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The upper bound for the first positive eigenvalue of Sub-Laplacian on a compact strictly pseudoconvex hypersurface

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Abstract: Let (M^{2n+1}, θ) be a compact strictly pseudoconvex real hypersurfaces equipped with the pseudohermitian structure θ, and λ_1 be the first positive eigenvalue of sub-Laplacian Δ_b on (M^{2n+1}, θ). In this paper, we will give the upper bound of λ_1 under certain conditions that "$\mathrm{Re}\Delta_b\left(\rho_j + \rho_{\bar{j}}\right)\left(2\tilde{\Delta}_\rho \rho_j + |\partial\rho|^2_\rho n^{-1}\rho^k\rho_{jk}\right) \le 0$ (for some j)" or "$\rho_{j\bar{k}} = \delta_{jk}$" holds, and apply these results to the ellipsoids furthermore.

Keywords: sub-Laplacian; upper bound; the first positive eigenvalue; compact real hypersurfaces; ellipsoids
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1. Introduction

Let (M^{2n+1}, θ) be a compact strictly pseudoconvex pseudohermitian manifold with real dimension $2n + 1 \ge 3$. Denote the tangential Cauchy–Riemann operator as $\bar{\partial}_b : L^2(M) \to L^2_{0,1}(M)$, and the formal adjoint with respect to the volume measure $dv = \theta \wedge (d\theta)^n$ as $\bar{\partial}^*_b$. The Kohn-Laplacian acting on functions is given by $\Box_b = \bar{\partial}^*_b \bar{\partial}_b$ and the sub-Laplacian is given by $\Delta_b = 2\mathrm{Re}\,\Box_b$.

Recall the Dirichlet problem of the Laplace operator Δ in R^n. Let $D \subseteq R^n$ be a bounded domain with C^1 boundary ∂D, and Δ be the Laplace operator. Consider the Dirichlet problem:

$$\begin{cases} \Delta u(x) = \lambda u(x), & x \in D, \\ u(x) = 0, & x \in \partial D. \end{cases}$$

The constant λ, which causes the fact that there exist nontrivial solutions u to this problem, is the eigenvalue of Δ. And the nontrivial solutions u are the eigenfunctions of λ correspond to λ. By the related theory of partial differential equations, we know that every eigenvalue of Δ is positive, and the spectra of Δ are discrete and diverge to infinity. In addition, we can compute the first positive eigenvalue λ_1 by the Rayleigh formula

$$\lambda_1 = \min\{\int_D \Delta u \cdot u| \quad u \in H_0^1, ||u||_{L^2}^2 = 1\} = \min_{u \in H_0^1, u \not\equiv 0} \frac{\int_D \Delta u \cdot u}{||u||_{L^2}^2}.$$

Extend the Dirichlet problem in R^n to which on CR manifolds [1, 28], and let M be a strictly pseudoconvex CR manifold, and $\Omega \subset M$ be a smoothly bounded domain. Let θ be a contact form on M, such that the Levi form is positive definite. Denote the sub-Laplacian of the pseudohermitian manifold (M, θ) as Δ_b. Then, the Dirichlet problem is as follows:

$$\begin{cases} \Delta_b u(x) = \lambda u(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{1.1}$$

Likewise, the number $\lambda \in R$ is an eigenvalue of (1.1) if there is a function $u \not\equiv 0$ satisfies (1.1), and u is the corresponding eigenfunction. The Dirichlet problem of Kohn-Laplacian $\Box_b$ is similar.

Laplacians and the corresponding spectral theory have been more and more concerned, and thus many academics dedicate to the related study and have worked out numerous interesting results, such as [11, 17, 27, 30, 33] and all that. Specially, there exists closed relation between the spectrum of Laplacians and the geometric properties of the underlying manifolds; refer to [13, 14, 29, 34] and so forth, including the classic Lichnerowicz-Obata theorem.

Our work is about the estimate on the first positive eigenvalue of the Laplacians on CR manifolds, and there have been a lot of results on the related study by many academics so far. For the first positive eigenvalue $\lambda_1(\Box_b)$ of Kohn-Laplacian, the lower bound was studied in [9, 18, 23, 24], and the upper bound was studied in [3, 21, 23]. The lower bound for the first positive eigenvalue $\lambda_1(\Delta_b)$ of sub-Laplacian was studied in [8,10,14,15,19,25] and so on, and about the upper bound, the authors proved that the first positive eigenvalue of sub-Laplacian on the CR sphere achieves its maximum when its pseudohermitian structure is the standard contact form in [2].

In this paper, our work is to study the upper bound of $\lambda_1(\Delta_b)$ on a compact strictly pseudoconvex pseudohermitian manifold in $\mathbb{C}^{n+1}$ by precise calculation. Let ρ be a smooth, strictly plurisubharmonic function in $\mathbb{C}^{n+1}$, and $\nu > 0$. Equipped $M = \rho^{-1}(\nu)$ with the usual pseudohermitian structure $\theta = \iota^*(i/2)(\bar{\partial}\rho - \partial\rho)$ induced by ρ and $d\theta = ih_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}}$ where $(h_{\alpha\bar{\beta}})$ is positive definite, so that M is a a compact strictly pseudoconvex pseudohermitian manifold with the volume form $dv = \theta \wedge (d\theta)^n$ in the sense of [34]. Moreover, a Kähler metric $\rho_{j\bar{k}}dz^j d\bar{z}^k$ is induced by ρ naturally in a neighborhood U of M. Let $[\rho^{j\bar{k}}]^t$ be the inverse of $H(\rho)$. The length of a smooth function u on U, which is denoted by ∂u, is given by

$$|\partial u|_\rho^2 = \rho^{j\bar{k}} u_j \bar{u}_{\bar{k}}$$

in the Kähler metric.

We define the degenerate differential operator $\tilde{\Delta}_\rho$ as

$$\tilde{\Delta}_\rho = \left(\frac{1}{|\partial\rho|_\rho^2} \rho^j \rho^{\bar{k}} - \rho^{j\bar{k}} \right) \partial_j \partial_{\bar{k}},$$

and obtain our first mian result:

Theorem 1.1. *Let ρ be a smooth, strictly plurisubharmonic function defined on an open set U of C^{n+1}, M be a compact connected regular level set of ρ, and λ_1 be the first positive eigenvalue of sub-Laplacian Δ_b on M. Assume that for some j,*

$$Re\Delta_b\left(\rho_j + \rho_{\bar{j}}\right)\left(2\tilde{\Delta}_\rho \rho_j + \frac{|\partial\rho|_\rho^2}{n}\rho^k \rho_{jk}\right) \leq 0,$$

then,

$$\lambda_1 \leq \frac{n}{|\partial\rho|_\rho^2}(z_0) \leq \max_M \frac{n}{|\partial\rho|_\rho^2}, \tag{1.2}$$

and if the equality holds, $|\partial\rho|_\rho^2$ must be a constant on M.

Next, we study the case when $\rho_{j\bar{k}} = \delta_{jk}$ and we find the following result:

Theorem 1.2. *Let ρ be a smooth, strictly plurisubharmonic function defined on an open set U of C^{n+1}, M be a compact connected regular level set of ρ, and λ_1 be the first positive eigenvalue of sub-Laplacian Δ_b on M. Suppose that $\rho_{j\bar{k}} = \delta_{jk}$, then,*

$$\lambda_1 \leq \frac{2n}{v(M)} \int_M \frac{1}{|\partial\rho|_\rho^2}, \tag{1.3}$$

and if the equality holds, $|\partial\rho|_\rho^2$ must be a constant on M.

Finally, we apply the above conclusions to ellipsoids.

Theorem 1.3. *Let $M_\nu = \rho^{-1}(\nu)$ (ν is a positive constant) be the ellipsoid, which is a compact regular level set of a real-valued strictly plurisubharmonic homogeneous quadratic polynomial $\rho(z)$ satisfying $\rho_{j\bar{k}} = \delta_{jk}$. Without loss of generality, we expressed $M_\nu = \rho^{-1}(\nu)$ as*

$$\rho(z) = |z|^2 + Re\sum_{j=1}^{n+1} A_j z_j^2 = \nu$$

with $0 \leq A_1 \leq A_2 \leq \cdots \leq A_{n+1} < 1$. Let $\lambda_1(M_\nu)$ be the first positive eigenvalue of sub-Laplacian Δ_b on M_ν. Then,

$$\lambda_1(M_\nu) \leq \min\left\{\frac{n(1+A_1)}{1-A_{n+1}}, \frac{2n}{\nu}\right\}.$$

And only when M_ν is a sphere of radius ν, $\lambda_1(M_\nu)$ can attain the upper bound $\min\left\{\frac{n(1+A_1)}{(1-A_{n+1})\nu}, \frac{2n}{\nu}\right\} = \frac{n}{\nu}$.

Our paper is organized as follows: In Section 2, we shall recall some notations and definitions about the sub-Laplacian Δ_b on compact real hypersurfaces and give a formula for Δ_b acting on functions; see Proposition 2.1. In Section 3, we shall give the upper bound of $\lambda_1(\Delta_b)$ under a certain condition (Theorem 1.1) and consider the case when $\rho_{j\bar{k}} = \delta_{jk}$ (Theorem 1.2). In Section 4, we shall discuss the upper bound of $\lambda_1(\Delta_b)$ on ellipsoids and generalize our discussion to Theorem 1.3.

2. Sub-Laplacian on compact real hypersurfaces

First of all, we recall some notations, definitions, and computations about the sub-Laplacian Δ_b on compact real hypersurfaces the same as those in [21].

Let M be a compact real hypersurface in $\mathbb{C}^{n+1}$ arising as a regular level set of a strictly plurisubharmonic function ρ:

$$M = \rho^{-1}(\nu) := \{Z \in U : \rho(Z) = \nu\}.$$

Here ρ is smooth on a neighborhood U of M and $d\rho \neq 0$ along M. Assume the complex Hessian $H(\rho) = [\rho_{j\bar{k}}]$ is positive definite, and thus ρ defines a Kähler metric $\rho_{j\bar{k}}dz^j d\bar{z}^k$ on U. Let $[\rho^{j\bar{k}}]^t$ be the inverse of $H(\rho)$. The length of a smooth function u on U, which is denoted by ∂u, is given by

$$|\partial u|_\rho^2 = \rho^{j\bar{k}} u_j \bar{u}_{\bar{k}}$$

in the Kähler metric.

Equip M with the pseudohermitian structure θ "induced" by ρ:

$$\theta = \iota^*(i/2)(\bar{\partial}\rho - \partial\rho).$$

The local admissible holomorphic coframe $\{\theta^\alpha : \alpha = 1, 2, \ldots, n\}$ on M is given by

$$\theta^\alpha = dz^\alpha - ih^\alpha\theta, \quad h^\alpha = |\partial\rho|_\rho^{-2}\rho^\alpha = |\partial\rho|_\rho^{-2}\rho_{\bar{j}}\rho^{\alpha\bar{j}}, \quad \alpha = 1, 2 \ldots n,$$

which is valid when $\rho_{n+1} \neq 0$. In [20], Li and Luk show us that at the point p with $\rho_{n+1} \neq 0$,

$$d\theta = ih_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}},$$

where the Levi matrix $[h_{\alpha\bar{\beta}}]$ is given explicitly:

$$h_{\alpha\bar{\beta}} = \rho_{\alpha\bar{\beta}} - \rho_\alpha\partial_{\bar{\beta}}\log\rho_{n+1} - \rho_{\bar{\beta}}\partial_\alpha\log\rho_{\overline{n+1}} + \rho_{n+1\overline{n+1}}\frac{\rho_\alpha\rho_{\bar{\beta}}}{|\rho_{n+1}|^2}.$$

And the inverse $[h^{\gamma\bar{\beta}}]$ of the Levi matrix is given by

$$h^{\gamma\bar{\beta}} = \rho^{\gamma\bar{\beta}} - \frac{\rho^\gamma\rho^{\bar{\beta}}}{|\partial\rho|_\rho^2}, \quad \rho^\gamma = \sum_{k=1}^{n+1}\rho_{\bar{k}}\rho^{\gamma\bar{k}}.$$

Moreover, let $[h_{\alpha\bar{\beta}}]$ be positive definite, and thus M is a compact strictly pseudoconvex hypersurface in the meaning of [34].

Denote the holomorphic frame dual to $\{\theta^\alpha\}$ as $\{Z_\alpha\}$:

$$Z_\alpha = \frac{\partial}{\partial z^\alpha} - \frac{\rho_\alpha}{\rho_{n+1}}\frac{\partial}{\partial z_{n+1}},$$

and $\omega_{\bar{\beta}}{}^{\bar{\sigma}}$ are the Tanaka-Webster connection forms, which are computed in [20, 34]:

$$\omega_{\bar{\beta}\alpha} = (Z_{\bar{\gamma}}h_{\alpha\bar{\beta}} - h_{\bar{\beta}}h_{\alpha\bar{\gamma}})\theta^{\bar{\gamma}} + h_\alpha h_{\gamma\bar{\beta}}\theta^\gamma + ih_{\alpha\bar{\sigma}}Z_{\bar{\beta}}h^{\bar{\sigma}}\theta, \quad h_\alpha = h_{\alpha\bar{\beta}}h^{\bar{\beta}}.$$

The Tanaka-Webster covariant derivatives are given by

$$\nabla_\alpha \nabla_{\bar\beta} f = Z_\alpha Z_{\bar\beta} f - \omega_{\bar\beta}{}^{\bar\sigma}(Z_\alpha) Z_{\bar\sigma} f.$$

In addition, the Reeb vector field is given by

$$T = i \sum_{j=1}^{n+1} \left(h^j \frac{\partial}{\partial z^j} - h^{\bar j} \frac{\partial}{\partial \bar z^j} \right), \quad h^j = \frac{\rho^j}{|\partial\rho|^2_\rho}.$$

According to the formula given by [21], suppose that U is an open set in a Kähler manifold and ρ is a Kähler potential on U . Let M be a smooth, compact, connected, regular level set of ρ, and $\Box_b$ be the Kohn-Laplacian defined on M with respect to $dv = \theta \wedge (d\theta)^n$, where $\theta = \frac{i}{2}(\bar\partial\rho - \partial\rho)$. Suppose that $(z^1, z^2, \ldots, z^{n+1})$ is a local coordinate system on an open set V. Define the vector fields

$$X_{jk} = \rho_k \partial_j - \rho_j \partial_k, \quad X_{\bar j \bar k} = \overline{X_{jk}}.$$

Then, Kohn-Laplacian $\Box_b$ acting on a smooth function f can be expressed as:

$$\Box_b f = -\frac{1}{2} |\partial\rho|^{-2}_\rho \rho^{p\bar k} \rho^{q\bar j} X_{pq} X_{\bar j \bar k} f, \tag{2.1}$$

and (2.1) can be written as

$$\Box_b f = \left(|\partial\rho|^{-2}_\rho \rho^k \rho^{\bar j} - \rho^{\bar j k} \right) f_{\bar j k} + \frac{n}{|\partial\rho|^2_\rho} \rho^{\bar k} f_{\bar k} \tag{2.2}$$

in local coordinates.

By calculating directly according to (2.2) and the fact that the sub-Laplacian $\Delta_b = 2\mathrm{Re}\Box_b$ when acting on smooth functions, we obtain the following result:

Proposition 2.1. *Under the conditions above, the sub-Laplacian Δ_b can be expressed as:*

$$\begin{aligned} \Delta_b f &= 2Re\Box_b f \\ &= 2\left(|\partial\rho|^{-2}_\rho \rho^k \rho^{\bar j} - \rho^{\bar j k} \right) f_{\bar j k} + \frac{n}{|\partial\rho|^2_\rho} (\rho^{\bar k} f_{\bar k} + \rho^k f_k) \\ &= 2\tilde\Delta_\rho f + \frac{n}{|\partial\rho|^2_\rho} (\rho^{\bar k} f_{\bar k} + \rho^k f_k). \end{aligned} \tag{2.3}$$

Here $\tilde\Delta_\rho = \left(|\partial\rho|^{-2}_\rho \rho^j \rho^{\bar k} - \rho^{j\bar k} \right) \partial_j \partial_{\bar k}$.

3. The upper bound for the first positive eigenvalue of sub-Laplacian on compact real hypersurfaces

In this section, we will give an estimate for the upper bound of the first positive eigenvalue of sub-Laplacian on compact real hypersurfaces.

It is known (see [1, 28]) that the positive eigenvalues of Δ_b satisfy:

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots ,$$

and $\lambda_k \to +\infty$ as $k \to \infty$. Denote the kernel of Δ_b by $E_0 = \mathrm{kel}(\Delta_b)$ and the eigenspace of Δ_b associated to the eigenvalue λ_k by E_k. Then,

$$L^2(M) = \bigoplus_{k=0}^{\infty} E_k.$$

Let m_k be the dimension of E_k, and $\{f_{k,j}\}_{j=1}^{m_k}$ be an orthonormal basis for E_k, thus we can prove the following proposition:

Proposition 3.1. *Let (M,θ) be an embedded compact strictly pseudoconvex pseudohermitian manifold and λ_1 be the first positive eigenvalue of sub-Laplacian Δ_b, then*

$$\lambda_1 = \inf \frac{\|\Delta_b u\|^2}{\int_M |d_b u|^2} = \inf \frac{\|\Delta_b u\|^2}{\int_M \left[|\bar{\partial}_b u|^2 + |\partial_b u|^2\right]}. \tag{3.1}$$

Proof: For any smooth function $f = \sum_{k=0}^{\infty} \sum_{j=1}^{m_k} a_{kj} f_{kj} \notin \mathrm{kel}(\Delta_b)$, and any real-valued function $u \notin \mathrm{kel}(\Delta_b)$, we have

$$\begin{aligned}
&\int_M u \Delta_b f \\
=&\int_M u \sum_{k=0}^{\infty} \sum_{j=1}^{m_k} \Delta_b a_{kj} f_{kj} \\
=&\int_M u \sum_{k=0}^{\infty} \sum_{j=1}^{m_k} \lambda_k a_{kj} f_{kj} \\
\geq&\int_M u \lambda_1 \sum_{k=0}^{\infty} \sum_{j=1}^{m_k} a_{kj} f_{kj} \\
=&\int_M u \left(\lambda_1 f\right).
\end{aligned}$$

Furthermore, for any real-valued function $u \notin \mathrm{kel}(\Delta_b)$,

$$\begin{aligned}
\|\Delta_b u\|^2 &= \int_M |\Delta_b u|^2 \\
&= \int_M \Delta_b u \cdot \Delta_b u \\
&\geq \int_M \lambda_1 u \left(\Delta_b u\right) \\
&= \lambda_1 \int_M u \Delta_b u \\
&= \lambda_1 \int_M |d_b u|^2 \\
&= \lambda_1 \int_M \left[|\bar{\partial}_b u|^2 + |\partial_b u|^2\right].
\end{aligned} \tag{3.2}$$

Thus,

$$\lambda_1 \leq \frac{\|\Delta_b u\|^2}{\int_M |d_b u|^2} = \frac{\|\Delta_b u\|^2}{\int_M \left[|\bar{\partial}_b u|^2 + |\partial_b u|^2\right]}.$$

Since u is arbitrary,

$$\lambda_1 \leq \inf \frac{||\Delta_b u||^2}{\int_M |d_b u|^2} = \inf \frac{||\Delta_b u||^2}{\int_M \left[|\bar{\partial}_b u|^2 + |\partial_b u|^2\right]}.$$

On the other hand, if $u \in E_1$,

$$\lambda_1 = \inf \frac{||\Delta_b u||^2}{\int_M |d_b u|^2} = \inf \frac{||\Delta_b u||^2}{\int_M \left[|\bar{\partial}_b u|^2 + |\partial_b u|^2\right]}.$$

Thus,

$$\lambda_1 = \inf \frac{||\Delta_b u||^2}{\int_M |d_b u|^2} = \inf \frac{||\Delta_b u||^2}{\int_M \left[|\bar{\partial}_b u|^2 + |\partial_b u|^2\right]}.$$

Then, we shall proof Theorem 1.1.
Proof: By calculating directly, we have

$$\Delta_b \rho_j = 2\tilde{\Delta}_\rho \rho_j + \frac{n}{|\partial\rho|^2_\rho}(\rho^{\bar{k}}\rho_{j\bar{k}} + \rho^k \rho_{jk}) = 2\tilde{\Delta}_\rho \rho_j + \frac{n}{|\partial\rho|^2_\rho}(\rho_j + \rho^k \rho_{jk}),$$

and

$$\Delta_b \rho_{\bar{j}} = 2\tilde{\Delta}_\rho \rho_{\bar{j}} + \frac{n}{|\partial\rho|^2_\rho}(\rho^{\bar{k}}\rho_{\bar{j}\bar{k}} + \rho^k \rho_{\bar{j}k}) = 2\tilde{\Delta}_\rho \rho_j + \frac{n}{|\partial\rho|^2_\rho}(\rho^{\bar{k}}\rho_{\bar{j}\bar{k}} + \rho_{\bar{j}}).$$

Furthermore,

$$\Delta_b \left(\rho_j + \rho_{\bar{j}}\right) = 2\tilde{\Delta}_\rho \left(\rho_j + \rho_{\bar{j}}\right) + \frac{n}{|\partial\rho|^2_\rho}(\rho_j + \rho^k \rho_{jk} + \rho^{\bar{k}}\rho_{\bar{j}\bar{k}} + \rho_{\bar{j}}).$$

Since $\Delta_b \rho_{\bar{j}} = \overline{\Delta_b \rho_j}$, in other words, $\Delta_b \left(\rho_j + \rho_{\bar{j}}\right)$ is real, thus,

$$\begin{aligned}
&|\Delta_b \left(\rho_j + \rho_{\bar{j}}\right)|^2 \\
=&\left(\Delta_b \left(\rho_j + \rho_{\bar{j}}\right)\right)^2 \\
=&\left(\Delta_b \left(\rho_j + \rho_{\bar{j}}\right)\right)\left(2\tilde{\Delta}_\rho \left(\rho_j + \rho_{\bar{j}}\right) + \frac{n}{|\partial\rho|^2_\rho}(\rho_j + \rho^k \rho_{jk} + \rho^{\bar{k}}\rho_{\bar{j}\bar{k}} + \rho_{\bar{j}})\right) \\
=&\frac{n}{|\partial\rho|^2_\rho}\left[\left(\rho_j + \rho_{\bar{j}}\right)\Delta_b \left(\rho_j + \rho_{\bar{j}}\right) + 2\frac{|\partial\rho|^2_\rho}{n}\tilde{\Delta}_\rho \left(\rho_j + \rho_{\bar{j}}\right)\Delta_b \left(\rho_j + \rho_{\bar{j}}\right) + \left(\rho^k \rho_{jk} + \rho^{\bar{k}}\rho_{\bar{j}\bar{k}}\right)\Delta_b \left(\rho_j + \rho_{\bar{j}}\right)\right] \\
=&\frac{n}{|\partial\rho|^2_\rho}\mathrm{Re}\left[\left(\rho_j + \rho_{\bar{j}}\right)\Delta_b \left(\rho_j + \rho_{\bar{j}}\right) + 2\Delta_b \left(\rho_j + \rho_{\bar{j}}\right)\left(2\tilde{\Delta}_\rho \rho_j + \frac{|\partial\rho|^2_\rho}{n}\rho^k \rho_{jk}\right)\right].
\end{aligned}$$

Under the condition

$$\mathrm{Re}\Delta_b \left(\rho_j + \rho_{\bar{j}}\right)\left(2\tilde{\Delta}_\rho \rho_j + \frac{|\partial\rho|^2_\rho}{n}\rho^k \rho_{jk}\right) \leq 0,$$

We obtain

$$|\Delta_b \left(\rho_j + \rho_{\bar{j}}\right)|^2 \leq \frac{n}{|\partial\rho|^2_\rho}\mathrm{Re}\left(\rho_j + \rho_{\bar{j}}\right)\Delta_b \left(\rho_j + \rho_{\bar{j}}\right) = \frac{n}{|\partial\rho|^2_\rho}\left(\rho_j + \rho_{\bar{j}}\right)\Delta_b \left(\rho_j + \rho_{\bar{j}}\right).$$

By the estimate (3.1),

$$\begin{aligned}
&\lambda_1 \int_M \left(\rho_j + \rho_{\bar{j}}\right) \Delta_b \left(\rho_j + \rho_{\bar{j}}\right) \\
=&\lambda_1 \int_M [|\bar{\partial}_b(\rho_j + \rho_{\bar{j}})|^2 + |\partial_b(\rho_j + \rho_{\bar{j}}|^2] \\
\leq& \int_M |\Delta_b \left(\rho_j + \rho_{\bar{j}}\right)|^2 \\
\leq& \int_M \frac{n}{|\partial\rho|_\rho^2} \left(\rho_j + \rho_{\bar{j}}\right) \Delta_b \left(\rho_j + \rho_{\bar{j}}\right).
\end{aligned} \tag{3.3}$$

Moreover, by applying the mean value theorem of the integral, there is $z_0 \in M$ such that

$$\begin{aligned}
0 \leq& \int_M \left(\frac{n}{|\partial\rho|_\rho^2} - \lambda_1\right) \left(\rho_j + \rho_{\bar{j}}\right) \Delta_b \left(\rho_j + \rho_{\bar{j}}\right) \\
=&\left(\frac{n}{|\partial\rho|_\rho^2}(z_0) - \lambda_1\right) \int_M \left(\rho_j + \rho_{\bar{j}}\right) \Delta_b \left(\rho_j + \rho_{\bar{j}}\right) \\
=&\left(\frac{n}{|\partial\rho|_\rho^2}(z_0) - \lambda_1\right) \int_M \left(\left|\bar{\partial}_b(\rho_j + \rho_{\bar{j}})\right|^2 + \left|\partial_b(\rho_j + \rho_{\bar{j}})\right|^2\right) \\
=&\left(\frac{n}{|\partial\rho|_\rho^2}(z_0) - \lambda_1\right) \int_M \left|d_b(\rho_j + \rho_{\bar{j}})\right|^2.
\end{aligned}$$

Therefore,

$$\lambda_1 \leq \frac{n}{|\partial\rho|_\rho^2}(z_0) \leq \max_M \frac{n}{|\partial\rho|_\rho^2}. \tag{3.4}$$

It is obviously that if the equality holds, $|\partial\rho|_\rho^2$ must be a constant on M.

Next, we will investigate the case when $\rho_{j\bar{k}} = \delta_{jk}$ and obtain Theorem 1.2. To prove Theorem 1.2, we first prove the following proposition.

Proposition 3.2. *Suppose ρ is a smooth strictly plurisubharmonic function defined on an open set $U \subset C^{n+1}$, M is a compact connected regular level set of ρ, and λ_1 is the first positive eigenvalue of Δ_b on M. Let $r(z)$ be the spectral radius of the matrix $[\rho^{j\bar{k}}(z)]$ and $s(z) = trace[\rho^{j\bar{k}}(z)] - r(z)$. Then,*

$$\lambda_1 \leq 2n^2 \frac{\int_M \frac{r(z)}{|\partial\rho|_\rho^2}}{\int_M s(z)}.$$

Proof: By (2.3),

$$\Delta_b(\bar{z}_j + z_j) = \frac{n}{|\partial\rho|_\rho^2} \left(\rho^{\bar{j}} + \rho^j\right),$$

and

$$||\Delta_b(\bar{z}_j + z_j)||^2 = n^2 \int_M \frac{\left|\rho^{\bar{j}} + \rho^j\right|^2}{|\partial\rho|_\rho^4} = 2n^2 \int_M \frac{|\rho^j|^2 + \mathrm{Re}(\rho^j)^2}{|\partial\rho|_\rho^4}. \tag{3.5}$$

By calculating directly,

$$|\bar{\partial}_b(\bar{z}_j + z_j)|^2 = |\partial_b(\bar{z}_j + z_j)|^2 = \delta_{\alpha j}\delta_{\beta j} \left(\rho^{\alpha\beta} - \frac{\rho^\alpha \rho^{\bar{\beta}}}{|\partial\rho|_\rho^2}\right) = \rho^{j\bar{j}} - \frac{|\rho^j|^2}{|\partial\rho|_\rho^2},$$

and

$$\int_M [|\bar{\partial}_b(\bar{z}_j + z_j)|^2 + |\partial_b(\bar{z}_j + z_j)|^2] = 2\int_M \left(\rho^{j\bar{j}} - \frac{|\rho^j|^2}{|\partial\rho|_\rho^2}\right).$$

According to the formula (3.1), it is easy to obtain

$$\lambda_1 \leq 2n^2 \frac{\int_M \frac{|\rho^j|^2 + \mathrm{Re}(\rho^j)^2}{|\partial\rho|_\rho^4}}{2\int_M \left(\rho^{j\bar{j}} - \frac{|\rho^j|^2}{|\partial\rho|_\rho^2}\right)} \leq 2n^2 \frac{\int_M \frac{|\rho^j|^2}{|\partial\rho|_\rho^4}}{\int_M \left(\rho^{j\bar{j}} - \frac{|\rho^j|^2}{|\partial\rho|_\rho^2}\right)}. \tag{3.6}$$

By the compute in [21],

$$\sum_{j=1}^{n+1} |\rho_j|^2 \leq r(z)|\partial\rho|_\rho^2.$$

Thus,

$$\begin{aligned}
\lambda_1 \leq \min_{1\leq j\leq n+1} 2n^2 &\frac{\int_M \frac{|\rho^j|^2}{|\partial\rho|_\rho^4}}{\int_M \left(\rho^{j\bar{j}} - \frac{|\rho^j|^2}{|\partial\rho|_\rho^2}\right)} \\
\leq& 2n^2 \frac{\sum_{j=1}^{n+1} \int_M \frac{|\rho^j|^2}{|\partial\rho|_\rho^4}}{\sum_{j=1}^{n+1} \int_M \left(\rho^{j\bar{j}} - \frac{|\rho^j|^2}{|\partial\rho|_\rho^2}\right)} \\
\leq& 2n^2 \frac{\int_M \frac{r(z)}{|\partial\rho|_\rho^2}}{\int_M s(z)}.
\end{aligned}$$

The following is the proof of Theorem 1.2.

Proof: **1)** As assumed conditions, $\rho_{j\bar{k}} = \delta_{jk}$; hence, $r(z) = 1$ and $s(z) = n$. Based on Proposition 3.2,

$$\lambda_1 \leq \frac{2n}{v(M)} \int_M \frac{1}{|\partial\rho|_\rho^2}.$$

2) If $\lambda_1 = \frac{2n}{v(M)} \int_M \frac{1}{|\partial\rho|_\rho^2}$, by formula (3.5), we have

$$||\Delta_b(\bar{z}_j + z_j)||^2 = 2n^2 \int_M \frac{|\rho^j|^2 + \mathrm{Re}(\rho^j)^2}{|\partial\rho|_\rho^4} \leq 4n^2 \int_M \frac{|\rho^j|^2}{|\partial\rho|_\rho^4}. \tag{3.7}$$

Therefore,

$$\sum_{j=1}^{n+1} ||\Delta_b(\bar{z}_j + z_j)||^2 \leq 4n^2 \int_M \frac{1}{|\partial\rho|_\rho^2} = 2n\lambda_1 v(M). \tag{3.8}$$

On the other hand, by formula (3.2),

$$\sum_{j=1}^{n+1} ||\Delta_b(\bar{z}_j + z_j)||^2 \geq 2\lambda_1 \sum_{j=1}^{n+1} \int_M \left(\rho^{j\bar{j}} - \frac{|\rho^j|^2}{|\partial\rho|_\rho^2}\right) = 2n\lambda_1 v(M). \tag{3.9}$$

Combine (3.8) with (3.9),

$$
\begin{aligned}
&\sum_{j=1}^{n+1} ||\Delta_b(\bar{z}_j + z_j)||^2 \\
=&2n\lambda_1 v(M) \\
=&2\lambda_1 \sum_{j=1}^{n+1} \int_M \left(\rho^{j\bar{j}} - \frac{|\rho^j|^2}{|\partial\rho|^2_\rho}\right) \\
=&\lambda_1 \sum_{j=1}^{n+1} \int_M \left(|\bar{\partial}_b(\bar{z}_j + z_j)|^2 + |\partial_b(\bar{z}_j + z_j)|^2\right).
\end{aligned} \tag{3.10}
$$

According to the inequality (3.2) or (3.1), it is easy to show that for any $j = 1, 2, \ldots, n+1$,

$$\Delta_b(\bar{z}_j + z_j) = \frac{n}{|\partial\rho|^2_\rho}\left(\rho^{\bar{j}} + \rho^j\right) \perp \mathrm{Ker}(\Delta_b),$$

and

$$\Delta_b(\bar{z}_j + z_j) = \frac{n}{|\partial\rho|^2_\rho}\left(\rho^{\bar{j}} + \rho^j\right) \in E_1,$$

here E_1 is the eigenspace of Δ_b corresponding to the eigenvalue λ_1. Thus,

$$
\begin{aligned}
&\Delta_b\left(\frac{n}{|\partial\rho|^2_\rho}\left(\rho^{\bar{j}} + \rho^j\right)\right) - \lambda_1\left(\frac{n}{|\partial\rho|^2_\rho}\left(\rho^{\bar{j}} + \rho^j\right)\right) \\
=&\Delta_b\left(\frac{n}{|\partial\rho|^2_\rho}\left(\rho^{\bar{j}} + \rho^j\right)\right) - \lambda_1\Delta_b(\bar{z}_j + z_j) \\
=&\Delta_b\left[\left(\frac{n}{|\partial\rho|^2_\rho}\left(\rho^{\bar{j}} + \rho^j\right)\right) - \lambda_1(\bar{z}_j + z_j)\right] \\
=&0.
\end{aligned} \tag{3.11}
$$

It means that

$$\left(\frac{n}{|\partial\rho|^2_\rho}\left(\rho^{\bar{j}} + \rho^j\right)\right) - \lambda_1(\bar{z}_j + z_j) \in \mathrm{Ker}(\Delta_b).$$

Hence, for the tangential vector fields $X_{lk} = \rho_k \frac{\partial}{\partial z_l} - \rho_l \frac{\partial}{\partial z_k}$ and $X_{\bar{l}\bar{k}} = \rho_{\bar{k}} \frac{\partial}{\partial z_{\bar{l}}} - \rho_{\bar{l}} \frac{\partial}{\partial z_{\bar{k}}}$ on M,

$$
\begin{aligned}
0 =& (X_{lk} + X_{\bar{l}\bar{k}})\left[\left(\frac{n}{|\partial\rho|^2_\rho}\left(\rho^{\bar{j}} + \rho^j\right)\right) - \lambda_1(\bar{z}_j + z_j)\right] \\
=&\left(\frac{n}{|\partial\rho|^2_\rho} - \lambda_1\right)\left[\delta_{jl}(\rho_k + \rho_{\bar{k}}) - \delta_{jk}(\rho_l + \rho_{\bar{l}})\right] + \frac{n}{|\partial\rho|^2_\rho}\left(\rho_k\rho_{jl} - \rho_l\rho_{jk} + \rho_{\bar{k}}\rho_{\bar{j}\bar{l}} - \rho_{\bar{l}}\rho_{\bar{j}\bar{k}}\right) \\
&+ \frac{n}{|\partial\rho|^4_\rho}\left(\rho^{\bar{j}} + \rho^j\right)(X_{lk} + X_{\bar{l}\bar{k}})\,|\partial\rho|^2_\rho.
\end{aligned} \tag{3.12}
$$

Observed that M is compact; let Z be the maximum point of $|\partial\rho|^2_\rho$, thus,

$$(X_{lk} + X_{\bar{l}\bar{k}})\,|\partial\rho|^2_\rho(Z) = 0.$$

Furthermore,

$$\left(\frac{n}{|\partial\rho|_\rho^2}-\lambda_1\right)\left[\delta_{jl}(\rho_k+\rho_{\bar{k}})-\delta_{jk}(\rho_l+\rho_{\bar{l}})\right]+\frac{n}{|\partial\rho|_\rho^2}\left(\rho_k\rho_{jl}-\rho_l\rho_{jk}+\rho_{\bar{k}}\rho_{\bar{j}\bar{l}}-\rho_{\bar{l}}\rho_{\bar{j}\bar{k}}\right)=0 \tag{3.13}$$

at Z.

Recall the inequality (3.6); we can easily find that the equation holds if and only if $\rho_j=\rho_{\bar{j}}$. Thus, $\rho_{jk}=\rho_{\bar{j}k}=\delta_{jk}$ and $\rho_{\bar{j}\bar{k}}=\bar{\rho}_{jk}=\delta_{jk}$. Substitute them into Eq (3.13),

$$\left(\frac{2n}{|\partial\rho|_\rho^2}-\lambda_1\right)\left(2\delta_{jl}\rho_k-2\delta_{jk}\rho_l\right)=0. \tag{3.14}$$

Since $|\partial\rho|\neq 0$, without loss of generality, we assume $\rho_1\neq 0$. Choose $k=1$ and $j=l=2$, then Eq (3.14) turn into

$$2\rho_1\left(\frac{2n}{|\partial\rho|_\rho^2}-\lambda_1\right)=0.$$

It implies that

$$\lambda_1=\frac{2n}{|\partial\rho|_\rho^2}(Z)=\min_{z\in M}\frac{2n}{|\partial\rho|_\rho^2}.$$

Namely,

$$\min_{z\in M}\frac{1}{|\partial\rho|_\rho^2}=\frac{1}{|\partial\rho|_\rho^2}(Z)=\frac{\lambda_1}{2n}. \tag{3.15}$$

On the other hand, by the assumption $\lambda_1=\frac{2n}{v(M)}\int_M\frac{1}{|\partial\rho|_\rho^2}$,

$$\frac{1}{v(M)}\int_M\frac{1}{|\partial\rho|_\rho^2}=\frac{\lambda_1}{2n}, \tag{3.16}$$

which is the average value of $\frac{1}{|\partial\rho|_\rho^2}$ on M. Combine (3.15) with (3.16), and we concluded that $|\partial\rho|_\rho^2$ must be a constant on M.

At the end of this section, let us observe an example that gives the upper for the first positive eigenvalue of Δ_b on a compact connected regular level set with Kähler potentials. This example has also been used to explore the upper bound for the first positive eigenvalue of Kohn-Laplacian [21].

Example 3.1. *Let ρ be defined and proper in a domain $U\subset C^{n+1}$ and be a strictly plurisubharmonic function of the expression*

$$\rho(z)=\ln(1+\|z\|^2)+\psi(z,\bar{z}),$$

where ψ is a real-valued pluriharmonic function. Suppose $M=\rho^{-1}(v)$ is a compact connected regular level set of ρ.

1) It is obtained by calculation that

$$\rho_{j\bar{k}}=\frac{1}{1+\|z\|^2}\left(\delta_{jk}-\frac{\bar{z}_jz_k}{1+\|z\|^2}\right),\quad \rho^{j\bar{k}}=(1+\|z\|^2)\left(\delta_{jk}+\bar{z}_kz_j\right).$$

Thus, the spectral radius of $\rho^{j\bar{k}}(z)$ is $r(z) = (1 + ||z||^2)^2$, and $s(z) = \text{trace}[\rho^{j\bar{k}}(z)] - r(z) = n(1 + ||z||^2)$. By Proposition 3.2 and the mean value theorem for integrals, there exists $z_0 \in M$ such that

$$\lambda_1 \leq 2n \frac{\int_M \frac{(1+||z||^2)^2}{|\partial\rho|_\rho^2}}{\int_M (1 + ||z||^2)} = 2n \frac{1 + ||z_0||^2}{|\partial\rho|_\rho^2(z_0)} \leq 2n \max_{z \in M} \frac{1 + ||z||^2}{|\partial\rho|_\rho^2}.$$

2) If $\psi = 0$ on M,

$$|\partial\rho|_\rho^2 = ||z||^2, \quad \tilde{\Delta}_\rho \rho_j = n\bar{z}_j,$$
$$\tilde{\Delta}_\rho(\rho_j + \rho_{\bar{j}}) = n(\bar{z}_j + z_j),$$
$$2\tilde{\Delta}_\rho \rho_j + \frac{|\partial\rho|_\rho^2}{n} \rho^k \rho_{jk} = \frac{2n^2(1 + ||z||^2) - ||z||^4}{n(1 + ||z||^2)} \bar{z}_j,$$
$$\Delta_b(\rho_j + \rho_{\bar{j}}) = n(\bar{z}_j + z_j) \left(\frac{1}{||z||^2} + \frac{2||z||^2}{1 + ||z||^2} \right).$$

Therefore,

$$\Delta_b(\rho_j + \rho_{\bar{j}}) \left(2\tilde{\Delta}_\rho \rho_j + \frac{|\partial\rho|_\rho^2}{n} \rho^k \rho_{jk} \right) = \bar{z}_j(\bar{z}_j + z_j) \left(\frac{1}{||z||^2} + \frac{2||z||^2}{1 + ||z||^2} \right) \frac{2n^2(1 + ||z||^2) - ||z||^4}{(1 + ||z||^2)},$$

and

$$\text{Re}\Delta_b(\rho_j + \rho_{\bar{j}}) \left(2\tilde{\Delta}_\rho \rho_j + \frac{|\partial\rho|_\rho^2}{n} \rho^k \rho_{jk} \right) = 2(\text{Re}z_j)^2 \left(\frac{1}{||z||^2} + \frac{2||z||^2}{1 + ||z||^2} \right) \frac{2n^2(1 + ||z||^2) - ||z||^4}{(1 + ||z||^2)}.$$

It is obvious that $2(\text{Re}z_j)^2 \geq 0$, $\frac{1}{||z||^2} + \frac{2||z||^2}{1+||z||^2} \geq 0$, and $1 + ||z||^2 \geq 0$. However, we cannot determine whether $2n^2(1 + ||z||^2) - ||z||^4 \leq 0$ on M. Thus, Theorem 1.1 fails in this example.

Since we have calculated that $r(z) = (1 + ||z||^2)^2 = e^{2\nu}$, $s(z) = n(1 + ||z||^2) = ne^\nu$, and $|\partial\rho|_\rho^2 = ||z||^2 = e^\nu - 1$ on M, by Proposition 3.2,

$$\lambda_1 \leq 2n \frac{\int_M \frac{e^{(2\nu)}}{e^\nu - 1}}{\int_M e^\nu} = \frac{2ne^\nu}{e^\nu - 1}.$$

We can deduce from the proof of Proposition 3.2 that λ_1 attains the upper bound if and only if $\text{Im}\rho^j = 0$. But $\rho^j = z_j(1 + ||z||^2)$ implies $\text{Im}\rho^j \not\equiv 0$ on M, thus, λ_1 cannot attain the upper bound $\frac{2ne^\nu}{e^\nu - 1}$.

3) We continue to discuss the case of $\psi = 0$. By direct calculation, $2n^2(1 + ||z||^2) - ||z||^4 = -e^{2\nu} + (2n^2 + 2) - 1 \leq 0$ only if $e^\nu \geq (n^2 + 1) + n\sqrt{n^2 + 2}$.

Choose $\nu \geq \ln[(n^2 + 1) + n\sqrt{n^2 + 2}]$ such that $\text{Re}\Delta_b(\rho_j + \rho_{\bar{j}}) \left(2\tilde{\Delta}_\rho \rho_j + \frac{|\partial\rho|_\rho^2}{n} \rho^k \rho_{jk} \right) \leq 0$, according to Theorem 1.1,

$$\lambda_1 \leq \max_M \frac{n}{|\partial\rho|_\rho^2}.$$

Observe that $|\partial\rho|_\rho^2 = ||z||^2 = e^\nu - 1$ is a constant on M, therefore,

$$\lambda_1 = \frac{n}{e^\nu - 1}.$$

4. The upper bound for the first positive eigenvalue of sub-Laplacian on real ellipsoids

We shall consider the upper bound for the first positive eigenvalue of sub-Laplacian on real ellipsoids and obtain the following results in this section.

Theorem 4.1. *Let $M_\nu = \rho^{-1}(\nu)$ be the ellipsoid, which is a compact regular level set of a real-valued strictly plurisubharmonic homogeneous quadratic polynomial $\rho(z)$ satisfying $\rho_{j\bar{k}} = \delta_{jk}$. Then,*

$$\lambda_1\left(M_\nu, \theta\right) \leq \frac{2n}{\nu}.$$

In addition, λ_1 cannot attain the upper $\frac{2n}{\nu}$ on the ellipsoid.

Proof: **1)** We proof the inequality firstly. By making a holomorphic unitary change of variables, the real ellipsoid M_ν can be represented as

$$\rho(z) = |z|^2 + \mathrm{Re}\sum_{j=1}^{n+1} A_j z_j^2 = \nu$$

with $0 \leq A_1 \leq A_2 \leq \cdots \leq A_{n+1} < 1$. Hence,

$$\rho_j = \bar{z}_j + A_j z_j, \qquad \rho_{\bar{j}} = z_j + A_j \bar{z}_j,$$

$$\rho_{j\bar{k}} = \rho^{j\bar{k}} = \delta_{jk}, \quad z_j\rho_j = |z_j|^2 + A_j z_j^2.$$

Thus,

$$\mathrm{Re}\sum_{j=1}^{n+1} z_j\rho_j = |z|^2 + \mathrm{Re}\sum_{j=1}^{n+1} A_j z_j^2 = \nu \qquad \text{on} \quad M_\nu.$$

By the calculation in [21],

$$\int_{M_\nu} \frac{A_j\bar{z}_j\rho_j}{|\partial\rho|_\rho^2} = \frac{1}{n}\int_{M_\nu} A_j\bar{z}_j\Box_b\bar{z}_j = \frac{A_j}{n}\int_{M_\nu} z_j\overline{\Box_b z_j} = 0.$$

Therefore,

$$\begin{aligned}
\int_{M_\nu}\frac{1}{|\partial\rho|_\rho^2} =& \frac{1}{\nu}\sum_{j=1}^{n+1}\int_{M_\nu}\frac{\mathrm{Re} z_j\rho_j}{|\partial\rho|_\rho^2} \\
=& \frac{1}{\nu}\sum_{j=1}^{n+1}\int_{M_\nu}\frac{\mathrm{Re}(z_j + A_j\bar{z}_j)\rho_j}{|\partial\rho|_\rho^2} \\
=& \frac{1}{\nu}\sum_{j=1}^{n+1}\int_{M_\nu}\frac{\mathrm{Re}\rho_{\bar{k}}\rho^{j\bar{k}}\rho_j}{|\partial\rho|_\rho^2} \\
=& \frac{1}{\nu}\int_{M_\nu}\frac{\mathrm{Re}|\partial\rho|_\rho^2}{|\partial\rho|_\rho^2} \\
=& \frac{\nu(M_\nu)}{\nu}.
\end{aligned} \tag{4.1}$$

According to Theorem 1.2, we have

$$\lambda_1(M_\nu) \le \frac{2n}{\nu}.$$

2) Now, we assume that $\lambda_1(M_\nu) = \frac{2n}{\nu}$ and will prove the further result in two ways.
Methods 1: Recall equality (3.13) and sum j from 1 to $n+1$. We have

$$\left(\frac{n}{|\partial\rho|^2_\rho} - \lambda_1\right)(\rho_k + \rho_{\bar{k}} - \rho_l - \rho_{\bar{l}}) + \sum_{j=1}^{n+1} \frac{n}{|\partial\rho|^2_\rho}\left(\rho_k\rho_{jl} - \rho_l\rho_{jk} + \rho_{\bar{k}}\rho_{\bar{j}\bar{l}} - \rho_{\bar{l}}\rho_{\bar{j}\bar{k}}\right) = 0 \tag{4.2}$$

at the maximum of $|\partial\rho|^2_\rho$ on M_ν.

According to [18], $|\partial\rho|^2_\rho$ attains its maximum at $Z = \left(0, 0, \cdots, \pm\frac{\sqrt{\nu}}{\sqrt{1+A_{n+1}}}\right)$ on M_ν. At $Z = \left(0, 0, \cdots, \frac{\sqrt{\nu}}{\sqrt{1+A_{n+1}}}\right)$, if "$k = l$", or "$k \neq n+1$ and $l \neq n+1$", (4.2) obviously holds. Thus, let $k = n+1$ and $l \neq n+1$. Substitute the following calculation

$$\rho_k + \rho_{\bar{k}} - \rho_l - \rho_{\bar{l}} = 2\nu\sqrt{1 + A_{n+1}},$$

and

$$\begin{aligned}
&\sum_{j=1}^{n+1}\left(\rho_k\rho_{jl} - \rho_l\rho_{jk} + \rho_{\bar{k}}\rho_{\bar{j}\bar{l}} - \rho_{\bar{l}}\rho_{\bar{j}\bar{k}}\right)\\
=&\sum_{j=1}^{n+1}\left(\rho_k\delta_{jl}A_j - \rho_l\delta_{jk}A_j + \rho_{\bar{k}}\delta_{jl}A_j - \rho_{\bar{l}}\delta_{jk}A_j\right)\\
=&2A_l\nu\sqrt{1 + A_{n+1}}
\end{aligned} \tag{4.3}$$

into (4.2), we obtained

$$2\nu\sqrt{1 + A_{n+1}}\left[\frac{n}{|\partial\rho|^2_\rho}(1 + A_l) - \lambda_1(M_\nu)\right] = 0.$$

Since $\sqrt{1 + A_{n+1}} \neq 0$, it must be true that

$$\frac{n}{|\partial\rho|^2_\rho}(1 + A_l) - \lambda_1(M_\nu) = 0.$$

It means $\lambda_1(M_\nu) = \frac{n}{|\partial\rho|^2_\rho}(1 + A_l)$ for $l = 1, 2, \cdots, n$. Thus, $A_1 = A_2 = \cdots = A_n$.

Apply above analysis to the case at one of the extremums $Z = \left(0, 0, \cdots, \pm\frac{\sqrt{\nu}}{\sqrt{1+A_n}}, 0\right)$, we find $A_1 = A_2 = \cdots = A_{n-1} = A_{n+1}$. Therefore, $A_1 = A_2 = \cdots = A_{n+1}$, and

$$\lambda_1(M_\nu) = n(1 + A_{n+1})\frac{1}{|\partial\rho|^2_\rho}(Z) = n(1 + A_{n+1})\min_{z\in M}\frac{1}{|\partial\rho|^2_\rho}. \tag{4.4}$$

Recall that $\lambda_1(M_\nu) = \frac{2n}{\nu(M)}\int_M \frac{1}{|\partial\rho|^2_\rho}$, and $1 + A_{n+1} \le 2$, we deduced from above that $A_1 = A_2 = \cdots = A_{n+1} = 1$. It contradicts $0 \le A_1 \le A_2 \le \cdots \le A_{n+1} < 1$.

In fact, inspired by the derivation in [18], we easily find that

$$|\partial\rho|^2_\rho = 2\nu - \sum_{j=1}^{n+1}(1 - A_j^2)|z_j|^2.$$

If $A_1 = A_2 = \cdots = A_{n+1} = 1$,

$$|\partial\rho|_\rho^2 \equiv 2\nu.$$

By (4.4),

$$\lambda_1(M_\nu) = \frac{n}{\nu}.$$

It contradicts the assumption $\lambda_1(M_\nu) = \frac{2n}{\nu}$. Thus, λ_1 cannot attain the upper bound $\frac{2n}{\nu}$ on ellipsoids.
Methods 2: According to Theorem 1.2, $\lambda_1(M_\nu) = \frac{2n}{\nu}$ if and only if $|\partial\rho|_\rho^2$ is a constant on M. Inspired by the derivation in [18], we find that

$$|\partial\rho|_\rho^2 = 2\nu - \sum_{j=1}^{n+1}(1 - A_j^2)|z_j|^2, \tag{4.5}$$

or

$$|\partial\rho|_\rho^2 = \nu + \mathrm{Re}\sum_{j=1}^{n+1} A_j(A_j|z_j|^2 - z_j^2). \tag{4.6}$$

Therefore, $|\partial\rho|_\rho^2$ is a constant on M if and only if $A_1 = A_2 = \cdots = A_{n+1} = 1$ or $A_1 = A_2 = \cdots = A_{n+1} = 0$.
Case 1: If $A_1 = A_2 = \cdots = A_{n+1} = 1$, by the above analysis in **Methods 1**, it contradicts the assumption that $\lambda_1(M_\nu) = \frac{2n}{\nu}$.
Case 2: If $A_1 = A_2 = \cdots = A_{n+1} = 0$, we have

$$|\partial\rho|_\rho^2 \equiv \nu.$$

By (4.4)

$$\lambda_1(M_\nu) = \frac{n}{\nu}.$$

It also contradicts the assumption that $\lambda_1(M_\nu) = \frac{2n}{\nu}$.

Summing up the above two cases, λ_1 cannot attain the upper bound $\frac{2n}{\nu}$ on ellipsoids.

Remark 4.1. **a**) In **Methods 1**, we can deduce that $A_1 = A_2 = \cdots = A_{n+1} = 1$ more easily and rapidly from the fact that $\lambda_1(M_\nu)$ attains its maximum if and only if $\mathrm{Im}\rho^j = 0 \quad (j = 1, 2, \cdots, n+1)$.

b) In fact, on the ellipsoid M_ν (including the case when $A_1 = A_2 = \cdots = A_{n+1} = 0$), $\mathrm{Im}\rho^j \neq 0$ ($j = 1, 2, \cdots, n+1$) leads to result that $\lambda_1(M_\nu)$ cannot attain the upper bound $\frac{2n}{\nu}$.

According to Theorem 4.1 and the Remark, we wondered if $\frac{2n}{\nu}$ is the supremum of λ_1 on ellipsoids. For the ellipsoids M_ν which are expressed as:

$$\rho(z) = |z|^2 + \mathrm{Re}\sum_{j=1}^{n+1} A_j z_j^2 = \nu,$$

we can calculate directly to show that

$$\Delta_b(\rho_j + \rho_{\bar{j}}) = (1 + A_j)\frac{n}{|\partial\rho|_\rho^2}(\rho_j + \rho_{\bar{j}}) = (1 + A_j)^2\frac{n}{|\partial\rho|_\rho^2}(z_j + z_{\bar{j}}), \tag{4.7}$$

and

$$\tilde{\Delta}_\rho \rho_j = 0.$$

Thus,

$$
\begin{aligned}
&\Delta_b\left(\rho_j+\rho_{\bar{j}}\right)\left(2\tilde{\Delta}_\rho\rho_j+\frac{|\partial\rho|_\rho^2}{n}\rho^k\rho_{jk}\right)\\
=&A_j(1+A_j)\rho_{\bar{j}}(\rho_j+\rho_{\bar{j}})\\
=&A_j(1+A_j)^2(z_j+A_jz_{\bar{j}})(z_j+z_{\bar{j}})\\
=&A_j(1+A_j)^2\left[(1+A_j)|z_j|^2+z_j^2+A_jz_{\bar{j}}^2\right].
\end{aligned}
\tag{4.8}
$$

Moreover,

$$
\begin{aligned}
&\mathrm{Re}\Delta_b\left(\rho_j+\rho_{\bar{j}}\right)\left(2\tilde{\Delta}_\rho\rho_j+\frac{|\partial\rho|_\rho^2}{n}\rho^k\rho_{jk}\right)\\
=&A_j(1+A_j)^2\left[(1+A_j)|z_j|^2+\mathrm{Re}(z_j^2+A_jz_{\bar{j}}^2)\right]\\
=&A_j(1+A_j)^2\left[(1+A_j)|z_j|^2+2\mathrm{Re}z_j^2-\mathrm{Re}(1-A_j)z_{\bar{j}}^2)\right]\\
=&A_j(1+A_j)^2\left[(1+A_j)|z_j|^2+(1+A_j)\mathrm{Re}z_j^2\right]\\
=&A_j(1+A_j)^3(|z_j|^2+\mathrm{Re}z_j^2).
\end{aligned}
\tag{4.9}
$$

Since $|z_j|^2+\mathrm{Re}z_j^2\geq 0$, and $|z_j|^2+\mathrm{Re}z_j^2\not\equiv 0$ on M_ν,

$$
\mathrm{Re}\Delta_b\left(\rho_j+\rho_{\bar{j}}\right)\left(2\tilde{\Delta}_\rho\rho_j+\frac{|\partial\rho|_\rho^2}{n}\rho^k\rho_{jk}\right)\leq 0
$$

if and only if $A_j=0$ for any $j=1,\cdots,n+1$. Therefore, by Theorem 1.1 and the proof of Theorem 4.1, we have

Corollary 4.1. *Let $M_\nu=\rho^{-1}(\nu)$ be a sphere of radius ν, where $\rho(z)=|z|^2$. Then,*

$$
\lambda_1(M_\nu)=\frac{n}{\nu}.
$$

In addition, we can deduce from the formula (4.7) that

$$
\lambda_1(M_\nu)\leq(1+A_j)\frac{n}{|\partial\rho|_\rho^2},
$$

for any $j=1,2,3,\cdots,n+1$. Therefore,

$$
\lambda_1(M_\nu)\leq(1+A_1)\frac{n}{|\partial\rho|_\rho^2}\leq(1+A_1)\max_{z\in M_\nu}\frac{n}{|\partial\rho|_\rho^2}=\frac{n(1+A_1)}{(1-A_{n+1})\nu}.
$$

Thus, we have the following corollary:

Corollary 4.2. *Let $M_\nu=\rho^{-1}(\nu)$ be the ellipsoid, which is a compact regular level set of a real-valued strictly plurisubharmonic homogeneous quadratic polynomial $\rho(z)=|z|^2+Re\sum_{j=1}^{n+1}A_jz_j^2$. Then,*

$$
\lambda_1(M_\nu)\leq\frac{n(1+A_1)}{(1-A_{n+1})\nu},
$$

and if for some j, $A_j>0$, M_ν is not the sphere, namely, $|\partial\rho|_\rho^2$ is not a constant along M_ν, and thus $\lambda_1(M_\nu)$ cannot attain the upper bound $\frac{n(1+A_1)}{(1-A_{n+1})\nu}$.

***Remark* 4.2.** The case when M_ν is not the sphere in Corollary 4.2 also tells us that the condition

$$\mathrm{Re}\Delta_b\left(\rho_j+\rho_{\bar{j}}\right)\left(2\tilde{\Delta}_\rho\rho_j+\frac{|\partial\rho|_\rho^2}{n}\rho^k\rho_{jk}\right)\le 0$$

in Theorem 1.1 cannot be relaxed.

By summarizing Theorem 4.1, Corollaries 4.1 and 4.2, we can easily obtain Theorem 1.3.

Finally, let us review the example given at the end of Section 3.

Example 4.1. *M is defined as which in Example 3.1. If $\psi = 0$, M is actually a sphere $\|z\|^2 = e^\nu - 1$. Applying Theorem 1.3 or Corollary 4.1,*

$$\lambda_1 = \frac{n}{e^\nu - 1}.$$

5. Conclusions

In this paper, we study the upper bound for the first positive eigenvalue of the sub-Laplacian Δ_b on a compact, strictly pseudoconvex hypersurface. First, we recalled some notations and definitions about the sub-Laplacian Δ_b on compact real hypersurfaces M and defined a degenerate differential operator $\tilde{\Delta}_\rho$, and then gave a formula for Δ_b acting on functions; see Proposition 2.1. Second, we gave a formula (3.1) for calculating the first positive eigenvalue of the sub-Laplacian by decomposition of the eigenspace. Then under a certain condition "$\mathrm{Re}\Delta_b\left(\rho_j+\rho_{\bar{j}}\right)\left(2\tilde{\Delta}_\rho\rho_j+\frac{|\partial\rho|_\rho^2}{n}\rho^k\rho_{jk}\right)\le 0$", by integration by parts and the mean value theorem of the integral, we proved that

$$\lambda_1(\Delta_b)\le\frac{n}{|\partial\rho|_\rho^2}(z_0)\le\max_M\frac{n}{|\partial\rho|_\rho^2},$$

and if the equality holds, $|\partial\rho|_\rho^2$ must be a constant on M (Theorem 1.1). Next, we considered the case when $\rho_{j\bar{k}} = \delta_{jk}$, and found that

$$\lambda_1(\Delta_b)\le\frac{2n}{\nu(M)}\int_M\frac{1}{|\partial\rho|_\rho^2},$$

and likewise, if the equality holds, $|\partial\rho|_\rho^2$ must be a constant on M (Theorem 1.2). Finally, we applied the above conclusions to ellipsoids and generalized Theorem 4.1, Corollaries 4.1 and 4.2 to Theorem 1.3, which shows that

$$\lambda_1(\Delta_b)\le\min\left\{\frac{n(1+A_1)}{1-A_{n+1}},\frac{2n}{\nu}\right\},$$

on ellipsoids M_ν, and only when M_ν is a sphere of radius ν, $\lambda_1(M_\nu)$ can attain the upper bound $\min\left\{\frac{n(1+A_1)}{(1-A_{n+1})\nu},\frac{2n}{\nu}\right\} = \frac{n}{\nu}$, see Theorem 1.3. At the end of Sections 3 and 4, to verify the correctness of our result, we provided an example that gives the upper for the first positive eigenvalue of Δ_b on a compact connected regular level set with Kähler potentials.

Integrated with the conclusions of [13,20,21], it is obvious that whether the sub-Laplacian or Kohn-Laplacian, the first positive eigenvalue λ_1 reaches its upper bound and lower bound if and only if M is a sphere, and $\lambda_1 = \frac{n}{\nu}$ with the radius ν in this case.

However, the results above were obtained under some restrictive conditions, and we expect to find more general cases. This is exactly what we are going to continue to study next.

Author contributions

Guijuan Lin: carried out the main reasoning and proof as well as calculations, and wrote the manuscript; Sujuan Long and Qiqi Zhang: completed the analysis of references and the collation of concepts, and also reviewed the manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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