AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(9): 25360–25375.
DOI: 10.3934/math.20241238
Received: 27 June 2024
Revised: 13 August 2024
Accepted: 19 August 2024
Published: 30 August 2024

Research article

The altered Hermite matrix: implications and ramifications

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Abstract: Matrix theory is essential for addressing practical problems and executing computational tasks. Matrices related to Hermite polynomials are essential due to their applications in quantum mechanics, numerical analysis, probability, and signal processing. Their orthogonality, recurrence relations, and spectral properties make them a valuable tool for both theoretical research and practical applications. From a different perspective, we introduced a variant of the Hermite matrix that incorporates triple factorials and demonstrated that this matrix satisfies various properties. By utilizing effective matrix algebra techniques, various algebraic properties of this matrix have been determined, including the product formula, inverse matrix and eigenvalues. Additionally, we extended this matrix to a more generalized form and derived several identities.

Keywords: Pascal-like matrix; Hermite polynomials; degenerate Hermite polynomials; factorization of matrix; Toeplitz matrix
Mathematics Subject Classification: 11C08, 05A10, 11B83, 15A23, 15B05

1. Introduction

In recent times, lower triangular matrices have sparked significant interest in various investigations. Initially, attention was drawn to the Pascal matrix and several of its generalizations; see [1, 2, 4–9, 13–17, 20–22, 34, 35, 42, 44–48]. The Pascal matrix, formed from the coefficients $\binom{n}{k}$, stands as one of the earliest documented instances of two-dimensional number arrays. Its origins trace back centuries, likely emerging as a tabular representation of coefficients in the expansion of $(x + y)^n$. Over time, this matrix has undergone extensive generalization through various mathematical approaches and interdisciplinary concepts [38–40]. One avenue of generalization involves expanding upon the fundamental recurrence relation of binomial coefficients. Another method employs divided differences to construct generalized binomial coefficients. Additionally, Pascal matrices have been interpreted as representations of linear operators acting on spaces of polynomials or formal power series. These diverse approaches illustrate the rich tapestry of mathematical exploration surrounding Pascal matrices.

Recently, several novel variants of numerous special numbers and polynomials have been introduced and studied using diverse methodologies [11, 12, 28–30, 36, 37, 41, 43]. Among these studies, the degenerate forms of Euler, Bernoulli, harmonic, and hyperharmonic numbers, as well as degenerate forms of Hermite polynomials and Fubini polynomials, have garnered significant attention [22–25]. Indeed, matrices with entries derived from the coefficients of special polynomials can offer intriguing insights into both the properties of the polynomials themselves and the structures of the resulting matrices. Polynomial matrices are widely studied in mathematics and find applications in various fields such as control theory, signal processing, and cryptography. For instance, when working with orthogonal polynomials like Legendre, Chebyshev, or Hermite polynomials, the coefficients of these polynomials possess interesting properties, and matrices constructed from these coefficients can exhibit certain patterns or symmetries [21, 34, 48]. These matrices often arise in problems involving approximation theory, numerical analysis, and differential equations. Exploring the connections between polynomial coefficients and matrix structures can lead to deeper understanding and applications in diverse areas of mathematics and its applications.

Numerous researchers have explored the wide array of characteristics displayed by these matrices, particularly focusing on their factorizations while also unveiling numerous combinatorial identities; see [1–3, 5–9, 13, 20, 26, 27, 30–35, 37, 42, 44–48]. Several of these matrices encompass variations and generalizations of binomial coefficients, see [4, 15, 26, 48]. Binomial coefficients are defined using factorials. The double factorial of a non-negative integer m, denoted as $m!!$, is defined as the product of all positive integers less than or equal to m that have the same parity (either all odd or all even). In other words,

$$m!! = \prod_{k=0}^{\lceil \frac{m}{2} \rceil - 1} (m - 2k).$$

Double factorials frequently emerge in integrals and power series, allowing for concise expressions of equations that would otherwise be verbose [18]. These numbers are also seen in the definition of modified Hermite polynomials [19]. The modified Hermite polynomials of two variables are defined by

$$\mathcal{H}_n^*(x, y) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} \sigma_{n-k} x^{\frac{1}{2}(n-k)} y^k \tag{1.1}$$

where

$$\sigma_k := \frac{1 + (-1)^k}{2}, \quad \left\{ {n \atop k} \right\} := \begin{cases} \frac{n!}{k!(n-k)!!}, & \text{if } n \geq k \geq 0 \\ 0, & \text{otherwise} \end{cases}.$$

Using these polynomials, a lower triangular matrix $T_n(x, y)$ of order $n + 1$ is defined with entries

$$(T_n(x, y))_{i,j} = \left\{ {i \atop j} \right\} \sigma_{i-j} x^{((i-j)/2)} y^j \text{ if } i \geq j,$$

see [48]. Thus, we can rewrite these entries as follows:

$$(T_n(x, y))_{i,j} = \left\{ {i \atop j} \right\} x^{((i-j)/2)} y^j [i - j \equiv 0 (\text{mod } 2)] \text{ if } i \geq j.$$

We can define double factorial binomial coefficients by substituting double factorials for regular factorials; see [10, 18]. Although these coefficients appear similar to classical binomial coefficients, they lack many of the properties of the latter. A notable difference is that, while classical binomial coefficients are always integers, double factorial binomial coefficients do not necessarily share this property. Consequently, a variant of the matrix $T_n(x, y)$ has been defined using double factorial binomial coefficients based on this idea, and it has been shown that this matrix satisfies several properties [26].

Higher factorials, or multifactorials, can indeed be defined. For example, $n!!!$, the triple factorial of n, is the product of positive integers less than or equal to n and congruent to n modulo 3. Based on these observations, it is also fascinating to explore how the results of the matrix might be affected if we substitute triple factorials for double factorials in the denominators of $\left\{ {i \atop j} \right\}$. Inspired by these works, it seems both instructive and intriguing to embark on research concerning matrices associated with altered Hermite polynomials. We define altered Hermite polynomials as follows:

Definition 1.1. *The altered Hermite polynomial, denoted by* $\mathsf{H}_r(x, y)$*, is a variant of the polynomial (1.1) defined for nonzero real numbers* x *and* y *as*

$$\mathsf{H}_r(x, y) = \sum_{i=0}^{r} \binom{r}{i}_t x^{\frac{r-i}{3}} y^i [r - i = 0(mod\ 3)],$$

where $\binom{r}{i}_t := \frac{r!}{i!(r-i)!!!}$.

In this paper, a square matrix associated with altered Hermite polynomials as its entries has been defined, and various properties of this matrix have been examined. We derive explicit expressions for the products, powers, and inverses of the matrices, as well as several factorization formulas.

2. The altered Hermite matrix

Using the definition of altered Hermite polynomial given in Definition 1.1, the i^{th} term of the r^{th} altered Hermite polynomial $\mathsf{H}_r(x, y)$ is denoted as

$$h_{r,i}(x, y) := \binom{r}{i}_t x^{\frac{r-i}{3}} y^i [r - i = 0(\text{mod } 3)], \quad \text{for } r \geq i \geq 0.$$

We construct a matrix $\mathcal{H}_n(x, y)$ with entries $h_{i,j}(x, y)$, where $h_{i,j}(x, y) = 0$ if $i < j$. Then the matrix $\mathcal{H}_6(x, y)$ will look as

$$\mathcal{H}_6(x, y) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & y & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y^2 & 0 & 0 & 0 & 0 \\ 2x & 0 & 0 & y^3 & 0 & 0 & 0 \\ 0 & 8xy & 0 & 0 & y^4 & 0 & 0 \\ 0 & 0 & 20xy^2 & 0 & 0 & y^5 & 0 \\ 40x^2 & 0 & 0 & 40xy^3 & 0 & 0 & y^6 \end{bmatrix}.$$

We will now delve into the properties and applications of the matrix $\mathcal{H}_n(x, y)$. The subsequent theorem elucidates its multiplication properties.

Theorem 2.1. *For positive integer n and real numbers x, y, z and w, we have*

$$\mathcal{H}_n(x,y)\mathcal{H}_n(w,z) = \mathcal{H}_n(x+wy^3, yz). \tag{2.1}$$

Proof. We prove the theorem by induction on n. It clearly holds for $n = 1$. Suppose it holds for $n-1$, and we want to prove it for n. We write $\mathcal{H}_n(x,y)$ in the following form:

$$\mathcal{H}_n(x,y) = \begin{bmatrix} \mathcal{H}_{n-1}(x,y) & 0 \\ Q_n(x,y) & y^n \end{bmatrix}$$

in which $Q_n(x,y)$ is a row matrix

$$Q_n(x,y) = \begin{bmatrix} \binom{n}{0}_t x^{\frac{n}{3}}[n=0(\text{mod } 3)] & \binom{n}{1}_t x^{\frac{n-1}{3}} y[n-1=0(\text{mod } 3)] & \cdots & \binom{n}{n-1}_t x^{\frac{1}{3}} y^{n-1}[1=0(\text{mod } 3)] \end{bmatrix}.$$

By matrix product we have,

$$\begin{aligned} \mathcal{H}_n(x,y)\mathcal{H}_n(w,z) &= \begin{bmatrix} \mathcal{H}_{n-1}(x,y) & 0 \\ Q_n(x,y) & y^n \end{bmatrix} \begin{bmatrix} \mathcal{H}_{n-1}(w,z) & 0 \\ Q_n(w,z) & z^n \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{H}_{n-1}(x,y)\mathcal{H}_{n-1}(w,z) & 0 \\ Q_n(x,y)\mathcal{H}_{n-1}(w,z) + y^n Q_n(w,z) & (yz)^n \end{bmatrix}. \end{aligned}$$

Using the induction hypothesis, we see that

$$\mathcal{H}_{n-1}(x,y)\mathcal{H}_{n-1}(w,z) = \mathcal{H}_{n-1}(x+wy^3, yz).$$

It is easy to see that,

$$Q_n(x,y)\mathcal{H}_{n-1}(w,z) + y^n Q_n(w,z) = Q_n(x+wy^3, yz).$$

Hence, we obtain,

$$\begin{aligned} \mathcal{H}_n(x,y)\mathcal{H}_n(w,z) &= \begin{bmatrix} \mathcal{H}_{n-1}(x,y)\mathcal{H}_{n-1}(w,z) & 0 \\ Q_n(x,y)\mathcal{H}_{n-1}(w,z) + y^n Q_n(w,z) & (yz)^n \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{H}_{n-1}(x+wy^3, yz) & 0 \\ Q_n(x+wy^3, yz) & (yz)^n \end{bmatrix} \\ &= \mathcal{H}_n(x+wy^3, yz). \end{aligned}$$

□

For $y = z = 1$, we obtain

$$\mathcal{H}_n(x,1)\mathcal{H}_n(w,1) = \mathcal{H}_n(x+w,1).$$

The inverse and power of matrix $\mathcal{H}_n(x,y)$ can be derived using Theorem 2.1.

Theorem 2.2. *Let y be a nonzero real number. Then we have*

$$\mathcal{H}_n^{-1}(x,y) = \mathcal{H}_n(-xy^{-3}, y^{-1}). \tag{2.2}$$

In particular,

$$\mathcal{H}_n^{-1}(1,1) = \mathcal{H}_n(-1,1). \tag{2.3}$$

Proof. By Theorem 2.1, we write

$$\begin{aligned}\mathcal{H}_n(x,y)\mathcal{H}_n(-xy^{-3},y^{-1}) &= \mathcal{H}_n(x-xy^{-3}y^3,yy^{-1})\\ &= \mathcal{H}_n(0,1)=I_{n+1}.\end{aligned}$$

Therefore, Equation (2.2) holds. By taking $x=y=1$ in (2.2), we get (2.3). □

Theorem 2.3. *For $k>0$, we have*

$$\mathcal{H}_n^k(x,y)=\mathcal{H}_n\left(x\sum_{l=0}^{k-1}y^{3l},y^k\right)=\begin{cases}\mathcal{H}_n\left(\frac{1-y^{3k}}{1-y^3}x,y^k\right), & \text{if } y\neq 1\\ \mathcal{H}_n(kx,y^k), & \text{if } y=1\end{cases}. \tag{2.4}$$

Proof. Taking $w=x$ and $z=y$ in (2.1), we have

$$\mathcal{H}_n^2(x,y)=\mathcal{H}_n\left(x(1+y^3),y^2\right).$$

Utilizing formula (2.1) once more, by multiplying $\mathcal{H}_n\left(x(1+y^3),y^2\right)$ and $\mathcal{H}_n(x,y)$, we obtain

$$\mathcal{H}_n^3(x,y)=\mathcal{H}_n\left(x(1+y^3+y^6),y^3\right).$$

Applying mathematical induction, we derive

$$\mathcal{H}_n^k(x,y)=\mathcal{H}_n\left(x(1+y^3+\cdots+y^{3k}),y^k\right).$$

After simplification, we arrive at Eq (2.4). □

We now aim to find the eigenvalues and eigenvectors of the matrix $\mathcal{H}_n(x,y)$ as defined.

Theorem 2.4. *Let $y\neq 1$. The eigenvalues of $\mathcal{H}_n(x,y)$ are $1,y,y^2,\dots,y^{n-1}$ and the columns of the matrix $\mathcal{H}_n\left(\frac{x}{1-y^3},1\right)$ represent the corresponding eigenvectors.*

Proof. The definition of eigenvalues and eigenvectors, along with the identity

$$\mathcal{H}_n(x,y)\mathcal{H}_n\Big(\frac{x}{1-y^3},1\Big)=\mathcal{H}_n\Big(\frac{x}{1-y^3},y\Big)$$

substantiates our claim. □

Corollary 2.5. *Let $Y=\mathrm{diag}\{1,y,y^2,\dots,y^{n-1}\}$. Then for $y\neq 1$, we have*

$$\mathcal{H}_n(x,y)=\mathcal{H}_n\Big(\frac{x}{1-y^3},1\Big)Y\mathcal{H}_n\Big(-\frac{x}{1-y^3},1\Big).$$

Let us consider the factorization of the matrix $\mathcal{H}_n(x,y)$ and observe some results with the following matrices: We will represent the matrices with entries as specified. We define $(n+1)\times(n+1)$ matrices $S_n(x,y)$ and $D_n(x,y)$ with entries

$$(S_n(x,y))_{i,j}=\begin{cases}1 & i=j=0\\ y & i=j\neq 0\\ x(i-1)(i-2)(S_n(x,y))_{i-3,j}, & i>j \text{ and } i\geq 3\\ 0, & i-j=1 \text{ or } 2 (\text{mod } 3) \text{ and } i<j\end{cases}$$

$$(D_n(x,y))_{i,j} = \begin{cases} 1 & i=j=0 \\ \frac{1}{y} & i=j\neq 0 \\ -(i-1)(i-2)\frac{x}{y}, & \text{for } j=i-3 \\ 0, & \text{otherwise} \end{cases}.$$

Additionally, we require the matrices,

$$\overline{\mathcal{H}}_k(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{H}_k(x,y) \end{bmatrix} \quad \text{of order } k+2 \quad \text{and}$$

$$U_n(x,y) = S_n(x,y), \quad U_k(x,y) = \begin{bmatrix} I_{n-k-1} & 0 \\ 0 & S_k(x,y) \end{bmatrix}, \quad 1 \leq k \leq n-1.$$

It is easy to see that

$$S_n^{-1}(x,y) = D_n(x,y).$$

Example 2.6.

$$S_6(x,y)D_6(x,y) =$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & y & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y & 0 & 0 & 0 & 0 \\ 2x & 0 & 0 & y & 0 & 0 & 0 \\ 0 & 6xy & 0 & 0 & y & 0 & 0 \\ 0 & 0 & 12xy & 0 & 0 & y & 0 \\ 40x^2 & 0 & 0 & 20xy & 0 & 0 & y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/y & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/y & 0 & 0 & 0 & 0 \\ -2x/y & 0 & 0 & 1/y & 0 & 0 & 0 \\ 0 & -6x/y & 0 & 0 & 1/y & 0 & 0 \\ 0 & 0 & -12x/y & 0 & 0 & 1/y & 0 \\ 0 & 0 & 0 & -20x/y & 0 & 0 & 1/y \end{bmatrix}$$

$$= I_7.$$

Lemma 2.7. *For $k > 0$, we have*

$$D_k(x,y)\mathcal{H}_k(x,y) = \overline{\mathcal{H}}_{k-1}(x,y), \tag{2.5}$$

or

$$\mathcal{H}_k(x,y) = S_k(x,y)\overline{\mathcal{H}}_{k-1}(x,y).$$

Proof. Verifying Eq (2.5) is quite straightforward using the definition of matrix multiplication. □

Example 2.8.

$$S_6(x,y)\overline{\mathcal{H}}_5(x,y)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & y & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y & 0 & 0 & 0 & 0 \\ 2x & 0 & 0 & y & 0 & 0 & 0 \\ 0 & 6xy & 0 & 0 & y & 0 & 0 \\ 0 & 0 & 12xy & 0 & 0 & y & 0 \\ 40x^2 & 0 & 0 & 20xy & 0 & 0 & y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y^2 & 0 & 0 & 0 \\ 0 & 2x & 0 & 0 & y^3 & 0 & 0 \\ 0 & 0 & 8xy & 0 & 0 & y^4 & 0 \\ 0 & 0 & 0 & 20xy^2 & 0 & 0 & y^5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & y & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y^2 & 0 & 0 & 0 & 0 \\ 2x & 0 & 0 & y^3 & 0 & 0 & 0 \\ 0 & 8xy & 0 & 0 & y^4 & 0 & 0 \\ 0 & 0 & 20xy^2 & 0 & 0 & y^5 & 0 \\ 40x^2 & 0 & 0 & 40xy^3 & 0 & 0 & y^6 \end{bmatrix}$$
$$= \mathcal{H}_6(x, y).$$

By employing Lemma 2.7 and the definition of the matrices $U_k(x, y)$, we can factorize $\mathcal{H}_n(x, y)$. The following theorem presents the factorization.

Theorem 2.9. *The matrix $\mathcal{H}_n(x, y)$ can be expressed as*

$$\mathcal{H}_n(x, y) = U_n(x, y)U_{n-1}(x, y) \cdots U_1(x, y).$$

Specifically,

$$\mathcal{H}_n = U_n U_{n-1} \cdots U_1$$

where $\mathcal{H}_n := \mathcal{H}_n(1, 1)$ and $U_k := U_k(1, 1)$ for $k = 1, 2, \ldots, n$.

According to Theorem 2.9, the factorization of the inverse of the matrix $\mathcal{H}_n(x, y)$ is given by

$$\mathcal{H}_n^{-1}(x, y) = U_1^{-1}(x, y)U_2^{-1}(x, y) \cdots U_n^{-1}(x, y),$$

where

$$U_k^{-1}(x, y) = \begin{bmatrix} I_{n-k-1} & 0 \\ 0 & D_k(x, y) \end{bmatrix}, \quad k = 1, 2, \ldots, n-1$$

and

$$U_n^{-1}(x, y) = D_n(x, y).$$

Example 2.10. *Given that*

$$\mathcal{H}_5(x, y) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & y & 0 & 0 & 0 & 0 \\ 0 & 0 & y^2 & 0 & 0 & 0 \\ 2x & 0 & 0 & y^3 & 0 & 0 \\ 0 & 8xy & 0 & 0 & y^4 & 0 \\ 0 & 0 & 20xy^2 & 0 & 0 & y^5 \end{bmatrix},$$

we can factorize this matrix using the defined matrices $U_k(x, y)$ for $k = 5, 4, 3, 2, 1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & y & 0 & 0 & 0 & 0 \\ 0 & 0 & y & 0 & 0 & 0 \\ 2x & 0 & 0 & y & 0 & 0 \\ 0 & 6xy & 0 & 0 & y & 0 \\ 0 & 0 & 12xy & 0 & 0 & y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & y & 0 & 0 & 0 \\ 0 & 0 & 0 & y & 0 & 0 \\ 0 & 2x & 0 & 0 & y & 0 \\ 0 & 0 & 6xy & 0 & 0 & y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & y & 0 & 0 \\ 0 & 0 & 0 & 0 & y & 0 \\ 0 & 0 & 2x & 0 & 0 & y \end{bmatrix}$$

$$\times\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&y&0\\0&0&0&0&0&y\end{bmatrix}\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&y\end{bmatrix}.$$

The matrices $\mathcal{H}_n(x^3, y)$ and $\mathcal{H}_n(-x^3, y)$ have the following factorizations, respectively, where the variables x and y are separated.

Theorem 2.11. *For $n > 0$, the equations are*

$$\mathcal{H}_n(x^3, y) = C_n(x)\mathcal{H}_n(1, 1)C_n^{-1}(x/y),$$
$$\mathcal{H}_n(-x^3, y) = C_n(x)\mathcal{H}_n(-1, 1)C_n^{-1}(x/y),$$

where $C_n(x) := \mathrm{diag}\{1, x, x^2, \ldots, x^n\}$ is a diagonal matrix.

Moreover, the factorizations of both $\mathcal{H}_n$ and $\mathcal{H}_n^{-1}$ involve a lower triangular Toeplitz matrix, as directly implied by their definitions.

Theorem 2.12. *The matrix $\mathcal{H}_n$ can be decomposed as follows:*

$$\mathcal{H}_n = J_n A_n J_n^{-1},$$

where $J_n := \mathrm{diag}\{0!, 1!, 2!, \ldots, n!\}$ and $A_n = [a_{ij}]$ with $a_{ij} = 1/(i - j)!!!$ for $i - j = 0(mod\ 3)$ and $a_{ij} = 0$ otherwise.

After some computations, we have $A_n^{-1} = [b_{ij}]$ where $b_{ij} = \frac{(-1)^{(i-j)/3}}{(i-j)!!!}$ for $i - j = 0(\mathrm{mod}\ 3)$ and $b_{ij} = 0$ otherwise.

3. Some applications of the altered Hermite matrix

Let us formulate a relation between the matrix $\mathcal{H}_n(x, 1)$ and the exponential of a special matrix. For any square matrix L, the exponential of L is defined as the matrix

$$e^L = I + L + \frac{L^2}{2!} + \frac{L^3}{3!} + \cdots + \frac{L^k}{k!} + \cdots .$$

Definition 3.1. *The matrix $L_n = [l_{i,j}]$ of order $n + 1$ is defined by*

$$l_{i,j} = \begin{cases} \binom{i}{j}_t, & \text{if } i = j + 3 \\ 0, & \text{otherwise} \end{cases}$$

for all $0 \le i, j \le n$.

We aim to prove that $\mathcal{H}_n(x, 1) = e^{xL_n}$. To establish this, we will demonstrate the following result: Let $\binom{i}{j}_{(t,k)} := \prod_{n=0}^{k-1} \binom{i - 3n}{i - 3n - 3}_t$ for a fixed nonnegative integer j.

Lemma 3.2. *The entries $(L_n^k)_{i,j}$ of the matrix L_n^k for positive integers k are defined as*

$$(L_n^k)_{i,j} = \begin{cases} \binom{i}{j}_{(t,k)}, & \text{if } i = j + 3k \\ 0, & \text{otherwise} \end{cases}.$$

Proof. The proof will proceed by induction on k. The base case is straightforward. Let's assume the inductive hypothesis for $(L_n^{k+1})_{i,j} = (L_n)_{i,j}(L_n^k)_{i,j}$. Then for $i \neq j+3k+3$, $(L_n^{k+1})_{i,j} = 0$. For $i = j+3k+3$, we have

$$(L_n^{k+1})_{i,j} = \binom{i}{r}_t \binom{r}{j}_{(t,k)} = \binom{j+3k+3}{r}_t \binom{r}{j}_{(t,k)} = \binom{i}{j}_{(t,k+1)}.$$

□

Theorem 3.3. *For $n \in \mathbb{N}, r \in \mathbb{Z}$ and $x \in \mathbb{R}$, we have*

$$\mathcal{H}_n(x,1) = e^{xL_n}.$$

Proof. Assume there exists a matrix M_n such that $\mathcal{H}_n(x,1) = e^{xM_n}$. Then, by differentiating both sides with respect to x and evaluating at $x = 0$, we obtain $\mathcal{H}'_n(x,1) \mid_{x=0} = M_n$. Therefore, there exists at most one matrix M_n such that $\mathcal{H}_n(x,1) = e^{xM_n}$. By calculating the derivative of the matrix $\mathcal{H}_n(x,1)$ with respect to x at $x = 0$, we observe that $M_n = L_n$, where L_n is defined as in Definition 3.1. From Lemma 3.2, $(L_n^k)_{i,j} = 0$ for $3k > n$, thus

$$e^{xL_n} = \sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} \frac{x^k}{k!} L_n^k.$$

Notice that $(e^{xL_n})_{i,j} = 0$ for $i < j$, and $(e^{xL_n})_{i,i} = 1$. For $i > j$ and $i = j+3k$, we have $(e^{xL_n})_{i,j} = \frac{x^k}{k!}(L_n^k)_{i,j} = \frac{x^k}{k!}\binom{i}{j}_{(t,k)} = (\mathcal{H}_n(x,1))_{i,j}$. □

At the conclusion of this section, we provide the explicit inverse of $I_n - a\mathcal{H}_n(x,1)$ for all $\mid a \mid< 1$.

Theorem 3.4. *For $\mid a \mid< 1$, the matrix $R_n(x) = (I_n - a\mathcal{H}_n(x,1))^{-1}$ is defined as follows*

$$(R_n(x))_{i,i} = \frac{1}{1-a}$$

for the main diagonal entries, and it is defined for $i > j$ as

$$(R_n(x))_{i,j} = (\mathcal{H}_n(x,1))_{i,j} Li_{j-i}(a)$$

where $Li_n(z)$ is the polylogarithm function.

Proof. For any $\mid a \mid< 1$, we have

$$R_n(x) = (I_n - a\mathcal{H}_n(x,1))^{-1} = \sum_{k=0}^{\infty} a^k \mathcal{H}_n(x,1)^k$$

and from Theorem 2.3, we can write

$$(R_n(x))_{i,j} = \sum_{k=0}^{\infty} a^k(\mathcal{H}_n(kx,1))_{i,j} = (\mathcal{H}_n(x,1))_{i,j} \sum_{k=0}^{\infty} a^k k^{i-j}.$$

Therefore, the proof can be completed by addressing the cases where $i = j$ and $i > j$. □

Example 3.5.

$$I_6 - a\mathcal{H}_6(x,1) = \begin{bmatrix} 1-a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-a & 0 & 0 & 0 & 0 \\ -2xa & 0 & 0 & 1-a & 0 & 0 & 0 \\ 0 & -8xa & 0 & 0 & 1-a & 0 & 0 \\ 0 & 0 & -20xa & 0 & 0 & 1-a & 0 \\ -40x^2a & 0 & 0 & -40xa & 0 & 0 & 1-a \end{bmatrix}.$$

The inverse of this matrix equals

$$\begin{bmatrix} \frac{1}{1-a} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{1-a} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{1-a} & 0 & 0 & 0 & 0 \\ 2xa\frac{1}{(1-a)^2} & 0 & 0 & \frac{1}{1-a} & 0 & 0 & 0 \\ 0 & 8xa\frac{1}{(1-a)^2} & 0 & 0 & \frac{1}{1-a} & 0 & 0 \\ 0 & 0 & 20xa\frac{1}{(1-a)^2} & 0 & 0 & \frac{1}{1-a} & 0 \\ 40x^2a\frac{a(a+1)}{(1-a)^3} & 0 & 0 & 40xa\frac{1}{(1-a)^2} & 0 & 0 & \frac{1}{1-a} \end{bmatrix}.$$

4. A generalization of the altered Hermite matrix

In this section, we introduce a matrix and derive several results from it.

Definition 4.1. *Let x and λ be arbitrary real numbers, and let n be a non-negative integer. Then*

$$x^{n|\lambda} = \begin{cases} x(x+\lambda)\cdots(x+(n-1)\lambda) & \text{if } n > 0 \\ 0 & \text{if } n = 0 \end{cases}.$$

Therefore, we obtain $x^{n|0} = x^n$ for $\lambda = 0$.

Lemma 4.2. (*[5, Lemma 1]) Let x, y, λ be real numbers and n be a positive integer. Then*

$$(x+y)^{n|\lambda} = \sum_{i=0}^{n} \binom{n}{i} x^{(n-i)|\lambda} y^{i|\lambda}.$$

Proof. See [5]. □

We consider the following matrix, which generalizes the altered Hermite matrix by incorporating the above lemma.

Definition 4.3. *The matrix* $\mathrm{H}_{n,\lambda}(x)$ *is defined by*

$$(\mathrm{H}_{n,\lambda}(x))_{i,j} := \binom{i}{j}_t x^{\lfloor \frac{i-j}{3} \rfloor|\lambda}[i-j=0(mod\ 3)].$$

Theorem 4.4. *Let* $n>0$. *Then*

$$\mathrm{H}_{n,\lambda}(x+y) = \mathrm{H}_{n,\lambda}(x)\mathrm{H}_{n,\lambda}(y).$$

Proof. The proof follows a similar approach to Theorem 2.1. □

Corollary 4.5. *For integers* j *and* k *with* $k>0$, *we have*

(i) $\mathrm{H}^{j}_{n,\lambda}(1) = \mathrm{H}_{n,\lambda}(j)$.
(ii) $\mathrm{H}^{k}_{n,\lambda}(j/k) = \mathrm{H}_{n,\lambda}(j)$.

Now, we extend Definition 4.3 to two variables, x and y.

Definition 4.6. *Let* x, y *and* λ *be real numbers, and let* n *be a positive integer. The matrix* $\mathrm{H}_{n,\lambda}(x,y)$ *is defined by*

$$(\mathrm{H}_{n,\lambda}(x,y))_{i,j} := \binom{i}{j}_t x^{\lfloor \frac{i-j}{3} \rfloor|\lambda} y^{j|\lambda}[i-j=0(mod\ 3)].$$

The following lemma directly follows from the above definition.

Lemma 4.7. $\mathrm{H}_{n,\lambda}(x,y)$ *can be expressed as*

$$\mathrm{H}_{n,\lambda}(x,y) = \mathrm{H}_{n,\lambda}(x)\mathrm{diag}\{1, y^{1|\lambda}, \cdots, y^{n|\lambda}\}.$$

Theorem 4.8. *For* $n>0$, *we have*

$$\mathrm{H}_{n,\lambda}(x+y,z) = \mathrm{H}_{n,\lambda}(x)\mathrm{H}_{n,\lambda}(y,z) = \mathrm{H}_{n,\lambda}(y)\mathrm{H}_{n,\lambda}(x,z).$$

Proof. By utilizing Theorem 4.4 and Lemma 4.7, we obtain the result. □

Let us generalize the altered Hermite matrix in to two variables associated with a sequence $\mathbf{b} = \{b_n\}_{n\geq 0}$.

Definition 4.9. *We define*

$$(\mathrm{H}_{n,\lambda}(x,y,\mathbf{b}))_{i,j} := b_j\binom{i}{j}_t x^{\lfloor \frac{i-j}{3} \rfloor|\lambda} y^{j|\lambda}[i-j=0(mod\ 3)].$$

Lemma 4.10. *The matrix* $\mathrm{H}_{n,\lambda}(x,y,\mathbf{b})$ *can be factorized as follows:*

$$\mathrm{H}_{n,\lambda}(x,y,\mathbf{b}) = \mathrm{H}_{n,\lambda}(x,y)\mathrm{diag}\{b_0, b_1, \cdots, b_n\}.$$

Proof. The proof can be straightforwardly accomplished using mathematical induction and Theorem 4.8. □

Theorem 4.11. *We can factorize* $\mathrm{H}_{n,\lambda}(x+y,z,\mathbf{b})$ *as follows:*

(i) $\mathrm{H}_{n,\lambda}(x+y,z,\mathbf{b}) = \mathrm{H}_{n,\lambda}(x)\mathrm{H}_{n,\lambda}(y,z,\mathbf{b})$.
(ii) $\mathrm{H}_{n,\lambda}(x+y,z,\mathbf{b}) = \mathrm{H}_{n,\lambda}(x)\mathrm{H}_{n,\lambda}(y,z)\mathrm{diag}\{b_0, b_1, \cdots, b_n\}$.

Proof. The proof follows by utilizing Theorems 4.4 and 4.8, along with Lemma 4.10. □

Proposition 4.12. *For any positive integer n and a real number x, we have*

$$(\mathrm{H}_{n,\lambda}(x) - I_{n+1})^{\lfloor \frac{n}{3} \rfloor} = M_n$$

where I_{n+1} is an identity matrix of order $n+1$ and the matrix M_n is a matrix of order $n+1$, has elements defined as follows:

- *For $n = 0(mod\ 3)$,*
$$(M_n)_{n,0} = n!\Big(\frac{x}{3}\Big)^{n/3}. \tag{4.1}$$
- *For $n = 1(mod\ 3)$,*
$$(M_n)_{n-1,0} = (n-1)!\Big(\frac{x}{3}\Big)^{(n-1)/3}, \quad (M_n)_{n,1} = (n)!\Big(\frac{x}{3}\Big)^{(n-1)/3}. \tag{4.2}$$
- *For $n = 2(mod\ 3)$*
$$\begin{aligned}(M_n)_{n-2,0} &= (n-2)!\Big(\frac{x}{3}\Big)^{(n-2)/3},\\ (M_n)_{n-1,1} &= (n-1)!\Big(\frac{x}{3}\Big)^{(n-2)/3},\\ (M_n)_{n,2} &= \frac{n!}{2}\Big(\frac{x}{3}\Big)^{(n-2)/3}\end{aligned}$$

with all other elements of M_n being zero.

Proof. We will prove it first for $n = 0(\mathrm{mod}\ 3)$. Let $n = 3m$. We aim to show that for each $1 \leq k \leq m$, all elements of the first $3k$ rows of the matrix M_{3k} defined as

$$M_{3k} := (\mathrm{H}_{3m,\lambda}(x) - I_{3m+1})^k \tag{4.3}$$

are zero, except for the $3k+1$-th row, where the first element is

$$(M_{3k})_{3k,0} = \binom{3}{0}_t\binom{6}{3}_t \cdots \binom{3k}{3k-3}_t x^k = (3k)!\Big(\frac{x}{3}\Big)^k. \tag{4.4}$$

All other elements in this $3k+1$-th row are zero.

The proof will proceed by induction on k. The base case $k = 1$ is straightforward. Suppose (4.3) and (4.4) hold for k. Then, using matrix multiplication, we proceed with the inductive step as follows:

$$\begin{aligned}&(\mathrm{H}_{3m,\lambda}(x) - I_{3m+1})^{k+1} = (\mathrm{H}_{3m,\lambda}(x) - I_{3m+1})M_{3k}\\ &= \begin{bmatrix} 0_{0,0} & 0_{0,1} & \cdots & \cdots & \cdots & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ *_{3k+2,0} & *_{3k+2,1} & \cdots & 0 & \cdots & \cdots & 0\\ \binom{3k+3}{0}_t x^{k|\lambda} & 0_{3k+2,1} & \cdots & \binom{3k+3}{3k}x & 0 & \cdots & 0\\ 0_{3k+4,0} & \binom{3k+4}{1}_t x^{(k+1)|\lambda} & \cdots & 0 & \binom{3k+4}{3k+1}_t x & 0 & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \times \begin{bmatrix} 0_{0,0} & 0_{0,1} & \cdots & 0_{0,2m}\\ \cdots & \cdots & \cdots & \cdots\\ 0_{3k-1,0} & 0_{3k-1,1} & \cdots & 0_{3k-1,3m}\\ (M_{3k})_{3k,0} & 0 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}\end{aligned}$$

$$= \begin{bmatrix} 0_{0,0} & 0_{0,1} & \cdots & 0_{0,2m} \\ \cdots & \cdots & \cdots & \cdots \\ 0_{3k+2,0} & 0_{3k+2,1} & \cdots & 0_{3k+2,3m} \\ (M_{3k+3})_{3k+3,0} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

where

$$(M_{3k+3})_{3k+3,0} = \binom{3k+3}{3k}_t x \cdot (M_{3k})_{3k,0} = \binom{3}{0}_t \binom{6}{3}_t \cdots \binom{3k}{3k-3}_t \binom{3k+3}{3k}_t x^{k+1}.$$

Therefore, (4.3) and (4.4) hold for $k+1$. By completing the induction up to $k = m$, we have proven that (4.1) holds true.

Applying the same procedure yields the results given by (4.2) and (4.3). □

5. Conclusions

Motivated by the works in [4, 26, 48], we have introduced a variant of the Hermite matrix that incorporates triple factorials and have shown that this matrix exhibits several notable properties. Through the application of advanced matrix algebra techniques, we have explored various algebraic characteristics of this matrix. Furthermore, we generalized this matrix and derived several identities related to it.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The author would like to thank the referees for their helpful suggestions and comments that significantly improved the presentation of this work.

Conflict of Interest

We have no conflict of interest to declare.

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