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Research article

The novel numerical solutions for time-fractional Fornberg-Whitham equation by using fractional natural transform decomposition method

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Abstract: The time-fractional partial differential equations were solved by the fractional natural transform decomposition method. Fractional derivatives are Caputo sense. The Fornberg-Whitham equation is a generalization of the Korteweg-de Vries (KdV) equation, which describes the propagation of long waves in shallow water. It includes higher-order dispersion terms, making it applicable to a wider range of dispersive media the fractional natural transform decomposition method (FNTDM) was also used to examine applications, and the solutions obtained by this method have been compared to those obtained by the variational iteration method, fractional variational iteration method, and homotopy perturbation method. In addition, the MAPLE package drew graphs of the solutions of nonlinear time-fractional partial differential equations, taking into account physics. The method described in this study exhibited a notable degree of computational precision and straightforwardness when used to the analysis and resolution of intricate phenomena pertaining to fractional nonlinear partial differential equations within the domains of science and technology.

Keywords: time-fractional partial differential equation; fractional natural transform decomposition method; Mittag-Leffler function; Adomian polynomials; variational iteration method
Mathematics Subject Classification: 35C05, 35R11, 65R10

1. Introduction

The subject matter of fractional calculus has been thoroughly examined and delineated by a multitude of eminent scholars. The formulation of unique conceptualizations of fractional calculus by the authors has subsequently established the fundamental principles of fractional analysis within the area. Fractional partial differential equations (FPDEs) are commonly utilized in the field of nonlinear model development and analysis of dynamical systems. The application of fractional calculus has been utilized in the evaluation and exploration of various fields, including chaos theory, financial models, disordered settings, and optics. Nonlinear difficulties in nature are primarily determined by solving fractional differential equations. Due to the inherent difficulty connected with obtaining analytical solutions for fractional differential equations that reflect nonlinear events, a wide array of analytical and numerical techniques are utilized [1–5].

FPDEs are utilized to describe a wide variety of phenomena in numerous scientific disciplines [6]. In [7], the finite difference methods, the Galerkin finite element methods, and the spectral methods for fractional partial differential equations (FPDEs), which are divided into the time-fractional, space-fractional, and space-time-fractional partial differential equations (PDEs) have been utilized to solve. The generalized physics laws involving fractional derivatives have been presented the new models and conceptions that can be used in complex systems having memory effects [8]. The existence and uniqueness of the solution of nonlinear fractional differential equations with Mittag-Leffler nonsingular kernel have been given [9]. The time-fractional Burgers Equation has been solved by using the improving homotopy analysis method [10]. The poor nutrition in the life cycle of humans has been examined in the fractional sense [11]. In [12], an efficient linear programming formulation has been proposed for a class of fractional-order optimal control problems with delay argument. Many researchers have examined the mathematical modeling of biological systems in the fractional sense [13]. The symmetric fractional derivative has been introduceds introduced and its properties are examined [14]. A path integral approach to quantum physics has been improved. Fractional path integrals over the paths of the Lévy flights have been described [15]. General information about the solutions of fractional mathematical models has been given [16]. Recent attention has been drawn to FPDEs as a result of their wide range of applications in the applied sciences, including control theory, image processing, signal processing and system identification, and fluid mechanics [16,17]. Since the most of nonlinear FPDEs cannot be solved analytically, a variety of numerical approaches have been created. There is the Adomian decomposition technique [18], Homotopy perturbation technique [19–21], collocation technique [22–24], Sumudu transform technique [25,26], differential transform technique [27–29], and variational iteration technique [30].

Fractional differential equations were most often made up of so-called Caputo-like operators with different kinds of kernels. In this case, we think we need to find the answer to the following question: Why do we have to use the same initial conditions as in the classical case for a fractional operator that doesn't yet have a very clear physical meaning? We think that the ideas that Liouville came up with in 1832 about the fractional calculus are still relevant today. For example, we need new types of fractional operators to solve real-world problems that can't be solved with other mathematical tools. We don't think that the answer to the question, "What is the most general fractional operator that can solve all kinds of complicated dynamical systems with different memory effects?" has been found yet. It's a big and interesting question that hasn't been answered yet. We think that the development of numerical methods for fractional operators is very important to the development of fractional calculus. This area needs a new point of view that gets around the problems caused by the memory effect [31].

In a dispersive medium, the evolution of a wave is described by the Fornberg-Whitham equation, which is a partial differential equation (PDE). Bengt Fornberg and Gerald B. Whitham introduced this concept in 1978. The Fornberg-Whitham equation is given by [32]

$$\begin{aligned} &D_t u(x,t) - c_0 D_{xxt} u(x,t) + c_1 D_x u(x,t), \\ &= c_2 u(x,t) D_{xxx} u(x,t) - c_3 u(x,t) D_x u(x,t) + 3 D_x u(x,t) D_{xx} u(x,t), \end{aligned} \tag{1}$$

where $u(x,t)$ is the fluid velocity, x is the spatial coordinate, t is time, x is space, and c_0, c_1, c_2, and c_3 are constants related to the dispersion properties of the medium. The Fornberg-Whitham equation represents an extension of the Korteweg-de Vries (KdV) equation, which provides a conceptual framework for understanding the dynamics of long waves in shallow water. The model incorporates dispersion terms of higher order, hence expanding its applicability to a broader spectrum of dispersive media [32].

The KdV equation is a highly significant PDE that provides a comprehensive description of the dynamics of specific categories of nonlinear waves. The concept of propagation of long, weakly nonlinear, and dispersive water waves in a canal was initially formulated by Korteweg and Vries in 1895. The KdV equation is given by [33]

$$D_t u(x,t) + c D_x u(x,t) + c_1 u(x,t) D_x u(x,t) + c_2 D_{xxx} u(x,t) = 0, \tag{2}$$

where $u(x,t)$ is the dependent variable representing the wave amplitude, x is the spatial coordinate, t is time, x is space, c is the phase speed of the wave, and c_1 and c_2 are constants related to the properties of the medium [33].

The KdV equation is a nonlinear dynamic equation that exhibits dispersiveness and integrability. This statement elucidates the phenomenon of waves that retain their form during propagation, demonstrating a harmonious interplay between nonlinear influences that induce wave distortion and dispersive influences that reinstate the wave's original shape. The KdV equation is renowned for its soliton solutions, which are singular wave solutions that arise from specific initial conditions and propagate without undergoing any transformative changes in their structure. Solitons are intrinsically stable and nonlinear entities that manifest in diverse physical systems, encompassing phenomena such as water waves, plasma physics, and nonlinear optics. In the study of solitons and nonlinear wave processes, the KdV equation is a fundamental model [33].

The present study focuses on the Klein-Gordon equation (KGE), which is a fundamental non-linear evolution equation that emerges within the framework of relativistic quantum mechanics. The nonrelativistic wave equation in quantum physics was established by Erwin Schrodinger and then examined in detail by renowned scientists O. Klein and W. Gordon in 1926. The quantum field equation (KGE) exhibits a wide range of applications in both classical field theory and quantum field theory. Furthermore, it has been widely employed in other domains of physical phenomena, including solid-state physics, dispersive wave phenomena, nonlinear optics, elementary particle behavior, and various classes of soliton solutions [34].

The Klein-Gordon equation is given by [34]

$$D_t u(x,t) = D_{xx} u(x,t) + a u(x,t) + b u^2(x,t) + c u^3(x,t), \tag{3}$$

where x is the spatial coordinate, t is time, x is space, and $a, b,$ and c are real constants.

In this paper, we consider the time-fractional Fornberg-Whitham equation (TFFWE) as follows:

$$D_t^{\alpha}u(x,t) - c_0 D_{xxt}u(x,t) + c_1 D_x u(x,t) \\ = c_2 u(x,t) D_{xxx}u(x,t) - c_3 u(x,t) D_x u(x,t) + 3 D_x u(x,t) D_{xx} u(x,t). \quad (4)$$

Also, we examine the time-fractional KdV equation as follows:

$$D_t^{\alpha}u(x,t) + c D_x u(x,t) + c_1 u(x,t) D_x u(x,t) + c_2 D_{xxx} u(x,t) = 0. \quad (5)$$

Besides, we analyze the time-fractional Klein-Gordon equation as follows:

$$D_t^{\alpha}u(x,t) = D_{xx}u(x,t) + a u(x,t) + b u^2(x,t) + c u^3(x,t). \quad (6)$$

The natural transform was used to solve the linear ordinary differential equations [35]. The natural transform was applied to Maxwell's equations by Silambarasn and Belgacem [36]. The Fourier integral is also used to derive the natural transform [37]. Rawashdeh and Maitama [38] first improved the natural decomposition method to solve nonlinear partial differential equations (NPDEs) in a variety of scientific fields [39–42]. In addition, the natural transform decomposition method (NTDM) has been used to solve NPDEs [43]. It has been proposed to use the fractional natural decomposition method (FNDM) [44]. Additionally, Rawashdeh proved three major theorems about FNDM [44]. Gao et al. used two current methods to obtain the numerical solution for the fractional Benney-Lin equation [45]. Momani et al. used the variational iteration technique to obtain numerical solutions to time-fractional KdV equations [33]. The fractional natural decomposition method (FNTDM) is utilized to examine the fractional KdV equation in [33]. As a result, the obtained numerical solutions are superior to their numerical solutions in [33].

The utilization of the natural transform (NT) in fractional calculus is a highly effective technique that presents numerous benefits in comparison to alternative integral transforms, such as the Laplace and Fourier transforms. Several advantages can be identified. The NT is a comprehensive extension of other widely recognized integral transforms, such as the Laplace and Fourier transforms. It possesses the capability to process a broader spectrum of functions and is especially well-suited for addressing fractional-order differential equations. The NT inherently includes fractional-order operators, which are prevalent in numerous real-world phenomena. This characteristic renders it a more appropriate instrument for representing intricate systems characterized by dynamics of non-integer order. The utilization of the NT has the potential to streamline the mathematical representation of an issue, hence facilitating its analysis and resolution in some instances. This holds particularly true for problems that involve differential equations of fractional order. The NT possesses a distinct physical interpretation due to its utilization of a fractional-order differential operator, hence facilitating comprehension of the fundamental dynamics inherent in a given system. The NT has exhibited a wide range of applications across diverse domains such as physics, engineering, biology, and finance, hence showcasing its adaptability and efficacy in addressing practical challenges. In general, the NT presents a distinct array of benefits that render it a desirable instrument in the realm of fractional calculus and its associated domains [46].

This paper introduces numerical approximation tools that have been specifically developed for the equations being examined. This research aims to solve the nonlinear time-fractional Fornberg-Whitham equation, nonlinear time-fractional Klein-Gordon equation, and nonlinear time-fractional KdV equation by using the FNTDM. These solutions represent the first application of their kind. This study aims to get innovative numerical solutions for the aforementioned equations by employing the recently developed hybrid approach. This study identifies some solutions that have not been previously examined in the current literature and provides a complete depiction of their graphical features.

The rest of the paper is structured as follows: Preliminaries have been introduced in Section 2. Fractional natural transform decomposition method are explained in Section 3. Applications have been given in Section 4. In Section 5, a result and discussion is introduced. Also, the conclusion is stated in Section 6.

2. Materials and methods

The section contains the general information about fractional calculus.

Definition 1.1. [47] The Mittag-Leffler function E_a is given by

$$E_a(x) = \sum_{n=0}^{\infty} \frac{x^a}{\Gamma(na+1)}. \tag{7}$$

Definition 1.2. [1] The Riemann-Liouville fractional integral operator of order $a \geq 0$, of a function $f \in C_\mu, \mu \geq -1$ is as Eq (2).

$$I^a f(x) = \begin{cases} \frac{1}{\Gamma(a)} \int_0^x (x-t)^{a-1} f(t)dt, & a > 0, x > 0, \\ I^0 f(x) = f(x), & a = 0, \end{cases} \tag{8}$$

where $\Gamma(.)$ is the Gamma function.

The following are two required properties of the operator I^a [27]:

For $f \in C_\mu, \mu, \gamma \geq -1,\ \alpha, \beta \geq 0$:

(1) $I^a I^\beta f(x) = I^\beta I^a f(x) = I^{a+\beta} f(x)$,

(2) $I^a x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$.

Definition 1.3. [1,6] The Caputo fractional derivative of a function $f(x)$ is given by Eq (3).

$$D^a f(x) = I^{a-n} D^n f(x) = \frac{1}{\Gamma(n-a)} \int_0^x (x-t)^{n-a-1} f^{(n)}(t)dt, \tag{9}$$

where $n-1 < a \leq n,\ n \in N,\ x > 0, f \in C_{-1}^n$.

The operator D^a must have two essential properties [27]:

(1) $D^a I^a f(x) = f(x)$,

(2) $I^a D^a f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{x^k}{k!},\ x > 0$.

Definition 1.4. [48] The natural transform of the function $f(t)$ is given as

$$N^+[f(t)] = Q(s,u) = \frac{1}{u} \int_0^\infty f(t) e^{-\frac{st}{u}} dt, \qquad s, u > 0, \tag{10}$$

where s and u are the transform variables.

Definition 1.5. [48] The inverse natural transform of the function is described via

$$N^{-}[Q(s,u)] = f(t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{\frac{st}{u}} Q(s,u)ds, \qquad s,u > 0, \tag{11}$$

where s and u are the natural transform variables.

Definition 1.6. [48] If n is any positive integer, where $n-1 \leq \alpha < n$ and $Q(s,u)$ is the natural transform of the function $f(t)$, then the natural transform $Q_{\alpha}^{c}(s,u)$ of the Caputo fractional derivative (CFD) of the function $f(t)$ of order α showed by $D^{\alpha}f(t)$ is defined by

$$\mathbb{N}^{+}[D^{\alpha}f(t)\,] = Q_{\alpha}^{c}(s,u) = \frac{s^{\alpha}}{u^{\alpha}}Q(s,u) - \sum_{m=0}^{n-1}\frac{s^{\alpha-(m+1)}}{u^{\alpha-m}}\left(D^{m}f(t)\right)_{t=0}. \tag{12}$$

Now consider the Table 1, which includes the natural fractional integral transform.

Table 1. [48] The natural fractional integral transforms of some basic functions.

$f(t)$	$\mathbb{N}^{+}[J^{\alpha}f(t)]$
1	$\frac{u^{\alpha}}{s^{\alpha+1}}$
t	$\frac{u^{\alpha+1}}{s^{\alpha+2}}$
$\frac{t^{n-1}}{(n-1)!}$, n=1,2,…	$\frac{u^{\alpha+n-1}}{s^{\alpha+n}}$
$\frac{t^{n-1}}{\Gamma(n)}$,n> 0	$\frac{u^{\alpha+n-1}}{s^{\alpha+n}}$
e^{at}	$\frac{u^{\alpha}}{s^{\alpha}(s-au)}$

3. Fractional natural transform decomposition method

Consider the nonlinear time-fractional partial differential equation (NTFPDE) with the initial condition

$$\begin{gathered} D_t^{a}u(x,t) + Lu(x,t) + Nu(x,t) = g(x,t), \\ n-1 < a \leq n, \qquad n \in \mathbb{N}, \\ u(x,0) = h(x), \end{gathered} \tag{13}$$

where $D_t^{a} = \frac{\partial^{a}}{\partial t^{a}}$ is the CFD operator, $g(x,t)$ is the source term, L is linear operator and N is nonlinear operator [27]. Applying natural transform to both sides of Eq (13), then Eq (14) is obtained as

$$\mathbb{N}^{+}[D_t^{a}u(x,t)] + \mathbb{N}^{+}[Lu(x,t) + Nu(x,t)] = \mathbb{N}^{+}[g(x,t)]. \tag{14}$$

It is obtained as a result of the natural transform property [47]

$$\frac{s^a}{u^a}\mathbb{N}^+[u(x,t)] - \frac{s^{a-1}}{u^a}u(x,0) = \mathbb{N}^+[g(x,t)] - \mathbb{N}^+[Lu(x,t) + Nu(x,t)]. \tag{15}$$

Rearranging Eq (15), then Eq (10) has been found by

$$\mathbb{N}^+[u(x,t)] = \frac{h(x)}{s} + \frac{u^a}{s^a}\mathbb{N}^+[g(x,t)] - \frac{u^a}{s^a}\mathbb{N}^+[Lu(x,t) + Nu(x,t)]. \tag{16}$$

The solution $u(x,t)$ is represented by the infinite series as in Eq (17).

$$\boldsymbol{u(x,t)} = \sum_{\boldsymbol{i=0}}^{\infty} \boldsymbol{u_i(x,t)}, \tag{17}$$

and the nonlinear terms $Nu(x,t)$ are written by the infinite series of Adomian polynomials as

$$Nu(x,t) = \sum_{i=0}^{\infty} A_i, \tag{18}$$

where

$$A_i = \frac{1}{i!}\left[\frac{d^i}{d\lambda^i}\left[N\sum_{i=0}^{\infty}\lambda^i v_i\right]\right]_{\lambda=0}, \qquad \mathrm{i} = 0,1,2\ \ldots \tag{19}$$

Substituting Eqs (18) and (19) into Eq (16), then Eq (20) is obtained as

$$\mathbb{N}^+\left[\sum_{i=0}^{\infty} u(x,t)\right] = \frac{h(x)}{s} + \frac{u^a}{s^a}\mathbb{N}^+[g(x,t)] - \frac{u^a}{s^a}\mathbb{N}^+\left[L\sum_{i=0}^{\infty} u_i(x,t) + \sum_{i=0}^{\infty} A_i\right]. \tag{20}$$

If both sides of Eq (20) are compared, then Eq (21) is obtained as

$$\begin{cases} \mathbb{N}^+[u_0(x,t)] = \dfrac{h(x)}{s} + \dfrac{u^a}{s^a}\mathbb{N}^+[g(x,t)], \\ \mathbb{N}^+[u_1(x,t)] = -\dfrac{u^a}{s^a}\mathbb{N}^+[Lu_0(x,t) + A_0]. \end{cases} \tag{21}$$

The general iteration formula is acquired as

$$\mathbb{N}^+[u_{i+1}(x,t)] = -\frac{u^a}{s^a}\mathbb{N}^+[Lu_i(x,t) + A_i], \qquad i \geq 1. \tag{22}$$

When the inverse NT (INT) is applied to Eqs (22), (17) and (18) are obtained as

$$u_0(x,t) = h(x) + \mathbb{N}^-\left[\frac{u^a}{s^a}\mathbb{N}^+[g(x,t)]\right], \tag{23}$$

$$u_{i+1}(x,t) = -\mathbb{N}^{-}\left[\frac{u^a}{s^a}\mathbb{N}^{+}[Lu_i(x,t) + A_i]\right]. \tag{24}$$

Finally, the approximate solution $u(x,t)$ is acquired as

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t). \tag{25}$$

Now, the FNTDM is used to obtain the numerical solutions to three equations that are already well-known.

4. Convergence analysis

Theorem 4.1. Let's assume that A is a Banach space. Then, the expansion result of $u(x,t)$ converges uncertainty; there becomes $0 < \kappa < 1$, so that $\|u_i(x,t)\| \leq \kappa\|u_{i-1}(x,t)\|$, for $\forall i \in N$.

Proof. Consider the subsequent succession

$$H_i(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots + u_i(x,t). \tag{26}$$

It is essential to confirm that successions of i-th partial sums form Cauchy series in Banach space. In this regard, we consider the following:

$$\begin{aligned} \|H_{i+1}(x,t) - H_i(x,t)\| \leq \|u_{i+1}(x,t)\| \leq \kappa\|u_i(x,t)\| \leq \kappa^2\|u_{i-1}(x,t)\| \\ \leq \cdots \leq \kappa^{i+1}\|u_0(x,t)\|. \end{aligned} \tag{27}$$

For every $\forall i,j \in N,\ i \leq j,$ it is obtained as

$$\begin{aligned} \left\|H_i(x,t) - H_j(x,t)\right\| \leq \left\|H_{j+1}(x,t) - H_j(x,t)\right\| + \left\|H_{j+2}(x,t) - H_{j+1}(x,t)\right\| \\ + \cdots + \|H_i(x,t) - H_{i+1}(x,t)\|. \end{aligned} \tag{28}$$

Using the triangle inequality, then the inequality (28) transforms into the inequality (29)

$$\begin{aligned} \left\|H_i(x,t) - H_j(x,t)\right\| \leq \left\|H_{j+1}(x,t) - H_j(x,t)\right\| + \left\|H_{j+2}(x,t) - H_{j+1}(x,t)\right\| \\ + \left\|H_{j+2}(x,t) - H_{j+1}(x,t)\right\|. \end{aligned} \tag{29}$$

The inequality (29) can be represented as

$$\left\|H_i(x,t) - H_j(x,t)\right\| \leq \kappa^{j+1}\|u_0(x,t)\| + \kappa^{j+2}\|u_0(x,t)\| + \cdots + \kappa^{i}\|u_0(x,t)\|. \tag{30}$$

Simplifying the inequality (30), then we have

$$\left\|H_i(x,t) - H_j(x,t)\right\| \leq \kappa^{j+1}\left(1 + \kappa + \kappa^2 + \cdots + \kappa^{i-j-1}\right)\|u_0(x,t)\|, \tag{31}$$

where $\left(\frac{1-\kappa^{i-j}}{1-\kappa}\right) = 1 + \kappa + \kappa^2 + \cdots + \kappa^{i-j-1}$.

Thus, inequality (32) is obtained as

$$\left\|H_i(x,t) - H_j(x,t)\right\| \le \kappa^{j+1}\left(\frac{1-\kappa^{i-j}}{1-\kappa}\right)\|u_0(x,t)\|. \tag{32}$$

Hence, it is acquired as $0 < \kappa < 1$, and $1 - \kappa^{i-j} \le 1$.
Using inequality (32), we have

$$\left\|H_i(x,t) - H_j(x,t)\right\| \le \frac{\kappa^{i+1}}{1-\kappa}\|u_0(x,t)\|. \tag{33}$$

Since $u_0(x,t)$ is bounded, it is obtained as

$$\lim_{i,j\to\infty}\left\|H_i(x,t) - H_j(x,t)\right\| = 0. \tag{34}$$

Thus, $\{H_i\}$ is a Cauchy series in Banach space. Because of this, it is concluded that Eq (25) converges.

5. Applications

Now we obtain the novel numerical solutions for the time-fractional Fornberg-Whitham equation, the time-fractional Klein-Gordon equation, the time-fractional KdV equation using the FNTDM.
Example 1. Examine the time-fractional Fornberg-Whitham equation [32]

$$\begin{cases} u_t^{\alpha}(x,t) - u_{xxt}(x,t) + u_x(x,t) = u(x,t)u_{xxx}(x,t) - u(x,t)u_x(x,t) + 3u_x(x,t)u_{xx}(x,t), \\ 0 < \alpha \le 1, \\ u(x,0) = e^{x/2}. \end{cases} \tag{35}$$

Applying NT to Eq (35) and utilizing the differential property of NT, we have

$$\begin{aligned}\frac{s^a}{u^a}\mathbb{N}^+[u(x,t)] - \frac{s^{a-1}}{u^a}u(x,0) = \mathbb{N}^+[&u_{xxt}(x,t) - u_x(x,t) + u(x,t)u_{xxx}(x,t) \\ &-u(x,t)u_x(x,t) + 3u_x(x,t)u_{xx}(x,t)].\end{aligned} \tag{36}$$

Rearranging Eq (36), then it is obtained as

$$\begin{aligned}\mathbb{N}^+[u(x,t)] = \frac{e^{x/2}}{s} + \frac{u^a}{s^a}\mathbb{N}^+[&u_{xxt}(x,t) - u_x(x,t) + u(x,t)u_{xxx}(x,t) \\ &-u(x,t)u_x(x,t) + 3u_x(x,t)u_{xx}(x,t)].\end{aligned} \tag{37}$$

When the INT is implemented to Eq (37), then it is obtained as

$$\begin{aligned}u(x,t) = \mathbb{N}^-\left[\frac{e^{\frac{x}{2}}}{s}\right] + \mathbb{N}^-\Big[&\frac{u^a}{s^a}\mathbb{N}^+[u_{xxt}(x,t) - u_x(x,t) + u(x,t)u_{xxx}(x,t) \\ &-u(x,t)u_x(x,t) + 3u_x(x,t)u_{xx}(x,t)]\Big].\end{aligned} \tag{38}$$

Using Adomian decomposition method, then we have

$$u_0(x,t) = \mathbb{N}^-\left[\frac{e^{x/2}}{s}\right] = e^{x/2}. \tag{39}$$

The general iteration formula is written as

$$\sum_{i=0}^{\infty} u_{i+1}(x,t) = \mathbb{N}^{-}\left[\frac{u^a}{s^a}\mathbb{N}^{+}\left[\sum_{i=0}^{\infty}(u_{xxt})_i - \sum_{i=0}^{\infty}(u_x)_i + \sum_{i=0}^{\infty} A_i - \sum_{i=0}^{\infty} B_i + 3\sum_{i=0}^{\infty} C_i\right]\right], \quad (40)$$

where A_i, B_i and C_i are Adomian polynomials, $i = 0,1,2,...$ These are found as

$$A_0(uu_{xxx}) = u_0 u_{0xxx}, \quad (41)$$

$$A_1(uu_{xxx}) = u_0 u_{1xxx} + u_1 u_{0xxx}, \quad (42)$$

$$A_2(uu_{xxx}) = u_1 u_{2xxx} + u_1 u_{1xxx} + u_2 u_{0xxx}, \quad (43)$$

$$B_0(uu_x) = u_0 u_{0x}, \quad (44)$$

$$B_1(uu_x) = u_0 u_{1x} + u_1 u_{0x}, \quad (45)$$

$$B_2(uu_x) = u_1 u_{2x} + u_1 u_{1x} + u_2 u_{0x}, \quad (46)$$

$$C_0(u_x u_{xx}) = u_{0x} u_{0xx}, \quad (47)$$

$$C_1(u_x u_{xx}) = u_{0x} u_{1xx} + u_{1x} u_{0xx}, \quad (48)$$

$$C_2(u_x u_{xx}) = u_{1x} u_{2xx} + u_{1x} u_{1xx} + u_{2x} u_{0xx}. \quad (49)$$

For $i = 0$ in Eq (40), it is obtained as

$$u_1(x,t) = \mathbb{N}^{-}\left[\frac{u^a}{s^a}\mathbb{N}^{+}[u_{0xxt} - u_{0x} + u_0 u_{0xxx} - u_0 u_{0x} + 3u_{0x} u_{0xx}]\right], \quad (50)$$

$$u_1(x,t) = -\frac{1}{2}e^{x/2}\mathbb{N}^{-}\left[\frac{u^a}{s^{a+1}}\right] = -\frac{1}{2}\frac{e^{\frac{x}{2}}t^{\alpha}}{\Gamma(\alpha+1)}.$$

For $i = 1$ in Eq (40), it is obtained as

$$u_2(x,t) = \mathbb{N}^{-}\left[\frac{u^a}{s^a}\mathbb{N}^{+}[u_{1xxt} - u_{1x} + u_0 u_{1xxx} + u_1 u_{0xxx} - u_0 u_{1x} - u_1 u_{0x} + 3u_{0x} u_{1xx} + 3u_{1x} u_{0xx}]\right], \quad (51)$$

$$u_2(x,t) = -\frac{1}{8}e^{\frac{x}{2}}\frac{t^{2\alpha-1}}{\Gamma(2\alpha)} + \frac{1}{4}e^{\frac{x}{2}}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}.$$

For $i = 2$ in Eq (40), it is obtained as

$$u_3(x,t) = \mathbb{N}^{-}\left[\frac{u^a}{s^a}\mathbb{N}^{+}[u_{2xxt} - u_{2x} + u_1 u_{2xxx} + u_1 u_{1xxx} + u_2 u_{0xxx} - u_1 u_{2x} - u_1 u_{1x} - u_2 u_{0x} + 3u_{1x} u_{2xx} + 3u_{1x} u_{1xx} + 3u_{2x} u_{0xx}]\right],$$

$$u_3(x,t) = -\frac{1}{32}e^{\frac{x}{2}}\frac{t^{3\alpha-2}}{\Gamma(3\alpha-1)} + \frac{1}{8}e^{\frac{x}{2}}\frac{t^{3\alpha-1}}{\Gamma(3\alpha)} - \frac{1}{8}e^{\frac{x}{2}}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}. \quad (52)$$

Thus, the numerical solution of Eq (35) is acquired as

$$u(x,t) = e^{x/2} - \frac{1}{2}e^{x/2}\frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{1}{8}e^{x/2}\frac{t^{2\alpha-1}}{\Gamma(2\alpha)} + \frac{1}{4}e^{x/2}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$
$$-\frac{1}{32}e^{x/2}\frac{t^{3\alpha-2}}{\Gamma(3\alpha-1)} + \frac{1}{8}e^{x/2}\frac{t^{3\alpha-1}}{\Gamma(3\alpha)} - \frac{1}{8}e^{x/2}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots \tag{53}$$

For $\alpha = 1$ in Eq (53), it is obtained as

$$u(x,t) = e^{\frac{x}{2}-\frac{2t}{3}}. \tag{54}$$

This is the exact solution of Eq (35). Thus, this approximation quickly converges to the exact solution.

Figures 1–4 show the graphs of Eq (53) for different values of α.

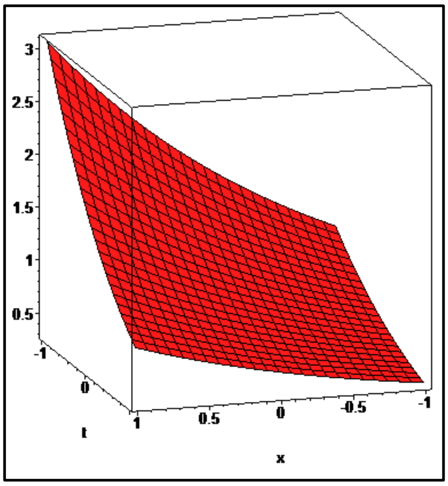


Figure 1. The variation of the exact solution.

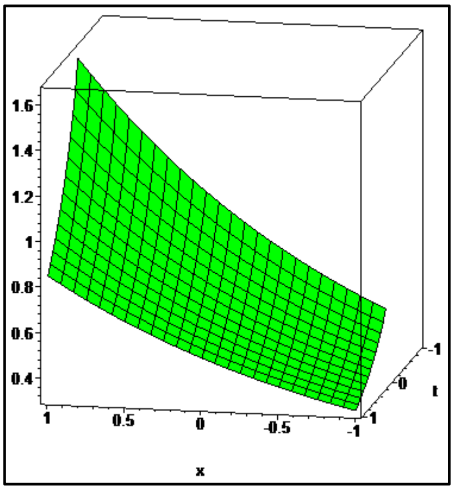


Figure 2. The variation of the numerical solution.

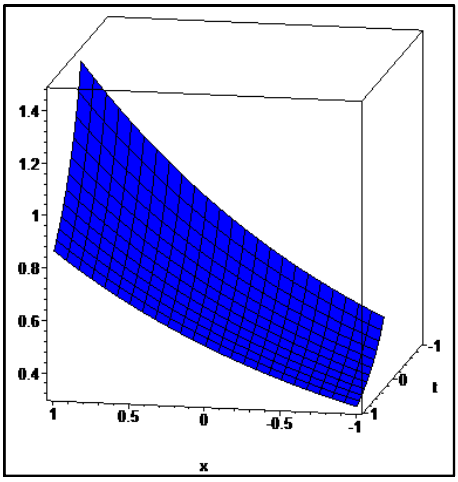


Figure 3. The variation of the numerical solution.

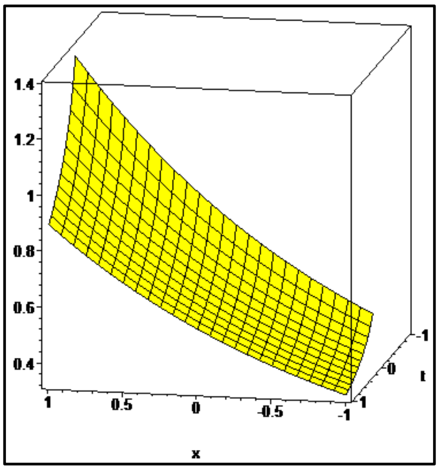


Figure 4. The variation of the numerical solution.

Figure 5 depicts the graph of FNTDM solutions for the distinct values of α and the exact solution to the Eq (35).

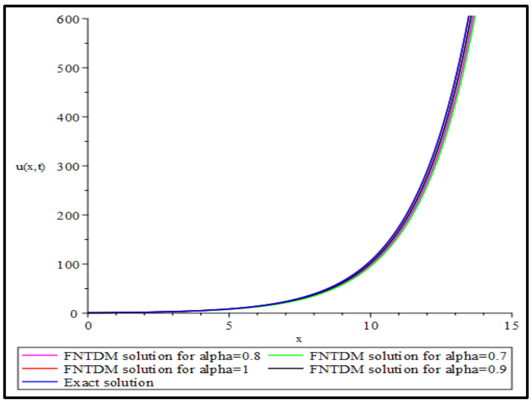


Figure 5. The comparison of the FNTDM solutions and the exact solution of Eq (35).

Example 2. Consider the time-fractional KdV equation [33]

$$\begin{cases} u_t^{\alpha}(x,t) + 6u(x,t)u_x(x,t) + u_{xxx}(x,t) = 0, \\ 0 < \alpha \le 1, t > 0, \\ u(x,0) = \frac{1}{2}\operatorname{sech}^2\left(\frac{x}{2}\right). \end{cases} \tag{55}$$

Applying the NT to Eq (55) and using the differential feature of the NT, it is obtained as

$$\frac{s^a}{u^a}\mathbb{N}^+[u(x,t)] - \frac{s^{a-1}}{u^a}u(x,0) = -\mathbb{N}^+[6u(x,t)u_x(x,t) + u_{xxx}(x,t)]. \tag{56}$$

Rearranging the Eq (56), it is acquired as

$$\mathbb{N}^+[u(x,t)] = \frac{\operatorname{sech}^2\left(\frac{x}{2}\right)}{2s} - \mathbb{N}^+[6u(x,t)u_x(x,t) + u_{xxx}(x,t)]. \tag{57}$$

Applying the INT to Eq (57), we obtain

$$u(x,t) = \mathbb{N}^-\left[\frac{\operatorname{sech}^2\left(\frac{x}{2}\right)}{2s}\right] - \mathbb{N}^-\left[\frac{u^a}{s^a}\mathbb{N}^+[6u(x,t)u_x(x,t) + u_{xxx}(x,t)]\right]. \tag{58}$$

As a result of using ADM, Eq (59) is obtained as

$$u_0(x,t) = \mathbb{N}^-\left[\frac{\operatorname{sech}^2\left(\frac{x}{2}\right)}{2s}\right] = \frac{1}{2}\operatorname{sech}^2\left(\frac{x}{2}\right). \tag{59}$$

Generally, the iteration formula in Eq (46) can be written out.

$$\sum_{i=0}^{\infty} u_{i+1}(x,t) = -\mathbb{N}^-\left[\frac{u^a}{s^a}\mathbb{N}^+[6\sum_{i=0}^{\infty} A_i + \sum_{i=0}^{\infty}(u_{xxx})_i]\right],\ i = 0,1,2,..., \tag{60}$$

where A_i is Adomian polynomial. These are explained in more detail below:

$$A_0(uu_x) = u_0u_{0x}, \tag{61}$$

$$A_1(uu_x) = u_0u_{1x} + u_1u_{0x}, \tag{62}$$

$$A_2(uu_x) = u_1u_{2x} + u_1u_{1x} + u_2u_{0x}. \tag{63}$$

For $i = 0$, it can be obtained in the form at Eq (64).

$$u_1(x,t) = -\mathbb{N}^-\left[\frac{u^a}{s^a}\mathbb{N}^+[6u_0u_{0x} + u_{0xxx}]\right], \tag{64}$$

$$u_1(x,t) = \frac{1}{2}\left(\frac{sinh\left(\frac{x}{2}\right)}{cosh^3\left(\frac{x}{2}\right)}\right)\mathbb{N}^-\left[\frac{u^a}{s^{a+1}}\right] = \frac{1}{2}\left(\frac{sinh\left(\frac{x}{2}\right)}{cosh^3\left(\frac{x}{2}\right)}\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)}. \tag{65}$$

For $i = 1$, it is acquired in the Eq (66) manner.

$$u_2(x,t) = -\mathbb{N}^-\left[\frac{u^a}{s^a}\mathbb{N}^+[6u_0u_{1x} + 6u_1u_{0x} + u_{1xxx}]\right]$$
$$= \frac{1}{4}\frac{\mathrm{t}^{2\alpha}}{\Gamma(2\alpha+1)}\left(\frac{2\cosh^2\left(\frac{\mathrm{x}}{2}\right) - 3}{\cosh^4(\frac{\mathrm{x}}{2})}\right). \tag{66}$$

For $i = 2,$ it is acquired through the Eq (67).

$$u_3(x,t) = \mathbb{N}^-\left[\frac{u^a}{s^a}\mathbb{N}^+[6u_1u_{2x} + 6u_1u_{1x} + 6u_2u_{0x} + u_{2xxx}]\right]$$
$$= \frac{3\mathrm{t}^{4\alpha}\Gamma(3\alpha+1)\sinh^2\left(\frac{\mathrm{x}}{2}\right)}{2\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)}\left(\frac{\cosh^2\left(\frac{\mathrm{x}}{2}\right) - 3}{\cosh^8(\frac{\mathrm{x}}{2})}\right)$$
$$+\frac{3\mathrm{t}^{3\alpha}\Gamma(2\alpha+1)\sinh\left(\frac{\mathrm{x}}{2}\right)}{4\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\left(\frac{2\cosh^2\left(\frac{\mathrm{x}}{2}\right) - 3}{\cosh^7(\frac{\mathrm{x}}{2})}\right)$$
$$+\frac{\mathrm{t}^{3\alpha}\sinh\left(\frac{\mathrm{x}}{2}\right)}{2\Gamma(3\alpha+1)}\left(\frac{\cosh^4\left(\frac{\mathrm{x}}{2}\right) - 12\cosh^2\left(\frac{\mathrm{x}}{2}\right) + 18}{\cosh^7(\frac{\mathrm{x}}{2})}\right). \tag{67}$$

Thus, the FNTDM solution of Eq (55) is obtained as

$$u(x,t) = \frac{1}{2}\mathrm{sech}^2\left(\frac{x}{2}\right) + \frac{1}{2}\left(\frac{sinh\left(\frac{x}{2}\right)}{cosh^3\left(\frac{x}{2}\right)}\right)\frac{\mathrm{t}^{\alpha}}{\Gamma(\alpha+1)}$$
$$+\frac{1}{4}\frac{\mathrm{t}^{2\alpha}}{\Gamma(2\alpha+1)}\left(\frac{2\cosh^2\left(\frac{\mathrm{x}}{2}\right) - 3}{\cosh^4(\frac{\mathrm{x}}{2})}\right)$$
$$+\frac{3}{2}\frac{\mathrm{t}^{4\alpha}\Gamma(3\alpha+1)\sinh^2\left(\frac{\mathrm{x}}{2}\right)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)}\left(\frac{\cosh^2\left(\frac{\mathrm{x}}{2}\right) - 3}{\cosh^8(\frac{\mathrm{x}}{2})}\right)$$
$$+\frac{3}{4}\frac{\mathrm{t}^{3\alpha}\Gamma(2\alpha+1)\sinh\left(\frac{\mathrm{x}}{2}\right)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\left(\frac{2\cosh^2\left(\frac{\mathrm{x}}{2}\right) - 3}{\cosh^7(\frac{\mathrm{x}}{2})}\right)$$
$$+\frac{1}{2}\frac{\mathrm{t}^{3\alpha}\sinh\left(\frac{\mathrm{x}}{2}\right)}{\Gamma(3\alpha+1)}\left(\frac{\cosh^4\left(\frac{x}{2}\right) - 12\cosh^2\left(\frac{\mathrm{x}}{2}\right) + 18}{\cosh^7(\frac{\mathrm{x}}{2})}\right). \tag{68}$$

Figures 6–9 show the graphs of Eq (68) for different values of α.

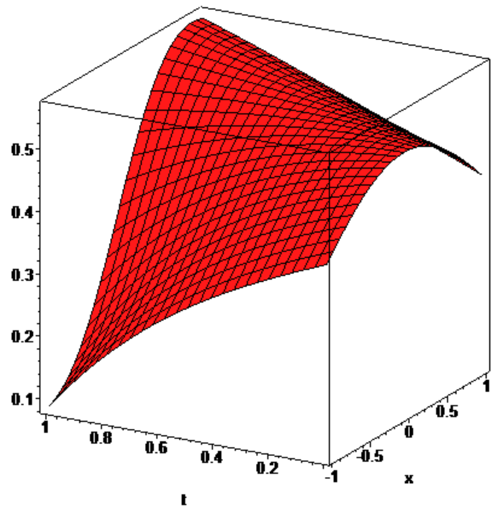


Figure 6. The variation of the numerical solution.

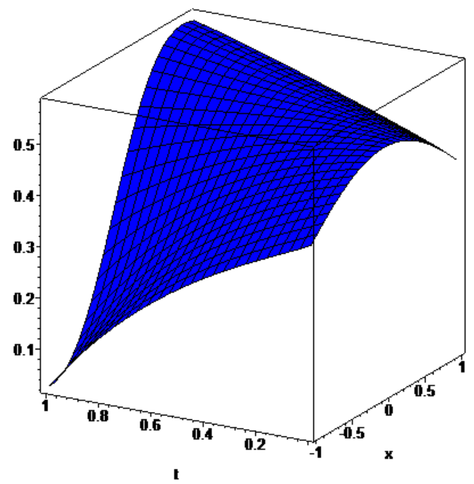


Figure 7. The variation of the numerical solution.

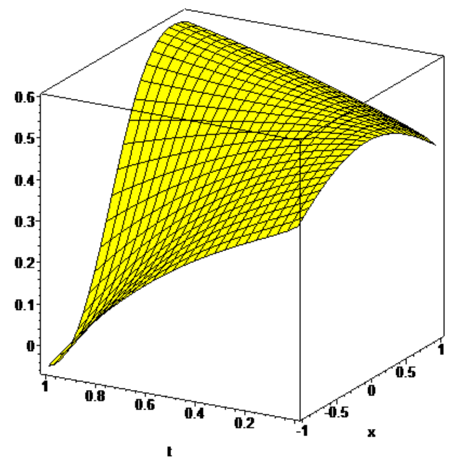


Figure 8. The variation of the numerical solution.

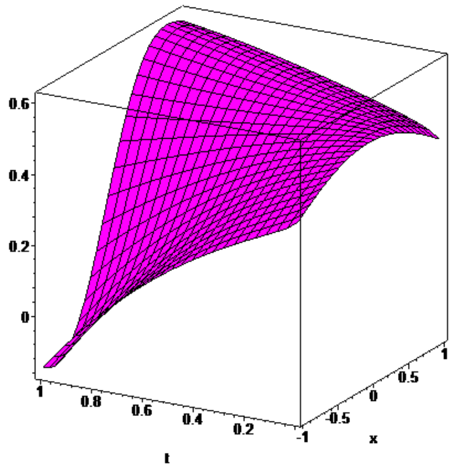


Figure 9. The variation of the numerical solution.

The form $u(x,t) = \frac{1}{2} sech^2(\frac{1}{2}(x-t))$ is the exact solution of the Eq (55). Figure 10 shows the graph of FNTDM solutions, and the exact solution of the problem of Eq (55).

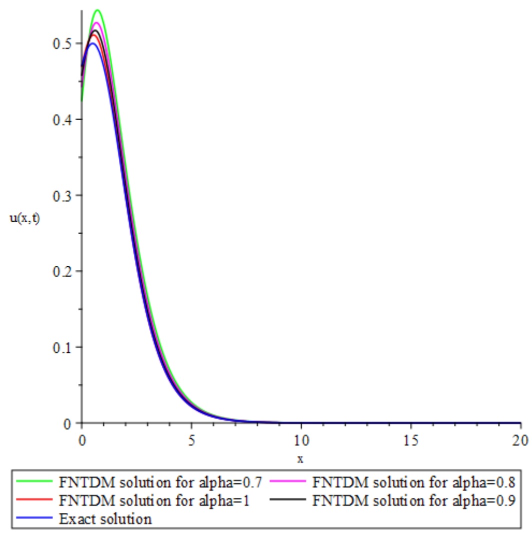


Figure 10. The comparison of the FNTDM solutions and the exact solution of the problem of Eq (55).

As shown in Table 2, the FNTDM solution outperforms the variational iteration method (VIM) solution in [33]. FNTDM's solution is also shown to be extremely close to the exact solution in Table 2. For FNTDM, the absolute error is extremely small, as shown in Table 4. The absolute errors incurred by the FNTDM solution, the VIM solution, and the exact solution are compared in Table 4, which displays the results of this comparison for each of the different values of x and t. From the data presented in Tables 2–4, it can be inferred that FNTDM is more efficient than VIM. In order to show the above results, a numerical experiment has been given to compare the approximate solution with the results from using VIM.

Table 2. Comparison of the exact solution, the fourth- order FNTDM solution and VIM solution for $\alpha = 1$.

x	t	**Exact solution**	**FNTDM**	**VIM**[33]
0.5	0.2	0.4944272562	0.4899360367	0.4659756825
0.5	0.4	0.4993756504	0.5066013315	0.4531705710
0.5	0.6	0.4993756504	0.5241301786	0.4305272602
0.5	0.8	0.4944272562	0.5463021537	0.3969809210
0.5	1.0	0.4847718146	0.5765495376	0.3514667239
1.0	0.2	0.4625037260	0.4287287736	0.3917081360
1.0	0.4	0.4783139560	0.4646411668	0.3867501153
1.0	0.6	0.4901639988	0.5036020808	0.3777335608
1.0	0.8	0.4975103744	0.5476061887	0.3640422286
1.0	1.0	0.5000000000	0.5980018021	0.3450598752
1.5	0.2	0.4102418342	0.3368813877	0.2989675701
1.5	0.4	0.4328624256	0.3770036331	0.3011816764
1.5	0.6	0.4533214172	0.4185649249	0.3052203864
1.5	0.8	0.4708639648	0.4610623277	0.3113688629
1.5	1.0	0.4847718146	0.5035846877	0.3199122694

Table 3. Comparison of the fourth-order FNTDM and VIM solutions for $\alpha = 0.9$.

x	t	**FNTDM**	**VIM [33]**
0.5	0.2	0.4935642639	0.4702688983
0.5	0.4	0.5117317003	0.4590684402
0.5	0.6	0.5329770263	0.4368821199
0.5	0.8	0.5618098819	0.4030135996
0.5	1.0	0.6015583798	0.3565801736
1.0	0.2	0.4366194018	0.3984853014
1.0	0.4	0.4761813095	0.3960603473
1.0	0.6	0.5190508465	0.3877651868
1.0	0.8	0.5676425491	0.3735652674
1.0	1.0	0.6230711465	0.3531318417
1.5	0.2	0.3457491949	0.3030335863
1.5	0.4	0.3889116028	0.3108887214
1.5	0.6	0.4313319727	0.3156795722
1.5	0.8	0.4728708698	0.3212977849
1.5	1.0	0.5126623876	0.3283282727

Table 4. Comparison of absolute error between FNTDM and VIM when $\alpha = 1$.

		t					
Methods	x	**0.00**	**0.02**	**0.04**	**0.06**	**0.08**	**0.10**
FNTDM	**0.00**	0.0000000000	0.0000250010	0.0001000166	0.0002250844	0.0004002665	0.0006256504
VIM		0.0000000000	0.0000250010	0.0001000166	0.0002250844	0.0004002665	0.0006256504
FNTDM	**0.02**	0.0000249976	0.0000000517	0.0000246111	0.0000987489	0.0002221497	0.0003946259
VIM		0.0000249976	0.0000999866	0.0002249976	0.0004001031	0.0006254007	0.0009010125
FNTDM	**0.04**	0.0000999633	0.0000248911	0.0000008250	0.0000223032	0.0000937902	0.0002131757
VIM		0.0000999633	0.0002248877	0.0003997866	0.0006247669	0.0008999613	0.0012255269
FNTDM	**0.06**	0.0002248146	0.0000998022	0.0000237362	0.0000041670	0.0000153344	0.0000815088
VIM		0.0002248146	0.0003995969	0.0006242510	0.0008989184	0.0012237657	0.0015989845
FNTDM	**0.08**	0.0003994139	0.0002245993	0.0000982686	0.0000193690	0.0000131270	0.0000002203
VIM		0.0003994139	0.0006239522	0.0008982042	0.0012223457	0.0015965772	0.0020211240
FNTDM	**0.10**	0.0006235700	0.0003991452	0.0002226913	0.0000928882	0.0000008442	0.0000319119
VIM		0.0006235700	0.0008977372	0.0012214058	0.0015947842	0.0020181065	0.0024916309

Example 3. Consider the time-fractional Klein-Gordon equation [34]

$$\begin{cases} u_t^{\alpha}(x,t) - u_{xx}(x,t) + u^2(x,t) = 0, \\ 0 < \alpha \leq 1, t \geq 0, \\ u(x,0) = 1 + sinx. \end{cases} \tag{69}$$

By using the NT on Eq (69) and making use of the differential property of the natural transform, Eq (70) is obtained as

$$\frac{s^{\alpha}}{u^{\alpha}}\mathbb{N}^{+}[u(x,t)] - \frac{s^{\alpha-1}}{u^{\alpha}}u(x,0) = \mathbb{N}^{+}[u_{xx}(x,t) - u^2(x,t)]. \tag{70}$$

After some rearrangement of Eq (70), the result can be written as Eq (71).

$$\mathbb{N}^{+}[u(x,t)] = \frac{1+sinx}{s} + \mathbb{N}^{+}[u_{xx}(x,t) - u^2(x,t)]. \tag{71}$$

By applying the INT to Eq (71), one can obtain the result as in Eq (72).

$$u(x,t) = \mathbb{N}^{-}\left[\frac{1+sinx}{s}\right] + \mathbb{N}^{-}\left[\frac{u^{a}}{s^{a}}\mathbb{N}^{+}[u_{xx}(x,t) - u^{2}(x,t)]\right]. \tag{72}$$

Equation (73) is obtained as a result of using ADM.

$$u_0(x,t) = \mathbb{N}^{-}\left[\frac{1+sinx}{s}\right] = 1 + sinx. \tag{73}$$

Generally, the iteration formula can be written as Eq (74).

$$\sum_{i=0}^{\infty} u_{i+1}(x,t) = \mathbb{N}^{-}\left[\frac{u^{a}}{s^{a}}\mathbb{N}^{+}[\sum_{i=0}^{\infty}(u_{xx})_i - \sum_{i=0}^{\infty} A_i]\right],\ i = 0,1,2,\dots, \tag{74}$$

where A_i is Adomian polynomial. This can be demonstrated using the following:

$$A_0(u^2) = {u_0}^2, \tag{75}$$

$$A_1(u^2) = 2u_0u_1, \tag{76}$$

$$A_2(u^2) = 2u_0u_2 + {u_1}^2. \tag{77}$$

For $i = 0$, it can be found as in Eq (78).

$$u_1(x,t) = \mathbb{N}^{-}\left[\frac{u^{a}}{s^{a}}\mathbb{N}^{+}[u_{0xx} - {u_0}^2]\right] \tag{78}$$

$$= (-3sinx - 1 - sin^2x)\mathbb{N}^{-}\left[\frac{u^{a}}{s^{a+1}}\right] \tag{79}$$

$$= (-3sinx - 1 - sin^2x)\frac{t^{\alpha}}{\Gamma(\alpha+1)}. \tag{80}$$

For $i = 1$, Eq (81) is acquired in this manner.

$$u_2(x,t) = \mathbb{N}^{-}\left[\frac{u^{a}}{s^{a}}\mathbb{N}^{+}[u_{1xx} - 2u_0u_1]\right] = \frac{t^{2\alpha}\mathrm{sinx}}{\Gamma(2\alpha+1)}(13 - 2\cos^2\mathrm{x} + 10\mathrm{sinx}). \tag{81}$$

For $i = 2$, Eq (82) is obtained in the form of

$$\begin{aligned} u_3(x,t) &= \mathbb{N}^{-}\left[\frac{u^{a}}{s^{a}}\mathbb{N}^{+}[u_{2xx} - 2u_0u_2 - {u_1}^2]\right] \\ &= (54\cos^3\mathrm{x} - 40\sin2\mathrm{x} - 46\mathrm{sinx} - 53\mathrm{cosx} - 46 + 50\cos^2\mathrm{x} - 4\cos^4\mathrm{x} \\ &\quad +24\mathrm{sinx}\cos^2\mathrm{x})\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{(-3\mathrm{sinx} - 1 - \sin^2\mathrm{x})^2 t^{3\alpha}\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}. \end{aligned} \tag{82}$$

Thus, the FNTDM solution of Eq (69) is obtained as

$$\begin{aligned} u(x,t) &= 1 + sinx + (-3sinx - 1 - sin^2x)\frac{t^{\alpha}}{\Gamma(\alpha+1)} \\ &\quad +(54\cos^3\mathrm{x} - 40\sin2\mathrm{x} - 46\mathrm{sinx} - 53\mathrm{cosx} - 46 + 50\cos^2\mathrm{x} - 4\cos^4\mathrm{x} \\ &\quad +24\mathrm{sinx}\cos^2\mathrm{x})\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{(3\mathrm{sinx} + 1 + \sin^2\mathrm{x})^2 t^{3\alpha}\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}. \end{aligned} \tag{83}$$

By substituting $\alpha = 1$ in Eq (83), it becomes Eq (84).

$$\begin{aligned} u(x,t) = 1 + sinx + (-3sinx - 1 - sin^2x)t + \frac{t^2 \mathrm{sinx}}{2}(13 - 2\cos^2 x + 10\mathrm{sinx}) \\ +(54\cos^3 x - 40\sin 2x - 46\mathrm{sinx} - 53\mathrm{cosx} - 46 + 50\cos^2 x \\ -4\cos^4 x + 24\mathrm{sinx}\cos^2 x)\frac{t^3}{6} + \frac{(3\mathrm{sinx}+1+\sin^2 x)^2 t^3}{3}. \end{aligned} \tag{84}$$

For different values of α, graphs of Eq (83) are shown in Figures 11–14.

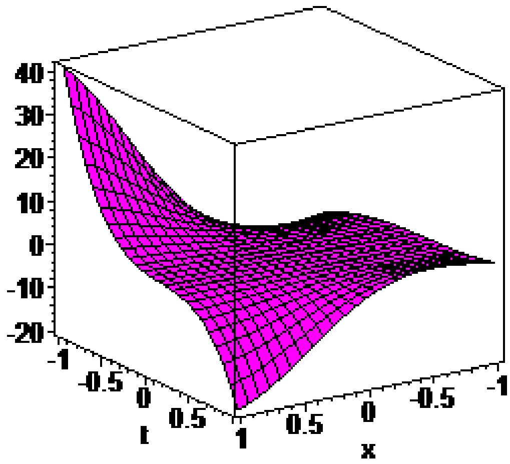


Figure 11. The variation of the exact solution.

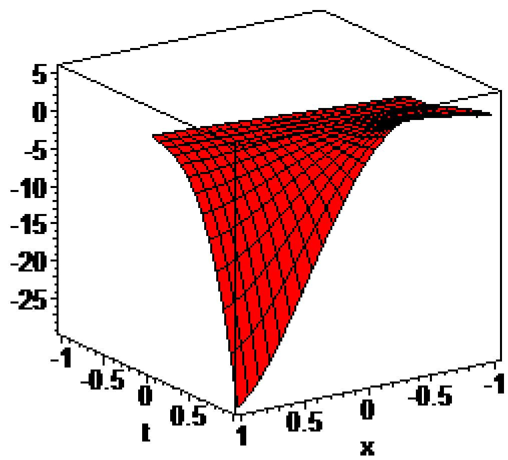


Figure 12. The variation of the numerical solution.

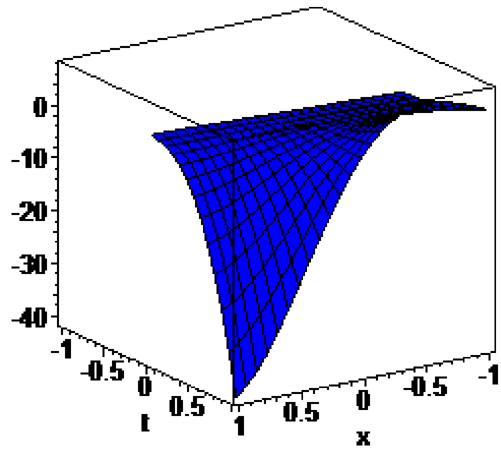


Figure 13. The variation of the numerical solution.

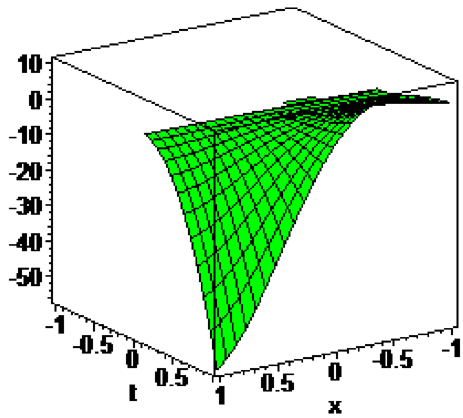
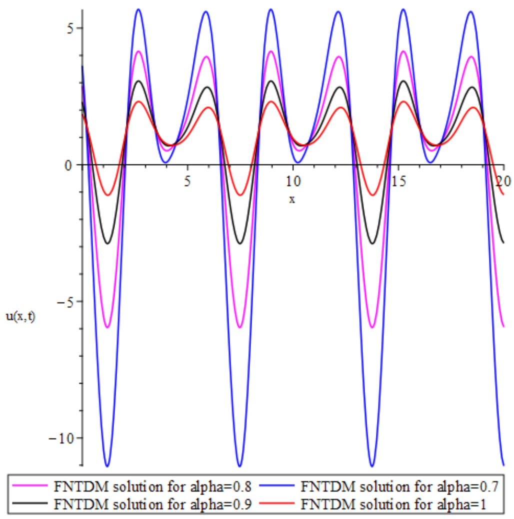


Figure 14. The variation of the numerical solution.

Figure 15 depicts the graph of FNTDM solutions for the given parameters, as well as the exact solution to the problem of Eq (69).



Figıre 15. The comparison of the FNTDM solutions and the exact solution of Eq (69).

Based on what is presented in Table 5, one can deduce that the FNTDM solution is close to homotopy perturbation method (HPM) solution. In addition, the numerical solution that was obtained by FVIM is very dissimilar to the solution that was obtained by FNTDM. According to what is shown in Table 5, FNTDM is significantly more impressive than fractional variational iteration method (FVIM). It has been given a numerical experiment to compare the approximate solution and results obtained by both employing HPM in [43] and employing FVIM in [34]. This is done so that the results shown above can be demonstrated.

Table 5. Comparison of the fourth-order FNTDM**,** fourth-order FVIM and HPM solutions for α=1.

x	t	**FNTDM**	**FVIM** [34]	**HPM** [43]
0.5	0.2	0.9936936760	1.0918618880	1.0252028450
0.5	0.4	0.1719163698	0.9204862583	0.3872139177
0.5	0.6	-1.6336542300	0.8513963520	-0.9483977960
0.5	0.8	-5.0707659740	0.7706898780	-3.4954888480
0.5	1.0	-10.7871667100	0.5644645330	-7.7679157920
1.0	0.2	1.1363780990	1.3260973680	1.1278077790
1.0	0.4	-0.0122573375	1.2818890290	-0.0304427681
1.0	0.6	-3.1903844890	1.4222229150	-3.9315959820
1.0	0.8	-9.3220562860	1.4604759670	-11.2300577100
1.0	1.0	-19.7725898200	1.1100251400	-23.6761734600
1.5	0.2	1.2572942580	1.4316136430	1.1290832020
1.5	0.4	0.2282734110	1.4636290930	-0.9566144330
1.5	0.6	-2.2946842630	1.6955418350	-6.4727800640
1.5	0.8	-7.5166954770	1.7293523710	-17.6325958400
1.5	1.0	-16.6428769500	1.1670611990	-36.6492438900

6. Result and discussion

Inferred from Table 2, it can be seen that the FNTDM solution for FPDEs is extremely close to the exact solution. The FNTDM solution for $\alpha = 1$ is illustrated in Table 2. From Table 2, one can deduce that the value of this solution rises when x remains the same and t is made greater. The FNTDM solution for the value $\alpha = 0.9$ is presented in Table 3. Table 4 demonstrates that absolute error is extremely low and that it approaches zero as the value of fractional order α is moved further away from one. Table 5 suggests that the FNTDM solution is close to the HPM solution. In addition, the numerical solution that was obtained by FNTDM is extremely dissimilar to that which was obtained by FVIM. According to Table 5, it can be seen that FNTDM is more advantageous than FVIM. Maple software is used to plot, for a variety of values of the parameter α, graphs of the numerical solutions that were obtained through FNTDM. When the alpha values are lowered, it can be inferred that the values of the FNTDM solution will rise as a result.

The complex process of classifying fractional operators spanned numerous periods, and numerous attempts have been made [49–54]. According to our assessment, there is currently no consensus on the fundamental classification criteria for fractional operators. The European Organization for Nuclear Research, known as CERN-recorded and-reported experimental findings have effectively diminished the mathematically sound structures of string theory and super-symmetry in favor of mainstream models based on much simpler notions and more physical concepts, such as gauge invariance. When creating models utilizing fractional operators, it may not always be beneficial to construct complicated fractional operators. In addition, there does not appear to be a unique fractional operator that can be used to explain all types of processes with distinct memory effects. We believe that the classification of fractional operators into classes is more reasonable from both experimental and mathematical perspectives [54].

Moreover, the various conceptions of singular and nonsingular fractional operators, along with their respective benefits and drawbacks, can exist under the umbrella of classes of operators and the powerful concept of memory.

7. Conclusions

During the 325-year history of fractional calculus, a critical mass of material has accumulated from both mathematical and applied perspectives, and the time has come to make a productive transition. Despite the fact that numerical methods for fractional differential equations have made significant contributions, they have not yet reached the level required to determine which fractional calculus model is most appropriate for a given set of real-world data [49–54]. We believe that one of the keys to the success of future theoretical and practical perspectives is to investigate the concept of diverse fractional calculus operator classes. This field will be significantly strengthened in the long run if both the benefits and limitations of specific fractional operator types are elucidated. Harmonizing the viewpoint that fractional operators should have a physical, biological, or economic meaning and should appear naturally in a set of real-world processes with the mathematical construction of fractional operators without reference to experimental data appears to be a difficult problem for fractional calculus researchers.

FNTDM is used to obtain numerical solutions to the three well-known nonlinear equations by applying the FNTDM algorithm. Table 4 shows that FNTDM outperforms VIM in terms of effectiveness. Both FVIM and HPM are shown in Table 5 to be less powerful than FNTDM. For the numerical solution of nonlinear time-fractional partial differential equations, it is concluded that FNTDM is the best, most effective and reliable tool. FNTDM has found to be an excellent algorithm. FNTDM has been utilized to apply for obtaining the numerical solutions of the three famous nonlinear equations. FNTDM is more effective than VIM as shown in the comparison in Table 4. Also, FNTDM is more powerful than both FVIM and HPM as shown in the comparison in Table 5. It is deduced that FNTDM is a superior, effective and reliable tool to determine the numerical solutions of the nonlinear time-fractional partial differential equations. It has been shown that FNTDM is an effective algorithm. In addition to that, it is demonstrated that this algorithm delivers the solution in the form of a series that rapidly and effectively converges to the exact solution being sought. In light of this, FNTDM is a dependable, efficient, and potent method for obtaining analytical solutions for various classes of linear and nonlinear time-fractional ordinary and PDEs.

Author contributions

Phd. Alkan contributed to the article in the areas of Conceptualization, Formal Analysis, Methodology, Software, Resources, Writing - original draft; Phd. Anac contributed to the article in the areas of Funding acquisition, Investigation, Supervision, Validation, Visualization, Writing–review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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