivative is more useful in modeling reality. The Erdélyi-Kober fractional derivative is often used in both mathematical texts and applications. The so-called Caputo type modification of the Erdélyi-Kober fractional derivative is a generalization of these types of fractional derivatives, and its operations attracts much attentions. Gorenflo, Luchko, and Mainardi first introduced and applied it to investigate of the scale-invariant solutions of the diffusion-wave equation in [3]. Kiryakova and Luchko investigated its general properties in the sense of multiple Erdélyi-Kober fractional derivatives and studied some examples of Cauchy problems of fractional differential equations involving these type operators in [4, 5]. For more meaningful results and useful applications of the Erdélyi-Kober fractional derivative, one can find in [6–9].

In this paper, we investigate the local well-posedness of the solution of the nonlinear problem (1.1) and prepare to establish the theoretical basis for finding efficient numerical approaches later. Some researchers with similar interest are finding solutions using numerical methods on diffusion models involving fractional derivatives. Recently, Hoang Luc N. and his collaborators studied a diffusion equation involving a regularized hyper-Bessel operator in [10], and Van Au V. and his collaborators established the existence and blowup results of a similar model with gradient nonlinearity in [11]. For more results, one can refer to [12–17] and the references therein. In order to show our results of the nonlinear problems, we should first point out the singularity of the initial datum is generated by the the Erdélyi-Kober fractional derivative [18], then construct and estimate the solution of the linear problem based on the Mittag–Leffler function. Through a direct observation with the estimates, the smooth effects of the Caputo type modification of the Erdélyi-Kober fractional derivative on the solution were confirmed in the inner of the domain. Based on these, by applying embedding theorem between Hilbert scales spaces and Lebesgue spaces, fixed point theory, and the Picard-iteration method, we establish the existence, uniqueness, and stability of the solution of fractional diffusion equation for the source term with two types different nonlinearities. Under this framework, the establishment of global well-posedness results of the problem (1.1) doesn't given and deserve to be considered with new ideas and approaches in another paper.

The rest of this paper is organized as follows: In §2, the basic knowledge of some interpolation function spaces and some established useful results are given. In §3, we construct a solution of the linear inhomogeneous diffusion equation and give related estimates in terms of Mittag-Leffler functions in Hilbert scales spaces. Meanwhile, the smooth effects of the Erdélyi-Kober operator are shown. In §4, the well-posedness is established for the nonlinear source function that satisfies the Lipschitz condition or with gradient nonlinearity.

2. Preliminary

First, we recall the results of the eigenvalue problem [19]

$$\begin{cases} -\Delta\Theta_i(x) = \lambda_i\Theta_i(x), & x \in \Omega, \\ \Theta_i(x) = 0, & x \in \partial\Omega. \end{cases} \tag{2.1}$$

Then there exists a system of standard and complete orthogonal basis $\{\Theta_i(x)\}_{i\in\mathbb{N}^+}$ in $\mathbb{L}^2(\Omega)$, the corresponding eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}^+}$ which satisfies $\lim_{i\to+\infty}\lambda_i = \infty$.

For any $\nu \in \mathbb{R}$, set

$$\mathbb{D}^\nu(\Omega) = \{u(x) = \sum_{i=1}^{\infty} u_i\Theta_i(x) \in \mathbb{L}^2(\Omega) : (\sum_{i=1}^{\infty} \lambda_i^{2\nu} u_i^2)^{\frac{1}{2}} < +\infty\}$$

is equipped with the norm

$$\|u\|_{D^\nu(\Omega)} = (\sum_{i=1}^{\infty} \lambda_i^{2\nu} u_i^2)^{\frac{1}{2}}.$$

It is obviously that $\mathbb{D}^0(\Omega) = \mathbb{L}^2(\Omega)$. Define

$$(-\Delta)^\nu u(x) = \sum_{i=1}^{\infty} \lambda_i^\nu u_i \Theta_i(x),$$

where $u_i = \int_\Omega u(x)\Theta_i(x)dx$. Since $\int_\Omega \Theta_i^2(x)dx = 1$, then it is easy to verify that

$$\|u\|_{D^\nu(\Omega)} = \|(-\Delta)^\nu u\|_{\mathbb{L}^2(\Omega)}.$$

In fact, $\mathbb{D}^\nu(\Omega)$ is a Hilbert scale space, for more introductions of these spaces, one can refers to [20].

In the following, we recall some embedding results between Hilbert scales spaces and Lebesgue spaces.

Lemma 2.1. *[11] Assume $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain, then*

$$L^q(\Omega) \hookrightarrow \mathbb{D}^\nu(\Omega), \quad \text{if} \quad \frac{-n}{4} < \nu \le 0,\ q \ge \frac{2n}{n-4\nu}. \tag{2.2}$$

$$L^q(\Omega) \hookleftarrow \mathbb{D}^\nu(\Omega), \quad \text{if} \quad 0 \le \nu < \frac{n}{4},\ q \le \frac{2n}{n-4\nu}. \tag{2.3}$$

Based on Lemma 2.1, btain the following nonlinear estimate, although the result had been established in [10] . Here we rewrite the proof by constructing an analytic function.

Lemma 2.2. *Prescribed $u, v \in \mathbb{D}^\nu(\Omega)$, $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain, and ν, ρ, p satisfy*

$$\nu < \rho \le \frac{n}{4} + \nu, \quad \frac{1}{2} \le \nu < \frac{n}{4} + \frac{1}{2},$$

$$max\{1, \frac{2n}{n-4(\nu-\rho)}\}p \le \frac{2n}{n-4(\nu-\frac{1}{2})},$$

then there exists

$$\||\nabla u|^p - |\nabla v|^p\|_{\mathbb{D}^{\nu-\rho}(\Omega)} \lesssim (\|u\|_{\mathbb{D}^\nu(\Omega)}^{p-1} + \|v\|_{\mathbb{D}^\nu(\Omega)}^{p-1}) \|u-v\|_{\mathbb{D}^\nu(\Omega)}. \tag{2.4}$$

Remark 2.3. *Here and in the following of this paper,* $|a| \lesssim |b|$ *means there exists a constant* $C > 0$ *such that* $|a| \leq C|b|$.

Proof. Consider the function

$$f(x) = (1-p)(x^p - 1) + xp(x^{p-2} - 1), \quad x \in (0, \infty), \quad p > 0, \tag{2.5}$$

then by computing the derivative, one has

$$f'(x) = (1-p)p[x^{p-2}(x-1) - (1-p)^{-1}].$$

Furthermore, for $0 < p < 1$, there is

$$f''(x) = (1-p)px^{p-3}[(p-1)x - p + 2].$$

with the zero point $x_0 = 1+\frac{1}{1-p}$, and $f'(x)$ is monotonically increasing for $x \in (0, 1+\frac{1}{p-1})$, monotonically decreasing for $x \in (1+\frac{1}{p-1}, \infty)$. Hence

$$f'(x) \leq f'(1 + \frac{1}{1-p}) = p[(1 + \frac{1}{1-p})^{p-2} - 1] < 0.$$

This yields that $f(x)$ is monotonically decreasing for $x \in (0, +\infty)$, and then $f(x) \leq 0$ since $f(1) = 0$. For $p = 1$, we obtain $f(x) = 1 - x \leq 0$ for $x \in (0, +\infty)$. For $p > 1$, we also have $f'(x) < 0$, and $f(x)$ is monotonically decreasing for $x \in (0, \infty)$ by a similar analysis, then $f(x) \leq 0$ is also holding. Therefore, there exists $f(x) \leq 0$ for any $x \in (0, \infty)$, $p > 0$.

Set $x = |\nabla u|\,(|\nabla v|)^{-1}$, and substitute it into $f(x) \leq 0$ in terms of (2.5), then we derive

$$(1-p)(|\nabla u|^p\,(|\nabla v|)^{-p} - 1) + p\,|\nabla u|\,(|\nabla v|)^{-1}(|\nabla u|^{p-2}\,(|\nabla v|)^{2-p} - 1) \leq 0,$$

which is equivalent to

$$||\nabla u|^p - |\nabla v|^p| \leq \left|p(|\nabla u| - |\nabla v|)(|\nabla u|^{p-1} + |\nabla v|^{p-1})\right|. \tag{2.6}$$

Take any $q \in [\max\{1, \frac{2n}{n-4(\nu-\rho)}\}p, \frac{2n}{n-4(\nu-\frac{1}{2})}]$ and integrate (2.6) on Ω, we have

$$\begin{aligned} \||\nabla u|^p - |\nabla v|^p\|_{L^{\frac{q}{p}}(\Omega)} &= [\int_\Omega |(|\nabla u|^p - |\nabla v|^p)|^{\frac{q}{p}}\,dx]^{\frac{p}{q}} \\ &\leq p[\int_\Omega |(|\nabla u| - |\nabla v|)A|^{\frac{q}{p}}\,dx]^{\frac{p}{q}} \\ &\leq p[\int_\Omega (|\nabla u| - |\nabla v|)^q dx]^{\frac{1}{q}}[\int_\Omega A^{\frac{q}{p-1}} dx]^{\frac{p-1}{q}} \\ &= p\,\||\nabla u| - |\nabla v|\|_{L^q(\Omega)}\,[\int_\Omega A^{\frac{q}{p-1}} dx]^{\frac{p-1}{q}} \end{aligned} \tag{2.7}$$

in terms of of hölder inequality, where

$$A = |\nabla u|^{p-1} + |\nabla v|^{p-1}. \tag{2.8}$$

Besides, it follows Minkovsiki inequality and (2.8) that

$$
\begin{aligned}
[\int_\Omega A^{\frac{q}{p-1}}dx]^{\frac{p-1}{q}} &= [\int \left|(|\nabla u|^{p-1}+|\nabla v|^{p-1})\right|^{\frac{q}{p-1}}dx]^{\frac{p-1}{q}} \\
&\le [\int_\Omega (|\nabla u|^{p-1})^{\frac{q}{p-1}}dx]^{\frac{p-1}{q}} + [\int_\Omega (|\nabla v|^{p-1})^{\frac{q}{p-1}}dx]^{\frac{p-1}{q}} \\
&\le \||\nabla u|\|^{p-1}_{L^q(\Omega)} + \||\nabla v|\|^{p-1}_{L^q(\Omega)},
\end{aligned} \tag{2.9}
$$

and applying Cauchy inequality, we have

$$
\begin{aligned}
||\nabla u|-|\nabla v|| &= \Big(\sum_{i=1}^n u_{x_i}^2 + \sum_{i=1}^n v_{x_i}^2 - 2((\sum_{i=1}^n u_{x_i}^2)(\sum_{i=1}^n v_{x_i}^2))\Big)^{\frac{1}{2}} \\
&\lesssim (\sum_{i=1}^n u_{x_i}^2 + \sum_{i=1}^n v_{x_i}^2 - 2\sum_{i=1}^n u_{x_i}v_{x_i})^{\frac{1}{2}} \\
&= |\nabla u - \nabla v|.
\end{aligned} \tag{2.10}
$$

Then, substituting (2.9) and (2.10) into (2.7), we arrive at

$$
\||\nabla u|^p - |\nabla v|^p\|_{L^{\frac{q}{p}}(\Omega)} \lesssim (\|\nabla u\|^{p-1}_{L^q(\Omega)} + \|\nabla v\|^{p-1}_{L^q(\Omega)})\,\|\nabla u - \nabla v\|_{L^q(\Omega)}. \tag{2.11}
$$

Based on (2.11) and Lemma 2.1 (2.3), one has

$$
\begin{aligned}
&\||\nabla u|^p - |\nabla v|^p\|_{L^{\frac{q}{p}}(\Omega)} \\
\lesssim&(\|\nabla u\|^{p-1}_{\mathbb{D}^{\nu-\frac{1}{2}}(\Omega)} + \|\nabla v\|^{p-1}_{\mathbb{D}^{\nu-\frac{1}{2}}(\Omega)})\,\|\nabla u - \nabla v\|_{\mathbb{D}^{\nu-\frac{1}{2}}(\Omega)} \\
\lesssim&(\|\nabla u\|^{p-1}_{\mathbb{D}^{\nu}(\Omega)} + \|\nabla v\|^{p-1}_{\mathbb{D}^{\nu}(\Omega)})\,\|\nabla u - \nabla v\|_{\mathbb{D}^{\nu}(\Omega)}.
\end{aligned} \tag{2.12}
$$

Then, (2.5) is derived in terms of (2.12) and Lemma 2.1 (2.2) for $\frac{2n}{n-4(\nu-\rho)} \le \frac{q}{p}$.

Next, we recall some definitions of Erdélyi-Kober fractional integral and differential operators.

Definition 2.4. *[1, 2] For a function $f(t) \in \mathbb{C}_\mu$, $\mu \in \mathbb{R}$, then the integral*

$$
I_\beta^{\alpha,\gamma} f(t) = \frac{t^{-\beta(\alpha+\gamma)}}{\Gamma(\gamma)} \int_0^t (t^\beta - \tau^\beta)^{\gamma-1}\tau^{\beta\alpha} f(\tau)d(\tau^\beta)
$$

is called the Erdélyi-Kober fractional integral of $f(t)$ with arbitrary parameters $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^+$ and $\gamma \in \mathbb{R}^+$, where the weighted space of continuous functions $f(t)$ is defined by

$$
\mathbb{C}_\mu^{(k)} = \{f(t) = t^p \tilde{f}(t) : p > \mu, \tilde{f}(t) \in \mathbb{C}^{(k)}[0,\infty)\}, k \in \mathbb{N}.
$$

The Erdélyi-Kober fractional integral is defined as the identity operator for $\gamma = 0$ and is reduced to the well-known Riemann-Liouville fractional integral with a power weight for $\alpha = 0, \beta = 1$.

Definition 2.5. *[4, 5] The Riemann-Liouille type modification of the Erdélyi-Kober fractional derivative of a function $f(t) \in \mathbb{C}_\mu^{(1)}$ with the order γ is defined by*

$$D_\beta^{\alpha,\gamma} f(t) = (\alpha + \frac{1}{\beta} t \frac{d}{dt}) I_\beta^{\alpha+\gamma,1-\gamma} f(t), \gamma \in (0,1].$$

Compared with the definition of the Riemann-Liouille type modification of the Erdélyi-Kober fractional derivative of a function $f(t) \in \mathbb{C}_\mu^{(1)}$ with the order γ, the Caputo type modification also can be defined.

Definition 2.6. *[4, 5] The Caputo type modification of the Erdélyi-Kober fractional derivative of a function $f(t) \in \mathbb{C}_\mu^{(1)}$ with the order γ is defined by*

$$ {}_*D_\beta^{\alpha,\gamma} f(t) = I_\beta^{\alpha+\gamma,1-\gamma} (\alpha + \frac{1}{\beta} t \frac{d}{dt}) f(t), \gamma \in (0,1].$$

The singular initial value problem of fractional differential equation with the Caputo-type modification of the Erdélyi-Kober derivative

$$\begin{cases} t^{-\beta\alpha} {}_*\mathbb{D}_\beta^{\alpha,\gamma} u(t) + \lambda u(t) = f(t), \quad t \in \mathbb{R}^+, \\ \lim\limits_{t\to 0} t^{\beta(\alpha+1)} u(t) = u_0 \end{cases} \tag{2.13}$$

had been studied and the following result can be directly derived from Theorem 3.4 in [18] with parameters $\alpha \in (-1, +\infty), \beta \in (0, \infty), \gamma \in (0, 1], \mu \in (0, 1]$.

Lemma 2.7. *[18] Given a function $f \in C_{\beta\delta}$, $\delta \geq \max\{0, -\alpha - \gamma\} - 1$, then there exists an explicit solution $u \in C_{\beta\delta}$ of the problem* (2.13)*, which is given in the form*

$$\begin{aligned} u(t) &= u_0 \Gamma(\gamma) t^{-\beta(\alpha+1)} E_{\gamma,\gamma}(-\lambda t^{\beta\gamma}) \\ &\quad + t^{-\beta(\alpha+\gamma)} \int_0^t (t^\beta - \tau^\beta)^{\gamma-1} \tau^{\beta(\alpha+\gamma)} E_{\gamma,\gamma}[-\lambda (t^\beta - \tau^\beta)^\gamma] f(\tau) d(\tau^\beta). \end{aligned} \tag{2.14}$$

At last, the Mittag-Leffler function is an entire function, which is represented by the convergent series

$$E_{\alpha,\beta}(t) = \sum_{i=0}^{\infty} \frac{t^i}{\Gamma(\alpha i + \beta)}, \qquad \Re(\alpha) > 0, \Re(\beta) > 0, \tag{2.15}$$

where $\alpha \in \mathbb{C}, \beta \in \mathbb{C}$, $\Re(\cdot)$ denoting the real part of a complex number. The asymptotic expansion of the Mittag-Leffler function is given in the following.

Lemma 2.8. *[21] Given $\alpha \in (0, 1), \beta \in \mathbb{R}$, and $\gamma \in (\frac{\pi\alpha}{2}, \pi)$, then there exists*

$$\left| E_{\alpha,\beta}(z) \right| \lesssim \frac{1}{1 + |z|} \tag{2.16}$$

for any $z \in \mathbb{C}$ such that $\gamma \leq |argz| < \pi$.

3. Estimates of the solution of the linear fractional diffusion equation

In this section, we consider the singular initial value problem of the linear fractional diffusion equation

$$\begin{cases} t^{-\beta\gamma}{}_{*}\mathbb{D}_{\beta}^{\alpha,\gamma}u(t,x) + (-\Delta)^{\mu}u(t,x) = f(t,x), \quad x \in \Omega, t \in \mathbb{R}^{+}, \\ u(t,x) = 0, \quad x \in \partial\Omega, t \in \mathbb{R}^{+}, \\ \lim_{t\to 0} t^{\beta(\alpha+1)}u(t,x) = u_0(x), \quad x \in \Omega \end{cases} \tag{3.1}$$

with the initial datum $u_0(x) \in \mathbb{D}^{\nu}(\Omega)$. In order to obtain the formal solution of (3.1), applying the method of separation of variables, we obtain the spectral problem

$$\begin{cases} (-\Delta)^{\mu}\Theta_i(x) = \lambda_i^{\mu}\Theta_i(x), \quad x \in \Omega, \\ \Theta_i(x) = 0, \quad x \in \partial\Omega, \end{cases}$$

and the corresponding Cauchy problem

$$\begin{cases} t^{-\beta\alpha}{}_{*}\mathbb{D}_{\beta}^{\alpha,\gamma}u(t) + \lambda_i^{\mu}u(t) = f_i(t), \quad t \in \mathbb{R}^{+}, \\ \lim\limits_{t\to 0} t^{\beta(\alpha+1)}u(t) = u_{0i}, \end{cases}$$

where $f_i(t) = \int_\Omega f(t,x)\Theta_i(x)dx$ and $u_{0i} = \int_\Omega u_0(x)\Theta_i(x)dx$.

Take

$$R_1(t)u_0(x) := \sum_{i=1}^{\infty} E_{\gamma,\gamma}(-\lambda_i^{\mu}t^{\beta\gamma})u_{0i}\Theta_i(x), \tag{3.2}$$

$$R_2(t,\tau)f(\tau,x) := \sum_{i=1}^{\infty}(t^{\beta} - \tau^{\beta})^{\gamma-1}E_{\gamma,\gamma}(-\lambda_i^{\mu}(t^{\beta} - \tau^{\beta})^{\gamma})f_i(\tau)\Theta_i(x), \tag{3.3}$$

then, in terms of (2.1) and Lemma 2.7 (2.14), we obtain a formal solution of the problem (3.1), which is represented by

$$u(t,x) = \Gamma(\gamma)t^{-\beta(\alpha+1)}R_1(t)u_0(x) + t^{-\beta(\alpha+\gamma)}\int_0^t \tau^{\beta(\alpha+\gamma)}R_2(t,\tau)f(\tau,x)d(\tau^{\beta}). \tag{3.4}$$

Lemma 3.1. *Given* $g \in \mathbb{D}^{\nu}(\Omega)$ *and* $0 \le \theta \le 1$, *then there exist*

$$\|R_1(t)g(x)\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)} \lesssim t^{-\theta\beta\gamma}\|g(x)\|_{\mathbb{D}^{\nu}(\Omega)}, \tag{3.5}$$

$$\|R_2(t,0)g(x)\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)} \lesssim t^{\beta((1-\theta)\gamma-1)}\|g(x)\|_{\mathbb{D}^{\nu}(\Omega)}. \tag{3.6}$$

Proof. In terms of (3.2), it is easy to derive that

$$\begin{aligned} \|R_1(t)g(x)\|^2_{\mathbb{D}^{\nu+\theta\mu}(\Omega)} &= \left\|(-\Delta)^{\nu+\theta\mu}R_1(t)g(x)\right\|^2_{L^2(\Omega)} \\ &= \sum_{i=1}^{\infty}\lambda_i^{2(\nu+\theta\mu)}[E_{\gamma,\gamma}(-\lambda_i^{\mu}t^{\beta\gamma})]^2 g_i^2. \end{aligned} \tag{3.7}$$

Based on (2.16), we have

$$
\begin{aligned}
\left|E_{\gamma,\gamma}(-\lambda_l^{\mu}t^{\beta\gamma})\right| \lesssim & \frac{1}{1+\lambda_i^{\mu}t^{\beta\gamma}} \\
= & (\frac{1}{1+\lambda_i^{\mu}t^{\beta\gamma}})^{(1-\theta)}(\frac{1}{1+\lambda_i^{\mu}t^{\beta\gamma}})^{\theta} \\
\lesssim & \lambda_i^{-\theta\mu}t^{-\theta\beta\gamma},
\end{aligned}
\tag{3.8}
$$

then, substituting (3.7) into (3.8), we obtain

$$
\begin{aligned}
\|R_1(t)g(x)\|^2_{\mathbb{D}^{\nu+\theta\mu}(\Omega)} \lesssim & \sum_{i=1}^{\infty} \lambda_i^{2(\nu+\theta\mu)}\lambda_i^{-2\theta\mu}t^{-2\theta\beta\gamma}g_i^2 \\
= & t^{-2\theta\beta\gamma}\|g(x)\|^2_{\mathbb{D}^{\nu}(\Omega)}.
\end{aligned}
$$

This yields (3.5).

Similarly, we derive

$$
\begin{aligned}
\|R_2(t,0)g(x)\|^2_{\mathbb{D}^{\nu+\theta\mu}(\Omega)} = & \sum_{i=1}^{\infty} \lambda_i^{2(\nu+\theta\mu)}t^{2\beta(\gamma-1)}[E_{\gamma,\gamma}(-\lambda_i^{\mu}t^{\beta\gamma})]^2 g_i^2 \\
\lesssim & \sum_{i=1}^{\infty} \lambda_i^{2(\nu+\theta\mu)}t^{2\beta(\gamma-1)-2\theta\beta\gamma}g_i^2\lambda_i^{-2\theta\mu} \\
= & t^{2\beta((1-\theta)\gamma-1)}\|g(x)\|^2_{\mathbb{D}^{\nu}(\Omega)}.
\end{aligned}
$$

Hence, (3.6) is holding.

Theorem 3.2 *Given* $\theta \in [0,1]$, $u_0(x) \in \mathbb{D}^{\nu}(\Omega)$, *and* $f(t,x) \in C((0,+\infty);\mathbb{D}^{\nu}(\Omega))$ *with a bounded norm in the sense of* $\sup_{\tau\in(0,t)} \left\|t^{\beta(\alpha+1+\theta\gamma)}f(t,x)\right\|_{\mathbb{D}^{\nu}(\Omega)} < M$ *for some positive constant M, then there exists a solution* $u \in C((0,+\infty);\mathbb{D}^{\nu+\theta\mu}(\Omega))$ *of the problem* (3.1)*, which is expressed by* (3.4) *and satisfies*

$$
\begin{aligned}
& \left\|t^{\beta(\alpha+1+\theta\gamma)}u(t,x)\right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)} \\
\lesssim & \|u_0(x)\|_{\mathbb{D}^{\nu}(\Omega)} + t^{(1-\theta)\beta\gamma}\sup_{\tau\in(0,t)}\left\|\tau^{\beta(\alpha+1+\theta\gamma)}f(\tau,x)\right\|_{\mathbb{D}^{\nu}(\Omega)}.
\end{aligned}
\tag{3.9}
$$

In particular, we have

$$
\lim_{t\to 0}\left\|t^{\beta(\alpha+1)}u(t,x)-u_0(x)\right\|_{\mathbb{D}^{\nu}(\Omega)} = 0. \tag{3.10}
$$

Moreover, there exists ${}_*\mathbb{D}^{\alpha,\gamma}_{\beta}u(t,x) \in C(0,+\infty;\mathbb{D}^{\nu+(\theta-1)\mu}(\Omega))$ *and*

$$
\begin{aligned}
& \left\|t^{\beta(\alpha+1+(\theta-1)\gamma)}{}_*\mathbb{D}^{\alpha,\gamma}_{\beta}u(t,x)\right\|_{\mathbb{D}^{\nu+(\theta-1)\mu}(\Omega)} \\
\lesssim & \|u_0(x)\|_{\mathbb{D}^{\nu}(\Omega)} + (1+t^{(1-\theta)\beta\gamma})\sup_{\tau\in(0,t)}\left\|\tau^{\beta(\alpha+1+\theta\gamma)}f(\tau,x)\right\|_{\mathbb{D}^{\nu}(\Omega)}.
\end{aligned}
\tag{3.11}
$$

Proof. Based on Lemma 3.1 (3.6), we have

$$
\begin{aligned}
&\left\| t^{\beta(\alpha+1+\theta\gamma)} t^{-\beta(\alpha+\gamma)} \int_0^t \tau^{\beta(\alpha+\gamma)} R_2(t,\tau) f(\tau,x) d(\tau^\beta) \right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)} \\
\lesssim & t^{\beta(1-\gamma(1-\theta))} \int_0^t \tau^{\beta(\alpha+\gamma)} \|R_2(t,\tau) f(\tau,x)\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)} d(\tau^\beta) \\
\lesssim & t^{\beta(\alpha+\gamma+1)} \int_0^t (\frac{\tau}{t})^{\beta(\gamma+\alpha)} (1-(\frac{\tau}{t})^\beta)^{\gamma(1-\theta)-1} \|f(\tau,x)\|_{\mathbb{D}^\nu(\Omega)} d((\frac{\tau}{t})^\beta) \\
\lesssim & t^{(1-\theta)\beta\gamma} \sup_{\tau\in(0,t)} \left\| \tau^{\beta(\alpha+1+\theta\gamma)} f(\tau,x) \right\|_{\mathbb{D}^\nu(\Omega)} \\
& \times \int_0^t (\frac{\tau}{t})^{\beta((1-\theta)\gamma-1)} (1-(\frac{\tau}{t})^\beta)^{\gamma(1-\theta)-1} d((\frac{\tau}{t})^\beta) \\
= & t^{(1-\theta)\beta\gamma} \sup_{\tau\in(0,t)} \left\| \tau^{\beta(\alpha+1+\theta\gamma)} f(\tau,x) \right\|_{\mathbb{D}^\nu(\Omega)} \int_0^1 s^{(1-\theta)\gamma-1} (1-s)^{\gamma(1-\theta)-1} ds \\
\lesssim & t^{(1-\theta)\beta\gamma} \sup_{\tau\in(0,t)} \left\| \tau^{\beta(\alpha+1+\theta\gamma)} f(\tau,x) \right\|_{\mathbb{D}^\nu(\Omega)} .
\end{aligned}
$$

Then, applying (3.4) and Lemma 3.1 (3.5), we obtain

$$
\begin{aligned}
&\left\| t^{\beta(\alpha+1+\theta\gamma)} u(t,x) \right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)} \\
\lesssim & \|u_0(x)\|_{\mathbb{D}^\nu(\Omega)} + \left\| t^{\beta(\alpha+1+\theta\gamma)} t^{-\beta(\alpha+\gamma)} \int_0^t \tau^{\beta(\alpha+\gamma)} R_2(t,\tau) f(\tau,x) d(\tau^\beta) \right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)} \\
\lesssim & \|u_0(x)\|_{\mathbb{D}^\nu(\Omega)} + t^{(1-\theta)\beta\gamma} \sup_{\tau\in(0,t)} \left\| \tau^{\beta(\alpha+1+\theta\gamma)} f(\tau,x) \right\|_{\mathbb{D}^\nu(\Omega)} .
\end{aligned}
$$

Hence, we derive that $u \in C(0,+\infty;\mathbb{D}^{\nu+\theta\mu}(\Omega))$ which satisfies (3.9) and (3.10). Furthermore, by a direct computation, there exists

$$
\begin{aligned}
&\left\| t^{\beta(\alpha+1+\theta\gamma)} (-\Delta)^\mu u(t,x) \right\|_{\mathbb{D}^{\nu+(\theta-1)\mu}(\Omega)} \\
= & \left\| t^{\beta(\alpha+1+\theta\gamma)} u(t,x) \right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)} \\
\lesssim & \|u_0(x)\|_{\mathbb{D}^\nu(\Omega)} + t^{(1-\theta)\beta\gamma} \sup_{\tau\in(0,t)} \left\| \tau^{\beta(\alpha+1+\theta\gamma)} f(\tau,x) \right\|_{\mathbb{D}^\nu(\Omega)} .
\end{aligned}
$$

Besides, in terms of the first Eq (3.1), we have

$$
\begin{aligned}
&\left\|t^{\beta(\alpha+1+(\theta-1)\gamma)}{}_*\mathbb{D}_\beta^{\alpha,\gamma}u(t,x)\right\|_{\mathbb{D}^{\nu+(\theta-1)\mu}(\Omega)}\\
=&\left\|t^{\beta(\alpha+1+\theta\gamma)}((-\Delta)^\mu u(t,x)+f(t,x))\right\|_{\mathbb{D}^{\nu+(\theta-1)\mu}(\Omega)}\\
\lesssim&\left\|t^{\beta(\alpha+1+\theta\gamma)}((-\Delta)^\mu u(t,x)\right\|_{\mathbb{D}^{\nu+(\theta-1)\mu}(\Omega)}+\left\|t^{\beta(\alpha+1+\theta\gamma)}f(t,x))\right\|_{\mathbb{D}^{\nu+(\theta-1)\mu}(\Omega)}\\
\lesssim&\|u_0(x)\|_{\mathbb{D}^\nu(\Omega)}+t^{(1-\theta)\beta\gamma}\sup_{\tau\in(0,t)}\left\|\tau^{\beta(\alpha+1+\theta\gamma)}f(\tau,x)\right\|_{\mathbb{D}^\nu(\Omega)}\\
&+\left\|t^{\beta(\alpha+1+\theta\gamma)}f(t,x))\right\|_{\mathbb{D}^{\nu+(\theta-1)\mu}(\Omega)}\\
\lesssim&\|u_0(x)\|_{\mathbb{D}^\nu(\Omega)}+(1+t^{(1-\theta)\beta\gamma})\sup_{\tau\in(0,t)}\left\|\tau^{\beta(\alpha+1)}f(\tau,x)\right\|_{\mathbb{D}^\nu(\Omega)}.
\end{aligned}
$$

Hence, we obtain (3.11) and complete the proof.

Remark 3.3. *It is easy to verify that* $f(t,x)\in C_{-\beta(\alpha+1+\theta\gamma)-\epsilon}((0,+\infty);\mathbb{D}^\nu(\Omega))$ *for any* $\epsilon>0$ *in Theorem 3.2, then* $f(t,x)\in C_{\beta\delta}((0,+\infty);\mathbb{D}^\nu(\Omega))$ *which is given in Theorem 2.8.*

Remark 3.4. *The condition* $\sup_{\tau\in(0,t)}\left\|t^{\beta(\alpha+1+\theta\gamma)}f(t,x)\right\|_{\mathbb{D}^\nu(\Omega)}<M$ *is natural for* $\alpha\in(-1,+\infty)$*,* $\beta\in(0,+\infty)$*,* $\gamma\in(0,1]$ *and* $\theta\in[0,1]$*, because* $\beta(\alpha+1+\theta\gamma)>0$ *for* $\theta=0$*.*

Remark 3.5. *The Caputo type modification of the Erdélyi-Kober fractional differential operator has smooth effects; the regularity of the solution is higher than the initial datum with* $\theta\mu$ *order.*

4. Local well-posedness for space-time fractional diffusion equation with nonlinear source term

First, we consider

$$
\begin{cases}
t^{-\beta\gamma}{}_*D_\beta^{\alpha,\gamma}u+(-\Delta)^\mu u=|\nabla u|^p, & x\in\Omega, t>0,\\
u(t,x)=0, \quad x\in\partial\Omega, t>0,\\
\lim_{t\to 0}t^{\beta(\alpha+1)}u(t,x)=u_0(x), & x\in\Omega
\end{cases}
\tag{4.1}
$$

with a gradient nonlinearity, where $\alpha\in(-1,+\infty)$, $\beta\in(0,\infty)$, $\gamma\in(0,1]$, $\mu\in(0,1)$, $\Omega\subseteq\mathbb{R}^n$, $n\in\mathbb{N}^+$. Given ν,θ,p satisfy

$$
\begin{aligned}
&0\le\theta\le 1,\quad \frac{1}{2}\le\nu<\frac{n}{4}+\frac{1}{2}, \nu\le\theta\mu<\frac{n}{4}+\nu,\\
&0<p\le\min\{\frac{2n}{m(n-4(\nu-\frac{1}{2}))},\frac{\alpha+\gamma+1}{\alpha+\theta\gamma+1}\},
\end{aligned}
\tag{4.2}
$$

where $m=\max\{1,\frac{2n}{n-4(\nu-\theta\mu)}\}$.

Based on Lemma 2.2, we show a similar result for the problem (4.1) in the same space.

Theorem 4.1. (Uniqueness and stability) *Under the conditions* (4.2)*, prescribed* $u_0\in\mathbb{D}^\nu(\Omega)$*, if* u *is a solution of the problem* (4.1)*, which satisfies*

$$
\sup_{t\in(0,T)}\left\|t^{\beta(\alpha+1+\theta\gamma)}u(t,x)\right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)}\le M \tag{4.3}
$$

for some given positive constants M and T, then the solution $u \in C((0,T];\mathbb{D}^{\nu+\theta\mu}(\Omega))$ *is uniqueness and stability.*

Proof. Assume $u_i(t,x)$ is a solution of the following problem

$$\begin{cases} t^{-\beta\gamma}{}_*D_\beta^{\alpha,\gamma}u_i + (-\Delta)^\mu u_i = |\nabla u_i|^p, \quad x\in\Omega, t>0,\\ u_i(t,x) = 0, \quad x\in\partial\Omega, t>0,\\ \lim_{t\to 0} t^{\beta(\alpha+1)}u_i(t,x) = u_{0i}(x), \quad x\in\Omega \end{cases} \tag{4.4}$$

for $u_{01}(x) = u_{02}(x) + \epsilon$.

Uniqueness If $\epsilon = 0$, and $u_1(t,x) \neq u_2(t,x)$, then set $U(t,x) = u_1(t,x) - u_2(t,x)$, we obtain

$$\begin{cases} t^{-\beta\gamma}{}_*D_\beta^{\alpha,\gamma}U + (-\Delta)^\mu U = |\nabla u_1|^p - |\nabla u_2|^p, \quad x\in\Omega, t>0,\\ U(t,x) = 0, \quad x\in\partial\Omega, t>0,\\ \lim_{t\to 0} t^{\beta(\alpha+1)}U(t,x) = 0, \quad x\in\Omega, \end{cases} \tag{4.5}$$

Under the condition $0 < p < \frac{\alpha+\gamma+1}{\alpha+1+\theta\gamma}$, there exist

$$\int_0^1 s^{\alpha+\gamma-p(\alpha+1+\theta\gamma)}(1-s)^{\gamma(1-\theta)-1}ds \lesssim 1. \tag{4.6}$$

According to Lemma 2.2, the nonlinear term satisfies

$$\||\nabla u_1|^p - |\nabla u_2|^p\|_{\mathbb{D}^\nu(\Omega)} < (\|u_1\|^{p-1}_{\mathbb{D}^{\nu+\theta\mu}(\Omega)} + \|u_2\|^{p-1}_{\mathbb{D}^{\nu+\theta\mu}(\Omega)})\,\|U\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)}. \tag{4.7}$$

Then, in terms of (4.6) and (4.7) and Lemma 3.1 (3.6), we have

$$\begin{aligned} &\left\|t^{\beta(\alpha+1+\theta\gamma)}U(t,x)\right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)}\\ \lesssim&\left\|t^{\beta(\alpha+1+\theta\gamma)}t^{-\beta(\alpha+\gamma)}\int_0^t \tau^{\beta(\alpha+\gamma)}R_2(t,\tau)(|\nabla u_1|^p - |\nabla u_2|^p)d(\tau^\beta)\right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)}\\ \lesssim& t^{\beta(\alpha+\gamma+1)}\int_0^t (\frac{\tau}{t})^{\beta(\gamma+\alpha)}(1-(\frac{\tau}{t})^\beta)^{\gamma(1-\theta)-1}\,\||\nabla u_1|^p - |\nabla u_2|^p\|_{\mathbb{D}^\nu(\Omega)}\,d((\frac{\tau}{t})^\beta)\\ \lesssim& t^{\beta(\alpha+\gamma+1)}\int_0^t (\frac{\tau}{t})^{\beta(\gamma+\alpha)}(1-(\frac{\tau}{t})^\beta)^{\gamma(1-\theta)-1}\\ &\times(\|u_1\|^{p-1}_{\mathbb{D}^{\nu+\theta\mu}(\Omega)} + \|u_2\|^{p-1}_{\mathbb{D}^{\nu+\theta\mu}(\Omega)})\,\|U\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)}\,d((\frac{\tau}{t})^\beta)\\ \lesssim& t^{\beta(\alpha+\gamma+1-p(\alpha+1+\theta\gamma))}\sup_{\tau\in(0,t)}\left\|\tau^{\beta(\alpha+1+\theta\gamma)}U(\tau,x)\right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)}\\ &\times\int_0^t (\frac{\tau}{t})^{\beta(\gamma+\alpha-p(\alpha+1+\theta\gamma))}(1-(\frac{\tau}{t})^\beta)^{\gamma(1-\theta)-1}d((\frac{\tau}{t})^\beta)\\ \lesssim& t^{\beta(\alpha+\gamma+1-p(\alpha+1+\theta\gamma))}\sup_{\tau\in(0,t)}\left\|\tau^{\beta(\alpha+1+\theta\gamma)}U(\tau,x)\right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)}. \end{aligned} \tag{4.8}$$

Besides, for some small $T_0 \in [0,1)$, there exists $T \in [0,T_0]$ such that

$$\left\|T^{\beta(\alpha+1+\theta\gamma)}U(T,x)\right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)} = \sup_{t\in[0,T_0]}\left\|\tau^{\beta(\alpha+1+\theta\gamma)}U(\tau,x)\right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)}.$$

Then, for all $t \in [0, T_0]$, it follows

$$\left\| t^{\beta(\alpha+1+\theta\gamma)} U(t,x) \right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)} \leq \frac{1}{2} \left\| t^{\beta(\alpha+1+\theta\gamma)} U(t,x) \right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)}$$

in terms of (4.8). This yields

$$\sup_{t\in[0,T]} \left\| t^{\beta(\alpha+1+\theta\gamma)} U(t,x) \right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)} = 0,$$

which means $U(t,x) \equiv 0$ for all $t \in (0, T]$. Then the uniqueness of the solution is established.
Stability If $|\epsilon| \ll 1$, set $U(t,x) = u_1(t,x) - u_2(t,x)$, we consider

$$\begin{cases} t^{-\beta\gamma} {}_*D_\beta^{\alpha,\gamma} U + (-\Delta)^\mu U = |\nabla u_1|^p - |\nabla u_2|^p, \quad x \in \Omega, t > 0, \\ U(t,x) = 0, \quad x \in \partial\Omega, t > 0, \\ \lim_{t\to 0} t^{\beta(\alpha+1)} U(t,x) = \epsilon, \quad x \in \Omega. \end{cases} \tag{4.9}$$

Then, by a direct computation, we have

$$\begin{aligned} &\left\| t^{\beta(\alpha+1+\theta\gamma)} U(t,x) \right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)} \\ \lesssim &\epsilon + t^{\beta(\alpha+\gamma+1-p(\alpha+1+\theta\gamma))} \sup_{\tau\in(0,t)} \left\| \tau^{\beta(\alpha+1+\theta\gamma)} U(\tau,x) \right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)}. \end{aligned} \tag{4.10}$$

Similarly, we can find a small $T \in (0,1)$ such that

$$\sup_{\tau\in(0,T)} \left\| t^{\beta(\alpha+1+\theta\gamma)} U(t,x) \right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)} \lesssim \epsilon. \tag{4.11}$$

This yields that the solution of the problem (4.1) continuously depends on the initial datum.
Theorem 4.2. (Existence) *Under the conditions* (4.2)*, assume $u_0 \in \mathbb{D}^\nu(\Omega)$, then there exists a solution $u \in C((0,T]; \mathbb{D}^{\nu+\theta\mu}(\Omega))$ of the problem* (4.1) *which satisfies* (4.3) *with $M = M(\|u_0(x)\|_{\mathbb{D}^\nu(\Omega)})$. Besides, if $p < 1$, then there exists ${}_*\mathbb{D}_\beta^{\alpha,\gamma} u(t,x) \in C((0,T]; \mathbb{D}^{\nu+(\theta-1)\mu}(\Omega))$.*
Proof. Denotes a set $\mathbb{S} = \{u(t,x) | \sup_{t\in(0,T]} \left\| t^{\beta(\alpha+1+\theta\gamma)} u(t,x) \right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)} < M\}$ for some $M > 0$ and $T > 0$. Consider the sequence $\{u_j(t,x)\}_{j\in\mathbb{N}} \subset \mathbb{S}$ expressed by

$$u_1(t,x) = \Gamma(\gamma) t^{-\beta(\alpha+1)} R_1(t) u_0(x),$$

$$u_{j+1}(t,x) = u_1(t,x) + t^{-\beta(\alpha+\gamma)} \int_0^t \tau^{\beta(\alpha+\gamma)} R_2(t,\tau) |\nabla u_j(\tau,x)|^p d(\tau^\beta), \ j \in \mathbb{N}.$$

Set $v_i(t,x) = t^{\beta\alpha+1+\theta\gamma} u_i(t,x)$ in the following.Then applying Lemma 3.1 (3.5), we have

$$\|v_1(t,x)\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)} \lesssim \|u_0(x)\|_{\mathbb{D}^\nu(\Omega)} < M. \tag{4.12}$$

This yields $u_1 \in \mathbb{S}$.

In the following, by use of induction methods, we prove $u_j \in S$ for $j \geq 2$. By use of a direct computation, there exists

$$\begin{aligned}&\left\|v_{j+1}(t,x)\right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)}\leq\|u_0(x)\|_{\mathbb{D}^{\nu}(\Omega)}\\&+t^{\beta(\alpha+\gamma+1)}\int_0^t(\frac{\tau}{t})^{\beta(\gamma+\alpha)}(1-(\frac{\tau}{t})^{\beta})^{\gamma(1-\theta)-1}\left\||\frac{\nabla v_j(\tau,x)}{\tau^{\beta(\alpha+1+\theta\gamma)}}|^p\right\|_{\mathbb{D}^{\nu}(\Omega)}d((\frac{\tau}{t})^{\beta}).\end{aligned}\tag{4.13}$$

Taking $u=u_j, v=0$ in Lemma 2.2, we have

$$\left\||\nabla u_j(\tau,x)|^p\right\|_{\mathbb{D}^{\nu}(\Omega)}\lesssim\left\|u_j(\tau,x)\right\|^p_{\mathbb{D}^{\nu+\theta\mu}(\Omega)}.\tag{4.14}$$

Substituting (4.14) into (4.13), it follows that

$$\begin{aligned}&\left\|v_{j+1}(t,x)\right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)}\\&\leq\|u_0(x)\|_{\mathbb{D}^{\nu}(\Omega)}+t^{\beta(\alpha+\gamma+1-p(\alpha+1+\theta\gamma))}\\&\quad\times\int_0^t(\frac{\tau}{t})^{\beta(\gamma+\alpha-p(\alpha+1+\theta\gamma))}(1-(\frac{\tau}{t})^{\beta})^{\gamma(1-\theta)-1}\\&\quad\times\left\|v_j(\tau,x)\right\|^p_{\mathbb{D}^{\nu+\theta\mu}(\Omega)}d((\frac{\tau}{t})^{\beta})\\&\lesssim\|u_0(x)\|_{\mathbb{D}^{\nu}(\Omega)}+t^{\beta(\alpha+\gamma+1-p(\alpha+1+\theta\gamma))}\sup_{\tau\in(0,t)}\left\|v_j(\tau,x)\right\|^p_{D^{\nu}(\Omega)}.\end{aligned}\tag{4.15}$$

Since $u_j\in\mathbb{S}$, and

$$\sup_{t\in(0,T)}\left\|v_j(t,x)\right\|_{D^{\nu+\theta\mu}(\Omega)}<M.\tag{4.16}$$

Then, in terms of (4.15) and (4.16), there exists some small T such that

$$\begin{aligned}&\left\|v_{j+1}(t,x)\right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)}\\&\lesssim\|u_0(x)\|_{\mathbb{D}^{\nu}(\Omega)}+T^{\beta(\alpha+\gamma+1-p(\alpha+1+\theta\gamma))}M\\&<M.\end{aligned}$$

This yields $u_{j+1}\in\mathbb{S}$. In terms of induction methods, we confirm $\{u_j\}_{j\in\mathbb{N}}\in\mathbb{S}$.

In the following, we show $\{u_j\}_{j\in\mathbb{N}}\in\mathbb{S}$ is a Cauchy convergent sequence. Consider

$$
\begin{aligned}
&\left\|v_{j+1}(t,x)-v_j(t,x)\right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)}\\
\lesssim &t^{\beta(\alpha+\gamma+1-p(\alpha+1+\theta\gamma))}\int_0^t(\frac{\tau}{t})^{\beta(\gamma+\alpha-p(\alpha+1+\theta\gamma))}(1-(\frac{\tau}{t})^{\beta})^{\gamma(1-\theta)-1}\\
&\times\left\|(|\nabla v_j(\tau,x)|^p-|\nabla v_{j-1}(\tau,x)|^p)\right\|_{\mathbb{D}^{\nu}(\Omega)}d((\frac{\tau}{t})^{\beta})\\
\lesssim &t^{\beta(\alpha+\gamma+1-p(\alpha+1+\theta\gamma))}\int_0^t(\frac{\tau}{t})^{\beta(\gamma+\alpha-p(\alpha+1+\theta\gamma))}(1-(\frac{\tau}{t})^{\beta})^{\gamma(1-\theta)-1}(\left\|v_j(\tau,x)\right\|^{p-1}_{\mathbb{D}^{\nu+\theta\mu}(\Omega)}\\
&+\left\|v_{j-1}(\tau,x)\right\|^{p-1}_{\mathbb{D}^{\nu+\theta\mu}(\Omega)})\left\|v_j(\tau,x)-v_{j-1}(\tau,x)\right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)}d((\frac{\tau}{t})^{\beta})\\
\lesssim &t^{\beta(\alpha+\gamma+1-p(\alpha+1+\theta\gamma))}\int_0^t(\frac{\tau}{t})^{\beta(\gamma+\alpha-p(\alpha+1+\theta\gamma))}(1-(\frac{\tau}{t})^{\beta})^{\gamma(1-\theta)-1}\\
&\times\left\|v_j(\tau,x)-v_{j-1}(\tau,x)\right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)}d((\frac{\tau}{t})^{\beta})\\
\lesssim &t^{\beta(\alpha+\gamma+1-p(\alpha+1+\theta\gamma))}\sup_{\tau\in(0,t)}\left\|v_j(\tau,x)-v_{j-1}(\tau,x)\right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)}.
\end{aligned}
\tag{4.17}
$$

Then, for some small T, (4.17) becomes into

$$
\begin{aligned}
&\left\|v_{j+1}(t,x)-v_j(t,x)\right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)}\\
\lesssim &T^{\beta(\alpha+\gamma+1)}\sup_{t\in(0,T)}\left\|v_j(t,x)-v_{j-1}(t,x)\right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)}\\
\le &\frac{1}{2}\sup_{t\in(0,T)}\left\|v_j(t,x)-v_{j-1}(t,x)\right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)}.
\end{aligned}
\tag{4.18}
$$

This implies $\{v_j\}_{j\in\mathbb{N}^+}$ is a Cauchy convergent sequence, which implies that there exists a $u\in\mathbb{S}$ such that

$$
\begin{aligned}
&\lim_{j\to\infty}\sup_{t\in(0,T)}\left\|v_j(t,x)-v(t,x)\right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)}\\
=&\lim_{j\to\infty}\sup_{t\in(0,T)}\left\|t^{\beta(\alpha+1+\theta\gamma)}(u_j(t,x)-u(t,x))\right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)}=0.
\end{aligned}
$$

Then, we derive

$$
\begin{aligned}
u(t,x)&=\lim_{j\to\infty}u_j(t,x)\\
&=u_1(t,x)+t^{\beta(\alpha+1)}\int_0^t(\frac{\tau}{t})^{\beta(\alpha+\gamma)}R_{2,\sigma}(t,\tau)|\nabla u(\tau,x)|^p d(\tau^{\beta}).
\end{aligned}
$$

Hence, we established the existence of the solution to the problem (4.1).

Moreover, by using a similar analysis in (4.8) with the first equation of (4.1), we obtain

$$
\begin{aligned}
&\left\| t^{\beta(\alpha+1+(\theta-1)\gamma)}{}_{*}\mathbb{D}_{\beta}^{\alpha,\gamma} u(t,x) \right\|_{\mathbb{D}^{\nu+(\theta-1)\mu}(\Omega)} \\
&= \left\| t^{\beta(\alpha+1+\theta\gamma)} ((-\Delta)^{\mu} u(t,x) + |\nabla u(t,x)|^{p}) \right\|_{\mathbb{D}^{\nu+(\theta-1)\mu}(\Omega)} \\
&\lesssim \|u_0(x)\|_{\mathbb{D}^{\nu}(\Omega)} + T^{\beta(1-p)(\alpha+1+\theta\gamma)} \sup_{t\in(0,T)} \left\| t^{\beta(\alpha+1+\theta\gamma)} u(t,x) \right\|^{p}_{\mathbb{D}^{\nu+(2\theta-1)\mu}(\Omega)} \\
&\lesssim \|u_0(x)\|_{\mathbb{D}^{\nu}(\Omega)}
\end{aligned}
$$

for some small $T > 0$.

Finally, we complete the proof of Theorem 4.2.

At last, we consider the problem (1.1) with the nonlinear source term $f(u, \nabla u)$, which satisfies $f(0,0) = 0$, and the Lipschitz condition, that is

$$
\|f(u_1, \nabla u_1) - f(u_2, \nabla u_2)\|_{\mathbb{D}^{\nu}(\Omega)} \lesssim \|u_1 - u_2\|_{\mathbb{D}^{\nu}(\Omega)} . \tag{4.19}
$$

Based on Theorem 3.2, applying the fixed point theorem or a similar method used in Theorems 4.1 and 4.2, we give the following results.

Theorem 4.3. **(Uniqueness and stability)** *Given* $u_0 \in \mathbb{D}^{\nu}(\Omega)$ *and* $\theta \in [0,1)$*, prescribed* $u \in C((0,T]; \mathbb{D}^{\nu+\theta\mu}(\Omega))$ *is a solution of the problem* (1.1) *under the condition* (4.19)*, which satisfies*

$$
\mathop{sup}_{t\in(0,T)} \left\| t^{\beta(\alpha+1+\theta\gamma)} u(t,x) \right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)} \leq M \tag{4.20}
$$

for some positive constants M and T, then the solution is unique and stable.

Proof. Assume $u_i(t,x)$, $i = 1,2$ is a solution of the following problem

$$
\begin{cases}
t^{-\beta\gamma}{}_{*}D_{\beta}^{\alpha,\gamma} u_i + (-\Delta)^{\mu} u_i = f(u_i, \nabla u_i), \quad x \in \Omega, t \in \mathbb{R}^{+}, \\
u_i(t,x) = 0, \quad x \in \partial\Omega, t \in \mathbb{R}^{+}, \\
\lim_{t\to 0} t^{\beta(\alpha+1)} u_i(t,x) = u_{0i}(x), \quad x \in \Omega
\end{cases} \tag{4.21}
$$

for $u_{01}(x) = u_{02}(x) + \epsilon$. Set $U(t,x) = u_1(t,x) - u_2(t,x)$, then there exists

$$
\begin{cases}
t^{-\beta\gamma}{}_{*}D_{\beta}^{\alpha,\gamma} U + (-\Delta)^{\mu} U = f(u_1, \nabla u_2) - f(u_1, \nabla u_2), \quad x \in \Omega, t \in \mathbb{R}^{+}, \\
U(t,x) = 0, \quad x \in \partial\Omega, t \in \mathbb{R}^{+}, \\
\lim_{t\to 0} t^{\beta(\alpha+1)} U(t,x) = \epsilon, \quad x \in \Omega.
\end{cases} \tag{4.22}
$$

Uniqueness If $\epsilon = 0$, and $u_1(t,x) \neq u_2(t,x)$, then there exists a nonzero solution U solves the problem

$$
\begin{cases}
t^{-\beta\gamma}{}_{*}D_{\beta}^{\alpha,\gamma} U + (-\Delta)^{\mu} U = f(u_1, \nabla u_2) - f(u_1, \nabla u_2), \quad x \in \Omega, t \in \mathbb{R}^{+}, \\
U(t,x) = 0, \quad x \in \partial\Omega, t \in \mathbb{R}^{+}, \\
\lim_{t\to 0} t^{\beta(\alpha+1)} U(t,x) = 0, \quad x \in \Omega.
\end{cases} \tag{4.23}
$$

According to the Lipschitz condition (4.19), the nonlinear term satisfies

$$
\|f(u_1, \nabla u_2) - f(u_1, \nabla u_2)\|_{\mathbb{D}^{\nu}(\Omega)} \lesssim \|U\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)} \tag{4.24}
$$

Then, in terms of (3.9) in Lemma 3.2, we have

$$\left\|t^{\beta(\alpha+1+\theta\gamma)}U(t,x)\right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)} \lesssim t^{(1-\theta)\beta\gamma} \sup_{\tau\in(0,t)} \left\|\tau^{\beta(\alpha+1+\theta\gamma)}U(\tau,x)\right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)} \tag{4.25}$$

Besides, for some small $T_0 \in [0,1)$, there exists $T \in [0,T_0]$ such that

$$\left\|T^{\beta(\alpha+1+\theta\gamma)}U(T,x)\right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)} = \sup_{t\in(0,T_0]} \left\|t^{\beta(\alpha+1+\theta\gamma)}U(t,x)\right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)}$$

Then, for all $t \in [0,T]$, it follows

$$\left\|t^{\beta(\alpha+1+\theta\gamma)}U(t,x)\right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)} \leq \frac{1}{2}\left\|t^{\beta(\alpha+1+\theta\gamma)}U(t,x)\right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)}$$

in terms of (4.24). This yields

$$\sup_{t\in(,T]} \left\|t^{\beta(\alpha+1+\theta\gamma)}U(t,x)\right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)} = 0$$

which means $U(t,x) \equiv 0$ for all $t \in (0,T]$. This is a contradiction with $u_1(t,x) \neq u_2(t,x)$.Then the uniqueness of the solution is established.

Stability Consider the problem (4.22) for $|\epsilon| \ll 1$, by a similar computation as deriving (3.9), we have

$$\left\|t^{\beta(\alpha+1+\theta\gamma)}U(t,x)\right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)} \lesssim \epsilon + t^{(1-\theta)\beta\gamma} \sup_{\tau\in(0,t)} \left\|\tau^{\beta(\alpha+1+\theta\gamma)}U(\tau,x)\right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)} \tag{4.26}$$

Then, based on the condition (4.20) and $\theta \in (0,1)$, we can find a small $T \in (0,1)$ such that

$$\sup_{\tau\in(0,T]} \left\|t^{\beta(\alpha+1+\theta\gamma)}U(t,x)\right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)} \lesssim \epsilon. \tag{4.27}$$

Finally, we complete the proof of the stability of the solution.

Theorem 4.4. **(Existence)** *Given* $u_0 \in \mathbb{D}^{\nu}(\Omega)$ *and* $\theta \in [0,1)$, *then there exists a solution* $u \in C((0,T];\mathbb{D}^{\nu+\theta\mu}(\Omega))$ *of the problem* (1.1)*, which satisfies* (4.20) *for* $M = M(\|u_0(x)\|_{\mathbb{D}^{\nu}(\Omega)})$. *Moreover, there exists* ${}_*\mathbb{D}_{\beta}^{\alpha,\gamma}u(t,x) \in C((0,T];\mathbb{D}^{\nu+(\theta-1)\mu}(\Omega))$.

Proof. Using a set $\mathbb{S}$ as defined in Theorem 4.2 for some $M > 0$ and $T > 0$. Define a mapping F by

$$Fu = u_1(t,x) + t^{-\beta(\alpha+\gamma)} \int_0^t \tau^{\beta(\alpha+\gamma)} R_2(t,\tau) f(u,\nabla u) d(\tau^{\beta}),$$

where

$$u_1(t,x) = \Gamma(\gamma) t^{-\beta(\alpha+1)} R_1(t) u_0(x).$$

Then applying Lemma 3.2 (3.9), we have

$$\left\|t^{\beta(\alpha+1+\theta\gamma)}Fu(t,\cdot)\right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)} \lesssim \|u_0(x)\|_{\mathbb{D}^{\nu}(\Omega)} < M \tag{4.28}$$

for some small $T > 0$. This means mapping F maps $\mathbb{S}$ into itself.

In the following, consider $Fu_1 - Fu_2$ for any $u_1, u_2 \in \mathbb{S}$. Based on Lemma 3.2 (3.9) and the Lipschitz condition (4.19), we have

$$
\begin{aligned}
&\left\|t^{\beta(\alpha+1+\theta\gamma)}(Fu-Fv)(t,x)\right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)}\\
\lesssim & t^{(1-\theta)\beta\gamma}\left\|t^{\beta(\alpha+1+\theta\gamma)}(u-v)(t,x)\right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)}\\
< & \frac{1}{2}\left\|t^{\beta(\alpha+1+\theta\gamma)}(u-v)(t,x)\right\|_{\mathbb{D}^{\nu+\theta\mu}(\Omega)}
\end{aligned}
\tag{4.29}
$$

for some small T, which implies mapping F is a contraction.

In terms of (4.28) and (4.29), we confirm that the mapping F has one fixed point in $\mathbb{S}$, and the point is the solution of the problem; then we established the existence of the solution.

Moreover, by use of (3.11) and (4.19), we obtain

$$
\begin{aligned}
&\left\|t^{\beta(\alpha+1+(\theta-1)\gamma)}{}_*\mathbb{D}_\beta^{\alpha,\gamma}u(t,x)\right\|_{\mathbb{D}^{\nu+(\theta-1)\mu}(\Omega)}\\
\lesssim & \|u_0(x)\|_{\mathbb{D}^\nu(\Omega)}+(1+t^{(1-\theta)\beta\gamma})\sup_{\tau\in(0,t)}\left\|\tau^{\beta(\alpha+1+\theta\gamma)}f(u,\nabla u)\right\|_{\mathbb{D}^\nu(\Omega)}\\
\lesssim & \|u_0(x)\|_{\mathbb{D}^\nu(\Omega)}+(1+T^{(1-\theta)\beta\gamma})\sup_{t\in(0,T)}\left\|\tau^{\beta(\alpha+1+\theta\gamma)}u(\tau,x)\right\|_{\mathbb{D}^\nu(\Omega)}\\
\lesssim & \|u_0(x)\|_{\mathbb{D}^\nu(\Omega)}.
\end{aligned}
$$

for some $T>0$. Finally, we complete the proof of Theorem 4.4.

5. Conclusions

This research on the initial boundary value problem of nonlinear fractional diffusion equation with the Caputo-type modification of the Erdélyi-Kober fractional derivative is an continuation of the work [17]. Through meticulous calculations, the smooth effects of the Caputo-type modification of the Erdélyi-Kober fractional derivative are established for the first time. Then based on this and the embedding theorem between Hilbert scales spaces and Lebesgue spaces, the well-posedness results are obtained with the nonlinear source term satisfying the Lipschitz condition or the gradient nonlinearity. Compared with the diffusion problems involving a regularized hyper-Bessel operator considered in [10, 11], we improved the interior regularity of the solution $u\in C((0,T];\mathbb{D}^{\nu+\theta\mu}(\Omega))$ with order $\theta\mu$ if the initial datum $u_0\in\mathbb{D}^\nu(\Omega)$ in our research. These results seem to be meaningful in potential applications and numerical calculations since the Caputo-type fractional models are easy to be interpreted in physical reality, whose initial datum is described with functions and their integer order derivatives, not any other fractional order derivatives.

Author contributions

All authors contributed equally to this work.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no competing interests.

References

1. V. Kiryakova, *Generalized fractional calculus and applications*, In: Pitman Research Notes in Mathematics, 1994.
2. Podlubny, I., *Fractional differential equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Elsevier, 1998.
3. R. Gorenflo, Y. Luchko, F. Mainardi, Wright functions as scale-invariant solutions of the diffusion-wave equation, *J. Comput. Appl. Math.*, **11** (2000), 175–191. https://doi.org/10.1016/S0377-0427(00)00288-0
4. V. Kiryakova, Y. Luchko, Riemann-Liouville and Caputo type multiple Erdélyi-Kober operators, *Open Phys.*, **11** (2013), 1314–1336. https://doi.org/10.2478/s11534-013-0217-1
5. V. Kiryakova,Y. Luchko, Multiple Erdélyi-Kober integrals and derivatives as operators of generalized fractional calculus, In: *Basic theory*, Boston: De Gruyter, 2019, 127–158. https://doi.org/10.1515/9783110571622-006
6. N. Bouteraa, Mustafa Inc, M. S. Hashemi, S. Benaicha, Study on the existence and nonexistence of solutions for a class of nonlinear Erdélyi-Kober type fractional differential equation on unbounded domain, *J. Geom. Phys.*, **178** (2022), 104546. https://doi.org/10.1016/j.geomphys.2022.104546
7. A-M. Hanna Latif, M. Al-Kandari, Y. Luchko, Operational method for solving fractional differential equations with the left-and right-hand sided Erdélyi-Kober fractional derivatives, *Fract. Calc. Appl. Anal.*, **23** (2020), 103–125. https://doi.org/10.1515/fca-2020-0004
8. Y. Luchko, J. Trujillo, Caputo-type modification of the Erdélyi-Kober fractional derivative, *Fract. Calc. Appl. Anal.*, **10** (2007), 249–267.
9. A. Yacine, M. Titraoui, New class of boundary value problem for nonlinear fractional differential equations involving Erdélyi-Kober derivative, *Commun. Math.*, **27** (2019), 113–141. https://doi.org/10.2478/cm-2019-0011
10. N. Hoang Luc, D. O'Regan, A. T. Nguyen, Solutions of a nonlinear diffusion equation with a regularized Hyper-Bessel operator, *Fractal Fract.*, **6** (2022), 530. https://doi.org/10.3390/fractalfract6090530
11. V. Van Au J. Singh, A. T. Nguyen, Well-posedness results and blow-up for a semi-linear time fractional diffusion equation with variable coefficients, *Electron. Res. Arch.*, **29** (2021), 3581–3607. https://doi.org/10.3934/era.2021052

12. G. Pagnini, Erdélyi-Kober fractional diffusion, *Fract. Calc. Appl. Anal.*, **15** (2012), 117–127. https://doi.org/10.2478/s13540-012-0008-1

13. L. Plociniczak, M. Świtala, Existence and uniqueness results for a time-fractional nonlinear diffusion equation, *J. Math. Anal. Appl.*, **462** (2018), 1425–1434. https://doi.org/10.1016/j.jmaa.2018.02.050

14. L. Plociniczak, M. Świtala, Numerical scheme for Erdélyi-Kober type fractional diffusion equation using Galerkin-Hermite method, *Fract. Calc. Appl. Anal.*, **25** (2022), 1651–1687. https://doi.org/10.1007/s13540-022-00063-x

15. L. Plociniczak, S. Sobieszek, Numerical scheme for integro-differential equations with Erdélyi-Kober type fractional operator, *Numer. Algor.*, **76** (2017), 125–150. https://doi.org/10.1007/s11075-016-0247-z

16. P. Souplet, Recent results and open problems on parabolic equations with gradient nonlinearities, *Electron. J. Differ. Eq.*, **20** (2001), 1–19.

17. K. Zhang, Existence results for a generalization of the time-fractional diffusion equation with variable coefficients, *Bound. Value Probl.*, **10** (2019), 2019. https://doi.org/10.1186/s13661-019-1125-0

18. K. Zhang, Existence and uniqueness of positive solution of a nonlinear differential equation with higher order Erdélyi-Kober operators, *AIMS Math.*, **9** (2024), 1358–1372. https://doi.org/10.3934/math.2024067

19. R. Courant, D. Hilbert, *Methods of mathematical physics*, John Wiley & Sons, 1989.

20. W. McLean, Strongly elliptic systems and boundary integral equations, Cambridge University Press, 2000.

21. H. J. Haubold, A. M. Mathai, R. K. Saxena, Mittag-Leffler functions and their applications, *J. Appl. Math.*, **2011** (2011), 298628. https://doi.org/10.1155/2011/298628