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Research article

Dual Toeplitz operators on the orthogonal complement of the harmonic Bergman space

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Abstract: This paper aimed to give some partial answers to the zero-product problem and commutativity problem concerning dual Toeplitz operators with nonharmonic symbols on the orthogonal complement of the harmonic Bergman space. Using the symbol map, we described the necessary condition for $S_{\varphi_1}S_{\varphi_2}\cdots S_{\varphi_N} = 0$ with radial symbols. Furthermore, we established the sufficient and necessary conditions for $S_\varphi S_\psi = S_\psi S_\varphi$ with $\varphi(z) = az^{p_1}\overline{z}^{q_1} + bz^{p_2}\overline{z}^{q_2}$ and $\psi(z) = z^s\overline{z}^t$.

Keywords: dual Toeplitz operator; harmonic Bergman space; zero-product problem; commutativity
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1. Introduction

Throughout this paper, we use $\mathbb{D}$ to denote the open unit disk on the complex plane $\mathbb{C}$. Let

$$dA(z) = \frac{1}{\pi}dxdy$$

be the Lebesgue area measure on $\mathbb{D}$, normalized so that the measure of $\mathbb{D}$ is 1. It is well-known that $L^2(\mathbb{D}, dA)$ is the Hilbert space of square integrable functions with the inner product

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}dA(z). \tag{1.1}$$

The Bergman space L_a^2 is the closed subspace of $L^2(\mathbb{D}, dA)$ consisting of analytic functions on $\mathbb{D}$, which is a reproducing kernel Hilbert space and its reproducing kernel (at $\lambda \in \mathbb{D}$) is given by

$$K_z(w) = \frac{1}{(1-\overline{z}w)^2}, \qquad w \in \mathbb{D}. \tag{1.2}$$

Letting P be the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto L_a^2, we have

$$Pf(z) = \langle f, K_z \rangle \tag{1.3}$$

for every $f \in L^2(\mathbb{D}, dA)$ and $z \in \mathbb{D}$. For more details about the function theory and operator theory on the Bergman space, one can consult Zhu's book [1].

The harmonic Bergman space L_h^2 is the closed subspace of $L^2(\mathbb{D}, dA)$ consisting of all complex-valued harmonic functions on $\mathbb{D}$. Observe that L_h^2 can be decomposed as

$$L_h^2 = zL_a^2 \oplus \overline{L_a^2}, \tag{1.4}$$

where $\overline{L_a^2}$ denotes the complex conjugate of L_a^2. It is easy to check that the function

$$R_z(w) = K_z(w) + \overline{K_z(w)} - 1, \quad (w \in \mathbb{D}) \tag{1.5}$$

is the reproducing kernel (at $\lambda \in \mathbb{D}$) for the harmonic Bergman space L_h^2. Denoting the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto L_h^2 by Q, then

$$Qf(z) = \langle f, R_z \rangle \tag{1.6}$$

for each $f \in L^2(\mathbb{D}, dA)$ and $z \in \mathbb{D}$. Using (1.5), we obtain that

$$Qf(z) = Pf(z) + \overline{P(\overline{f})(z)} - (Pf)(0) \tag{1.7}$$

for all $f \in L^2(\mathbb{D}, dA)$ and $z \in \mathbb{D}$. According to (1.7), routine calculations yield that

$$(I - Q)(z^n \overline{z}^m) = \begin{cases} z^n \overline{z}^m - \frac{m-n+1}{m+1} \overline{z}^{m-n}, & m > n, \\ z^n \overline{z}^m - \frac{n-m+1}{n+1} z^{n-m}, & m \leqslant n. \end{cases} \tag{1.8}$$

Additionally, we refer to the paper [2] for more knowledge about the harmonic Bergman space.

The Toeplitz operator and the Hankel operator with symbol $\varphi \in L^\infty(\mathbb{D}, dA)$ (the collection of all essentially bounded functions on the unit disk) on the harmonic Bergman space L_h^2 are defined by

$$T_\varphi f = Q(\varphi f) \tag{1.9}$$

and

$$H_\varphi f = (I - Q)(\varphi f), \tag{1.10}$$

respectively. Under the decomposition

$$L^2(\mathbb{D}, dA) = L_h^2 \oplus (L_h^2)^\perp,$$

the multiplication operator M_φ with symbol φ can be represented as

$$M_\varphi = \begin{pmatrix} T_\varphi & H_{\overline{\varphi}}^* \\ H_\varphi & S_\varphi \end{pmatrix}, \tag{1.11}$$

where the operator S_φ is defined by

$$S_\varphi f = (I - Q)f, \quad f \in (L_h^2)^\perp. \tag{1.12}$$

In fact, S_φ is bounded and linear on $(L_h^2)^\perp$ when $\varphi \in L^\infty(\mathbb{D}, dA)$. The operator S_φ is called the *dual Toeplitz operator* with symbol φ. The following elementary properties of dual Toeplitz operators on $(L_a^2)^\perp$ can be founded in [3]:

(1) $\|S_\varphi\| \leqslant \|\varphi\|_\infty$;

(2) $S_\varphi^* = S_{\overline{\varphi}}$;

(3) $S_{\alpha\varphi+\beta\psi} = \alpha S_\varphi + \beta S_\psi$ for all $\varphi, \psi \in L^\infty(\mathbb{D}, dA)$ and all complex constants α, β;

one can easily verify that the above conclusions also hold for dual Toeplitz operators on $(L_h^2)^\perp$ using the definition of S_φ.

The concept of "*dual Toeplitz operator*" was first introduced and investigated by Stroethoff and Zheng on the orthogonal complement of the Bergman space; see [3, 4]. Since then, researchers have extended the spectral theory and algebraic properties of dual Toeplitz operators on $(L_a^2)^\perp$ established in [3] to the setting of dual Toeplitz operators on the orthogonal complements of various function spaces. For instance, the Bergman space over the unit ball ([5–7]), the Bergman space over the polydisk ([8]), the Dirichlet space ([9–11]), the harmonic Bergman space ([12, 13]), and the harmonic Dirichlet space ([14]).

Recently, the investigation concerning dual Toeplitz operators on the orthogonal complement of the harmonic Bergman space $(L_h^2)^\perp$ has attracted the attention of many scholars. In 2015, Yang and Lu [13] obtained a complete characterization for the commuting dual Toeplitz operators on $(L_h^2)^\perp$ with bounded harmonic symbols. However, the corresponding commutativity problem for dual Toeplitz operators with nonharmonic symbols is still open. In 2021, Peng and Zhao [12] characterized the boundedness, compactness, spectral structure, and algebraic properties of dual Toeplitz operators on $(L_h^2)^\perp$. In addition, Wang and Zhao [15] established a necessary and sufficient condition for dual Toeplitz operators with nonharmonic symbols of the form

$$\varphi(z) = az^{n_1}\overline{z}^{m_1} + bz^{n_2}\overline{z}^{m_2}$$

to be hypo-normal on $(L_h^2)^\perp$, where n_1, n_2, m_1, m_2 are nonnegative integers and a, b are complex numbers.

Although many scholars have studied the properties of dual Toeplitz operators, there are few results on dual Toeplitz operators with nonharmonic symbols. In this paper, we try to study when the product of two dual Toeplitz operators with radial symbols equals zero and when two dual Toeplitz operators with nonharmonic symbols commute on the orthogonal complement of the harmonic Bergman space. As the function theory of $(L_h^2)^\perp$ is much more complicated than that of $(L_a^2)^\perp$, it is quite difficult to solve the zero-product problem and the commutativity problem mentioned above in general cases. In order to seek the breakthrough point of those two problems, in the present paper we consider some special radial symbols and quasi-homogeneous symbols, and give certain partial answers.

The organization of this paper is as follows. In Section 2, we will show that there exists an index k such that $\varphi_k = 0$ a.e. on $\mathbb{D}$ if

$$\prod_{k=1}^{N} S_{\varphi_k} = 0$$

in the case of

$$\varphi_k(z) = \sum_{m=0}^{\infty} a_{k,m}|z|^m, \quad k = 1, 2, \cdots, N, \tag{1.13}$$

where N is an arbitrary positive integer and each $a_{k,m}$ is a constant. Moreover, in Section 3 we give a characterization for

$$S_\varphi S_\psi = S_\psi S_\varphi$$

on $(L_h^2)^\perp$ if

$$\varphi(z) = az^{p_1}\overline{z}^{q_1} + bz^{p_2}\overline{z}^{q_2}$$

and

$$\psi(z) = z^s\overline{z}^t,$$

where $a, b \in \mathbb{C}$ such that $a \neq b$ and p_1, p_2, q_1, q_2, s, t are all nonnegative integers.

2. The zero-product problem for dual Toeplitz operators

In this section, we investigate the zero-product problem concerning dual Toeplitz operators with some special symbols on the orthogonal complement of the harmonic Bergman space $(L_h^2)^\perp$ via the symbol map, which was established in Lemma 2.1 [12, Theorem 4.1]. For the sake of completeness, we state the result on the symbol map of dual Toeplitz operators on $(L_h^2)^\perp$ as follows:

Lemma 2.1. *There is a contractive C^*-homomorphism ρ from the dual Toeplitz algebra $\mathfrak{T}(L^\infty(\mathbb{D}, dA))$ to $L^\infty(\mathbb{D}, dA)$ such that $\rho(S_\varphi) = \varphi$ for each $\varphi \in L^\infty(\mathbb{D}, dA)$.*

Let us begin with dual Toepitz operators with bounded harmonic symbols.

Proposition 2.2. *Suppose that $\varphi_1, \varphi_2, \cdots, \varphi_N$ are N bounded harmonic functions on the unit disk $\mathbb{D}$. If*

$$S_{\varphi_1}S_{\varphi_2}\cdots S_{\varphi_N} = 0,$$

then there exists $k \in \{1, 2, \cdots, N\}$ such that $\varphi_k = 0$.

Proof. By Lemma 2.1, we have that

$$\varphi_1\varphi_2\cdots\varphi_N = 0$$

if

$$S_{\varphi_1}S_{\varphi_2}\cdots S_{\varphi_N} = 0.$$

Then, applying the uniqueness theorem of harmonic functions, we deduce that there exist some $k \in \{1, 2, \cdots, N\}$ such that $\varphi_k = 0$. □

In the following proposition, we solve the zero-product problem for two dual Toeplitz operators with symbols of the form $\sum_{n=0}^{\infty} c_n|z|^n$, where each c_n is a complex constant.

Proposition 2.3. *Suppose that*

$$\varphi(z) = \sum_{n=0}^{\infty} a_n |z|^n$$

and

$$\psi(z) = \sum_{m=0}^{\infty} b_m |z|^m$$

are two bounded functions on the unit disk $\mathbb{D}$. If $S_\varphi S_\psi = 0$, then $\varphi = 0$ or $\psi = 0$.

Proof. If $S_\varphi S_\psi = 0$, then we have by Lemma 2.1 that $\varphi\psi = 0$. Note that

$$\varphi(z)\psi(z) = \sum_{k=0}^{\infty} c_k |z|^k, \tag{2.1}$$

where

$$c_k = \sum_{l=0}^{k} a_l b_{k-l}.$$

Thus, we have $c_k = 0$ for all $k \geqslant 0$. In particular, $c_0 = a_0 b_0 = 0$. Then, we consider the following three cases:

Case 1. $a_0 = 0, b_0 \neq 0$. In this case, we have that

$$0 = c_1 = a_1 b_0 + a_0 b_1 \Longrightarrow a_1 = 0; \tag{2.2}$$

$$0 = c_2 = a_2 b_0 + a_1 b_1 + a_0 b_2 \Longrightarrow a_2 = 0. \tag{2.3}$$

Continuing this process, we obtain that

$$0 = a_k b_0 + a_{k-1} b_1 + \cdots + a_1 b_{k-1} + a_0 b_k \Longrightarrow a_k = 0 \tag{2.4}$$

for $k = 1, 2, \cdots$. This yields that $a_n = 0$ for all $n \geqslant 0$.

Case 2. $a_0 \neq 0, b_0 = 0$. In this case, we get that

$$0 = c_1 = a_1 b_0 + a_0 b_1 \Longrightarrow b_1 = 0; \tag{2.5}$$

$$0 = c_2 = a_2 b_0 + a_1 b_1 + a_0 b_2 \Longrightarrow b_2 = 0. \tag{2.6}$$

Using the same argument as the one used in Case 1, we conclude that

$$0 = a_k b_0 + a_{k-1} b_1 + \cdots + a_1 b_{k-1} + a_0 b_k \Longrightarrow b_k = 0 \tag{2.7}$$

for $k = 1, 2, \cdots$. It follows that $b_m = 0$ for all $m \geqslant 0$.

Case 3. $a_0 = 0$ and $b_0 = 0$. In this case, we have

$$\varphi(z) = \sum_{n=1}^{\infty} a_n |z|^n$$

and

$$\psi(z) = \sum_{m=1}^{\infty} b_m |z|^m.$$

Then, repeating the arguments used in the proof of Cases 1 and 2, we can show that $\varphi = 0$ or $\psi = 0$.

This completes the proof of Proposition 2.3. □

The next theorem shows that the previous proposition can be generalized to the case of the product of arbitrary finitely many dual Toeplitz operators.

Theorem 2.4. *Suppose that*

$$\varphi_k(z) = \sum_{m=0}^{\infty} a_{k,m}|z|^m, \quad k = 1, 2, \cdots, N, \tag{2.8}$$

which are N bounded functions on the unit disk $\mathbb{D}$*. If*

$$S_{\varphi_1} S_{\varphi_2} \cdots S_{\varphi_N} = 0,$$

then there exist some $k \in \{1, 2, \cdots, N\}$ *such that* $\varphi_k = 0$.

Proof. Since

$$S_{\varphi_1} S_{\varphi_2} \cdots S_{\varphi_N} = 0,$$

we again conclude by Lemma 2.1 that

$$\varphi_1\varphi_2 \cdots \varphi_N = 0$$

on the disk $\mathbb{D}$. Observe that

$$\varphi_1\varphi_2 \cdots \varphi_N = \varphi_1(\varphi_2\varphi_3 \cdots \varphi_N)$$

and $\varphi_2\varphi_3 \cdots \varphi_N$ can be written as follows:

$$(\varphi_2\varphi_3 \cdots \varphi_N)(z) = \sum_{l=0}^{\infty} c_l|z|^l. \tag{2.9}$$

Applying the conclusion of Proposition 2.3, we obtain that

$$\varphi_1 = 0 \quad \text{or} \quad \varphi_2\varphi_3 \cdots \varphi_N = 0.$$

If $\varphi_1 = 0$, then we are done. Otherwise, we have that

$$0 = \varphi_2\varphi_3 \cdots \varphi_N = \varphi_2(\varphi_3 \cdots \varphi_N). \tag{2.10}$$

By using the same method as the one used in the previous paragraph, we deduce that

$$\varphi_2 = 0 \quad \text{or} \quad \varphi_3\varphi_4 \cdots \varphi_N = 0.$$

Then, repeating this process yields the desired result. □

3. The commutativity problem for dual Toeplitz operators

In this section, we mainly study the commuting dual Toeplitz operators with some special quasi-homogeneous symbols on the orthogonal complement of the harmonic Bergman space $(L_h^2)^\perp$. To do so, we need the following lemma, which can be proven by direct calculations.

Lemma 3.1. *For any positive integers s, m, n with $m > n$, we have*

(1) $Q(|z|^s z^n \overline{z}^m) = \frac{2(m-n+1)}{s+2m+2}\overline{z}^{m-n}$;

(2) $Q(|z|^s \overline{z}^n) = \frac{2(n+1)}{s+2n+2}\overline{z}^n$;

(3) $Q(|z|^s z^n) = \frac{2(n+1)}{s+2n+2}z^n$;

(4) $Q(z^n \overline{z}^m) = \frac{m-n+1}{m+1}\overline{z}^{m-n}$;

(5) $Q(z^m \overline{z}^n) = \frac{m-n+1}{m+1}z^{m-n}$.

To study the commuting problem for the dual Toeplitz operators on the orthogonal complement of the harmonic Bergman space, we first consider the simplest radial symbol $|z|^n$.

Proposition 3.2. *Suppose that*

$$\varphi(z) = |z|^s \quad \textit{and} \quad \psi(z) = |z|^t,$$

where s, t are positive integers. Then,

$$S_\varphi S_\psi = S_\psi S_\varphi$$

if, and only if, $\varphi = \psi$.

Proof. The sufficiency is obvious, and we need only to prove the necessity. By (1.8) in Section 1, we see that

$$(z\overline{z}^2 - \frac{2}{3}\overline{z}) \in (L_h^2)^\perp.$$

Since

$$S_\varphi S_\psi = S_\psi S_\varphi,$$

we have

$$S_\varphi S_\psi\Big(z\overline{z}^2 - \frac{2}{3}\overline{z}\Big) = S_\psi S_\varphi\Big(z\overline{z}^2 - \frac{2}{3}\overline{z}\Big). \tag{3.1}$$

Elementary calculations give us that

$$\begin{aligned} S_\varphi\Big(z\overline{z}^2 - \frac{2}{3}\overline{z}\Big) &= (I-Q)\Big(|z|^s z\overline{z}^2 - \frac{2}{3}|z|^s\overline{z}\Big) \\ &= |z|^s z\overline{z}^2 - \frac{2}{3}|z|^s\overline{z} - \Big(\frac{4}{s+6} - \frac{8}{3s+12}\Big)\overline{z} \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} S_\psi S_\varphi\Big(z\overline{z}^2 - \frac{2}{3}\overline{z}\Big) &= (I-Q)\Big[|z|^{s+t} z\overline{z}^2 - \frac{2}{3}|z|^{s+t}\overline{z} - \Big(\frac{4}{s+6} - \frac{8}{3s+12}\Big)|z|^t\overline{z}\Big] \\ &= |z|^{s+t} z\overline{z}^2 - \frac{2}{3}|z|^{s+t}\overline{z} - \Big(\frac{4}{s+6} - \frac{8}{3s+12}\Big)|z|^t\overline{z} - \frac{4}{s+t+6}\overline{z} \\ &\quad + \frac{8}{3(s+t+4)}\overline{z} + \Big(\frac{4}{s+6} - \frac{8}{3s+12}\Big)\frac{4}{t+4}\overline{z}. \end{aligned} \tag{3.3}$$

Similarly, we have that

$$\begin{aligned} S_\varphi S_\psi\Big(z\overline{z}^2-\frac{2}{3}\overline{z}\Big) &= (I-Q)\Big[|z|^{s+t}z\overline{z}^2-\frac{2}{3}|z|^{s+t}\overline{z}-\Big(\frac{4}{t+6}-\frac{8}{3t+12}\Big)|z|^s\overline{z}\Big] \\ &= |z|^{s+t}z\overline{z}^2-\frac{2}{3}|z|^{s+t}\overline{z}-\Big(\frac{4}{t+6}-\frac{8}{3t+12}\Big)|z|^s\overline{z}-\frac{4}{s+t+6}\overline{z} \\ &\quad +\frac{8}{3(s+t+4)}\overline{z}+\Big(\frac{4}{t+6}-\frac{8}{3t+12}\Big)\frac{4}{s+4}\overline{z}. \end{aligned} \tag{3.4}$$

Combining (3.1), (3.3), and (3.4) gives

$$\begin{aligned} 0 =&\Big(\frac{4}{s+6}-\frac{8}{3s+12}\Big)|z|^t\overline{z}-\Big(\frac{4}{t+6}-\frac{8}{3t+12}\Big)|z|^s\overline{z} \\ &-\Big[\Big(\frac{4}{s+6}-\frac{8}{3s+12}\Big)\frac{4}{t+4}-\Big(\frac{4}{t+6}-\frac{8}{3t+12}\Big)\frac{4}{s+4}\Big]\overline{z}. \end{aligned} \tag{3.5}$$

If $s \neq t$, then the coefficients of $|z|^t\overline{z}$ and $|z|^s\overline{z}$ are zero. This implies that

$$\frac{4}{s+6}-\frac{8}{3s+12}=0 \tag{3.6}$$

and

$$\frac{4}{t+6}-\frac{8}{3t+12}=0. \tag{3.7}$$

It follows that $s = t = 0$, which is a contradiction, completing the proof. □

The next theorem shows that Proposition 3.2 can be extended to a general case.

Theorem 3.3. *Let*

$$\varphi(z) = z^p\overline{z}^q \quad \text{and} \quad \psi(z) = z^s\overline{z}^t,$$

where p, q, s, t are all nonnegative integers. Then, $S_\varphi S_\psi = S_\psi S_\varphi$ if, and only if, one of the following conditions holds:

(1) *$\varphi = c\psi$ for some constant c;*
(2) *φ and ψ are both analytic;*
(3) *φ and ψ are both co-analytic;*
(4) *Either φ or ψ is constant.*

In order to simplify the proof of Theorem 3.3, we require the following lemma:

Lemma 3.4. *Let*

$$f(x) = a\frac{x-s+t}{x+t+1}-a\frac{x(x-s+t)}{(x+1)(x+t)}-b\frac{x-p+q}{x+q+1}+b\frac{x(x-p+q)}{(x+1)(x+q)}, \tag{3.8}$$

where $s, t, p, q \geqslant 0$, and $a, b \in \mathbb{C}$. If there exist some M such that $f(x) \equiv 0$ when $x > M$, then $at-bq = 0$.

Proof. Since

$$\frac{x-s+t}{x+t+1}-\frac{x(x-s+t)}{(x+1)(x+t)}=\frac{(x-s+t)t}{(x+t+1)(x+1)(x+t)} \tag{3.9}$$

and

$$\frac{x-p+q}{x+q+1}-\frac{x(x-p+q)}{(x+1)(x+q)}=\frac{(x-p+q)q}{(x+q+1)(x+1)(x+q)}, \tag{3.10}$$

we have

$$\begin{aligned} 0&=a\frac{(x-s+t)t}{(x+t+1)(x+1)(x+t)}-b\frac{(x-p+q)q}{(x+q+1)(x+1)(x+q)}\\ &=\frac{1}{x+1}\Big[a\frac{(x-s+t)t}{(x+t+1)(x+t)}-b\frac{(x-p+q)q}{(x+q+1)(x+q)}\Big]\\ &=\frac{a(x-s+t)(x+q+1)(x+q)t-b(x-p+q)(x+t+1)(x+t)q}{(x+1)(x+t+1)(x+t)(x+q+1)(x+q)},\quad x>M. \end{aligned} \tag{3.11}$$

Let

$$g(x)=a(x-s+t)(x+q+1)(x+q)t-b(x-p+q)(x+t+1)(x+t)q. \tag{3.12}$$

Then, g is a polynomial and (3.11) implies that g has infinitely many zeros. Thus, $g\equiv 0$ and, hence, the coefficient of x^3 is zero, i.e., $at-bq=0$. □

Now we are ready to prove Theorem 3.3.

Proof of Theorem 3.3. The sufficiency is obvious, so we only need to show the necessity. Since

$$S_\varphi S_\psi=S_\psi S_\varphi \quad\text{and}\quad z\overline{z}^m-\frac{m}{m+1}\overline{z}^{m-1}\in(L_h^2)^\perp,$$

we have

$$S_\varphi S_\psi\Big(z\overline{z}^m-\frac{m}{m+1}\overline{z}^{m-1}\Big)=S_\psi S_\varphi\Big(z\overline{z}^m-\frac{m}{m+1}\overline{z}^{m-1}\Big) \tag{3.13}$$

for all integers m satisfying

$$(m-1)>\max\{s-t,s-t+p-q,0\}.$$

Notice that

$$\begin{aligned} S_{z^s\overline{z}^t}\Big(z\overline{z}^m-\frac{m}{m+1}\overline{z}^{m-1}\Big)&=(I-Q)\Big(z^{s+1}\overline{z}^{m+t}-\frac{m}{m+1}z^s\overline{z}^{m+t-1}\Big)\\ &=z^{s+1}\overline{z}^{m+t}-\frac{m}{m+1}z^s\overline{z}^{m+t-1}-Q(z^{s+1}\overline{z}^{m+t})+Q\Big(\frac{m}{m+1}z^s\overline{z}^{m+t-1}\Big)\\ &=z^{s+1}\overline{z}^{m+t}-\frac{m}{m+1}z^s\overline{z}^{m+t-1}-\frac{m-s+t}{m+t+1}\overline{z}^{m-s+t-1}+\frac{m(m-s+t)}{(m+1)(m+t)}\overline{z}^{m-s+t-1}\\ &=z^{s+1}\overline{z}^{m+t}-\frac{m}{m+1}z^s\overline{z}^{m+t-1}-\Big[\frac{m-s+t}{m+t+1}-\frac{m(m-s+t)}{(m+1)(m+t)}\Big]\overline{z}^{m-s+t-1}, \end{aligned} \tag{3.14}$$

where the third equality follows from Lemma 3.1. Moreover, we have

$$
\begin{aligned}
&S_{z^p\overline{z}^q}S_{z^s\overline{z}^t}\Big(z\overline{z}^m-\frac{m}{m+1}\overline{z}^{m-1}\Big)\\
&=(I-Q)\Big\{z^{s+p+1}\overline{z}^{m+t+q}-\frac{m}{m+1}z^{s+p}\overline{z}^{m+t+q-1}-\Big[\frac{m-s+t}{m+t+1}-\frac{m(m-s+t)}{(m+1)(m+t)}\Big]z^p\overline{z}^{m-s+t+q-1}\Big\}\\
&=z^{s+p+1}\overline{z}^{m+t+q}-\frac{m}{m+1}z^{s+p}\overline{z}^{m+t+q-1}-\Big[\frac{m-s+t}{m+t+1}-\frac{m(m-s+t)}{(m+1)(m+t)}\Big]z^p\overline{z}^{m-s+t+q-1}\\
&\quad-Q(z^{s+p+1}\overline{z}^{m+t+q})+\frac{m}{m+1}Q(z^{s+p}\overline{z}^{m+t+q-1})+\Big[\frac{m-s+t}{m+t+1}-\frac{m(m-s+t)}{(m+1)(m+t)}\Big]Q(z^p\overline{z}^{m-s+t+q-1})\\
&=z^{s+p+1}\overline{z}^{m+t+q}-\frac{m}{m+1}z^{s+p}\overline{z}^{m+t+q-1}-\Big[\frac{m-s+t}{m+t+1}-\frac{m(m-s+t)}{(m+1)(m+t)}\Big]z^p\overline{z}^{m-s+t+q-1}\\
&\quad-\frac{m-s+t-p+q}{m+t+q+1}\overline{z}^{m-s+t-p+q-1}+\frac{m(m-s+t-p+q)}{(m+1)(m+t+q)}\overline{z}^{m-s+t-p+q-1}\\
&\quad+\Big[\frac{m-s+t}{m+t+1}-\frac{m(m-s+t)}{(m+1)(m+t)}\Big]\frac{m-s+t-p+q}{m-s+t+q}\overline{z}^{m-s+t-p+q-1}.
\end{aligned}
\tag{3.15}
$$

On the other hand,

$$
\begin{aligned}
&S_{z^s\overline{z}^t}S_{z^p\overline{z}^q}\Big(z\overline{z}^m-\frac{m}{m+1}\overline{z}^{m-1}\Big)\\
&=(I-Q)\Big\{z^{s+p+1}\overline{z}^{m+t+q}-\frac{m}{m+1}z^{s+p}\overline{z}^{m+t+q-1}-\Big[\frac{m-p+q}{m+q+1}-\frac{m(m-p+q)}{(m+1)(m+q)}\Big]z^s\overline{z}^{m-p+t+q-1}\Big\}\\
&=z^{s+p+1}\overline{z}^{m+t+q}-\frac{m}{m+1}z^{s+p}\overline{z}^{m+t+q-1}-\Big[\frac{m-p+q}{m+q+1}-\frac{m(m-p+q)}{(m+1)(m+q)}\Big]z^s\overline{z}^{m-p+t+q-1}\\
&\quad-\frac{m-s+t-p+q}{m+t+q+1}\overline{z}^{m-s+t-p+q-1}+\frac{m(m-s+t-p+q)}{(m+1)(m+t+q)}\overline{z}^{m-s+t-p+q-1}\\
&\quad+\Big[\frac{m-p+q}{m+q+1}-\frac{m(m-p+q)}{(m+1)(m+q)}\Big]\frac{m-s+t-p+q}{m-p+t+q}\overline{z}^{m-s+t-p+q-1}.
\end{aligned}
\tag{3.16}
$$

It follows from (3.13) that

$$
\begin{aligned}
0=&(S_{z^p\overline{z}^q}S_{z^s\overline{z}^t}-S_{z^s\overline{z}^t}S_{z^p\overline{z}^q})\Big(z\overline{z}^m-\frac{m}{m+1}\overline{z}^{m-1}\Big)\\
=&\Big[\frac{m-p+q}{m+q+1}-\frac{m(m-p+q)}{(m+1)(m+q)}\Big]z^s\overline{z}^{m-p+t+q-1}-\Big[\frac{m-s+t}{m+t+1}-\frac{m(m-s+t)}{(m+1)(m+t)}\Big]z^p\overline{z}^{m-s+t+q-1}\\
&+\Big[\frac{m-s+t}{m+t+1}-\frac{m(m-s+t)}{(m+1)(m+t)}\Big]\frac{m-s+t-p+q}{m-s+t+q}\overline{z}^{m-s+t-p+q-1}\\
&-\Big[\frac{m-p+q}{m+q+1}-\frac{m(m-p+q)}{(m+1)(m+q)}\Big]\frac{m-s+t-p+q}{m-p+t+q}\overline{z}^{m-s+t-p+q-1}\\
=&\Big[\frac{m-p+q}{m+q+1}-\frac{m(m-p+q)}{(m+1)(m+q)}\Big]|z|^{2s}\overline{z}^{m-p+t+q-s-1}-\Big[\frac{m-s+t}{m+t+1}-\frac{m(m-s+t)}{(m+1)(m+t)}\Big]|z|^{2p}\overline{z}^{m-s+t+q-p-1}\\
&+\Big[\frac{m-s+t}{m+t+1}-\frac{m(m-s+t)}{(m+1)(m+t)}\Big]\frac{m-s+t-p+q}{m-s+t+q}\overline{z}^{m-s+t-p+q-1}\\
&-\Big[\frac{m-p+q}{m+q+1}-\frac{m(m-p+q)}{(m+1)(m+q)}\Big]\frac{m-s+t-p+q}{m-p+t+q}\overline{z}^{m-s+t-p+q-1}.
\end{aligned}
\tag{3.17}
$$

Next, we need to consider the following five cases:

Case 1. $p \neq s$, $p \neq 0$, and $s \neq 0$. Then, the coefficients of $|z|^{2p}\overline{z}^{m-s+t+q-p-1}$ and $|z|^{2s}\overline{z}^{m-s+t+q-p-1}$ are both zero, which implies that

$$\begin{cases} \frac{m-s+t}{m+t+1} - \frac{m(m-s+t)}{(m+1)m+t} = 0, \\ \frac{m-p+q}{m+q+1} - \frac{m(m-p+q)}{(m+1)(m+q)} = 0. \end{cases} \tag{3.18}$$

By Lemma 3.4, we obtain that $t = q = 0$. Hence, (2) holds.

Case 2. $p = s$, $p \neq 0$, and $s \neq 0$. In this case, we have

$$\begin{aligned} 0 =& (S_{z^p\overline{z}^q}S_{z^s\overline{z}^t} - S_{z^s\overline{z}^t}S_{z^p\overline{z}^q})\Big(z\overline{z}^m - \frac{m}{m+1}\overline{z}^{m-1}\Big) \\ =& \Big[\frac{m-p+q}{m+q+1} - \frac{m(m-p+q)}{(m+1)(m+q)} - \frac{m-s+t}{m+t+1} + \frac{m(m-s+t)}{(m+1)(m+t)}\Big]|z|^{2s}\overline{z}^{m-p+t+q-s-1} \\ &+ \Big[\frac{m-s+t}{m+t+1} - \frac{m(m-s+t)}{(m+1)(m+t)}\Big]\frac{m-s+t-p+q}{m-s+t+q}\overline{z}^{m-s+t-p+q-1} \\ &- \Big[\frac{m-p+q}{m+q+1} - \frac{m(m-p+q)}{(m+1)(m+q)}\Big]\frac{m-s+t-p+q}{m-p+t+q}\overline{z}^{m-s+t-p+q-1}. \end{aligned} \tag{3.19}$$

This gives that the coefficient of $|z|^{2s}\overline{z}^{m-p+t+q-s-1}$ is zero, i.e.,

$$\frac{m-p+q}{m+q+1} - \frac{m(m-p+q)}{(m+1)(m+q)} - \frac{m-s+t}{m+t+1} + \frac{m(m-s+t)}{(m+1)(m+t)} = 0. \tag{3.20}$$

Using Lemma 3.4, we conclude that $t = q$. Hence, (1) holds.

Case 3. $p \neq 0$ and $s = 0$. Then, the coefficient of $|z|^{2p}\overline{z}^{m-s+t+q-p-1}$ is zero. Hence, we have

$$\frac{m-s+t}{m+t+1} - \frac{m(m-s+t)}{(m+1)(m+t)} = 0. \tag{3.21}$$

Applying Lemma 3.4, we get $t = 0$, i.e., ψ is constant.

Case 4. $s \neq 0$ and $p = 0$. Then, the coefficients of $|z|^{2s}\overline{z}^{m-s+t+q-p-1}$ are zero, which is equivalent to

$$\frac{m-p+q}{m+q+1} - \frac{m(m-p+q)}{(m+1)(m+q)} = 0. \tag{3.22}$$

It follows from Lemma 3.4 that $q = 0$. This implies that φ is constant.

Case 5. $s = p = 0$. This case is trivial.

This completes the proof of Theorem 3.3. □

In the rest of this section, we will study the commutativity problem for dual Toeplitz operators with symbols of the form $az^{p_1}\overline{z}^{q_1} + bz^{p_2}\overline{z}^{q_2}$ and $z^s\overline{z}^t$, where p_1, p_2, q_1, q_2, s, and t are all nonnegative integers. To this end, we still require a number of lemmas as follows:

Lemma 3.5. *Let*

$$\varphi(z) = az^{p_1}\overline{z}^{q_1} + bz^{p_2}\overline{z}^{q_2}, \quad \psi(z) = z^s\overline{z}^t,$$

where s, t, p_1, p_2, q_1, q_2 are nonnegative integers. For $m \in \mathbb{N}$ large enough, we have

$$\begin{aligned}
&(S_\varphi S_\psi - S_\psi S_\varphi)\Big(z\overline{z}^m - \frac{m}{m+1}\overline{z}^{m-1}\Big)\\
&= a\Big[\frac{m-p_1+q_1}{m+q_1+1} - \frac{m(m-p_1+q_1)}{(m+1)(m+q_1)}\Big]|z|^{2s}\overline{z}^{m-s+t-p_1+q_1-1}\\
&\quad - a\Big[\frac{m-s+t}{m+t+1} - \frac{m(m-s+t)}{(m+1)(m+t)}\Big]|z|^{2p_1}\overline{z}^{m-s+t-p_1+q_1-1}\\
&\quad + a\Big[\frac{m-s+t}{m+t+1} - \frac{m(m-s+t)}{(m+1)(m+t)}\Big]\frac{m+q_1-p_1-s+t}{m+q_1-s+t}\overline{z}^{m-s+t-p_1+q_1-1}\\
&\quad - a\Big[\frac{m-p_1+q_1}{m+q_1+1} - \frac{m(m-p_1+q_1)}{(m+1)(m+q_1)}\Big]\frac{m+q_1-p_1-s+t}{m+q_1-p_1+t}\overline{z}^{m-s+t-p_1+q_1-1}\\
&\quad + b\Big[\frac{m-p_2+q_2}{m+q_2+1} - \frac{m(m-p_2+q_2)}{(m+1)(m+q_2)}\Big]|z|^{2s}\overline{z}^{m-s+t-p_2+q_2-1}\\
&\quad - b\Big[\frac{m-s+t}{m+t+1} - \frac{m(m-s+t)}{(m+1)(m+t)}\Big]|z|^{2p_2}\overline{z}^{m-s+t-p_2+q_2-1}\\
&\quad + b\Big[\frac{m-s+t}{m+t+1} - \frac{m(m-s+t)}{(m+1)(m+t)}\Big]\frac{m+q_2-p_2-s+t}{m+q_2-s+t}\overline{z}^{m-s+t-p_2+q_2-1}\\
&\quad - b\Big[\frac{m-p_2+q_2}{m+q_2+1} - \frac{m(m-p_2+q_2)}{(m+1)(m+q_2)}\Big]\frac{m+q_2-p_2-s+t}{m+q_2-p_2+t}\overline{z}^{m-s+t-p_2+q_2-1}.
\end{aligned} \tag{3.23}$$

Proof. This can be proven easily by elementary computations. □

Lemma 3.6. *Let φ and ψ be the functions as in Lemma 3.5. For $m \in \mathbb{N}$ large enough, we have*

$$\begin{aligned}
&(S_{\overline{\varphi}} S_{\overline{\psi}} - S_{\overline{\psi}} S_{\overline{\varphi}})\Big(z\overline{z}^m - \frac{m}{m+1}\overline{z}^{m-1}\Big)\\
&= \overline{a}\Big[\frac{m-q_1+p_1}{m+p_1+1} - \frac{m(m-q_1+p_1)}{(m+1)(m+p_1)}\Big]|z|^{2t}\overline{z}^{m-t+s-q_1+p_1-1}\\
&\quad - \overline{a}\Big[\frac{m-t+s}{m+s+1} - \frac{m(m-t+s)}{(m+1)(m+s)}\Big]|z|^{2q_1}\overline{z}^{m-t+s-q_1+p_1-1}\\
&\quad + \overline{a}\Big[\frac{m-t+s}{m+s+1} - \frac{m(m-t+s)}{(m+1)(m+s)}\Big]\frac{m+p_1-q_1-t+s}{m+p_1-t+s}\overline{z}^{m-t+s-q_1+p_1-1}\\
&\quad - \overline{a}\Big[\frac{m-q_1+p_1}{m+p_1+1} - \frac{m(m-q_1+p_1)}{(m+1)(m+p_1)}\Big]\frac{m+p_1-q_1-t+s}{m+p_1-q_1+s}\overline{z}^{m-t+s-q_1+p_1-1}\\
&\quad + \overline{b}\Big[\frac{m-q_2+p_2}{m+p_2+1} - \frac{m(m-q_2+p_2)}{(m+1)(m+p_2)}\Big]|z|^{2t}\overline{z}^{m-t+s-q_2+p_2-1}\\
&\quad - \overline{b}\Big[\frac{m-t+s}{m+s+1} - \frac{m(m-t+s)}{(m+1)(m+s)}\Big]|z|^{2q_2}\overline{z}^{m-t+s-q_2+p_2-1}\\
&\quad + \overline{b}\Big[\frac{m-t+s}{m+s+1} - \frac{m(m-t+s)}{(m+1)(m+s)}\Big]\frac{m+p_2-q_2-t+s}{m+p_2-t+s}\overline{z}^{m-t+s-q_2+p_2-1}\\
&\quad - \overline{b}\Big[\frac{m-q_2+p_2}{m+p_2+1} - \frac{m(m-q_2+p_2)}{(m+1)(m+p_2)}\Big]\frac{m+p_2-q_2-t+s}{m+p_2-q_2+s}\overline{z}^{m-t+s-q_2+p_2-1}.
\end{aligned} \tag{3.24}$$

Proof. This is a direct conclusion of Lemma 3.5. □

In view of Lemma 3.5, we conclude that

$$\begin{aligned} 0 &= \alpha_1 x^{2s+m-s+t-p_1+q_1-1} + \alpha_2 x^{2p_1+m-s+t-p_1+q_1-1} + \alpha_3 x^{m-s+t-p_1+q_1-1} \\ &\quad + \alpha_4 x^{2s+m-s+t-p_2+q_2-1} + \alpha_5 x^{2p_2+m-s+t-p_2+q_2-1} + \alpha_6 x^{m-s+t-p_2+q_2-1} \\ &= x^{m-s+t-p_1+q_1-1}(\alpha_1 x^{2s} + \alpha_2 x^{2p_1} + \alpha_3 + \alpha_4 x^{2s+p_1-q_1-p_2+q_2} \\ &\quad + \alpha_5 x^{2p_2+p_1-q_1-p_2+q_2} + \alpha_6 x^{p_1-q_1-p_2+q_2}) \end{aligned} \tag{3.25}$$

for all $x \in (0,1)$ if $S_\varphi S_\psi = S_\psi S_\varphi$, where the coefficients are given by:

$$\begin{aligned} \alpha_1 &= a\frac{m-p_1+q_1}{m+q_1+1} - a\frac{m(m-p_1+q_1)}{(m+1)(m+q_1)}, \quad \alpha_2 = -a\frac{m-s+t}{m+t+1} + a\frac{m(m-s+t)}{(m+1)(m+t)}, \\ \alpha_3 &= a\Big[\frac{m-s+t}{m+t+1} - \frac{m(m-s+t)}{(m+1)(m+t)}\Big]\frac{m+q_1-p_1-s+t}{m+q_1-s+t} \\ &\quad - a\Big[\frac{m-p_1+q_1}{m+q_1+1} - \frac{m(m-p_1+q_1)}{(m+1)(m+q_1)}\Big]\frac{m+q_1-p_1-s+t}{m+q_1-p_1+t}, \\ \alpha_4 &= b\frac{m-p_2+q_2}{m+q_2+1} - b\frac{m(m-p_2+q_2)}{(m+1)(m+q_2)}, \quad \alpha_5 = -b\frac{m-s+t}{m+t+1} + b\frac{m(m-s+t)}{(m+1)(m+t)}, \\ \alpha_6 &= b\Big[\frac{m-s+t}{m+t+1} - \frac{m(m-s+t)}{(m+1)(m+t)}\Big]\frac{m+q_2-p_2-s+t}{m+q_2-s+t} \\ &\quad - b\Big[\frac{m-p_2+q_2}{m+q_2+1} - \frac{m(m-p_2+q_2)}{(m+1)(m+q_2)}\Big]\frac{m+q_2-p_2-s+t}{m+q_2-p_2+t}. \end{aligned}$$

Notice that (3.25) is equivalent to

$$0 = \alpha_1 x^{2s} + \alpha_2 x^{2p_1} + \alpha_3 + \alpha_4 x^{2s+p_1-q_1-p_2+q_2} + \alpha_5 x^{2p_2+p_1-q_1-p_2+q_2} + \alpha_6 x^{p_1-q_1-p_2+q_2}. \tag{3.26}$$

Similarly, we have by Lemma 3.6 that

$$\begin{aligned} 0 &= \beta_1 x^{2t+m-t+s-q_1+p_1-1} + \beta_2 x^{2q_1+m-t+s-q_1+p_1-1} + \beta_3 x^{m-t+s-q_1+p_1-1} \\ &\quad + \beta_4 x^{2t+m-t+s-q_2+p_2-1} + \beta_5 x^{2q_2+m-t+s-q_2+p_2-1} + \beta_6 x^{m-t+s-q_2+p_2-1} \\ &= x^{m-t+s-q_1+p_1-1}(\beta_1 x^{2t} + \beta_2 x^{2q_1} + \beta_3 + \beta_4 x^{2t-p_1+q_1+p_2-q_2} \\ &\quad + \beta_5 x^{2q_2-p_1+q_1+p_2-q_2} + \beta_6 x^{-p_1+q_1+p_2-q_2}) \end{aligned} \tag{3.27}$$

for all $x \in (0,1)$ whenever

$$S_{\overline{\varphi}} S_{\overline{\psi}} = S_{\overline{\psi}} S_{\overline{\varphi}},$$

where

$$\begin{aligned} \beta_1 &= \overline{a}\frac{m-q_1+p_1}{m+p_1+1} - \overline{a}\frac{m(m-q_1+p_1)}{(m+1)(m+p_1)}, \quad \beta_2 = -\overline{a}\frac{m-t+s}{m+s+1} + \overline{a}\frac{m(m-t+s)}{(m+1)(m+s)}, \\ \beta_3 &= \overline{a}\Big[\frac{m-t+s}{m+s+1} - \frac{m(m-t+s)}{(m+1)(m+s)}\Big]\frac{m+p_1-q_1-t+s}{m+p_1-t+s} \\ &\quad - \overline{a}\Big[\frac{m-q_1+p_1}{m+p_1+1} - \frac{m(m-q_1+p_1)}{(m+1)(m+p_1)}\Big]\frac{m+p_1-q_1-t+s}{m+p_1-q_1+s}, \\ \beta_4 &= \overline{b}\frac{m-q_2+p_2}{m+p_2+1} - \overline{b}\frac{m(m-q_2+p_2)}{(m+1)(m+p_2)}, \quad \beta_5 = -\overline{b}\frac{m-t+s}{m+s+1} + \overline{b}\frac{m(m-t+s)}{(m+1)(m+s)}, \end{aligned}$$

$$\beta_6 = \overline{b}\Big[\frac{m-t+s}{m+s+1} - \frac{m(m-t+s)}{(m+1)(m+s)}\Big]\frac{m+p_2-q_2-t+s}{m+p_2-t+s}$$
$$-\overline{b}\Big[\frac{m-q_2+p_2}{m+p_2+1} - \frac{m(m-q_2+p_2)}{(m+1)(m+p_2)}\Big]\frac{m+p_2-q_2-t+s}{m+p_2-q_2+s}.$$

Clearly, (3.27) is equivalent to

$$0 = \beta_1 x^{2t} + \beta_2 x^{2q_1} + \beta_3 + \beta_4 x^{2t-p_1+q_1+p_2-q_2} + \beta_5 x^{2q_2-p_1+q_1+p_2-q_2} + \beta_6 x^{-p_1+q_1+p_2-q_2}. \tag{3.28}$$

Using the same method as the one used in the proof of Lemma 3.4, we list without proof five lemmas as follows:

Lemma 3.7. *Let*

$$\begin{aligned} f(x) = &\Big[\frac{x-s+t}{x+t+1} - \frac{x(x-s+t)}{(x+1)(x+t)}\Big]\frac{x+q_2-p_2-s+t}{x+q_2-s+t} \\ &-\Big[\frac{x-p_2+q_2}{x+q_2+1} - \frac{x(x-p_2+q_2)}{(x+1)(x+q_2)}\Big]\frac{x+q_2-p_2-s+t}{x+q_2-p_2+t}, \end{aligned} \tag{3.29}$$

where s, t, p_2, q_2 are all nonnegative numbers. If there exist some M such that $f(x) \equiv 0$ when $x > M$, then $t = q_2$.

Lemma 3.8. *Let*

$$\begin{aligned} f(x) =& a\Big[\frac{x-s+t}{x+t+1} - \frac{x(x-s+t)}{(x+1)(x+t)}\Big]\frac{x+q_1-p_1-s+t}{x+q_1-s+t} \\ &- a\Big[\frac{x-p_1+q_1}{x+q_1+1} - \frac{x(x-p_1+q_1)}{(x+1)(x+q_1)}\Big]\frac{x+q_1-p_1-s+t}{x+q_1-p_1+t} \\ &+ b\Big[\frac{x-s+t}{x+t+1} - \frac{x(x-s+t)}{(x+1)(x+t)}\Big]\frac{x+q_2-p_2-s+t}{x+q_2-s+t} \\ &- b\Big[\frac{x-p_2+q_2}{x+q_2+1} - \frac{x(x-p_2+q_2)}{(x+1)(x+q_2)}\Big]\frac{x+q_2-p_2-s+t}{x+q_2-p_2+t}, \end{aligned} \tag{3.30}$$

where s, t, p_1, p_2, q_1, q_2 are all nonnegative numbers. If there exist some M such that $f(x) \equiv 0$ when $x > M$, then

$$at - aq_1 + bt - bq_2 = 0.$$

Lemma 3.9. *Let*

$$\begin{aligned} f(x) =& a\Big[\frac{x-s+t}{x+t+1} - \frac{x(x-s+t)}{(x+1)(x+t)}\Big]\frac{x+q_1-p_1-s+t}{x+q_1-s+t} \\ &- a\Big[\frac{x-p_1+q_1}{x+q_1+1} - \frac{x(x-p_1+q_1)}{(x+1)(x+q_1)}\Big]\frac{x+q_1-p_1-s+t}{x+q_1-p_1+t} \\ &+ b\Big[\frac{x-p_2+q_2}{x+q_2+1} - \frac{x(x-p_2+q_2)}{(x+1)(x+q_2)}\Big], \end{aligned} \tag{3.31}$$

where s, t, p_1, p_2, q_1, q_2 are all nonnegative numbers. If there exist some M such that $f(x) \equiv 0$ when $x > M$, then

$$at - aq_1 + bq_2 = 0.$$

Lemma 3.10. *Let*

$$
\begin{aligned}
f(x) = & - a\Big[\frac{x-t+s}{m+s+1} - \frac{x(x-t+s)}{(x+1)(x+s)}\Big] \\
& + b\Big[\frac{x-q_2+p_2}{x+p_2+1} - \frac{x(x-q_2+p_2)}{(x+1)(x+p_2)}\Big] \\
& - b\Big[\frac{x-t+s}{x+s+1} - \frac{x(x-t+s)}{(x+1)(x+s)}\Big],
\end{aligned}
\tag{3.32}
$$

where s, t, p_2, q_2 are all nonnegative numbers. If there exist some M such that $f(x) \equiv 0$ when $x > M$, then

$$as - bp_2 + bs = 0.$$

Lemma 3.11. *Let*

$$
\begin{aligned}
f(x) = & - a\Big[\frac{x-s+t}{x+t+1} - \frac{x(x-s+t)}{(x+1)(x+t)}\Big] \\
& + a\Big[\frac{x-s+t}{x+t+1} - \frac{x(x-s+t)}{(x+1)(x+t)}\Big]\frac{x+q_1-p_1-s+t}{x+q_1-s+t} \\
& - a\Big[\frac{x-p_1+q_1}{x+q_1+1} - \frac{x(x-p_1+q_1)}{(x+1)(x+q_1)}\Big]\frac{x+q_1-p_1-s+t}{x+q_1-p_1+t} \\
& + b\Big[\frac{x-s+t}{x+t+1} - \frac{x(x-s+t)}{(x+1)(x+t)}\Big]\frac{x+q_2-p_2-s+t}{x+q_2-s+t} \\
& - b\Big[\frac{x-p_2+q_2}{x+q_2+1} - \frac{x(x-p_2+q_2)}{(x+1)(x+q_2)}\Big]\frac{x+q_2-p_2-s+t}{x+q_2-p_2+t},
\end{aligned}
\tag{3.33}
$$

where s, t, p_1, p_2, q_1, q_2 are all nonnegative numbers. If there exist some M such that $f(x) \equiv 0$ when $x > M$, then

$$aq_1 - bt + bq_2 = 0.$$

We are now in the position to discuss the commutativity problem for dual Toeplitz operators with symbols

$$\varphi(z) = az^{p_1}\overline{z}^{q_1} + bz^{p_2}\overline{z}^{q_2}$$

and

$$\psi(z) = z^s\overline{z}^t.$$

Before giving a complete answer to such a problem, we need to analyze various situations for the integers s, t, p_j, and q_j with $j = 1, 2$. Based on Theorem 3.3, we assume that a and b are both nonzero in the following:

Proposition 3.12. *Let*

$$\varphi(z) = az^{p_1}\overline{z}^{q_1} + bz^{p_2}\overline{z}^{q_2}$$

and

$$\psi(z) = z^s\overline{z}^t,$$

where $a, b \in \mathbb{C}$, s, p_1, p_2 are positive and t, q_1, q_2 are nonnegative. In the case of $s = p_1 = p_2$, we have that

$$S_\varphi S_\psi = S_\psi S_\varphi$$

if, and only if, $\varphi = c\psi$ for some constant c.

Proof. Clearly, we only need to prove the necessity. Based on Lemma 3.5, we will consider the following four cases:

Case 1. $p_1 - q_1 - p_2 + q_2 = 0$. Since $p_1 = p_2$, we have $q_1 = q_2$, which implies that

$$\varphi(z) = (a+b)z^{p_1}\overline{z}^{q_1}.$$

It follows from Theorem 3.3 that $q_1 = t$.

Case 2. $p_1 - q_1 - p_2 + q_2 = -2s$. By (3.26), we have that $\alpha_6 = 0$ and $\alpha_1 + \alpha_2 = 0$. Now, combining Lemmas 3.4 and 3.7 gives $t = q_1 = q_2$.

Case 3. $p_1 - q_1 - p_2 + q_2 = 2s$. Using (3.26) again, we deduce that $\alpha_3 = 0$ and $\alpha_4 + \alpha_5 = 0$. Thus, we obtain by Lemmas 3.4 and 3.7 that $t = q_1 = q_2$.

Case 4. $p_1 - q_1 - p_2 + q_2$ is not equal to 0, not equal to $-2s$, and not equal to $2s$. From (3.26), we conclude that $\alpha_3 = 0$ and $\alpha_6 = 0$, which implies $t = q_1 = q_2$.

This finishes the proof of Proposition 3.12. □

Proposition 3.13. *Let*

$$\varphi(z) = az^{p_1}\overline{z}^{q_1} + bz^{p_2}\overline{z}^{q_2}$$

and

$$\psi(z) = z^s\overline{z}^t,$$

where $a, b \in \mathbb{C}$ such that $a \neq b$, s, p_1, p_2 are positive and t, q_1, q_2 are nonnegative. If $s = p_1 \neq p_2$, then

$$S_\varphi S_\psi = S_\psi S_\varphi$$

if, and only if, $t = q_1 = q_2 = 0$.

Proof. In order to show the necessity, we need to consider the following five cases:

Case 1. $p_1 - q_1 - p_2 + q_2 = 0$. In this case, (3.26) can be rewritten as

$$\begin{aligned} 0 &= \alpha_1 x^{2s} + \alpha_2 x^{2s} + \alpha_3 + \alpha_4 x^{2s} + \alpha_5 x^{2p_2} + \alpha_6 \\ &= (\alpha_1 + \alpha_2 + \alpha_4)x^{2s} + \alpha_5 x^{2p_2} + (\alpha_3 + \alpha_6). \end{aligned} \tag{3.34}$$

Thus, $\alpha_5 = 0$ and $\alpha_3 + \alpha_6 = 0$. Applying Lemmas 3.4 and 3.8, we obtain $t = 0$ and

$$(a+b)t - aq_1 - bq_2 = 0. \tag{3.35}$$

If $q_1 = 0$ or $q_2 = 0$, then $t = q_1 = q_2 = 0$ follows immediately.

Next, we consider the case that $q_1 \neq 0$ and $q_2 \neq 0$. Using

$$S_\varphi S_\psi = S_\psi S_\varphi$$

if, and only if,

$$S_{\overline{\varphi}} S_{\overline{\psi}} = S_{\overline{\psi}} S_{\overline{\varphi}},$$

we have by (3.28) that

$$\begin{aligned} 0 &= \beta_1 x^{2t} + \beta_2 x^{2q_1} + \beta_3 + \beta_4 x^{2t-p_1+q_1+p_2-q_2} + \beta_5 x^{2q_2-p_1+q_1+p_2-q_2} + \beta_6 x^{-p_1+q_1+p_2-q_2} \\ &= \beta_2 x^{2q_1} + \beta_5 x^{2q_2} + (\beta_1 + \beta_3 + \beta_4 + \beta_6). \end{aligned} \tag{3.36}$$

Now we are going to analyze the following two sub-cases:

Sub-case 1.1. If $q_1 = q_2$, then $p_1 - q_1 - p_2 + q_2 = 0$ implies $p_1 = p_2$. This is a contradiction.

Sub-case 1.2. If $q_1 \neq q_2$, then $\beta_2 = \beta_5 = 0$. However, Lemma 3.4 tells us that $s = 0$, which is also a contradiction.

Therefore, we obtain that $t = q_1 = q_2 = 0$ in Case 1.

Case 2. $p_1 - q_1 - p_2 + q_2 = 2s$. In this case, we have $\alpha_3 = \alpha_4 = \alpha_5 = 0$. It follows from Lemmas 3.4 and 3.7 that $t = q_1 = q_2 = 0$.

Case 3. $p_1 - q_1 - p_2 + q_2 = -2s$. In this case, $\alpha_6 = 0$ and $\alpha_3 + \alpha_4 = 0$. Thus, we have $t = q_2$ and $at - aq_1 + bq_2 = 0$ by Lemmas 3.7 and 3.9. Next, we are going to show that $t = 0$. Suppose not, we assume that $t > 0$. Using (3.28), we have

$$\begin{aligned} 0 &= \beta_1 x^{2t} + \beta_2 x^{2q_1} + \beta_3 + \beta_4 x^{2t-p_1+q_1+p_2-q_2} + \beta_5 x^{2q_2-p_1+q_1+p_2-q_2} + \beta_6 x^{-p_1+q_1+p_2-q_2} \\ &= \beta_1 x^{2t} + \beta_2 x^{2q_1} + \beta_3 + \beta_4 x^{2t+2s} + \beta_5 x^{2t+2s} + \beta_6 x^{2s}. \end{aligned} \tag{3.37}$$

Sub-case 3.1. If $t + s \neq q_1$, then $\beta_4 + \beta_5 = 0$. However, Lemma 3.4 gives that $s = p_2$, which contradicts the assumption that $s \neq p_2$.

Sub-case 3.2. If $t + s = q_1$, then $\beta_2 + \beta_4 + \beta_5 = 0$. It follows from Lemma 3.10 that

$$-as + bp_2 - bs = 0. \tag{3.38}$$

Substituting $q_1 = s + t$, $s = p_1$ and $q_2 = t$ into the equation $p_1 - q_1 - p_2 + q_2 = -2s$ gives $p_2 = 2s$. Thus, (3.38) is reduced to $(b - a)s = 0$, but $b - a \neq 0$ yields $s = 0$, which contradicts that $s > 0$.

Consequently, we get $t = 0$ and $q_1 = q_2 = 0$ in Case 3.

Case 4. $p_1 - q_1 - p_2 + q_2 = -2p_2$. By (3.26), we have

$$0 = (\alpha_1 + \alpha_2)x^{2s} + \alpha_4 x^{2s-2p_2} + \alpha_6 x^{-2p_2} + (\alpha_3 + \alpha_5). \tag{3.39}$$

It follows that $\alpha_1 + \alpha_2 = 0, \alpha_4 = 0, \alpha_6 = 0$. Thus, we obtain by Lemmas 3.4 and 3.7 that $t = q_1 = q_2 = 0$.

Case 5. $p_1 - q_1 - p_2 + q_2$ is not equal to $0, 2s, -2s$, or $-2p_2$. In this case, we have that $\alpha_3 = \alpha_4 = \alpha_6 = 0$. By Lemmas 3.4 and 3.7, we also obtain that $t = q_1 = q_2 = 0$.

This completes the proof of Proposition 3.13. □

Proposition 3.14. *Let*

$$\varphi(z) = az^{p_1}\overline{z}^{q_1} + bz^{p_2}\overline{z}^{q_2}$$

and

$$\psi(z) = z^s\overline{z}^t,$$

where $a, b \in \mathbb{C}$, s, p_1, p_2 are positive and t, q_1, q_2 are nonnegative. In the case of $p_1 = p_2 \neq s$, we have that $S_\varphi S_\psi = S_\psi S_\varphi$ if, and only if, $t = q_1 = q_2 = 0$.

Proof. Based on Lemma 3.5, we need to consider the following six cases:

Case 1. $p_1 - q_1 - p_2 + q_2 = 0$. Since $p_1 = p_2$, we have $q_1 = q_2$ and $\varphi(z) = (a + b)z^{p_1}\overline{z}^{q_2}$. It follows from Theorem 3.3 that $t = q_1 = q_2 = 0$.

Case 2. $p_1 - q_1 - p_2 + q_2 = 2s$. In this case, using (3.26) we obtain

$$0 = (\alpha_1 + \alpha_6)x^{2s} + \alpha_2 x^{2p_1} + \alpha_3 + \alpha_4 x^{4s} + \alpha_5 x^{2p_1+2s}. \tag{3.40}$$

This implies that $\alpha_1 + \alpha_6 = 0, \alpha_3 = 0$, and $\alpha_5 = 0$. Applying Lemmas 3.4, 3.7, and 3.9, we obtain $t = 0, t = q_1$, and $aq_1 + bt - bq_2 = 0$. Hence, $t = q_1 = q_2 = 0$.

Case 3. $p_1 - q_1 - p_2 + q_2 = 2p_1$. In this case, we have $\alpha_3 = \alpha_4 = 0$. By Lemmas 3.4 and 3.7, we obtain that $t = q_1$ and $q_2 = 0$. If $t > 0$, then we have by (3.28) that

$$0 = (\beta_1 + \beta_2)x^{2t} + \beta_3 + \beta_4 x^{2t-2p_1} + (\beta_5 + \beta_6)x^{-2p_1}. \tag{3.41}$$

Thus, $\beta_5 + \beta_6 = 0$. Using Lemma 3.9, we get $p_2 = 0$, which is a contradiction. Hence, $t = 0$ and $q_1 = q_2 = 0$.

Case 4. $p_1 - q_1 - p_2 + q_2 = -2s$. By (3.26), we obtain that $\alpha_2 = \alpha_6 = 0$ and $\alpha_3 + \alpha_4 = 0$. Combining Lemmas 3.4, 3.7, and 3.9 gives $t = 0, t = q_2$, and $at - aq_1 + bq_2 = 0$. This yields that $t = q_1 = q_2 = 0$.

Case 5. $p_1 - q_1 - p_2 + q_2 = -2p_1$. Now we have $\alpha_1 = \alpha_6 = 0$. By Lemmas 3.4 and 3.7, we obtain that $q_1 = 0$ and $t = q_2$. Next, we will show $t = q_2 = 0$. If not, we obtain by (3.28) that $\beta_4 + \beta_5 = 0$, but Lemma 3.4 gives that $s = p_2$, which is a contradiction. Therefore, $t = q_1 = q_2 = 0$.

Case 6. $p_1 - q_1 - p_2 + q_2$ is not equal to $0, 2s, -2s, 2p_1$, or $-2p_1$. In this case, we have $\alpha_3 = \alpha_6 = 0$. Applying Lemma 3.4, we obtain $q_1 = t = q_2$, which means $\varphi(z) = (a + b)z^{p_1}\overline{z}^t$. It follows from Theorem 3.3 again that $t = q_1 = q_2 = 0$.

This finishes the proof of Proposition 3.14. □

Proposition 3.15. *Let*

$$\varphi(z) = az^{p_1}\overline{z}^{q_1} + bz^{p_2}\overline{z}^{q_2}$$

and

$$\psi(z) = z^s\overline{z}^t,$$

where $a, b \in \mathbb{C}$, s, p_1, p_2 are positive and t, q_1, q_2 are nonnegative. If p_1, p_2, s are different from each other, then

$$S_\varphi S_\psi = S_\psi S_\varphi$$

if, and only if, $t = q_1 = q_2 = 0$.

Proof. We first show that $p_1 - q_1 - p_2 + q_2 \neq 0$. If not, then we have by (3.26) that

$$\begin{aligned} 0 &= \alpha_1 x^{2s} + \alpha_2 x^{2p_1} + \alpha_3 + \alpha_4 x^{2s+p_1-q_1-p_2+q_2} + \alpha_5 x^{2p_2+p_1-q_1-p_2+q_2} + \alpha_6 x^{p_1-q_1-p_2+q_2} \\ &= (\alpha_1 + \alpha_4)x^{2s} + \alpha_2 x^{2p_1} + \alpha_5 x^{2p_2} + (\alpha_3 + \alpha_6). \end{aligned} \tag{3.42}$$

It follows that $\alpha_2 = \alpha_1 + \alpha_4 = 0$. Combining this with Lemma 3.4, we obtain that $t = 0$ and $aq_1 + bq_2 = 0$. If $q_1 = q_2$, then $p_1 - q_1 - p_2 + q_2 = 0$ gives $p_1 = p_2$. This contradicts the hypothesis that $p_1 \neq p_2$. If $q_1 \neq q_2$, then we have by (3.28) that $\beta_2 = 0$. Then, it follows from Lemma 3.4 that $s = 0$, which is a contradiction. Based on Lemma 3.5, we need only to consider the following five cases:

Case 1. $p_1 - q_1 - p_2 + q_2 = 2s$. In this case, we have $\alpha_3 = \alpha_1 + \alpha_6 = 0$. It follows from Lemmas 3.7 and 3.9 that $t = q_1$ and $aq_1 + bt - bq_2 = 0$. If $t > 0$, then we obtain by (3.28) that

$$\begin{aligned} 0 &= \beta_1 x^{2t} + \beta_2 x^{2q_1} + \beta_3 + \beta_4 x^{2t-p_1+q_1+p_2-q_2} + \beta_5 x^{2q_2-p_1+q_1+p_2-q_2} + \beta_6 x^{-p_1+q_1+p_2-q_2} \\ &= (\beta_1 + \beta_2)x^{2t} + \beta_4 x^{2t-2s} + \beta_5 x^{2q_2-2s} + \beta_6 x^{-2s} + \beta_3. \end{aligned} \tag{3.43}$$

If $q_2 = 0$, then $\beta_5 + \beta_6 = 0$. It follows from Lemma 3.9 that $-bs + bs - bp_2 = 0$, i.e., $p_2 = 0$. This is a contradiction. If $q_2 \neq 0$, then $\beta_6 = 0$. By Lemma 3.7, we deduce that $s = p_2$, which is also a contradiction. Thus, $t = 0$. Moreover, $t = q_1 = q_2 = 0$.

Case 2. $p_1 - q_1 - p_2 + q_2 = 2p_1$. In this case, we have $\alpha_3 = \alpha_4 = 0$. Using Lemmas 3.4 and 3.7, we obtain that $t = q_1$ and $q_2 = 0$. Now we are going to show $t = 0$. Suppose that $t > 0$. It follows from (3.28) that $\beta_5 + \beta_6 = 0$. Using Lemma 3.9, we conclude that $p_2 = 0$, which is a contradiction. So, we have $t = q_1 = q_2 = 0$.

Case 3. $p_1 - q_1 - p_2 + q_2 = -2s$. In this case, we have

$$\begin{aligned} 0 &= \alpha_1 x^{2s} + \alpha_2 x^{2p_1} + \alpha_3 + \alpha_4 x^{2s+p_1-q_1-p_2+q_2} + \alpha_5 x^{2p_2+p_1-q_1-p_2+q_2} + \alpha_6 x^{p_1-q_1-p_2+q_2} \\ &= \alpha_1 x^{2s} + \alpha_2 x^{2p_1} + \alpha_5 x^{2p_2-2s} + \alpha_6 x^{-2s} + (\alpha_3 + \alpha_4), \end{aligned} \tag{3.44}$$

which gives that $\alpha_6 = \alpha_3 + \alpha_4 = 0$.

Using Lemmas 3.7 and 3.9, we obtain that $t = q_2$ and $at - aq_1 + bq_2 = 0$. If $t = 0$, then we deduce that $q_1 = q_2 = 0$ immediately. If $t > 0$, then substituting $t = q_2$ into (3.28), we get

$$\begin{aligned} 0 &= \beta_1 x^{2t} + \beta_2 x^{2q_1} + \beta_3 + \beta_4 x^{2t-p_1+q_1+p_2-q_2} + \beta_5 x^{2q_2-p_1+q_1+p_2-q_2} + \beta_6 x^{-p_1+q_1+p_2-q_2} \\ &= \beta_1 x^{2t} + \beta_2 x^{2q_1} + \beta_3 + (\beta_4 + \beta_5) x^{2t+2s} + \beta_6 x^{2s}. \end{aligned} \tag{3.45}$$

If $q_1 = 0$, then $\beta_2 + \beta_3 = 0$. Lemma 3.9 implies that $-as + as - ap_1 = 0$, i.e., $p_1 = 0$, which is a contradiction. If $q_1 \neq 0$, then $\beta_3 = 0$. It follows from Lemma 3.7 that $s = p_1$, which contradicts that $s \neq p_1$. Therefore, we have $t = q_1 = q_2 = 0$.

Case 4. $p_1 - q_1 - p_2 + q_2 = -2p_2$. Using (3.26), we have that $\alpha_1 = \alpha_6 = 0$. Moreover, it follows from Lemmas 3.4 and 3.7 that $q_1 = 0$ and $t = q_2$. If $t = 0$, then $t = q_1 = q_2 = 0$. If $t > 0$, then using (3.28), we get that $\beta_4 + \beta_5 = 0$. However, Lemma 3.4 implies $s = p_2$, which is a contradiction.

Case 5. $p_1 - q_1 - p_2 + q_2$ is not equal to $2s$, $2p_1$, $-2s$, or $-2p_2$. In this case, we have $\alpha_3 = \alpha_6 = 0$. It follows from Lemma 3.7 that $t = q_1 = q_2$. If $t \neq 0$, then we have

$$\begin{aligned} 0 &= \beta_1 x^{2t} + \beta_2 x^{2q_1} + \beta_3 + \beta_4 x^{2t-p_1+q_1+p_2-q_2} + \beta_5 x^{2q_2-p_1+q_1+p_2-q_2} + \beta_6 x^{-p_1+q_1+p_2-q_2} \\ &= (\beta_1 + \beta_2) x^{2t} + (\beta_4 + \beta_5) x^{2t-p_1+q_1+p_2-q_2} + \beta_6 x^{-p_1+q_1+p_2-q_2} + \beta_3. \end{aligned} \tag{3.46}$$

If $-p_1 + q_1 + p_2 - q_2 = 2t$, then $\beta_4 + \beta_5 = 0$. Applying Lemma 3.4, we have $s = p_2$, which is a contradiction. If $-p_1 + q_1 + p_2 - q_2 \neq 2t$, then $\beta_6 = 0$. Using Lemma 3.7, we obtain that $s = p_2$, which contradicts that $s \neq p_2$. Consequently, $t = q_1 = q_2 = 0$.

This finishes the proof of Proposition 3.15. □

Proposition 3.16. *Let*

$$\varphi(z) = az^{p_1}\bar{z}^{q_1} + bz^{p_2}\bar{z}^{q_2}$$

and

$$\psi(z) = z^s\bar{z}^t,$$

where $a, b \in \mathbb{C}$. *If* $s = p_1 = 0$ *and* $p_2 \geqslant 1$, *then*

$$S_\varphi S_\psi = S_\psi S_\varphi$$

if, and only if, ψ *is constant.*

Proof. To show the necessity, we need to discuss the following two cases:

Case 1. $p_1 - q_1 - p_2 + q_2 = -2p_2$. In this case, we have $\alpha_4 + \alpha_6 = 0$. By Lemma 3.9, we have that $t = 0$ and ψ is a constant function.

Case 2. $p_1 - q_1 - p_2 + q_2 \neq -2p_2$. In this case, we have $\alpha_5 = 0$. Moreover, it follows from Lemma 3.4 that $t = 0$. This also implies that ψ is constant. □

Proposition 3.17. *Let*

$$\varphi(z) = az^{p_1}\overline{z}^{q_1} + bz^{p_2}\overline{z}^{q_2}$$

and

$$\psi(z) = z^s\overline{z}^t,$$

where $a, b \in \mathbb{C}$*. If* $p_1 = p_2 = 0$ *and* $s \geqslant 1$*, then*

$$S_\varphi S_\psi = S_\psi S_\varphi$$

if, and only if, φ *is constant.*

Proof. Based on Lemma 3.5, we need to analyze the following three cases:

Case 1. $p_1 - q_1 - p_2 + q_2 = 0$. Since $p_1 = p_2 = 0$, we have $q_1 = q_2$. By (3.26), we obtain that

$$\begin{aligned} 0 &= \alpha_1 x^{2s} + \alpha_2 x^{2p_1} + \alpha_3 + \alpha_4 x^{2s+p_1-q_1-p_2+q_2} + \alpha_5 x^{2p_2+p_1-q_1-p_2+q_2} + \alpha_6 x^{p_1-q_1-p_2+q_2} \\ &= (\alpha_1 + \alpha_4)x^{2s} + (\alpha_2 + \alpha_3 + \alpha_5 + \alpha_6). \end{aligned} \tag{3.47}$$

This yields that $\alpha_1 + \alpha_4 = 0$. Moreover, Lemma 3.4 implies that $aq_1 + bq_2 = 0$, i.e., $(a + b)q_1 = 0$. If $a + b = 0$, then

$$\varphi(z) = (a + b)\overline{z}^{q_1} = 0.$$

If $q_1 = 0$, then

$$\varphi(z) = (a + b)\overline{z}^{q_1} = a + b.$$

Thus, φ is a constant function.

Case 2. $p_1 - q_1 - p_2 + q_2 = 2s$. In this case, we have

$$\begin{aligned} 0 &= \alpha_1 x^{2s} + \alpha_2 x^{2p_1} + \alpha_3 + \alpha_4 x^{2s+p_1-q_1-p_2+q_2} + \alpha_5 x^{2p_2+p_1-q_1-p_2+q_2} + \alpha_6 x^{p_1-q_1-p_2+q_2} \\ &= (\alpha_1 + \alpha_5 + \alpha_6)x^{2s} + \alpha_4 x^{4s} + (\alpha_2 + \alpha_3). \end{aligned} \tag{3.48}$$

It follows that $\alpha_4 = \alpha_2 + \alpha_3 = 0$. By Lemma 3.4, we get that $q_2 = 0$. Furthermore, we obtain by Lemma 3.9 that $-at + at - aq_1 = 0$, which implies $q_1 = 0$. Thus, φ is constant.

Case 3. $p_1 - q_1 - p_2 + q_2$ is not equal to $2s$ and not equal to 0. In this case, we have $\alpha_1 = \alpha_5 + \alpha_6 = 0$. Now, Lemma 3.4 gives $q_1 = 0$ and Lemma 3.9 implies that $-bt + bt - bq_2 = 0$. This yields that $q_2 = 0$, so φ is a constant function. □

Proposition 3.18. *Let*

$$\varphi(z) = az^{p_1}\overline{z}^{q_1} + bz^{p_2}\overline{z}^{q_2}$$

and

$$\psi(z) = z^s\overline{z}^t,$$

where $a, b \in \mathbb{C}$. *If* $s = 0$, $p_1 \neq 0$, *and* $p_2 \neq 0$, *then*

$$S_\varphi S_\psi = S_\psi S_\varphi$$

if, and only if, ψ *is a constant function.*

Proof. To complete the proof, we need only to show the necessity and discuss the following three cases:

Case 1. $p_1 - q_1 - p_2 + q_2 = 0$. In this case, we have

$$\begin{aligned} 0 &= \alpha_1 x^{2s} + \alpha_2 x^{2p_1} + \alpha_3 + \alpha_4 x^{2s+p_1-q_1-p_2+q_2} + \alpha_5 x^{2p_2+p_1-q_1-p_2+q_2} + \alpha_6 x^{p_1-q_1-p_2+q_2} \\ &= \alpha_2 x^{2p_1} + \alpha_5 x^{2p_2} + (\alpha_1 + \alpha_3 + \alpha_4 + \alpha_6). \end{aligned} \tag{3.49}$$

If $p_1 = p_2$, then $p_1 - q_1 - p_2 + q_2 = 0$ implies that $q_1 = q_2$. Thus,

$$\varphi(z) = (a + b) z^{p_1} \overline{z}^{q_1}.$$

By Theorem 3.3, we have $t = 0$. If $p_1 \neq p_2$, then $\alpha_2 = \alpha_5 = 0$. Lemma 3.4 implies that $t = 0$. Thus, ψ is constant.

Case 2. $p_1 - q_1 - p_2 + q_2 = 2p_1$. In this case, we have $\alpha_5 = 0$. It follows from Lemma 3.4 that $t = 0$, which means that ψ is constant.

Case 3. $p_1 - q_1 - p_2 + q_2$ is not equal to 0 and not equal to $2p_1$. Under this assumption, we have that $\alpha_4 + \alpha_6 = 0$. Using Lemma 3.9, we obtain that $bq_2 + bt - bq_2 = 0$, i.e., $t = 0$. So, ψ is constant. □

Proposition 3.19. *Let*

$$\varphi(z) = a z^{p_1} \overline{z}^{q_1} + b z^{p_2} \overline{z}^{q_2}$$

and

$$\psi(z) = z^s \overline{z}^t,$$

where $a, b \in \mathbb{C}$. *In the cases of* $p_1 = 0$, $s \neq 0$, *and* $p_2 \neq 0$, *we have that*

$$S_\varphi S_\psi = S_\psi S_\varphi$$

if, and only if, $q_1 = 0$ *and one of the following conditions holds:*

(1) $s = p_2$ *and* $t = q_2$;

(2) $s \neq p_2$ *and* $t = q_2 = 0$.

Proof. Based on Lemma 3.5, we need to consider the following four cases:

Case 1. $p_1 - q_1 - p_2 + q_2 = 0$. In this case, we obtain by (3.26) that

$$\begin{aligned} 0 &= \alpha_1 x^{2s} + \alpha_2 x^{2p_1} + \alpha_3 + \alpha_4 x^{2s+p_1-q_1-p_2+q_2} + \alpha_5 x^{2p_2+p_1-q_1-p_2+q_2} + \alpha_6 x^{p_1-q_1-p_2+q_2} \\ &= (\alpha_1 + \alpha_4) x^{2s} + \alpha_5 x^{2p_2} + (\alpha_2 + \alpha_3 + \alpha_6). \end{aligned} \tag{3.50}$$

This yields that $\alpha_2 + \alpha_3 + \alpha_6 = 0$. Using Lemma 3.11, we have that

$$aq_1 + bq_2 - bt = 0. \tag{3.51}$$

Let us show $q_1 = 0$ first. If $q_1 > 0$, then we observe that t and q_2 cannot be equal to 0 at the same time. Now we are going to discuss the following four sub-cases:

Sub-case 1.1. $t = 0$ and $q_2 \neq 0$. Then, it follows from (3.28) that

$$\begin{aligned} 0 &= \beta_1 x^{2t} + \beta_2 x^{2q_1} + \beta_3 + \beta_4 x^{2t-p_1+q_1+p_2-q_2} + \beta_5 x^{2q_2-p_1+q_1+p_2-q_2} + \beta_6 x^{-p_1+q_1+p_2-q_2} \\ &= \beta_2 x^{2q_1} + \beta_5 x^{2q_2} + (\beta_1 + \beta_3 + \beta_4 + \beta_6). \end{aligned} \tag{3.52}$$

If $q_1 = q_2$, then $p_1 - q_1 - p_2 + q_2 = 0$ implies that $p_1 = p_2$. This is a contradiction. If $q_1 \neq q_2$, then $\beta_2 = 0$. Applying Lemma 3.4 gives $s = 0$, which is also impossible.

Sub-case 1.2. $t \neq 0$ and $q_2 = 0$. Substituting $p_1 = 0$ and $q_2 = 0$ into $p_1 - q_1 - p_2 + q_2 = 0$ yields $q_1 + p_2 = 0$. This is impossible, since $q_1 \geqslant 0$ and $p_2 \geqslant 1$.

Sub-case 1.3. $t = q_2$ and $q_2 \neq 0$. Using (3.51), we deduce that $q_1 = 0$. This is a contradiction.

Sub-case 1.4. $t \neq q_2, t \neq 0$ and $q_2 \neq 0$. It follows from (3.28) that

$$\begin{aligned} 0 &= \beta_1 x^{2t} + \beta_2 x^{2q_1} + \beta_3 + \beta_4 x^{2t-p_1+q_1+p_2-q_2} + \beta_5 x^{2q_2-p_1+q_1+p_2-q_2} + \beta_6 x^{-p_1+q_1+p_2-q_2} \\ &= (\beta_1 + \beta_4) x^{2t} + \beta_2 x^{2q_1} + \beta_5 x^{2q_2} + (\beta_3 + \beta_6). \end{aligned} \tag{3.53}$$

If $t = q_1$, then $\beta_5 = 0$. Applying Lemma 3.4 gives that $s = 0$. This contradicts the assumption that $s \neq 0$. If $t \neq q_1$, then $\beta_1 + \beta_4 = 0$. It follows from Lemma 3.4 that $ap_1 + bp_2 = 0$. Combining this with $p_1 = 0$ yields $p_2 = 0$. This contradicts the assumption that $p_2 \neq 0$.

From Sub-cases 1.1–1.4, we conclude that $q_1 = 0$. Combining this with (3.51) implies that $t = q_2$. Furthermore, if $s = p_2$, then we obtain (1); otherwise, if $s \neq p_2$, then $\alpha_5 = 0$. It follows from Lemma 3.4 that $t = q_2 = 0$, which gives (2).

Case 2. $p_1 - q_1 - p_2 + q_2 = -2s$. In this case, we obtain by (3.26) that

$$\begin{aligned} 0 &= \alpha_1 x^{2s} + \alpha_2 x^{2p_1} + \alpha_3 + \alpha_4 x^{2s+p_1-q_1-p_2+q_2} + \alpha_5 x^{2p_2+p_1-q_1-p_2+q_2} + \alpha_6 x^{p_1-q_1-p_2+q_2} \\ &= \alpha_1 x^{2s} + \alpha_5 x^{2p_2-2s} + \alpha_6 x^{-2s} + (\alpha_2 + \alpha_3 + \alpha_4). \end{aligned} \tag{3.54}$$

Next, we will consider the following three sub-cases:

Sub-case 2.1. $p_2 = 2s$. It follows that $\alpha_1 + \alpha_5 = \alpha_6 = 0$. From Lemma 3.4, we get that

$$t = q_2 \tag{3.55}$$

and

$$aq_1 - bt = 0. \tag{3.56}$$

If $t = 0$, then q_1 must be zero, since $a \neq 0$. If $t \neq 0$, then $q_1 \neq 0$. By (3.28), we obtain that $\beta_3 = 0$. Using Lemma 3.7, we obtain that $s = p_1$, which is a contradiction. Hence, we have $t = 0$. Combining this and (3.55) gives that $t = q_1 = q_2 = 0$.

Sub-case 2.2. $p_2 = s$. In this case, we have that $\alpha_1 = \alpha_6 = 0$. Using Lemmas 3.4 and 3.7, we obtain $q_1 = 0$ and $t = q_2$.

Subcase 2.3. $p_2 \neq 2s$ and $p_2 \neq s$. In this case, we have $\alpha_1 = \alpha_5 = \alpha_6 = 0$. It follows from Lemmas 3.4 and 3.7 that $t = q_1 = q_2 = 0$.

Case 3. $p_1 - q_1 - p_2 + q_2 = -2p_2$. It follows that $\alpha_1 = 0$. Using Lemma 3.4, we have $q_1 = 0$. Then we can deduce that (1) or (2) holds by using Theorem 3.3.

Case 4. $p_1 - q_1 - p_2 + q_2$ is not equal to 0, $-2s$, or $-2p_2$. In this case, we obtain that $\alpha_2 + \alpha_3 = 0$. Lemma 3.9 implies that $-at + at - aq_1 = 0 = -aq_1$, which yields $q_1 = 0$. Finally, using Theorem 3.3 again, we get the desired results.

This completes the proof of Proposition 3.19. □

Now we are ready to state and prove the main result of this section.

Theorem 3.20. *Let*

$$\varphi(z) = az^{p_1}\overline{z}^{q_1} + bz^{p_2}\overline{z}^{q_2}$$

and

$$\psi(z) = z^s\overline{z}^t,$$

where $a, b \in \mathbb{C}\backslash\{0\}$ *such that* $a \neq b$, p_j, q_j, *s, and t are nonnegative integers,* $j = 1, 2$. *Then,*

$$S_\varphi S_\psi = S_\psi S_\varphi$$

if, and only if, one of the following conditions holds:
(i) ψ *is a constant function;*
(ii) *Both* φ *and* ψ *are analytic;*
(iii) *Both* φ *and* ψ *are co-analytic;*
(iv) *There exist* $\alpha, \beta \in \mathbb{C}$, *not both zero, such that* $\varphi = \alpha\psi + \beta$.

Proof. Obviously, it is sufficient to show the necessity. To do so, we divide the proof into nine steps as follows:

(1) $s = p_1 = p_2$ (Proposition 3.12);
(2) $s = p_1$ and $p_1 \neq p_2$ (Proposition 3.13);
(3) $p_1 = p_2$ and $p_2 \neq s$ (Proposition 3.14);
(4) s, p_1, p_2 are pairwise different (Proposition 3.15);
(5) $s = p_1 = 0$ and $p_2 \neq 0$ (Proposition 3.16);
(6) $p_1 = p_2 = 0$ and $s \neq 0$ (Proposition 3.17);
(7) $s = 0$, $p_1 \neq 0$, and $p_2 \neq 0$ (Proposition 3.18);
(8) $p_1 = 0$, $s \neq 0$, and $p_2 \neq 0$ (Proposition 3.19);
(9) $s = p_1 = p_2 = 0$ (this situation is trivial).

This finishes the proof of Theorem 3.20. □

4. Conclusions

In this research, we conduct a study of dual Toeplitz operators on the orthogonal complement of the harmonic Bergman space and obtain that:

(1) Suppose that

$$\varphi_k(z) = \sum_{m=0}^{\infty} a_{k,m}|z|^m, \quad (k = 1, 2, \cdots, N),$$

which are N bounded functions on the unit disk $\mathbb{D}$. If

$$S_{\varphi_1} S_{\varphi_2} \cdots S_{\varphi_N} = 0,$$

then there exist some $k \in \{1, 2, \cdots, N\}$ such that $\varphi_k = 0$;

(2) Let

$$\varphi(z) = az^{p_1}\overline{z}^{q_1} + bz^{p_2}\overline{z}^{q_2}$$

and

$$\psi(z) = z^s\overline{z}^t,$$

where $a, b \in \mathbb{C}\backslash\{0\}$ such that $a \neq b$, p_j, q_j, s, and t are nonnegative integers, $j = 1, 2$. Then,

$$S_\varphi S_\psi = S_\psi S_\varphi$$

if, and only if, one of the following conditions holds:

(i) ψ is a constant function;

(ii) Both φ and ψ are analytic;

(iii) Both φ and ψ are co-analytic;

(iv) There exist $\alpha, \beta \in \mathbb{C}$, not both zero, such that $\varphi = \alpha\psi + \beta$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that she has no conflict of interest.

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