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Research article

Some properties of weaving K-frames in n-Hilbert space

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Abstract: K-frames are more generalized than ordinary frames, particularly in terms of their weaving properties. The study of weaving K-frames in Hilbert space has already been explored. Given the significance of n-Hilbert spaces in functional analysis, it is essential to study weaving K-frames in n-Hilbert spaces. In this paper, we introduced the notion of weaving K-frames in n-Hilbert spaces and obtained some new properties for these frames using operator theory methods. First, the concept of weaving K-frames in n-Hilbert spaces is developed, and examples are given. By virtue of auxiliary operators, such as the preframe operator, analysis operator, and frame operator, some new properties and characterizations of these frames are presented, and several new methods for their construction are given. Stability and perturbation results are discussed and new inequalities are established as applications.

Keywords: n-Hilbert space; weaving frames; weaving K-frames; K-woven frames; perturbation
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1. Introduction

Frame theory is based on the development of wavelet theory. Frames have gradually become an important tool in signal processing due to the need to solve increasingly complex real problems. A variety of new generalized frames have also emerged, and many researchers have studied K-frames in Hilbert spaces [1, 2]. For instance, Bemrose et al. [3] introduced weaving frames in Hilbert spaces. There are also many results on weaving K-frames in Hilbert spaces [4, 5].

S. Gähler [6], Diminnie et al. [7], H. Gunawan [8], and A. Misiak [9] introduced the concept of linear 2-normed spaces, 2-inner product spaces, n-normed spaces, and n-inner product spaces for $n \geq 2$.

For the last 30 years, research on 2-Hilbert space, and n-Hilbert space has been an important topic in the field of functional analysis. However, because wavelet theory and frame theory were developed relatively recently, and the classical results of frame theory are established in Hilbert spaces, few works have used frame theory for studying frames in n-Hilbert spaces. Wavelet theory researchers need to further explore this area.

Recently, A. Akbar et al. [10] studied frames in a 2-inner product space. P. Ghosh et al. [11] presented the notion of frames in n-Hilbert spaces.

Now, various generalized frames, such as G-frames, K-frames, and weaving (or woven) frames in Hilbert spaces are a hot topic in frame theory. For instance, Li et al. [13] discussed weaving g-frames in Hilbert spaces.

The motivation of this article is to study weaving K-frames in n-Hilbert spaces, we still introduce and characterize the concept of weaving K-frames in n-Hilbert spaces and present several new methods for their construction. We then present some stability and perturbation results for weaving K-frames in n-Hilbert spaces.

The systematic study of the theory of various generalized frames in n-Hilbert spaces, and in particular, the definition of various generalized frames in n-Hilbert spaces, and the study of the characterization, perturbation, stability, and constructive properties of these generalized frames, will enrich and expand the theory of frames.

Throughout this paper, suppose that H denotes a separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle$, and $B(H)$ denotes the space of all bounded linear operators on H. We also denote $R(T)$ as a range set of T, where $T \in B(H)$. Let $\mathbb{N}$ be an index set of natural numbers, and $\ell^2(\mathbb{N})$ denotes the space of square, summable scalar-valued sequences with the index set $\mathbb{N}$. For a given number, $m \in \mathbb{N}$, let $[m] = \{1, 2, \ldots, m\}$ and $[m]^c = \{m+1, m+2, \ldots\}$. As usual, we denote the set of all bounded linear operators from H to another Hilbert space K by $B(H, K)$, and if $H = K$, then $B(H, K)$ is abbreviated to $B(H)$.

2. Preliminaries

Lemma 2.1. [14] *Let $T_1, T_2 \in B(H)$. Then, there are the following equivalent statements:*

(*i*) *For some $\alpha > 0$, $T_1 T_1^* \leq \alpha^2 T_2 T_2^*$;*

(*ii*) *$R(T_1) \subseteq R(T_2)$;*

(*iii*) *$T_1 = T_2 W$ for some $W \in B(H)$.*

Lemma 2.2. [15] *Let H_1, H_2 be two Hilbert spaces and $T_1 \in B(H_1, H_2)$, where $R(T_1)$ is closed. Then, there exists $T_1^+ : H_2 \to H_1$, the pseudo-inverse of T_1, such that $T_1 T_1^+ x = x$, $\forall x \in R(T_1)$.*

Definition 2.1. [8] *Let $n \in \mathbb{N}$ and X be a linear space of dimensions $d \geq n$. let $\|\cdot, \cdots, \cdot\| : X^n \to \mathbb{R}$ be a function such that for every $v, w_1, w_2, \cdots, w_n \in X$ and $\alpha \in \mathbb{R}$, there is*

(i) $\|w_1, w_2, \cdots, w_n\| = 0$ if and only if $w_1, w_2, \cdots, w_n$ are linearly dependent;

(ii) $\|w_1, w_2, \cdots, w_n\|$ is invariant under any permutations of $w_1, w_2, \cdots, w_n$;

(iii) $\|\alpha w_1, w_2, \cdots, w_n\| = |\alpha| \|w_1, w_2, \cdots, w_n\|$, $\alpha \in \mathbb{R}$;

(iv) $\|w_1 + v, w_2, \cdots, w_n\| \leq \|w_1, w_2, \cdots, w_n\| + \|v, w_2, \cdots, w_n\|$.

The function $\|\cdot, \cdots, \cdot\| : X^n \to \mathbb{R}$ is called an n-norm on X, and the pair (X, $\|\cdot, \cdots, \cdot\|$) is called an (real) n-normed space.

Remark 2.1. Gähler introduced the concept of n-norm to generalize the notion of length, area, and volume in a real vector space (see [6]).

Definition 2.2. [9] *Let $n \in \mathbb{N}$ and X be a linear space of dimensions $d \geq n$ and let $\langle \cdot, \cdot | \cdot, \cdots, \cdot \rangle : X^{n+1} \to \mathbb{R}$ be a function such that for every $w, v, w_1, w_2, \cdots, w_n \in X$ and $\alpha \in \mathbb{R}$, we have* *(i)* $\langle w_1, w_1|w_2, \cdots, w_n\rangle \geq 0$ *and* $\langle w_1, w_1|w_2, \cdots, w_n\rangle = 0$ *if and only if* $w_1, w_2, \cdots, w_n$ *are linearly dependent;*

(ii) $\langle w, v|w_2, \cdots, w_n\rangle = \langle w, v|w_{i_2}, \cdots, w_{i_n}\rangle$ *for every permutation* $(i_2, \cdots, i_n)$ *of* $(2, \cdots, n)$*;*

(iii) $\langle w, v|w_2, \cdots, w_n\rangle = \langle v, w|w_2, \cdots, w_n\rangle$*;*

(iv) $\langle \alpha w, v|w_2, \cdots, w_n\rangle = \alpha\langle w, v|w_2, \cdots, w_n\rangle, \quad for\, every\ \alpha \in \mathbb{R}$*;*

(v) $\langle w + v, w_1|w_2, \cdots, w_n\rangle = \langle w, w_1|w_2, \cdots, w_n\rangle + \langle v, w_1|w_2, \cdots, w_n\rangle$.

The function $\langle\cdot, \cdot|\cdot, \cdots, \cdot\rangle : X^{n+1} \to \mathbb{R}$ *is called an n-inner product. Here, the pair* $(X, \langle\cdot, \cdot|\cdot, \cdots, \cdot\rangle)$ *is called a (real) n-inner product space.*

Lemma 2.3. [9] *Let X be an n-inner product space. Then,*

$$\|w_1, w_2, \cdots, w_n\| = \sqrt{\langle w_1, w_1|w_2, \cdots, w_n\rangle}$$

defines an n-norm, for which

$$\langle w, v|w_2, \cdots, w_n\rangle = \frac{1}{4}\left(\|w + v, w_2, \cdots, w_n\|^2 + \|w - v, w_2, \cdots, w_n\|^2\right)$$

and

$$\|w + v, w_2, \cdots, w_n\|^2 + \|w - v, w_2, \cdots, w_n\|^2 = 2(\|w, w_2, \cdots, w_n\|^2 + \|v, w_2, \cdots, w_n\|^2)$$

hold for all $w, v, w_1, w_2, \cdots, w_n \in X$.

Remark 2.2. [16] Any inner product space $(X, \langle\cdot, \cdot|\cdot, \cdots, \cdot\rangle)$ can be equipped with the standard n-inner product

$$\langle w, v|w_2, \cdots, w_n\rangle = \det\begin{pmatrix} \langle w, v\rangle & \langle w, w_2\rangle & \cdots & \langle w, w_n\rangle \\ \langle w_2, v\rangle & \langle w_2, w_2\rangle & \cdots & \langle w_2, w_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle w_n, v\rangle & \langle w_n, w_2\rangle & \cdots & \langle w_n, w_n\rangle \end{pmatrix}$$

and its induced n-norm.

$$\|w_1, \cdots, w_n\| = \sqrt{\det\left(\left\langle w_i, w_j\right\rangle\right)}.$$

Definition 2.3. [17] *Let* $(X, \langle\cdot, \cdot|\cdot, \cdots, \cdot\rangle)$ *be an n-inner product space and* $\{e_i\}_{i=1}^n$ *be linearly independent vectors in X. Then, for a given set* $F = \{a_2, \cdots, a_n\} \subset X$, *if* $\langle e_i, e_j|a_2, \cdots, a_n\rangle = \delta_{i,j}, i, j \in \{1, 2, \cdots, n\}$, *where*

$$\delta_{i,j} = \begin{cases} 1, & if\, i = j \\ 0, & if\, i \neq j \end{cases}$$

the family $\{e_i\}_{i=1}^n$ *is said to be F-orthogonal. If an F-orthogonal set is countable, we can arrange it in the form of a sequence* $\{e_i\}$ *and call it an F-orthogonal sequence.*

Remark 2.3. It was shown in [18] that $\ell^2(\mathbb{N})$ has its natural n-norm, which can be viewed as a generalization of its usual norm. It was proven in [17] that $\ell^2(\mathbb{N})$ has an F-orthonormal basis $\{e_j\}_{j=1}^{\infty}$.

Definition 2.4. [8] *A sequence* $\{x_k\}$ *in a linear n-normed space X is said to be convergent to some* $x \in X$ *if for every* $c_2, \cdots, c_n \in X$, $\lim_{k\to\infty} \|x_k - x, c_2, \cdots, c_n\| = 0$*, and it is called a Cauchy sequence if* $\lim_{l,k\to\infty} \|x_l - x_k, c_2, \cdots, c_n\| = 0$ *for every* $c_2, \cdots, c_n \in X$. *The space X is said to be complete if every Cauchy sequence in this space is convergent with X. An n-inner product space is called an n-Hilbert space if it is complete with respect to its induced norm.*

In order to construct the workspace for our discussion, let H be an n-Hilbert space; consider $C = \{c_2, c_3, \cdots, c_n\}$, where $c_2, c_3, \cdots, c_n$ are fixed elements in H. Let L_C be the linear subspace of H spanned by the non-empty finite set C. Then, the quotient space H/L_C is a normed linear space with respect to the norm

$$\|f + L_C\|_C = \|f, c_2 \ldots c_n\|, \; for\ all f \in H.$$

Let M_C be the orthogonal complement of L_C, that is, $H = L_C \oplus M_C$. Define $\langle f, g\rangle_C = \langle f, g\,|c_2, \cdots, c_n\rangle$ on H. Then, $\langle\cdot,\cdot\rangle_C$ is a semi-inner product for H, and this semi-inner product induces an inner product on the quotient space H/L_C, which is given by

$$\langle f + L_C, g + L_C\rangle_C = \langle f, g\rangle_C = \langle f, g\,|c_2, \cdots, c_n\rangle \;\;, for\ all f, g \in H.$$

Now, by identifying H/L_C with M_C in an obvious way, we obtain an inner product on M_C. Now, for every $f \in M_C$, we define $\|f\|_C = \sqrt{\langle f, f\rangle_C}$, and $(M_C, \|\cdot\|_C)$ is a norm space. Let H_C be the completion of the inner product space M_C.

Remark 2.4. In fact, when given an inner product space $(V, \langle\cdot,\cdot\rangle)$ and a linear independent set $\{c_1, c_2, \cdots, c_n\}$ in V, we can, in general, derive a new inner product of $\langle\cdot,\cdot\rangle^*$ from the given inner product $\langle\cdot,\cdot\rangle$ by first defining an n-inner product on V and then defining the new inner product $\langle\cdot,\cdot\rangle^*$ on V with respect to $\{c_1, c_2, \cdots, c_n\}$ (see [19]).

Remark 2.5. For any n-inner product space with $n \geq 2$, we can derive an inner product from the n-inner product so that one can develop the notion of orthogonality and the Fourier series theory in an n-inner product space just as in an inner product space (see [20]).

Definition 2.5. [11] *Let H be an n-Hilbert space and $c_2, \cdots, c_n \in H$. If there exists a constant* $0 < A \leq B < \infty$ *such that*

$$A\|f, c_2 \ldots c_n\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i\,|c_2, \cdots, c_n\rangle|^2 \leq B\|f, c_2 \ldots c_n\|^2, \; for\ all f \in H$$

then $\{f_i\}_{i=1}^{\infty}$ in H is said to be an $(A,\ B)$ frame associated with $(c_2, \cdots, c_n)$ for H with lower- and upper-frame bounds of A and B.

If $\{f_i\}_{i=1}^{\infty}$ only satisfies the right-hand side of the inequality, then $\{f_i\}_{i=1}^{\infty}$ is called a Bessel sequence associated with $(c_2, \cdots, c_n)$ for H.

Let $\{f_i\}_{i=1}^{\infty}$ be an (A, B) frame associated with $(c_2, \cdots, c_n)$ for H, with the frame bounds $A,\ B$. Then, the preframe operator for $\{f_i\}_{i=1}^{\infty}$ is

$$T_C : \ell^2(\mathbb{N}) \to H_C, T_C(\{a_i\}_{i=1}^{\infty}) = \sum_{i=1}^{\infty} a_i f_i.$$

The analysis operator for $\{f_i\}_{i=1}^{\infty}$ is

$$T_C^* : H_C \to \ell^2(\mathbb{N}), T_C^*(f) = \{\langle f, f_i | c_2 \ldots c_n\rangle\}_{i=1}^{\infty},$$

and the frame operator S_C for $\{f_i\}_{i=1}^{\infty}$ is

$$S_C : H_C \to H_C, S_C(f) = \sum_{i=1}^{\infty} \langle f, f_i\,|c_2, \cdots, c_n\rangle f_i,$$

for all $f \in H_C$.

It is easy to prove the following fact.

Let H be a n-Hilbert space. $c_2, \cdots, c_n \in H$, we say $\{f_i\}_{i=1}^{\infty}$ in H is an $(A,\ B)$ frame associated with $(c_2, \cdots, c_n)$ for H, with lower- and upper-frame bounds of A and B if and only if it is an $(A,\ B)$ frame associated with $(c_2, \cdots, c_n)$ for H_C, with lower- and upper-frame bounds of A and B.

In what follows, we use $(H, \langle \cdot, \cdot | \cdot, \cdots, \cdot \rangle)$ to denote an n-Hilbert space and I_H to denote the identity operator on H. Let $B(H_C)$ be the space of all bounded linear operators on H_C.

3. Weaving K-frames in n-Hilbert spaces

Definition 3.1. [12] *Let $K \in B(H_C)$. A sequence $\{f_i\}_{i=1}^{\infty} \subseteq H$ is said to be an $(A,\ B)$ K-frame associated with $(c_2, \cdots, c_n)$ for H if there are constants $0 < A \leq B < +\infty$ such that*

$$A\|K^* f, c_2, \cdots, c_n\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i | c_2, \cdots, c_n \rangle|^2 \leq B\|f, c_2, \cdots, c_n\|^2 \tag{3.1}$$

for all $f \in H_C$, where K^* denotes the adjoint operator of K.

Definition 3.2. *Let $\left\{ \left\{ f_{1j} \right\}_{j=1}^{+\infty}, \ldots, \left\{ f_{mj} \right\}_{j=1}^{+\infty} \right\}$ be a collection of K-frames associated with $(c_2, \cdots, c_n)$ for H. $\left\{ \left\{ f_{1j} \right\}_{j=1}^{+\infty}, \ldots, \left\{ f_{mj} \right\}_{j=1}^{+\infty} \right\}$ is said to be $(A_F,\ B_F)$ K-woven frame associated with $(c_2, \cdots, c_n)$ for H if there are constants of A_F and B_F such that for every partition $\{\sigma_1, \sigma_2, \cdots, \sigma_m\}$ of $\mathbb{N}$, $\left\{ \left\{ f_{1j} \right\}_{j \in \sigma_1}, \ldots, \left\{ f_{mj} \right\}_{j \in \sigma_m} \right\}$ is an $(A_F,\ B_F)$ K-frame associated with $(c_2, \cdots, c_n)$ for H with lower- and upper-K-frame bounds of A_F and B_F, respectively. Each collection $\left\{ \left\{ f_{1j} \right\}_{j \in \sigma_1}, \ldots, \left\{ f_{mj} \right\}_{j \in \sigma_m} \right\}$ is called a weaving associated with $(c_2, \cdots, c_n)$ for H.*

$\left\{ \left\{ f_{1j} \right\}_{j=1}^{+\infty}, \ldots, \left\{ f_{mj} \right\}_{j=1}^{+\infty} \right\}$ is said to be A_F tight K-woven frame associated with $(c_2, \cdots, c_n)$ for H, if $A_F = B_F$, and is said to be Parseval K-woven frame associated with $(c_2, \cdots, c_n)$ for H, if $A_F = B_F = 1$.

Example 3.1. *We consider the simple case when $K = I_H$ in Definitions 3.1 and 3.2. When given the subset E of $\mathbb{N}$, a family $\mathcal{F} = \left\{ \left\{ f_{ij} \right\}_{j=1}^{\infty} \right\}_{i \in E}$ of sequences $f_i (i \in E)$ in an n-Hilbert space H is present, and for every partition $\sigma = \{\sigma_i\}_{i \in E}$ of $\mathbb{N}$, let Δ be a bijection from $\Sigma(\sigma) := \cup_{i \in E} \{(i, j) : j \in \sigma_i\}$ onto $\mathbb{N}$. According to this bijection Δ, for each $n \in \mathbb{N}$, there exists a unique element (i, j) of $\Sigma(\sigma)$ such that $j \in \sigma_i$ and $n = \Delta(i, j)$. We define $f_n^{\sigma,\Delta} = f_{ij}$ and then obtain the sequence $\mathcal{F}^{\sigma,\Delta} = \{f_n^{\sigma,\Delta}\}_{n=1}^{\infty}$, denoted by $\left\{ f_{ij} \right\}_{j \in \sigma_i, i \in E}$ (or $\cup_{i \in E} \left\{ f_{ij} \right\}_{j \in \sigma_i}$ for short.) We call the sequence $\mathcal{F}^{\sigma,\Delta}$* **a woven sequence when** $K = I_H$ *of the family $\mathcal{F}$ with respect to the partition σ and the bijection Δ.*

For example, when $E = \mathbb{N}$, $\sigma_i = \{2i-1, 2i\} (i \in \mathbb{N})$, we obtain a partition $\sigma = \{\sigma_i\}_{i \in E}$ of $\mathbb{N}$. By listing the elements of $\Sigma(\sigma) = \cup_{i \in \mathbb{N}} \{(i, 2i-1), (i, 2i)\}$ according to the direction shown in Figure 1, we obtain a bijection Δ from $\Sigma(\sigma)$ onto $\mathbb{N}$. In this case, the woven sequence $\mathcal{F}^{\sigma,\Delta}$ of the family $\mathcal{F}$ with respect to the partition σ and the bijection Δ is as follows:

$$\mathcal{F}^{\sigma,\Delta} = \{f_{1,1}, f_{1,2}, f_{2,3}, f_{2,4}, f_{3,5}, f_{3,6}, f_{4,7}, f_{4,8}, \ldots, f_{i,2i-1}, f_{i,2i}, \ldots, \} := \left\{ f_{ij} \right\}_{j \in \sigma_i, i \in \mathbb{N}},$$

where $f_{i,j} = f_{ij}$.

Clearly, for any two bijections $\Delta_k (k = 1, 2)$ from $\Sigma(\sigma)$ onto $\mathbb{N}$, $\mathcal{F}^{\sigma,\Delta_1}$ is a frame (or Bessel sequence) associated to $(c_2, \cdots, c_n)$ for H if and only if $\mathcal{F}^{\sigma,\Delta_2}$ is a frame (or Bessel sequence) associated with $(c_2, \cdots, c_n)$ for H too.

$(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),\cdots,(1,2i-1),(1,2i),\cdots$
$(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),\cdots,(2,2i-1),(2,2i),\cdots$
$(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),\cdots,(3,2i-1),(3,2i),\cdots$
$\cdots\cdots$
$(i,1),(i,2),(i,3),(i,4),(i,5),(i,6),\cdots,(i,2i-1),(i,2i),\cdots$
$\cdots\cdots$

Figure 1. A woven sequence when $K = I_H$.

4. Some characterizations of weaving K-frames in n-Hilbert spaces

Theorem 4.1. *For every $i \in [m]$, let $\{f_{ij}\}_{j=1}^{\infty}$ be (A_i, B_i) K-frame associated with $(c_2, \cdots, c_n)$ for H. Then, for every partition $\{\sigma_1, \sigma_2, \cdots, \sigma_m\}$ of $\mathbb{N}$, $\cup_{i\in[m]}\{f_{ij}\}_{j\in\sigma_i}$ is a $\left(\sum_{i\in[m]} B_i\right)$ Bessel sequence associated with $(c_2, \cdots, c_n)$ for H.*

Proof. For every partition $\{\sigma_1, \sigma_2, \cdots, \sigma_m\}$ of $\mathbb{N}$, according to Definitions 3.2 and 3.1, we have

$$\left(\sum_{i\in[m]} B_i\right)\|f, c_2, \cdots, c_n\|^2 \geq \sum_{i\in[m]}\sum_{j\in\sigma_i} |\left\langle f, f_{ij}|c_2, \cdots, c_n\right\rangle|^2$$

for all $f \in H_C$. □

Theorem 4.2. *For every $i \in [m]$, let $\{f_{ij}\}_{j=1}^{\infty}$ be (A_i, B_i) K-frame associated with $(c_2, \cdots, c_n)$ for H. Then, there are the following equivalent statements:*

(i) For every partition $\{\sigma_1, \sigma_2, \cdots, \sigma_m\}$ of $\mathbb{N}$, let $L_\sigma \in B(\ell^2(\mathbb{N}), H_C)$ be defined by $L_\sigma(e_j) = f_{ij}$ if $j \in \sigma_i$ $(i = 1\cdots m)$, and there is $A_F > 0$, so that for every partition $\{\sigma_i\}_{i\in[m]}$ of $\mathbb{N}$,

$$A_F KK^* \leq L_\sigma L_\sigma^* \tag{4.1}$$

holds true, where $\{e_j\}_{j=1}^{\infty}$ is an F-orthonormal basis for $\ell^2(\mathbb{N})$.

(ii) $\left\{\left\{f_{1j}\right\}_{j=1}^{+\infty}, \ldots, \left\{f_{mj}\right\}_{j=1}^{+\infty}\right\}$ is K-woven sequence associated with $(c_2, \cdots, c_n)$ for H.

Proof. $(ii) \Rightarrow (i)$: For every partition $\{\sigma_1, \sigma_2, \cdots, \sigma_m\}$ of $\mathbb{N}$, let T_σ be the preframe operator of $\cup_{i\in[m]}\{f_{ij}\}_{j\in\sigma_i}$. Let $L_\sigma = T_\sigma$. Then, there is $L_\sigma(e_i) = T_\sigma(e_i) = f_{ij}$ for every $j \in \sigma(i \in [m])$. Let A_F be the lower K-frame bound for $\{\{f_{ij}\}_{j\in 1}^{\infty} : i \in [m]\}$.

Then, we have

$$\begin{aligned} &A_F \langle KK^* f, f|c_2, \cdots, c_n\rangle \\ &\leq \sum_{i\in[m]}\sum_{j\in\sigma_i} |\left\langle f, f_{ij}|c_2, \cdots, c_n\right\rangle|^2 \\ &= \langle L_\sigma L_\sigma^* f, f|c_2, \cdots, c_n\rangle. \end{aligned}$$

For every $f \in H_C$.

Then, there is $A_F KK^* \leq L_\sigma L_\sigma^*$.

$(i) \Rightarrow (ii)$: According to Theorem 4.1, the positive number $\sum_{i\in[m]} B_i$ is an upper K-frame bound.

For every partition $\{\sigma_1, \sigma_2, \cdots, \sigma_m\}$ of $\mathbb{N}$, then, by virtue of Eq (4.1) and the definition of L_σ in (i), we have

$$\begin{aligned}&A_F\|K^*f, c_2, \cdots, c_n\|^2\\&\leq \langle L_\sigma L_\sigma^* f, f|c_2, \cdots, c_n\rangle\\&= \sum_{j\in\mathbb{N}} |\left\langle L_\sigma^* f, e_j|c_2, \cdots, c_n\right\rangle|^2 \quad forevery\\&= \sum_{i\in[m]}\sum_{j\in\sigma_i} |\left\langle f, f_{ij}|c_2, \cdots, c_n\right\rangle|^2\end{aligned}$$

f$\in H_C$.

So, we obtain the lower K-frame inequality. Then, $\left\{\left\{f_{1j}\right\}_{j=1}^{+\infty}, \ldots, \left\{f_{mj}\right\}_{j=1}^{+\infty}\right\}$ is a K-woven sequence associated with $(c_2, \cdots, c_n)$ for H. □

Theorem 4.3. *Let* $\left\{\left\{f_{1j}\right\}_{j=1}^{+\infty}, \ldots, \left\{f_{mj}\right\}_{j=1}^{+\infty}\right\}$ *be a collection of K-frames associated with* $(c_2, \cdots, c_n)$ *for H. Then, the following statements are equivalent:*

(i) For all $T_1 \in B(H_C)$, $\left\{\{T_1(f_{1j})\}_{j=1}^{\infty}, \cdots, \{T_1(f_{mj})\}_{j=1}^{\infty}\right\}$ *is a* T_1K*-woven sequence associated with* $(c_2, \cdots, c_n)$ *for H.*

(ii) $\left\{\left\{f_{1j}\right\}_{j=1}^{+\infty}, \ldots, \left\{f_{mj}\right\}_{j=1}^{+\infty}\right\}$ *is a K-woven sequence associated with* $(c_2, \cdots, c_n)$ *for H.*

Proof. $(ii) \Rightarrow (i)$: Let the K-frame bounds for $\left\{\left\{f_{1j}\right\}_{j=1}^{+\infty}, \ldots, \left\{f_{mj}\right\}_{j=1}^{+\infty}\right\}$ be $(A_F,\ B_F)$.

For every partition $\{\sigma_1, \sigma_2, \cdots, \sigma_m\}$ of $\mathbb{N}$, according to Definitions 3.2 and 3.1, there is

$$\sum_{i\in[m]}\sum_{j\in\sigma_i} |\left\langle f, T_1(f_{ij})|c_2, \cdots, c_n\right\rangle|^2 \leq B_F\|T_1\|^2\|f, c_2, \cdots, c_n\|^2,\ for\ allf \in H_C.$$

Similarly, we have

$$\begin{aligned}&\sum_{i\in[m]}\sum_{j\in\sigma_i} |\left\langle f, T_1(f_{ij})|c_2, \cdots, c_n\right\rangle|^2 \geq A_F\|K^*T_1^* f, c_2, \cdots, c_n\|^2\\&= A_F\|(T_1K)^* f, c_2, \cdots, c_n\|^2\end{aligned}$$

for all $f \in H_C$.

It follows that, $\left\{\{T_1(f_{1j})\}_{j=1}^{\infty}, \cdots, \{T_1(f_{mj})\}_{j=1}^{\infty}\right\}$ is a $\left(A_F,\ B_F\|T_1\|^2\right)$ T_1K-woven sequence associated with $(c_2, \cdots, c_n)$ for H.

$(i) \Rightarrow (ii)$: Let $T_1 = I_H$. Then, $\{\{f_{ij}\}_{j\in I}^{\infty} : i \in [m]\}$ is K-woven sequence associated with $(c_2, \cdots, c_n)$ for H. □

5. Construction of weaving K-frames in n-Hilbert spaces

It was shown in [18] that $\ell^2((\mathbb{N})$ has its natural n-norm, which can be viewed as a generalization of its usual norm. It was proven in [17] that $\ell^2(\mathbb{N})$ has an F-orthonormal basis $\{e_j\}_{j=1}^{\infty}$.

Theorem 5.1. *Taking $p, q \in [m]$, where p, q are fixed elements in $[m]$, the following statements are equivalent:*

(i) *There exists a Bessel sequence $\left\{g_j\right\}_{j\in N}$ associated with $(c_2, \cdots, c_n)$ for H such that for all $\sigma \subset \mathbb{N}$, there is*

$$Kf = \sum_{j\in\sigma} \left\langle f, g_j | c_2, \cdots, c_n \right\rangle f_{pj} + \sum_{j\in\sigma^c} \left\langle f, g_j | c_2, \cdots, c_n \right\rangle f_{qj}, \; for \; all f \in H_C. \tag{5.1}$$

(ii) *Two K-frames $\left\{f_{pj}\right\}_{j\in\mathbb{N}}$ and $\left\{f_{qj}\right\}_{j\in\mathbb{N}}$ are K-woven sequence associated with $(c_2, \cdots, c_n)$ for H.*

Proof. (ii) $\Rightarrow$ (i). Let A_F be a lower K-frame bound of $\left\{f_{pj}\right\}_{j\in N} \cup \left\{f_{qj}\right\}_{j\in N}$. For all $\sigma \subset \mathbb{N}$, let T_F be the preframe operator of the Bessel sequence $\left\{f_{pj}\right\}_{j\in\sigma} \cup \left\{f_{qj}\right\}_{j\in\sigma^c}$. Then, $T_F\left(e_j\right) = f_{pj}$ if $j \in \sigma$, and $T_F\left(e_j\right) = f_{qj}$ if $j \in \sigma^c$, where $\left\{e_j\right\}_{j\in\mathbb{N}}$ is the F-orthonormal basis of $\ell^2(\mathbb{N})$ (see Definition 2.3 and Remark 2.4).

Since

$$A_F \langle KK^* f, f | c_2, \cdots, c_n \rangle \leq \langle T_F {T_F}^* f, f | c_2, \cdots, c_n \rangle, for \; all \; f \in H_C,$$

then we have $KK^* \leq \frac{1}{A_F} T_F {T_F}^*$ According to Lemma 2.1, there exists $W \in B(H_C, \ell^2(\mathbb{N}))$ such that $K = T_F W$. By taking $g_j = W^* e_j$ for $j \in \mathbb{N}$, then $\left\{g_j\right\}_{j\in N}$ is a Bessel sequence associated with $(c_2, \cdots, c_n)$ for H. Then, we have

$$\begin{aligned}
Kf &= T_F W f \\
&= T_F \left(\sum_{j\in\sigma} \left\langle f, W^* e_j | c_2, \cdots, c_n \right\rangle e_j + \sum_{j\in\sigma^c} \left\langle f, W^* e_j | c_2, \cdots, c_n \right\rangle e_j \right) \\
&= \sum_{j\in\sigma} \left\langle f, g_j | c_2, \cdots, c_n \right\rangle f_{pj} + \sum_{j\in\sigma^c} \left\langle f, g_j | c_2, \cdots, c_n \right\rangle f_{qj}
\end{aligned}$$

for all $f \in H_C$.

(i) $\Rightarrow$ (ii). Let B_2 be the Bessel bound of $\left\{g_j\right\}_{j\in\mathbb{N}}$. By virtue of Eq (5.1), we have

$$\begin{aligned}
\langle Kg, f | c_2, \cdots, c_n \rangle &= \sum_{j\in\sigma} \left\langle g, g_j | c_2, \cdots, c_n \right\rangle \left\langle f_{pj}, f | c_2, \cdots, c_n \right\rangle \\
&+ \sum_{j\in\sigma^c} \left\langle g, g_j | c_2, \cdots, c_n \right\rangle \left\langle f_{qj}, f | c_2, \cdots, c_n \right\rangle \\
&= \left\langle g, \sum_{j\in\sigma} \left\langle f, f_{pj} | c_2, \cdots, c_n \right\rangle g_j \right\rangle + \left\langle g, \sum_{j\in\sigma^c} \left\langle f, f_{qj} | c_2, \cdots, c_n \right\rangle g_j \right\rangle
\end{aligned}$$

for all $f, g \in H_C$.

It follows that, $K^* f = \sum_{j\in\sigma} \left\langle f, f_{pj} | c_2, \cdots, c_n \right\rangle g_j + \sum_{j\in\sigma^c} \left\langle f, f_{qj} | c_2, \cdots, c_n \right\rangle g_j$.

Thus,

$$\begin{aligned}\|K^*f, c_2, \cdots, c_n\|^2 &= \sup_{\|g,c_2,\cdots,c_n\|=1} |\langle K^*f, g, c_2, \cdots, c_n\rangle|^2 \\ &\leq 2B_2\left(\sum_{j\in\sigma}\left|\left\langle f, f_{pj}|c_2, \cdots, c_n\right\rangle\right|^2 + \sum_{j\in\sigma^c}|\left\langle f, f_{qj}|c_2, \cdots, c_n\right\rangle|^2\right).\end{aligned}$$

□

Let $\left\{f_{pj}\right\}_{j\in\mathbb{N}}$ and $\left\{f_{qj}\right\}_{j\in\mathbb{N}}$ be two Bessel sequences associated with $(c_2, \cdots, c_n)$ for H. For every $\sigma \subset \mathbb{N}$, define $S^{\sigma}_{F_p,F_q} : H_C \to H_C$ by

$$S^{\sigma}_{F_p,F_q}f = \sum_{j\in\sigma}\left\langle f, f_{pj}|c_2, \cdots, c_n\right\rangle f_{pj} + \sum_{j\in\sigma^c}\left\langle f, f_{qj}|c_2, \cdots, c_n\right\rangle f_{qj}, \quad for\ every\ f \in H_C. \tag{5.2}$$

Then, $S^{\sigma}_{F_p,F_q}$ is a positive and self-adjoint operator.

Theorem 5.2. *Taking $p, q \in [m]$, where p, q are fixed elements in $[m]$, let $\left\{f_{pj}\right\}_{j\in\mathbb{N}}$ be an $\left(A_p, B_p\right)$ K-frame associated with $(c_2, \cdots, c_n)$ for H, and let $\left\{f_{qj}\right\}_{j\in\mathbb{N}}$ be an $\left(A_q, B_q\right)$K-frame associated with $(c_2, \cdots, c_n)$ for H. If there are constants $\lambda, \mu \in [0, 1)$ such that $\|S^{\sigma}_{F_p,F_q}f - K^*f, c_2, \cdots, c_n\| \leq \lambda\|S^{\sigma}_{F_p,F_q}f, c_2, \cdots, c_n\| + \mu\|K^*f, c_2, \cdots, c_n\|$ holds for all $f \in H_C$, then $\left\{f_{pj}\right\}_{j\in\mathbb{N}}$ and $\left\{f_{qj}\right\}_{j\in\mathbb{N}}$ are $\left(\frac{(1-\mu)^2}{(1+\lambda)^2\|T_F\|^2},\ B_p + B_q\right)$ K-woven sequence associated with $(c_2, \cdots, c_n)$ for H, where T_F is the preframe operator of the Bessel sequence $\left\{f_{pj}\right\}_{j\in\sigma} \cup \left\{f_{qj}\right\}_{j\in\sigma^c}$.*

Proof. For all $f \in H_C$, there is

$$\begin{aligned}&\|S^{\sigma}_{F_p,F_q}f, c_2, \cdots, c_n\| \\ &\geq \|K^*f, c_2, \cdots, c_n\| - \|S^{\sigma}_{F_p,F_q}f - K^*f, c_2, \cdots, c_n\| \\ &\geq (1-\mu)\|K^*f, c_2, \cdots, c_n\| - \lambda\|S^{\sigma}_{F_p,F_q}f, c_2, \cdots, c_n\|.\end{aligned}$$

So, $\frac{1-\mu}{1+\lambda}\|K^*f, c_2, \cdots, c_n\| \leq \|S^{\sigma}_{F_p,F_q}f, c_2, \cdots, c_n\|$; then, by Eq (5.2) and the definition of T_F, we have

$$\begin{aligned}&\tfrac{(1-\mu)^2}{(1+\lambda)^2}\tfrac{1}{\|T_F\|^2}\|K^*f, c_2, \cdots, c_n\|^2 \\ &\leq \tfrac{1}{\|T_F\|^2}\|S^{\sigma}_{F_p,F_q}f, c_2, \cdots, c_n\|^2 \\ &= \tfrac{1}{\|T_F\|^2}\|\sum_{j\in\sigma}\left\langle f, f_{pj}|c_2, \cdots, c_n\right\rangle f_{pj} + \sum_{j\in\sigma^c}\left\langle f, f_{qj}|c_2, \cdots, c_n\right\rangle f_{qj}\|^2 \\ &\leq (B_p + B_q)\|f, c_2, \cdots, c_n\|^2,\ for\ all\ f \in H_C.\end{aligned}$$

□

Theorem 5.3. *Taking $i = p, q, r \in [m]$, where p, q, r are fixed elements in $[m]$, let $\left\{f_{ij}\right\}_{j\in\mathbb{N}}$ be an (A_i, B_i) K-frame associated with $(c_2, \cdots, c_n)$ for H, and let T_i be the preframe operator. Let $\left\{f_{pj}\right\}_{j\in\mathbb{N}}$ and $\left\{f_{qj}\right\}_{j\in\mathbb{N}}$ be K-woven sequence associated with $(c_2, \cdots, c_n)$ for H, with lower K-frame bounds of A^{pq}, and let $\left\{f_{pj}\right\}_{j\in\mathbb{N}}$ and $\left\{f_{rj}\right\}_{j\in\mathbb{N}}$ be K-woven sequence associated with $(c_2, \cdots, c_n)$ for H with lower K-frame bounds of A^{pr}. If $K \in B(H_C)$ is a positive and closed range operator, and if $Kf = \sum_{j\in N}\left\langle f, f_{qj}|c_2, \cdots, c_n\right\rangle f_{rj}$ holds for all $f \in H_C$ and there is $A^{pq} + A^{pr} > \left(B_p + 2\sqrt{B_qB_r}\right)\|K^+\|^2$, then $\left\{f_{pj}\right\}_{j\in\mathbb{N}}$ and $\left\{f_{qj} + f_{rj}\right\}_{j\in\mathbb{N}}$ are $\left(A^{pq} + A^{pr} - B_p + 2\sqrt{B_qB_r}\|K^+\|^2,\ B_p + 2\left(B_q + B_r\right)\right)$ K-woven sequence associated with $(c_2, \cdots, c_n)$ for $R(K)$.*

Proof. For all $\sigma \subset \mathbb{N}$, let $T^{\sigma}_{f_q}\left(\left\{a_j\right\}_{j\in N}\right) = \sum_{j\in\sigma} a_j f_{qj}$ and $T^{\sigma}_{f_r}\left(\left\{a_j\right\}_{j\in N}\right) = \sum_{j\in\sigma} a_j f_{rj}$ for any $\left\{a_j\right\}_{j\in N} \in \ell^2(\mathbb{N})$. Then, there is $\|T^{\sigma}_{f_q}\| \le \|T_q\|$ and $\|T^{\sigma}_{f_r}\| \le \|T_r\|$.

Hence,

$$\begin{aligned}
&(A^{pq} + A^{pr} - (B_p + 2\sqrt{B_q B_r})\|K^{+}\|^2)\|K^* f, c_2, \cdots, c_n\|^2,\\
&\le (A^{pq} + A^{pr})\|K^* f, c_2, \cdots, c_n\|^2 - B_p\|f, c_2, \cdots, c_n\|^2\\
&-\|f, c_2, \cdots, c_n\|\|T^{\sigma}_{f_r} T^*_q f - T^{\sigma}_{f_q} T^*_r f, c_2, \cdots, c_n\|\\
&\le \sum_{j\in\sigma}\left|\left\langle f, f_{pj}|c_2, \cdots, c_n\right\rangle\right|^2 + \sum_{j\in\sigma^c}\left|\left\langle f, f_{qj} + f_{rj}|c_2, \cdots, c_n\right\rangle\right|^2
\end{aligned}$$

for all $f \in R(K)$.

The proof for concluding that the upper bound is $B_p + 2\left(B_q + B_r\right)$, which is similar. □

Theorem 5.4. *Taking $p, q \in [m]$, where p, q are fixed elements in $[m]$, let $\left\{f_{pj}\right\}_{j\in\mathbb{N}}$ be an (A_p, B_p) K-frame associated with $(c_2, \cdots, c_n)$ for H, and let $\left\{f_{qj}\right\}_{j\in\mathbb{N}}$ be an (A_q, B_q) K-frame associated with $(c_2, \cdots, c_n)$ for H, if there are constants of $0 \le \lambda, \mu < 1$ such that*

$$\begin{aligned}
&\left(\sum_{j\in\mathbb{I}}|\left\langle f, f_{qj} - f_{pj}|c_2, \cdots, c_n\right\rangle|^2\right)^{1/2} \le \lambda\left(\sum_{j\in\mathbb{I}}|\left\langle f, f_{pj}|c_2, \cdots, c_n\right\rangle|^2\right)^{1/2}\\
&+\mu\left(\sum_{j\in\mathbb{I}}|\left\langle f, f_{qj}|c_2, \cdots, c_n\right\rangle|^2\right)^{1/2}
\end{aligned} \tag{5.3}$$

for all I$\subset \mathbb{N}$.

Then, $\left\{f_{pj}\right\}_{j\in\mathbb{N}}$ and $\left\{f_{qj}\right\}_{j\in\mathbb{N}}$ are $\left(\left(A_p \min\left\{1, (\frac{1-\lambda}{1-\mu})^2\right\}\right),\ B_p + B_q\right)$ K-woven sequence associated with $(c_2, \cdots, c_n)$ for H.

Proof. For all $\sigma \subset \mathbb{N}$, according to Eq (5.3), we have

$$\begin{aligned}
&\left(\sum_{j\in\sigma^c}\left|\left\langle f, f_{qj}|c_2, \cdots, c_n\right\rangle\right|^2\right)^{1/2}\\
&\ge (1-\lambda)\left(\sum_{j\in\sigma^c}|\left\langle f, f_{pj}|c_2, \cdots, c_n\right\rangle|^2\right)^{1/2} - \mu\left(\sum_{j\in\sigma^c}|\left\langle f, f_{qj}|c_2, \cdots, c_n\right\rangle|^2\right)^{1/2}
\end{aligned}$$

for all $f \in H_C$.

Then, we have

$$\left(\tfrac{1-\lambda}{1+\mu}\right)^2\sum_{j\in\sigma^c}\left|\left\langle f, f_{pj}|c_2, \cdots, c_n\right\rangle\right|^2 \le \sum_{j\in\sigma^c}\left|\left\langle f, f_{qj}|c_2, \cdots, c_n\right\rangle\right|^2.$$

Hence,

$$\begin{aligned}
&\left(A_p \min\left\{1, (\tfrac{1-\lambda}{1+\mu})^2\right\}\right)\|K^* f, c_2 \ldots c_n\|^2\\
&\le \sum_{j\in\sigma}\left|\left\langle f, f_{pj}|c_2, \cdots, c_n\right\rangle\right|^2 + \left(\tfrac{1-\lambda}{1+\mu}\right)^2\sum_{j\in\sigma^c}\left|\left\langle f, f_{pj}|c_2, \cdots, c_n\right\rangle\right|^2\\
&\le \left(B_p + B_q\right)\|f, c_2, \cdots, c_n\|^2, for\ all f \in H_C.
\end{aligned}$$

□

Theorem 5.5. *Taking $p, q \in [m]$, where p, q are fixed elements in $[m]$, let $\left\{f_{pj}\right\}_{j\in\mathbb{N}}$ and $\left\{f_{qj}\right\}_{j\in\mathbb{N}}$ be $(A_F,\ B_F)$ K-woven sequence associated with $(c_2, \cdots, c_n)$ for H, and let $\left\{g_{pj}\right\}_{j\in\mathbb{N}}$ and $\left\{g_{qj}\right\}_{j\in\mathbb{N}}$ be$(A_G,\ B_G)$ K-woven sequence associated with $(c_2, \cdots, c_n)$ for H. For all $\sigma \subset \mathbb{N}$, let T_F^σ be the preframe operators of $\left\{f_{pj}\right\}_{j\in\sigma} \cup \left\{f_{qj}\right\}_{j\in\sigma^c}$ and let T_G^σ be the preframe operators of $\left\{g_{pj}\right\}_{j\in\sigma} \cup \left\{g_{qj}\right\}_{j\in\sigma^c}$. If there are constants of $0 \le \lambda, \mu < 2$ such that*

$$\begin{aligned}&\sum_{j\in\sigma}\left|\left\langle f, f_{pj} - g_{pj}|c_2, \cdots, c_n\right\rangle\right|^2 + \sum_{j\in\sigma^c}\left|\left\langle f, f_{pj} - g_{qj}|c_2, \cdots, c_n\right\rangle\right|^2\\ &\le \lambda\left(\sum_{j\in\sigma}\left|\left\langle f, f_{pj}|c_2, \cdots, c_n\right\rangle\right|^2 + \sum_{j\in\sigma^c}\left|\left\langle f, f_{qj}|c_2, \cdots, c_n\right\rangle\right|^2\right)\\ &+\mu\left(\sum_{j\in\sigma}\left|\left\langle f, g_{pj}|c_2, \cdots, c_n\right\rangle\right|^2 + \sum_{j\in\sigma^c}\left|\left\langle f, g_{qj}|c_2, \cdots, c_n\right\rangle\right|^2\right)\end{aligned}$$

for all $\sigma \subset \mathbb{N}$ and for all $f \in H_C$, then $\left\{f_{pj} + g_{pj}\right\}_{j\in\mathbb{N}}$ and $\left\{f_{qj} + g_{qj}\right\}_{j\in\mathbb{N}}$ are $(((2-\lambda)A_F + (2-\mu)A_G, 2(B_F + B_G))$ K-woven sequence associated with $(c_2, \cdots, c_n)$ for H.

Proof. According to the assumption, for all $\sigma \subset \mathbb{N}$, we have $\|T_F^\sigma f - T_G^\sigma f, c_2, \cdots, c_n\|^2 \le \lambda\|T_F^\sigma f, c_2, \cdots, c_n\|^2 + \mu\|T_G^\sigma f, c_2, \cdots, c_n\|^2$ for all $f \in H_C$; thus,

$$\begin{aligned}&\sum_{j\in\sigma}\left|\left\langle f, f_{pj} + g_{pj}|c_2, \cdots, c_n\right\rangle\right|^2 + \sum_{j\in\sigma^c}\left|\left\langle f, f_{qj} + g_{qj}|c_2, \cdots, c_n\right\rangle\right|^2\\ &\ge 2\|T_F^\sigma f, c_2, \cdots, c_n\|^2 + 2\|T_G^\sigma f, c_2, \cdots, c_n\|^2\\ &-\lambda\|T_F^\sigma f, c_2, \cdots, c_n\|^2 - \mu\|T_G^\sigma f, c_2, \cdots, c_n\|^2\\ &\ge ((2-\lambda)A_F + (2-\mu)A_G)\|K^* f, c_2, \cdots, c_n\|^2\end{aligned}.$$

□

6. Some stability and perturbation results on weaving K-frames in n-Hilbert spaces

Theorem 6.1. *For all $i \in [m]$, let $\left\{f_{ij}\right\}_{j\in\mathbb{N}}$ be an (A_i, B_i) K-frame associated to $(c_2, \cdots, c_n)$ for H, and let T_{f_i} be the preframe operator. For all $\sigma \subset \mathbb{N}$, let $T_{f_i}^\sigma$ be $T_{f_i}^\sigma(\{a_j\}_{j\in N}) = \sum_{j\in\sigma} a_j f_{i,j}$, and $R\left(T_{f_i}^\sigma\right) \subset R(K)\,(i \in [m])$. Suppose that $R\left(T_{f_i}^\sigma\right) \subset R(K)\,(i \in [m])$. If there are the constants $\alpha_i, \beta_i, \gamma_i \ge 0\,(i \in [m])$ such that*

$$\sum_{i\in[m]\setminus\{n\}}\left(\sqrt{B_n} + \sqrt{B_i}\right)\left(\alpha_i\sqrt{B_n} + \beta_i\sqrt{B_i} + \gamma_i\right)\|\widetilde{K^*}^{-1}\|^2 < A_n$$

and

$$\begin{aligned}&\lambda_i\left\|\sum_{j\in N} a_j f_{nj}, c_2, \cdots, c_n\right\| + \mu_i\left\|\sum_{j\in N} a_j f_{ij}, c_2, \cdots, c_n\right\| + \gamma_i\left(\sum_{j\in N}|a_j|^2\right)^{1/2}\\ &\ge \left\|\sum_{j\in N} a_j(f_{nj} - f_{ij}), c_2, \cdots, c_n\right\|\\ &(\ i\in [m]\setminus\{n\})\end{aligned} \tag{6.1}$$

for some fixed $n \in [m]$ and for any sequence of scalars $\left\{a_j\right\}_{c\in\mathbb{N}} \in \ell^2(\mathbb{N})$, then $\left\{\left\{f_{1j}\right\}_{j\in\mathbb{N}}, \cdots, \left\{f_{mj}\right\}_{j\in\mathbb{N}}\right\}$ is $\left(\left(A_n - \sum_{i\in[m]\setminus\{n\}}(\sqrt{B_n + B_i})(\lambda_i\sqrt{B_n} + \mu_i\sqrt{B_i} + \gamma_i)\|\widetilde{K^*}^{-1}\|^2\right)^2,\ \sum_{i\in[m]} B_i\right)$ K-woven sequence associated *with* $(c_2, \cdots, c_n)$ *for H, where* $\widetilde{K^*} : Ker^{\perp}(K^*) \to R(K^*)$ *is the restriction of* K^* *on* $Ker^{\perp}(K^*)$.

Proof. There is $\|T_{f_n}\| \leq \sqrt{B_n}$ and $\|T_{f_i}\| \leq \sqrt{B_i}$ for a fixed $n \in [m]$ and $i \in [m] \setminus \{n\}$. By using Eq (6.1), we have

$$\begin{aligned}&\|T_{f_n}\{a_j\}_{j\in N} - T_{f_i}\{a_j\}_{j\in N}, c_2, \cdots, c_n\|\\ &\leq \left(\lambda_i \sqrt{B_n} + \mu_i \sqrt{B_i} + \gamma_i\right) \|\{a_j\}_{j\in N}\|_{\ell^2(N)}\end{aligned}$$

for all $\sigma \subset \mathbb{N}, \left\{a_j\right\}_{j\in N} \in \ell^2(\mathbb{N})$, and for all $i \in [m] \setminus \{n\}$.

Therefore, $\|T_{f_n} - T_{f_i}\| \leq \lambda_i \sqrt{B_n} + \mu_i \sqrt{B_n} + \gamma_i$. Then, for every partition $\{\sigma_i\}_{i\in[m]}$ of N, we have

$$\begin{aligned}&\|T^{\sigma_i}_{f_n} T^*_{f_n} - T^{\sigma_i}_{f_i} T^*_{f_i}\|\\ &\leq \left(\|T_{f_n}\| + \|T_{f_i}\|\right)\|T_{f_n} - T_{f_i}\|\\ &\leq \left(\sqrt{B_n} + \sqrt{B_i}\right)\left(\lambda_i \sqrt{B_n} + \mu_i \sqrt{B_n} + \gamma_i\right)\end{aligned}$$

It follows that

$$\begin{aligned}&\sum_{i\in[m]} \sum_{j\in\sigma_i} \left|\left\langle g, f_{ij} | c_2, \cdots, c_n\right\rangle\right|^2\\ &\geq \left|\sum_{j\in N} \left\langle g, f_{nj} | c_2, \cdots, c_n\right\rangle\right|^2\\ &-\|g, c_2, \cdots, c_n\| \sum_{i\in[m]\setminus\{n\}} \left\|\sum_{j\in\sigma_i} \left(\left\langle g, f_{nj} | c_2, \cdots, c_n\right\rangle f_{nj} - \left\langle g, f_{ij} | c_2, \cdots, c_n\right\rangle f_{ij}\right), c_2, \cdots, c_n\right\|^2\\ &\geq A_n \|K^* g, c_2, \cdots, c_n\|^2 - \sum_{i\in[m]\setminus\{n\}} (\sqrt{B_n + B_i})(\lambda_i \sqrt{B_n} + \mu_i \sqrt{B_i} + \gamma_i)\|g, c_2, \cdots, c_n\|^2\\ &\geq \left(A_n - \sum_{i\in[m]\setminus\{n\}} (\sqrt{B_n + B_i})(\lambda_i \sqrt{B_n} + \mu_i \sqrt{B_i} + \gamma_i)\|\widetilde{K^*}^{-1}\|\right)^2 \|K^* g, c_2, \cdots, c_n\|^2\end{aligned}$$

for all $g \in Ker^{\perp}(K^*)$.

Then, for all $f \in H_C$, $f = f_{k_1} + f_{k_2}$ holds, where $f_{k_1} \in Ker(K^*)$, and $f_{k_2} \in Ker^{\perp}(K^*)$ and *for all* $\sigma \subset N$ and *for all* $i \in [m]$, $R\left(T^{\sigma_i}_{f_n}\right) \subset R(K)$ holds. So, we have

$$\begin{aligned}&\left(A_n - \sum_{i\in[m]\setminus\{n\}} (\sqrt{B_n + B_i})(\lambda_i \sqrt{B_n} + \mu_i \sqrt{B_i} + \gamma_i)\|\widetilde{K}^{*-1}\|\right)^2 \|K^* f, c_2, \cdots, c_n\|^2\\ &\leq \sum_{i\in[m]} \sum_{j\in\sigma_i} \left|\left\langle f_{k_2}, f_{ij} | c_2, \cdots, c_n\right\rangle\right|^2\\ &= \sum_{i\in[m]} \sum_{j\in\sigma_i} \left|\left\langle f, f_{ij} | c_2, \cdots, c_n\right\rangle\right|^2\end{aligned}$$

and the upper K-frame bound $\sum_{i\in[m]} B_i$ of $\left\{f_{ij}\right\}_{j\in N. i\in[m]}$ is obvious. This completes the proof. □

Theorem 6.2. *Let* $\left\{\left\{f_{1j}\right\}_{j\in\sigma_1}, \ldots, \left\{f_{mj}\right\}_{j\in\sigma_m}\right\}$ *be* (A_F, B_F) *K-woven sequence associated with* $(c_2, \cdots, c_n)$ *for H, and let* $T_1 \in B(H_C)$*,* $T_1 K = K T_1$*, and* T_1 *have a closed range. If* $R(K^*) \subset R(T_1)$*, then* $\left\{\{T_1(f_{1j})\}_{j\in\sigma_1}, \cdots, \{T_1(f_{mj})\}_{j\in\sigma_m}\right\}$ *is* $\left(A_F \|T^+\|^{-2}, \quad B_F \|T_1\|^2\right)$ *K-woven sequence associated with* $(c_2, \cdots, c_n)$ *for H.*

Proof. For every partition $\{\sigma_1, \sigma_2, \cdots, \sigma_m\}$ of $\mathbb{N}$, then

$$\sum_{i\in[m]}\sum_{j\in\sigma_i} |\left\langle f, T_1(f_{ij})|c_2, \cdots, c_n\right\rangle|^2 \leq B_F\|T_1\|^2\|f, c_2, \cdots, c_n\|^2 ,$$

holds for all $f \in H_C$.

Since $T_1K = KT_1$, then $K^*T_1^* = T_1^*K^*$ holds. By Lemma 2.2 and the facts that T_1 has a closed range and $R(K^*) \subset R(T_1)$,

$$\begin{aligned}&\|K^*f, c_2, \cdots, c_n\|^2\\ &\leq \|(T_1^+)\|^2\|K^*T_1^*f, c_2, \cdots, c_n\|^2, for\ all\ f \in H_C\end{aligned}$$

holds true. Thus, we have

$$\sum_{i\in[m]}\sum_{j\in\sigma_i} |\left\langle f, T_1(f_{ij})|c_2, \cdots, c_n\right\rangle|^2 \geq A_F\|T_1^+\|^{-2}\|K^*f, c_2, \cdots, c_n\|^2.$$

□

Theorem 6.3. *Let $K \in B(H_C)$ have a closed range. Let $\left\{\left\{f_{1j}\right\}_{j=1}^{+\infty}, \ldots, \left\{f_{mj}\right\}_{j=1}^{+\infty}\right\}$ be $(A_F,\ B_F)$ K-woven sequence associated with $(c_2, \cdots, c_n)$ for H. Then, $\left\{\left\{K^*f_{1j}\right\}_{j=1}^{+\infty}, \ldots, \left\{K^*f_{mj}\right\}_{j=1}^{+\infty}\right\}$ is $\left(A_F\|K^+\|^{-2},\ \ B_F\|K\|^{-2}\right)$ K^*-woven sequence associated with $(c_2, \cdots, c_n)$ for H.*

Proof. According to Lemma 2.2,

$$\|K^+\|^2\|K^*g, c_2, \cdots, c_n\|^2 \geq \|(K^+)^*K^*g, c_2, \cdots, c_n\|^2 = \|g, c_2, \cdots, c_n\|^2$$

holds true. For every partition $\{\sigma_1, \sigma_2, \cdots, \sigma_m\}$ of $\mathbb{N}$, we have

$$\begin{aligned}&A_F\|K^+\|^{-2}\|Kf, c_2 \ldots c_n\|^2\\ &\leq \sum_{i\in[m]}\sum_{j\in\sigma_i} |\left\langle f, K^*f_{ij}|c_2, \cdots, c_n\right\rangle|^2\\ &\leq B_F\|K\|^2\|f, c_2, \cdots, c_n\|^2,\ \ for\ all\ f \in H_C\end{aligned}.$$

□

Theorem 6.4. *Taking $p, q \in [m]$, where p, q are fixed elements in $[m]$, let $\{f_{pj}\}_{j\in\mathbb{N}}$ be an (A_p, B_p) K-frame associated with $(c_2, \cdots, c_n)$ for H, let $\{f_{qj}\}_{j\in\mathbb{N}}$ be an (A_q, B_q) K-frame associated with $(c_2, \cdots, c_n)$ for H, and let them be $(A_F,\ B_F)$ K-woven sequence associated with $(c_2, \cdots, c_n)$ for H. Let $T_i \in B(H_C)$ be surjective, and $T_iK = KT_i (i = p, q)$. If $Ker(K^*) \subset Ker(T_i^*)$ for $i = p, q$, and $\|T_p^+\|\|T_q - T_p\|\|\widetilde{K^*}^{-1}\| \leq \sqrt{A_F/D_q}$, then $\{T_pf_{pj}\}_{j\in\mathbb{N}}$ and $\{T_qf_{qj}\}_{j\in\mathbb{N}}$ are $\left(\left(\sqrt{A_F}\|U_p^+\|^{-1} - \sqrt{D_q}\|U_q - U_p\|^{-1}\|\widetilde{K^*}^{-1}\|\right)^2,\ \ D_p\|T_p|^2 + D_q\|T_q\|^2\right)$ K-woven sequence associated with $(c_2, \cdots, c_n)$ for H, where $\widetilde{K^*} : Ker^{\perp}(K^*) \to R(K^*)$ is the restriction of K^* on $Ker^{\perp}(K^*)$.*

Proof. For all $\sigma \subset \mathbb{N}$, we have

$$\left(\sum_{j\in\sigma}\left|\left\langle g,T_pf_{pj}|c_2,\cdots,c_n\right\rangle\right|^2+\sum_{j\in\sigma^c}\left|\left\langle g,T_qf_{qj}|c_2,\cdots,c_n\right\rangle\right|^2\right)^{1/2}$$
$$\geq\left(\sum_{j\in\sigma}\left|\left\langle T_p^*g,f_{pj}|c_2,\cdots,c_n\right\rangle\right|^2+\sum_{j\in\sigma^c}\left|\left\langle T_p^*g,f_{qj}|c_2,\cdots,c_n\right\rangle\right|^2\right)^{1/2}$$
$$-\left(\sum_{j\in\sigma^c}\left|\left\langle\left(T_q^*-T_p^*\right)g,f_{qj}|c_2...c_n\right\rangle\right|^2\right)^{1/2}$$
$$\geq\sqrt{A_F}\|K^*T_p^*g,c_2,\cdots,c_n\|-\sqrt{D_q}\|T_q^*-T_p^*\|\|g,c_2,\cdots,c_n\|$$
$$=\left(\sqrt{A_F}\|T_p^*K^*g,c_2,\cdots,c_n\|-\sqrt{D_q}\|T_q^*-T_p^*\|\|\widetilde{K^*}^{-1}\|\right)\|K^*g,c_2,\cdots,c_n\|$$

for all $g\in Ker^{\perp}(K^*)$.

For for all $f\in H_C$, we have $f=f_{k_1}+f_{k_2}$, where $f_{k_1}\in Ker(K^*)$ and $f_{k_2}\in Ker^{\perp}(K^*)$, and since $Ker(K^*)\subset Ker^{\perp}\left(U_i^*\right)(i=p,q)$, then

$$\left(\sqrt{A_F}\|T_p^+\|^{-1}-\sqrt{B_q}\|T_q-T_p\|\|\widetilde{K^*}^{-1}\|\right)^2\|K^*f,c_2,\cdots,c_n\|^2$$
$$=\left(\sqrt{A_F}\|T_p^+\|^{-1}-\sqrt{B_q}\|T_q-T_p\|\|\widetilde{K^*}^{-1}\|\right)^2\|K^*f_{k_2},c_2,\cdots,c_n\|^2$$
$$\leq\sum_{j\in\sigma}\left|\left\langle f,T_pf_{pj}|c_2,\cdots,c_n\right\rangle\right|^2+\sum_{j\in\sigma^c}\left|\left\langle f,T_qf_{qj}|c_2,\cdots,c_n\right\rangle\right|^2$$

and

$$\left(B_p\|T_p\|^2+B_q\|T_q\|^2\right)\|f,c_2,\cdots,c_n\|^2$$
$$\geq\sum_{j\in N}\left|\left\langle f,T_pf_{pj}|c_2,\cdots,c_n\right\rangle\right|^2+\sum_{j\in N}\left|\left\langle f,T_qf_{qj}|c_2,\cdots,c_n\right\rangle\right|^2$$
$$\geq\sum_{j\in\sigma}\left|\left\langle f,T_pf_{pj}|c_2,\cdots,c_n\right\rangle\right|^2+\sum_{j\in\sigma^c}\left|\left\langle f,T_qf_{qj}|c_2,\cdots,c_n\right\rangle\right|^2,\quad\forall f\in H_C$$

holds true. □

Theorem 6.5. *Taking $p,q\in[m]$, where p,q are fixed elements in $[m]$, suppose that two K-frames $\left\{f_{pj}\right\}_{j\in\mathbb{N}}$ and $\left\{f_{qj}\right\}_{j\in\mathbb{N}}$ are $(C_F,\ D_F)$ K-woven sequence, let S_F^{σ} be the frame operator of $\left\{f_{pj}\right\}_{j\in\sigma}\cup\left\{f_{qj}\right\}_{j\in\sigma^c}$, and let $U\in B(H_C)$ be a positive operator. If $US_F^{\sigma}=S_F^{\sigma}U$, then $\left\{f_{pj}+Uf_{pj}\right\}_{j\in\mathbb{N}}$ and $\left\{f_{qj}+Uf_{qj}\right\}_{j\in\mathbb{N}}$ is $\left(C_F,D_F\|I_H+U\|^2\right)$ K-woven sequence associated with $(c_2,\cdots,c_n)$ for H.*

Proof. For all $\sigma \subset \mathbb{N}$,

$$\begin{aligned}&\sum_{j\in\sigma}\left|\left\langle f, f_{pj}+Uf_{pj}|c_2,\cdots,c_n\right\rangle\right|^2+\sum_{j\in\sigma^c}\left|\left\langle f, f_{qj}+Uf_{qj}|c_2,\cdots,c_n\right\rangle\right|^2\\&\le D_F\|I_H+U\|^2\|f,c_2,\cdots,c_n\|^2\end{aligned}$$

holds true for all $f \in H_C$.

Since U is a positive operator with $US_F^\sigma = S_F^\sigma U$, we can prove that $US_F^\sigma \ge 0$ and $S_F^\sigma U^* \ge 0$. Then,

$$\begin{aligned}&\sum_{j\in\sigma}\left\langle f, f_{pj}+Uf_{pj}|c_2,\cdots,c_n\right\rangle\left(f_{pj}+Uf_{pj}\right)\\&+\sum_{j\in\sigma^c}\left\langle f, f_{qj}+Uf_{qj}|c_2,\cdots,c_n\right\rangle\left(f_{qj}+Uf_{qj}\right)\\&=S_F^\sigma f+US_F^\sigma f+S_F^\sigma U^*f+US_F^\sigma U^*f\ge S_F^\sigma f\end{aligned}$$

for every $f \in H_C$. Then,

$$\begin{aligned}&\sum_{j\in\sigma}\left|\left\langle f, f_{pj}+Uf_{pj}|c_2,\cdots,c_n\right\rangle\right|^2+\sum_{j\in\sigma^c}\left|\left\langle f, f_{qj}+Uf_{qj}|c_2,\cdots,c_n\right\rangle\right|^2\\&\ge\left\langle S_F^\sigma f, f|c_2,\cdots,c_n\right\rangle\ge C_F\|K^*f,c_2,\cdots,c_n\|^2\end{aligned}$$

holds true. □

Theorem 6.6. *Taking $p, q \in [m]$, where p, q are fixed elements in $[m]$, let $\{f_{pj}\}_{j\in\mathbb{N}}$ and $\{f_{qj}\}_{j\in\mathbb{N}}$ be $(C_F,\ D_F)$ K-woven sequence associated with $(c_2,\cdots,c_n)$ for H, and let $\{g_{pj}\}_{j\in\mathbb{N}}$ and $\{g_{qj}\}_{j\in\mathbb{N}}$ be $(C_G,\ D_G)$ K-woven sequence associated with $(c_2,\cdots,c_n)$ for H. For $\forall\sigma\subset\mathbb{N}$, let T_F^σ be the preframe operators of $F=\left\{f_{pj}\right\}_{j\in\sigma}\cup\left\{f_{qj}\right\}_{j\in\sigma^c}$, and let T_G^σ be the preframe operators of $G=\left\{g_{pj}\right\}_{j\in\sigma}\cup\left\{g_{qj}\right\}_{j\in\sigma^c}$. Let $U_p, U_q \in B(H_C)$ be co-isometrics $KU_i = U_iK$, where $i = p, q$, if $T_F^{\sigma^*}T_G^\sigma = 0$. Then, $\{U_pf_{pj}+U_qg_{pj}\}_{j\in\mathbb{N}}$ and $\{U_pf_{qj}+U_qg_{qj}\}_{j\in\mathbb{N}}$ are $\left(C_F+C_G,\quad 2\left(D_F\|U_p\|^2+D_G\|U_q\|^2\right)\right)$ K-woven sequence associated with $(c_2,\cdots,c_n)$ for H.*

Proof.

$$\begin{aligned}&\sum_{j\in\sigma}\left|\left\langle f, U_pf_{pj}+U_qg_{pj}|c_2,\cdots,c_n\right\rangle\right|^2+\sum_{j\in\sigma^c}\left|\left\langle f, U_pf_{qj}+U_qg_{qj}|c_2,\cdots,c_n\right\rangle\right|^2\\&\le 2\left(\|T_F^\sigma U_p^*f,c_2,\cdots,c_n\|^2+\|T_G^\sigma U_q^*f,c_2,\cdots,c_n\|^2\right)\\&\le 2\left(D_F\|U_p\|^2+D_G\|U_q\|^2\right)\|f,c_2,\cdots,c_n\|^2\end{aligned}$$

for all $f \in H_C$.

By virtue of $T_{\mathrm{G}}^{\sigma^*}T_{\mathrm{F}}^\sigma = T_{\mathrm{F}}^{\sigma^*}T_{\mathrm{G}}^\sigma = 0$, then

$$\begin{aligned}&(C_F+C_G)\|K^*f,c_2,\cdots,c_n\|^2\\&=C_F\|K^*U_p^*f,c_2,\cdots,c_n\|^2+C_G\|K^*U_q^*f,c_2,\cdots,c_n\|^2\\&\le\left\langle\left(T_F^\sigma U_p^*f+T_G^\sigma U_q^*f\right)^*\left(T_F^\sigma U_p^*+T_G^\sigma U_q^*\right)f,f|c_2\ldots c_n\right\rangle\\&=\sum_{j\in\sigma}\left|\left\langle f, U_pf_{pj}+U_qg_{pj}|c_2,\cdots,c_n\right\rangle\right|^2+\sum_{j\in\sigma^c}\left|\left\langle f, U_pf_{qj}+U_qg_{qj}|c_2,\cdots,c_n\right\rangle\right|^2\end{aligned}$$

for all $f \in H_C$ holds true. □

7. Applications

Based on the notion and results of weaving K-frames in n-Hilbert space, as an application, we now establish new inequalities on weaving K-frames in n-Hilbert space.

Taking $p, q \in [m]$, where p, q are fixed elements in $[m]$, when given a weaving K-frame $\{g_{pj}\}_{j\in\sigma} \cup \{g_{qj}\}_{j\in\sigma^c}$ associated to $(c_2, \cdots, c_n)$ for H, recall that a Bessel sequence, $F = \{f_j\}_{j\in\mathbb{N}}$, for H is said to be a K-dual of $\{g_{pj}\}_{j\in\sigma} \cup \{g_{qj}\}_{j\in\sigma^c}$ associated to $(c_2, \cdots, c_n)$ for H if

$$Kf = \sum_{j\in\sigma} \langle f, g_{pj}|c_2, \cdots, c_n\rangle f_j + \sum_{j\in\sigma^c} \langle f, g_{qj}|c_2, \cdots, c_n\rangle f_j, \quad for\ all\ f \in H_C.$$

For any $\sigma \subset \mathbb{N}$, for all $\{a_j\}_{j\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$, and for all $f \in H_C$, we define two bounded linear operators, T_1, $T_2 \in B(H_C)$, as follows:

$$\begin{aligned} T_1 f &= \sum_{j\in\sigma} a_j \langle f, g_{pj}|c_2, \cdots, c_n\rangle f_j + \sum_{j\in\sigma^c} a_j \langle f, g_{qj}|c_2, \cdots, c_n\rangle f_j \\ T_2 f &= \sum_{j\in\sigma} (1 - a_j)\langle f, g_{pj}|c_2, \cdots, c_n\rangle f_j + \sum_{j\in\sigma^c} (1 - a_j)\langle f, g_{qj}|c_2, \cdots, c_n\rangle f_j. \end{aligned} \tag{7.1}$$

Theorem 7.1. *Taking $p, q \in [m]$, where p, q are fixed elements in $[m]$, suppose that two K-frames $\{g_{pj}\}_{j\in\mathbb{N}}$ and $\{g_{qj}\}_{j\in\mathbb{N}}$ are K-woven sequence associated to $(c_2, \cdots, c_n)$ for H. Then, for any $\sigma \subset \mathbb{N}$, for all $\{a_j\}_{j\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$, and for all $f \in H_C$, we have*

$$\begin{aligned} \frac{3}{4}\|Kf, c_2, \cdots, c_n\|^2 &\le \left\| \sum_{j\in\sigma} a_j \langle f, g_{pj}|c_2, \cdots, c_n\rangle f_j + \sum_{j\in\sigma^c} a_j \langle f, g_{qj}|c_2, \cdots, c_n\rangle f_j \right\|^2 \\ &\quad + \mathrm{Re}\Big(\sum_{j\in\sigma} (1 - a_j)\langle f, g_{pj}|c_2, \cdots, c_n\rangle \langle f_j, Kf|c_2, \cdots, c_n\rangle \\ &\quad + \sum_{j\in\sigma^c} (1 - a_j)\langle f, g_{qj}|c_2, \cdots, c_n\rangle \langle f_j, Kf|c_2, \cdots, c_n\rangle \Big) \\ &\le \frac{3\|K\|^2 + \|T_1 - T_2\|^2}{4} \|f, c_2, \cdots, c_n\|^2, \end{aligned}$$

where T_1 and T_2 are given in Eq (7.1), and $\{f_j\}_{j\in\mathbb{N}}$ is a K-dual of $\{g_{pj}\}_{j\in\sigma} \cup \{g_{qj}\}_{j\in\sigma^c}$ associated with $(c_2, \cdots, c_n)$ for H.

Proof. The proof is divided into three steps:

Step 1.

Suppose that P, Q, $K \in B(H)$ and $P + Q = K$. Then, for each $f \in H$,

$$\|Pf\|^2 + \mathrm{Re}\langle Qf, Kf\rangle \ge \frac{3}{4}\|Kf\|^2.$$

Step 2.

For any $\sigma \subset \mathbb{N}$, for all $\{a_j\}_{j\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$, and for all $f \in H_C$, it is easy to check that $T_1 + T_2 = K$. By virtue of Step 1, we obtain

$$\left\|\sum_{j\in\sigma} a_j\langle f, g_{pj}|c_2,\cdots,c_n\rangle f_j + \sum_{j\in\sigma^c} a_j\langle f, g_{qj}|c_2,\cdots,c_n\rangle f_j, c_2,\cdots,c_n\right\|^2$$
$$+\operatorname{Re}\Big(\sum_{j\in\sigma}(1-a_j)\langle f, g_{pj}|c_2,\cdots,c_n\rangle\langle f_j, Kf|c_2,\cdots,c_n\rangle$$
$$+\sum_{j\in\sigma^c}(1-a_j)\langle f, g_{qj}|c_2,\cdots,c_n\rangle\langle f_j, Kf|c_2,\cdots,c_n\rangle\Big)$$
$$=\|T_1 f, c_2,\cdots,c_n\|^2+\operatorname{Re}\langle T_2 f, Kf|c_2,\cdots,c_n\rangle \geq \frac{3}{4}\|Kf, c_2,\cdots,c_n\|^2.$$

Step 3.

$$\left\|\sum_{j\in\sigma} a_j\langle f, g_{pj}|c_2,\cdots,c_n\rangle f_j + \sum_{j\in\sigma^c} a_j\langle f, g_{qj}|c_2,\cdots,c_n\rangle f_j, c_2,\cdots,c_n\right\|^2$$
$$+\operatorname{Re}\Big(\sum_{j\in\sigma}(1-a_j)\langle f, g_{pj}|c_2,\cdots,c_n\rangle\langle f_j, Kf|c_2,\cdots,c_n\rangle$$
$$+\sum_{j\in\sigma^c}(1-a_j)\langle f, g_{qj}|c_2,\cdots,c_n\rangle\langle f_j, Kf|c_2,\cdots,c_n\rangle\Big)$$
$$=\langle T_1 f, T_1 f|c_2,\cdots,c_n\rangle+\frac{1}{2}\langle T_2 f, Kf|c_2,\cdots,c_n\rangle+\frac{1}{2}\langle Kf, T_2 f|c_2,\cdots,c_n\rangle$$
$$=\frac{3}{4}\langle Kf, Kf|c_2,\cdots,c_n\rangle+\frac{1}{4}\langle (T_1-T_2)f, (T_1-T_2)f|c_2,\cdots,c_n\rangle$$
$$\leq \frac{3}{4}\|K\|^2\|f, c_2,\cdots,c_n\|^2+\frac{1}{4}\|T_1-T_2\|^2\|f, c_2,\cdots,c_n\|^2=\frac{3\|K\|^2+\|T_1-T_2\|^2}{4}\|f, c_2,\cdots,c_n\|^2,$$

and the proof is complete. □

8. Conclusions

In this paper, we develop the idea of weaving K-frames in n-Hilbert spaces and established some properties of these frames.

This work first introduces and discusses the concept of weaving K-frames in n-Hilbert spaces (Definitions 3.1 and 3.2) and gives examples (Example 3.1). Then, some characterization conditions of weaving K-frames in n-Hilbert space are proved by virtue of auxiliary operators, such as the preframe operator, analysis operator, and frame operator (Theorems 4.1–4.3). Then, several constructions of weaving K-frames in n-Hilbert spaces are offered by the same auxiliary operators, such as the preframe operator, analysis operator, and frame operator (Theorems 5.1–5.5). Finally, the perturbation and stability theorems of weaving K-frames in n-Hilbert spaces are discussed by virtue of the same auxiliary operators (Theorems 6.1–6.6). As applications, new inequalities on weaving K-frames in n-Hilbert spaces are established (Theorem 7.1). The obtained results further enriched the frame theory in n-Hilbert spaces.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that there are no conflicts of interest.

References

1. X. Xiao, Y. Zhu, L. Găvruţa, Some properties of K-frames in Hilbert spaces, *Results Math.*, **63** (2013), 1243–1255. https://doi.org/10.1007/s00025-012-0266-6
2. L. Găvruţa, Frames for operators, *Appl. Comput. Harmon. Anal.*, **32** (2012), 139–144. https://doi.org/10.1016/j.acha.2011.07.006
3. T. Bemrose, P. G. Cassaza, K. Grochenig, M. C. Lammers, R. G. Lynch, Weaving frames, *Oper. Matrices*, **10** (2016), 1093–1116. https://doi.org/10.7153/oam-10-61
4. Deepshikha, L. K. Vashisht, Weaving K-frames in Hilbert spaces, *Results Math.*, **73** (2018), 1–20. https://doi.org/10.1007/s00025-018-0843-4
5. Z. Xiang, Some new results of Weaving K-frames in Hilbert space, *Numer. Func. Anal. Opt.*, **42** (2021), 409–429. https://doi.org/10.1080/01630563.2021.1882488.
6. S. Gähler, Lineare 2-Normierte Raume, *Math. Nachr.*, **28** (1965), 1–43. https://doi.org/10.1002/mana.19640280102
7. C. Diminnie, S. Gähler, A. White, 2-Inner product spaces, *Demonstratio Math.*, **10** (1977), 169–188.
8. H. Gunawan, M. Mashadi, On n-normed spaces, *Int. J. Math. Math. Sci.*, **27** (2001), 631–639. https://doi.org/10.1155/S0161171201010675
9. A. Misiak, n-Inner product spaces, *Math. Nachr.*, **140** (1989), 299–319.
10. A. Arefijamaal, S. Ghadir, Frames in 2-Inner product spaces, *Iran. J. Math. Sci. Inform.*, **8** (2013), 123–130.
11. P. Ghosh, T. Samanta, Construction of frame relative to n-Hilbert space, *J. Linear Topol. Algebra*, **10** (2021), 117–130. https://doi.org/10.48550/arXiv.2101.01657
12. P. Ghosh, T. Samanta, Some properties of K-frame in n-Hilbert space, *arxiv preprint*, 2021. https://doi.org/10.48550/arXiv.2102.05255
13. D. Li, J. S. Leng, T. Huang, X. Li, On weaving g-frames for Hilbert spaces, *Complex Anal. Oper. Th.*, **14** (2020), 1–25. https://doi.org/10.1007/s11785-020-00991-7
14. R. G. Douglas, On majorization, factorization, and range inclusion of operators on Hilbert space, *Proc. Amer. Math. Soc.*, **17** (1966), 413–415. https://doi.org/10.1090/S0002-9939-1966-0203464-1
15. O. Christensen, An introduction to frames and Riesz bases, *Appl. Numer. Harmon. Anal.*, Birkhauser: Cham, Switzerland, 2016. https://doi.org/10.1007/978-3-319-25613-9

16. H. Gunawan, On convergence in n-inner product spaces, *Bull. Malays. Math. Sci. So.*, **25** (2002), 11–16.
17. A. Misiak, Orthogonality and orthonormality in n-inner product spaces, *Math. Nach.*, **143** (1989), 249–261. https://doi.org/10.1002/mana.19891430119
18. H. Gunawan, The space of p-summable sequences and its natural n-norm, *Bull. Aust. Math. Soc.*, **64** (2001), 137–147.
19. H. Gunawan, An inner product that makes a set of vectors orthonormal, *Austral. Math. Soc. Gaz.*, **28** (2001), 194–197.
20. H. Gunawan, Inner products on n-inner product spaces, *Soochow J. Math.*, **28** (2002), 389–398.