AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(9): 25319–25332.
DOI:10.3934/math.20241236
Received: 29 June 2024
Revised: 15 August 2024
Accepted: 23 August 2024
Published: 29 August 2024

Research article

Characterizing N-type derivations on standard operator algebras by local actions

Khalil Hadi Hakami[1], **Junaid Nisar**[2,*], **Kholood Alnefaie**[3] **and Moin A. Ansari**[1,*]

[1] Department of Mathematics, College of Science, Jazan University, P.O. Box. 114, Jazan 45142, Kingdom of Saudi Arabia

[2] Department of Applied Sciences, Symbiosis Institute of Technology, Symbiosis International (Deemed) University, Lavale, Pune, India

[3] Department of Mathematics, College of Science, Taibah University, Madinah 42353, Saudi Arabia

* **Correspondence:** Email: junaidnisar73@gmail.com, junaid.nisar@sitpune.edu.in, maansari@jazanu.edu.sa.

Abstract: On an infinite dimensional complex Hilbert space $\mathcal{H}$, we consider a standard operator algebra $\mathcal{S}$ with an identity operator I that is closed with respect to *adjoint* operation. $P_n(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n)$ is set of polynomials defined under indeterminates $\mathcal{X}_1, \mathcal{X}_2, \cdots, \mathcal{X}_n$ by n with multiplicative Lie products with set of positive integers $\mathbb{N}$. It is shown that a map $\Theta : \mathcal{S} \to \mathcal{S}$ satisfying

$$\Theta\left(P_n\left(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \ldots, \mathcal{D}_n\right)\right) = \sum_{i=1}^{n} P_n\left(\mathcal{D}_1, \ldots, \mathcal{D}_{i-1}, \Theta\left(\mathcal{D}_i\right), \mathcal{D}_{i+1}, \ldots, \mathcal{D}_n\right),$$

for any $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \ldots, \mathcal{D}_n \in \mathcal{S}$ with $\mathcal{D}_1\mathcal{D}_2\mathcal{D}_3 \ldots \mathcal{D}_n = 0$ can be represented as $d(x) + \tau(x)$ for every $x \in \mathcal{S}$, where $d : \mathcal{S} \to \mathcal{S}$ is an additive derivation with another map $\tau : \mathcal{S} \to \mathcal{Z}(\mathcal{S})$ that vanishes on each $(n-1)^{th}$ commutator $P_n(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \ldots, \mathcal{D}_n)$ with $\mathcal{D}_1\mathcal{D}_2\mathcal{D}_3 \cdots \mathcal{D}_n = 0$.

Keywords: Lie derivation; Lie triple derivation; adjoint operation; standard operator algebra
Mathematics Subject Classification: 16W25, 46K15, 47B47

1. Introduction

The set $\mathcal{Z}(\mathcal{S})$ is the center over the complex field $\mathbb{C}$ of a standard operator algebra $\mathcal{S}$ with a linear mapping $d : \mathcal{S} \to \mathcal{S}$. Here d is a derivation if the condition $d(\mathcal{D}\mathcal{E}) = d(\mathcal{D})\mathcal{E} + \mathcal{D}d(\mathcal{E})$ is satisfied by d for all $\mathcal{D}, \mathcal{E} \in \mathcal{S}$. If the requirement for linearity is relaxed to additivity, d is referred to as a derivation with additivity. The map $d : \mathcal{S} \to \mathcal{S}$ is known as Lie derivation with multiplication if it fulfills the condition $d([\mathcal{D},\mathcal{E}]) = [d(\mathcal{D}),\mathcal{E}] + [\mathcal{D}, d(\mathcal{E})]$ for all $\mathcal{D},\mathcal{E} \in \mathcal{S}$, where $[\mathcal{D},\mathcal{E}] = \mathcal{D}\mathcal{E} - \mathcal{E}\mathcal{D}$ denotes the Lie bracket. In a similar way, d is called a Lie triple derivation with multiplication if $d([[\mathcal{D},\mathcal{E}],\mathcal{F}]) = [[d(\mathcal{D}),\mathcal{E}],\mathcal{F}] + [[\mathcal{D}, d(\mathcal{E})],\mathcal{F}] + [[\mathcal{D},\mathcal{E}], d(\mathcal{F})]$ for all $\mathcal{D},\mathcal{E},\mathcal{F} \in \mathcal{S}$.

In the last few years, significant attention has been given to some special conditions that change a linear map into a derivation (class of derivation) including Lie derivation (see [1–5, 8, 10, 11, 15, 16]). Most of these studies have focused on identifying the specific conditions that allow Lie derivations or even simple derivations to be fully characterized by their action on certain sub-structures of these algebras. Numerous works have been published on the investigation of the local actions of Lie derivations in operator algebras. Lu and Jing [7] demonstrated that if X is a Banach space with dimension greater than two and $L : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ is a linear map such that $L([\mathcal{D},\mathcal{E}]) = [L(\mathcal{D}),\mathcal{E}] + [\mathcal{D}, L(\mathcal{E})]$ for all $\mathcal{D},\mathcal{E} \in \mathcal{B}(\mathcal{X})$ with $\mathcal{D}\mathcal{E} = 0$ (or $\mathcal{D}\mathcal{E} = \mathbb{P}$, where $\mathbb{P}$ is a fixed nontrivial idempotent), then there exists an operator $T \in \mathcal{B}(\mathcal{X})$ and a linear map $\phi : \mathcal{B}(\mathcal{X}) \to \mathbb{C}I$ that vanishes at all commutators $[\mathcal{D},\mathcal{E}]$ with $\mathcal{D}\mathcal{E} = 0$ (or $\mathcal{D}\mathcal{E} = \mathbb{P}$), such that $L(\mathcal{D}) = T\mathcal{D} - \mathcal{D}T + \phi(\mathcal{D})$ for all $\mathcal{D} \in \mathcal{B}(\mathcal{X})$. Similarly, Qi and Hou [9] characterized Lie derivations on any von Neumann algebra $\mathcal{S}$ without central summands of type I_1. They showed that if $L : \mathcal{S} \to \mathcal{S}$ is an additive map such that $L([\mathcal{D},\mathcal{E}]) = [L(\mathcal{D}),\mathcal{E}] + [\mathcal{D}, L(\mathcal{E})]$ for all $\mathcal{D},\mathcal{E} \in \mathcal{S}$ with $\mathcal{D}\mathcal{E} = 0$, then there exists an additive derivation $\delta : \mathcal{S} \to \mathcal{S}$ and an additive map $\phi : \mathcal{S} \to Z(\mathcal{S})$ that vanishes at all commutators $[\mathcal{D},\mathcal{E}]$ with $\mathcal{D}\mathcal{E} = 0$, such that $L = \delta + \phi$. In 2018, Liu [6] investigated the characterization of Lie triple derivations on von Neumann algebras without central abelian projections. He proved that if a linear map $L : \mathcal{S} \to \mathcal{S}$ satisfies $L([[\mathcal{D},\mathcal{E}],\mathcal{F}]) = [[L(\mathcal{D}),\mathcal{E}],\mathcal{F}] + [[\mathcal{D}, L(\mathcal{E})],\mathcal{F}] + [[\mathcal{D},\mathcal{E}], L(\mathcal{F})]$ for all $\mathcal{D},\mathcal{E},\mathcal{F} \in \mathcal{S}$ with $\mathcal{D}\mathcal{E} = 0$ (or $\mathcal{D}\mathcal{E} = \mathbb{P}$, where $\mathbb{P}$ is a fixed non-trivial projection in $\mathcal{S}$), then there exists an additive derivation $d : \mathcal{S} \to \mathcal{S}$ and an additive map $\gamma : \mathcal{S} \to Z(\mathcal{S})$ that vanishes on every second commutator $[[\mathcal{D},\mathcal{E}],\mathcal{F}]$ with $\mathcal{D}\mathcal{E} = 0$ (or $\mathcal{D}\mathcal{E} = \mathbb{P}$), such that $L(X) = d(X) + \gamma(X)$ for all $X \in \mathcal{S}$. In recent years, several researchers have explored Lie n-derivations across various types of algebras (see [12–14, 17] and related references). However, to date, no research work has been done based on local actions for derivations of Lie type within operator algebras, indicating a need for further investigation in this area.

Inspired by the aforementioned results, we investigate derivations of Lie type based on algebras like standard operator algebras. We demonstrate that derivations of Lie type on standard operator algebras exhibit a general form at zero products.

2. Notation and preliminaries

For a fixed positive integer n, where $n \geq 2$, we define polynomials sequence as

$$
\begin{aligned}
P_1(y_1) &= y_1, \\
P_2(y_1, y_2) &= [P_1(y_1), y_2] = [y_1, y_2], \\
P_3(y_1, y_2, y_3) &= [P_2(y_1, y_2), y_3] = [[y_1, y_2], y_3], \\
&\cdots\cdots \\
P_n(y_1, y_2, \ldots, y_n) &= [P_{n-1}(y_1, y_2, \ldots, y_{n-1}), y_n].
\end{aligned}
$$

The polynomial $P_n(y_1, y_2, \ldots, y_n)$ is known as $(n-1)^{th}$ commutator. A map $\Theta : \mathcal{S} \to \mathcal{S}$ that is additive is known as Lie n-derivation or n-type derivation if the following is satisfied:

$$\Theta\left(P_n\left(y_1, y_2, \ldots, y_n\right)\right) = \sum_{i=1}^{n} P_n\left(y_1, \ldots, y_{i-1}, \Theta\left(y_i\right), y_{i+1}, \ldots, y_n\right),$$

for all $y_1, y_2, \ldots, y_n \in \mathcal{S}$. More generally, by removing the additivity of Θ, we obtain that Θ is a nonlinear Lie n-derivation. The Lie derivation is a generalization of the Lie 2-derivation. Further, every Lie triple derivation is a generalization of Lie 3-derivation. A combination of these three derivations i.e., Lie derivations, Lie triple derivations, and Lie n-derivations are called derivations of Lie type. On a Hilbert space $\mathcal{H}$ over C, the set $\mathcal{B}(H)$ represents all algebras with bounded linear operators. The subalgebra of finite rank operators that is bounded is denoted by $\mathcal{F}(\mathcal{H})$. It is important to note that $\mathcal{F}(\mathcal{H})$ is a $*$-closed ideal within $\mathcal{B}(\mathcal{H})$. An algebra $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ is referred to as a standard operator algebra if $\mathcal{F}(H) \subseteq \mathcal{S}$. A projection $\mathbb{P}$ is an operator $\mathbb{P} \in \mathcal{B}(\mathcal{H})$ that satisfies the conditions $\mathbb{P}^* = \mathbb{P}$ and $\mathbb{P}^2 = \mathbb{P}$. An algebra $\mathcal{S}$ is called a prime algebras if $\mathcal{D}\mathcal{S}\mathcal{E} = 0$ gives either $\mathcal{D} = 0$ or $\mathcal{E} = 0$. Interestingly, all standard operator algebras are prime.

Now, considering a projection $\mathbb{P}_1 \in \mathcal{S}$ so that $\mathbb{P}_2 = I - \mathbb{P}_1$. $\mathcal{S}_{jk} = P_j\mathcal{S}P_k$ for $j = 1$ and $k = 2$. Then by the Peirce-decomposition of $\mathcal{S}$, we have $\mathcal{S} = \mathcal{S}_{11} \oplus \mathcal{S}_{12} \oplus \mathcal{S}_{21} \oplus \mathcal{S}_{22}$ and $S^*_{jk} \in \mathcal{S}_{kj}$ for any $S_{jk} \in \mathcal{S}_{jk}$. The following significant result is used frequently in the main results:

Lemma 2.1. *In a Hilbert space $\mathcal{H}$ over a complex field C that is closed with respect to adjoint operation, let $\mathcal{S}$ be a standard operator algebra with operator I. Further, let $\mathcal{D}_{11} \in \mathcal{S}_{11}$ and $\mathcal{D}_{22} \in \mathcal{S}_{22}$. If $\mathcal{D}_{11}\mathcal{E}_{12} = \mathcal{E}_{12}\mathcal{D}_{22}$ or $\mathcal{E}_{21}\mathcal{D}_{11} = \mathcal{D}_{22}\mathcal{E}_{21}$ for all $\mathcal{E}_{12} \in \mathcal{S}_{12}, \mathcal{E}_{21} \in \mathcal{S}_{21}$, then $\mathcal{D}_{11} + \mathcal{D}_{22} \in \mathcal{Z}(\mathcal{S})$.*

Proof. Let $\mathcal{D}_{11} \in \mathcal{S}_{11}$ and $\mathcal{D}_{22} \in \mathcal{S}_{22}$. Assume that $\mathcal{D}_{11}\mathcal{E}_{12} = \mathcal{E}_{12}\mathcal{D}_{22}$ for all $\mathcal{E}_{12} \in \mathcal{S}_{12}$. For any $\mathcal{X}_{11} \in \mathcal{S}_{11}$ and $\mathcal{X}_{12} \in \mathcal{S}_{12}$, we get

$$\mathcal{D}_{11}\mathcal{X}_{11}\mathcal{X}_{12} = \mathcal{X}_{11}\mathcal{X}_{12}\mathcal{D}_{22} = \mathcal{X}_{11}\mathcal{D}_{11}\mathcal{X}_{12}.$$

Since $\mathcal{S}$ is prime, we have $\mathcal{D}_{11}\mathcal{X}_{11} = \mathcal{X}_{11}\mathcal{D}_{11}$.

For any $\mathcal{X}_{12} \in \mathcal{S}_{12}$ and $\mathcal{X}_{22} \in \mathcal{S}_{22}$, we have

$$\mathcal{X}_{12}\mathcal{X}_{22}\mathcal{D}_{22} = \mathcal{D}_{11}\mathcal{X}_{12}\mathcal{X}_{22} = \mathcal{X}_{12}\mathcal{D}_{22}\mathcal{X}_{22}.$$

It follows from the primeness of $\mathcal{S}$ that $\mathcal{X}_{22}\mathcal{D}_{22} = \mathcal{D}_{22}\mathcal{X}_{22}$.

For any $\mathcal{X}_{12} \in \mathcal{S}_{12}$ and $\mathcal{X}_{21} \in \mathcal{S}_{21}$, we obtain

$$\mathcal{D}_{22}\mathcal{X}_{21}\mathcal{X}_{12} = \mathcal{X}_{21}\mathcal{X}_{12}\mathcal{D}_{22} = \mathcal{X}_{21}\mathcal{D}_{11}\mathcal{X}_{12}.$$

It follows that $\mathcal{D}_{22}\mathcal{X}_{21} = \mathcal{X}_{21}\mathcal{D}_{11}$.

For any $\mathcal{X} \in \mathcal{S}$, we have

$$\begin{aligned}
\left(\mathcal{D}_{11} + \mathcal{D}_{22}\right)\mathcal{X} &= \left(\mathcal{D}_{11} + \mathcal{D}_{22}\right)\left(\mathcal{X}_{11} + \mathcal{X}_{12} + \mathcal{X}_{21} + \mathcal{X}_{22}\right) \\
&= \mathcal{D}_{11}\mathcal{X}_{11} + \mathcal{D}_{11}\mathcal{X}_{12} + \mathcal{D}_{22}\mathcal{X}_{21} + \mathcal{D}_{22}\mathcal{X}_{22} \\
&= \mathcal{X}_{11}\mathcal{D}_{11} + \mathcal{X}_{12}\mathcal{D}_{22} + \mathcal{X}_{21}\mathcal{D}_{11} + \mathcal{X}_{22}\mathcal{D}_{22} \\
&= \left(\mathcal{X}_{11} + \mathcal{X}_{12} + \mathcal{X}_{21} + \mathcal{X}_{22}\right)\left(\mathcal{D}_{11} + \mathcal{D}_{22}\right) \\
&= \mathcal{X}\left(\mathcal{D}_{11} + \mathcal{D}_{22}\right),
\end{aligned}$$

which implies that $\mathcal{D}_{11} + \mathcal{D}_{22} \in \mathcal{Z}(\mathcal{S})$.

Similarly, we can prove that if $\mathcal{E}_{21}\mathcal{D}_{11} = \mathcal{D}_{22}\mathcal{E}_{21}$ for all $\mathcal{E}_{21} \in \mathcal{S}_{21}$, then $\mathcal{D}_{11} + \mathcal{D}_{22} \in \mathcal{Z}(\mathcal{S})$. □

3. Main result

Theorem 3.1. *On a Hilbert space $\mathcal{H}$ that is infinite-dimensional over a field of complex numbers C with an identity operator $\mathcal{I}$, let $\mathcal{S}$ be a standard operator algebra. If $\mathcal{S}$ is closed with respect to adjoint operation and map $\Theta : \mathcal{S} \to \mathcal{S}$ is defined as*

$$\Theta\left(P_n\left(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \ldots, \mathcal{D}_n\right)\right) = \sum_{i=1}^{n} P_n\left(\mathcal{D}_1, \ldots, \mathcal{D}_{i-1}, \Theta\left(\mathcal{D}_i\right), \mathcal{D}_{i+1}, \ldots, \mathcal{D}_n\right),$$

for any $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \ldots, \mathcal{D}_n \in \mathcal{S}$ with $\mathcal{D}_1\mathcal{D}_2\mathcal{D}_3 \ldots \mathcal{D}_n = 0$, then $\Theta(x) = d(x) + \tau(x)$ for every $x \in \mathcal{S}$, where the map $d : \mathcal{S} \to \mathcal{S}$ is an additive derivation and $\tau : \mathcal{S} \to \mathcal{Z}(\mathcal{S})$ is a map vanishing on each $(n-1)^{th}$ commutator $P_n\left(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \ldots, \mathcal{D}_n\right)$ with $\mathcal{D}_1\mathcal{D}_2\mathcal{D}_3 \ldots \mathcal{D}_n = 0$.

Lemma 3.1. *We have that* $\Theta(0) = 0$.

Proof. $\Theta(0) = \Theta(P_n(0, 0, \cdots, 0)) = P_n(\Theta(0), 0, \cdots, 0) + \cdots + P_n(0, 0, \cdots, \Theta(0)) = 0.$ □

Lemma 3.2. *For $\mathcal{D}_{ij} \in \mathcal{S}_{ij}(1 \le i \ne j \le 2)$, we have*

(1) $\Theta\left(\mathcal{D}_{ii} + \mathcal{D}_{ij}\right) - \Theta\left(\mathcal{D}_{ii}\right) - \Theta\left(\mathcal{D}_{ij}\right) \in \mathcal{Z}(\mathcal{S})$.
(2) $\Theta\left(\mathcal{D}_{ii} + \mathcal{D}_{ji}\right) - \Theta\left(\mathcal{D}_{ii}\right) - \Theta\left(\mathcal{D}_{ji}\right) \in \mathcal{Z}(\mathcal{S})$.

Proof. (1) Firstly, we show the result holds for $i = 1$ and $j = 2$. For that, we consider $T = \Theta\left(\mathcal{D}_{11} + \mathcal{D}_{12}\right) - \Theta\left(\mathcal{D}_{11}\right) - \Theta\left(\mathcal{D}_{12}\right)$. Since

$$\mathbb{P}_2\left(\mathcal{D}_{11} + \mathcal{D}_{12}\right)\mathbb{P}_1\mathbb{P}_2 \cdots \mathbb{P}_2 = \mathbb{P}_2\mathcal{D}_{11}\mathbb{P}_1\mathbb{P}_2 \cdots \mathbb{P}_2 = \mathbb{P}_2\mathcal{D}_{12}\mathbb{P}_1\mathbb{P}_2 \cdots \mathbb{P}_2 = 0,$$

we obtain

$$\begin{aligned}
\Theta\left(\mathcal{D}_{12}\right) &= \Theta\left(P_n\left(\mathbb{P}_2, \mathcal{D}_{11} + \mathcal{D}_{12}, \mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_2\right)\right) \\
&= P_n\left(\Theta\left(\mathbb{P}_2\right), \mathcal{D}_{11} + \mathcal{D}_{12}, \mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_2\right) + P_n\left(\mathbb{P}_2, \Theta\left(\mathcal{D}_{11} + \mathcal{D}_{12}\right), \mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_2\right) \\
&\quad + P_n\left(\mathbb{P}_2, \mathcal{D}_{11} + \mathcal{D}_{12}, \Theta\left(\mathbb{P}_1\right), \mathbb{P}_2, \ldots, \mathbb{P}_2\right) + \cdots + P_n(\mathbb{P}_2, \mathcal{D}_{11} + \mathcal{D}_{12}, \mathbb{P}_1, \ldots, \Theta(\mathbb{P}_2)).
\end{aligned}$$

Alternatively, making use of Lemma 3.1, we have

$$\begin{aligned}
\Theta\left(\mathcal{D}_{12}\right) &= \Theta\left(P_n\left(\mathbb{P}_2, \mathcal{D}_{11} + \mathcal{D}_{12}, \mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_2\right)\right) \\
&= \Theta\left(P_n\left(\mathbb{P}_2, \mathcal{D}_{11}, \mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_2\right)\right) + \Theta\left(P_n\left(\mathbb{P}_2, \mathcal{D}_{12}, \mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_2\right)\right) \\
&= P_n\left(\Theta\left(\mathbb{P}_2\right), \mathcal{D}_{11} + \mathcal{D}_{12}, \mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_2\right) + P_n\left(\mathbb{P}_2, \Theta\left(\mathcal{D}_{11}\right) + \Theta\left(\mathcal{D}_{12}\right), \mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_2\right) \\
&\quad + P_n\left(\mathbb{P}_2, \mathcal{D}_{11} + \mathcal{D}_{12}, \Theta\left(\mathbb{P}_1\right), \mathbb{P}_2, \ldots, \mathbb{P}_2\right) + \cdots + P_n(\mathbb{P}_2, \mathcal{D}_{11} + \mathcal{D}_{12}, \mathbb{P}_1, \mathbb{P}_2, \ldots, \Theta(\mathbb{P}_2)).
\end{aligned}$$

From the last two expressions, we find $P_n\left(\mathbb{P}_2, T, \mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_2\right) = 0$. This means that $\mathbb{P}_2 T \mathbb{P}_1 = \mathbb{P}_1 T \mathbb{P}_2 = 0$.

For any $\mathcal{X}_{12} \in \mathcal{S}_{12}$, we have

$$\mathcal{X}_{12}\left(\mathcal{D}_{11} + \mathcal{D}_{12}\right)\mathbb{P}_1\mathbb{P}_2 \cdots \mathbb{P}_2 = \mathcal{X}_{12}\mathcal{D}_{11}\mathbb{P}_1\mathbb{P}_2 \cdots \mathbb{P}_2 = \mathcal{X}_{12}\mathcal{D}_{12}\mathbb{P}_1\mathbb{P}_2 \cdots \mathbb{P}_2 = 0.$$

That means

$$\Theta\left(\mathcal{D}_{11}\mathcal{X}_{12}\right) = \Theta\left(P_n\left(\mathcal{X}_{12}, \mathcal{D}_{11} + \mathcal{D}_{12}, \mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_2\right)\right)$$

$$
\begin{aligned}
&= P_n(\Theta(\mathcal{X}_{12}), \mathcal{D}_{11}+\mathcal{D}_{12}, \mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_2) + P_n(\mathcal{X}_{12}, \Theta(\mathcal{D}_{11}+\mathcal{D}_{12}), \mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_2) \\
&\quad + P_n(\mathcal{X}_{12}, \mathcal{D}_{11}+\mathcal{D}_{12}, \Theta(\mathbb{P}_1), \mathbb{P}_2, \ldots, \mathbb{P}_2) + \cdots + P_n(\mathcal{X}_{12}, \mathcal{D}_{11}+\mathcal{D}_{12}, \mathbb{P}_1, \mathbb{P}_2, \ldots, \Theta(\mathbb{P}_2)).
\end{aligned}
$$

On the other hand, by using Lemma 3.1, we have

$$
\begin{aligned}
\Theta(\mathcal{D}_{11}\mathcal{X}_{12}) &= \Theta(P_n(\mathcal{X}_{12}, \mathcal{D}_{11}, \mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_2)) + \Theta(P_n(\mathcal{X}_{12}, \mathcal{D}_{12}, \mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_2)) \\
&= P_n(\Theta(\mathcal{X}_{12}), \mathcal{D}_{11}+\mathcal{D}_{12}, \mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_2) \\
&\quad + P_n(\mathcal{X}_{12}, \Theta(\mathcal{D}_{11}) + \Theta(\mathcal{D}_{12}), \mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_2) \\
&\quad + P_n(\mathcal{X}_{12}, \mathcal{D}_{11}+\mathcal{D}_{12}, \Theta(\mathbb{P}_1), \mathbb{P}_2, \ldots, \mathbb{P}_2) \\
&\quad + \cdots + P_n(\mathcal{X}_{12}, \mathcal{D}_{11}+\mathcal{D}_{12}, \mathbb{P}_1, \mathbb{P}_2, \ldots, \Theta(\mathbb{P}_2)).
\end{aligned}
$$

Comparing the above two equations, we obtain

$$P_n(\mathcal{X}_{12}, T, \mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_2) = 0.$$

On solving the above equation, we obtain $\mathbb{P}_1 T \mathbb{P}_1 \mathcal{X}_{12} = \mathcal{X}_{12}\mathbb{P}_2 T \mathbb{P}_2$ for all $\mathcal{X}_{12} \in \mathcal{S}_{12}$. Therefore, by using Lemma 2.1, we get $\mathbb{P}_1 T \mathbb{P}_1 + \mathbb{P}_2 T \mathbb{P}_2 \in \mathcal{Z}(\mathcal{S})$. Hence, $T = T_{12} + T_{21} \in \mathcal{Z}(\mathcal{S})$. That is

$$T = \Theta(\mathcal{D}_{11}+\mathcal{D}_{12}) - \Theta(\mathcal{D}_{11}) - \Theta(\mathcal{D}_{12}) \in \mathcal{Z}(\mathcal{S}).$$

In a similar way, it can be shown for $i = 2$ and $j = 1$.
(2) By using the same technique as in (1), one can easily show that

$$\Theta\left(\mathcal{D}_{ii}+\mathcal{D}_{ji}\right) - \Theta(\mathcal{D}_{ii}) - \Theta\left(\mathcal{D}_{ji}\right) \in \mathcal{Z}(\mathcal{S}).$$

□

Lemma 3.3. *For any $\mathcal{D}_{ij}, \mathcal{E}_{ij} \in \mathcal{S}_{ij} (1 \le i \ne j \le 2)$, we have*

$$\Theta\left(\mathcal{D}_{ij}+\mathcal{E}_{ij}\right) = \Theta\left(\mathcal{D}_{ij}\right) + \Theta\left(\mathcal{E}_{ij}\right).$$

Proof. Firstly, we show the above result is true for $i = 1$ and $j = 2$.
Since $(\mathcal{D}_{12}+\mathbb{P}_1)(\mathbb{P}_2+\mathcal{E}_{12})\mathbb{P}_1 \cdots \mathbb{P}_1 = 0$ for any $\mathcal{D}_{12}, \mathcal{E}_{12} \in \mathcal{S}_{12}$, we have

$$
\begin{aligned}
\Theta(\mathcal{D}_{12}+\mathcal{E}_{12}) &= \Theta(P_n(\mathcal{D}_{12}+\mathbb{P}_1, \mathbb{P}_2+\mathcal{E}_{12}, -\mathbb{P}_1, \ldots, -\mathbb{P}_1)) \\
&= P_n(\Theta(\mathcal{D}_{12}) + \Theta(\mathbb{P}_1), \mathbb{P}_2+\mathcal{E}_{12}, \mathbb{P}_1, \ldots, \mathbb{P}_1) \\
&\quad + P_n(\mathcal{D}_{12}+\mathbb{P}_1, \Theta(\mathbb{P}_2) + \Theta(\mathcal{E}_{12}), \mathbb{P}_1, \ldots, \mathbb{P}_1) \\
&\quad + P_n(\mathcal{D}_{12}+\mathbb{P}_1, \mathbb{P}_2+\mathcal{E}_{12}, \Theta(\mathbb{P}_1), \ldots, \mathbb{P}_1) \\
&\quad + P_n(\mathcal{D}_{12}+\mathbb{P}_1, \mathbb{P}_2+\mathcal{E}_{12}, \mathbb{P}_1, \ldots, \Theta(\mathbb{P}_1)) \\
&= \Theta(P_n(\mathcal{D}_{12}, \mathbb{P}_2, -\mathbb{P}_1, \ldots, -\mathbb{P}_1)) + \Theta(P_n(-\mathbb{P}_1, \mathbb{P}_2, -\mathbb{P}_1, \ldots, -\mathbb{P}_1)) \\
&\quad + \Theta(P_n(\mathcal{D}_{12}, \mathcal{E}_{12}, -\mathbb{P}_1, \ldots, -\mathbb{P}_1)) + \Theta(P_n(\mathbb{P}_1, \mathcal{E}_{12}, -\mathbb{P}_1, \ldots, -\mathbb{P}_1)) \\
&= \Theta(\mathcal{D}_{12}) + \Theta(\mathcal{E}_{12}).
\end{aligned}
$$

Secondly, it is easy to follow the same pattern to see that the result is also true for $i = 2$ and $j = 1$, that is

$$\Theta(\mathcal{D}_{21}+\mathcal{E}_{21}) = \Theta(\mathcal{D}_{21}) + \Theta(\mathcal{E}_{21}).$$

□

Lemma 3.4. $\mathbb{P}_1\Theta(\mathbb{P}_i)\mathbb{P}_1 + \mathbb{P}_2\Theta(\mathbb{P}_i)\mathbb{P}_2 \in \mathcal{Z}(\mathcal{S}), i = 1, 2.$

Proof. Let $\mathcal{D}_{12} \in \mathcal{S}_{12}$. Since $\mathcal{D}_{12}\mathbb{P}_1 \cdots \mathbb{P}_1 = 0$, by Lemma 3.3, we have

$$\begin{aligned}(-1)^{n+1}\Theta(\mathcal{D}_{12}) &= \Theta\left((-1)^{n+1}\mathcal{D}_{12}\right)\\ &= \Theta(P_n(\mathcal{D}_{12}, \mathbb{P}_1, \ldots, \mathbb{P}_1))\\ &= P_n(\Theta(\mathcal{D}_{12}), \mathbb{P}_1, \ldots, \mathbb{P}_1) + \sum_{k=2}^{n} P_n(\mathcal{D}_{12}, \mathbb{P}_1, \ldots, \overbrace{\Theta(\mathbb{P}_1)}^{k-th}, \ldots, \mathbb{P}_1)\\ &= (-1)^{n+1}\mathbb{P}_1\Theta(\mathcal{D}_{12})\mathbb{P}_2 + \mathbb{P}_2\Theta(\mathcal{D}_{12})\mathbb{P}_1 + (-1)^{n+1}\mathbb{P}_1\Theta(\mathbb{P}_1)\mathcal{D}_{12} - (-1)^{n+1}\mathcal{D}_{12}\Theta(\mathbb{P}_1)\mathbb{P}_2.\end{aligned}$$

By pre-multiplying by $\mathbb{P}_1$ and post-multiplying by $\mathbb{P}_2$ on both sides, we obtain

$$\mathbb{P}_1\Theta(\mathbb{P}_1)\mathbb{P}_1\mathcal{D}_{12} = \mathcal{D}_{12}\mathbb{P}_2\Theta(\mathbb{P}_1)\mathbb{P}_2.$$

By Lemma 2.1, we have $\mathbb{P}_1\Theta(\mathbb{P}_1)\mathbb{P}_1 + \mathbb{P}_2\Theta(\mathbb{P}_1)\mathbb{P}_2 \in \mathcal{Z}(\mathcal{S})$. Similarly, $\mathbb{P}_1\Theta(\mathbb{P}_2)\mathbb{P}_1 + \mathbb{P}_2\Theta(\mathbb{P}_2)\mathbb{P}_2 \in \mathcal{Z}(\mathcal{S})$. □

Now, $M = \mathbb{P}_1\Theta(\mathbb{P}_1)\mathbb{P}_2 - \mathbb{P}_2\Theta(\mathbb{P}_1)\mathbb{P}_1$. Define a map $\Omega : \mathcal{S} \to \mathcal{S}$ by $\Omega(\mathcal{D}) = \Theta(\mathcal{D}) - [\mathcal{D}, M]$ for all $\mathcal{D} \in \mathcal{S}$.

Lemma 3.5. Ω *has the following properties:*

(1) Ω *also satisfies*

$$\Omega(P_n(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \ldots, \mathcal{D}_n)) = \sum_{i=1}^{n} P_n(\mathcal{D}_1, \ldots, \mathcal{D}_{i-1}, \Omega(\mathcal{D}_i), \mathcal{D}_{i+1}, \ldots, \mathcal{D}_n), \tag{3.1}$$

for any $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \ldots, \mathcal{D}_n \in \mathcal{S}$ *with* $\mathcal{D}_1\mathcal{D}_2\ldots\mathcal{D}_n = 0$.

(2) $\Omega(\mathbb{P}_1) \in \mathcal{Z}(\mathcal{S})$.

(3) $\Omega(\mathbb{P}_2) \in \mathcal{Z}(\mathcal{S})$.

Proof. (1) Since Θ satisfies the condition (3.1), it is easy to see that Ω also satisfies the condition (3.1) with $\mathcal{D}_1\mathcal{D}_2\ldots\mathcal{D}_n = 0$.
(2) By using Peirce-decomposition:

$$\Theta(\mathbb{P}_1) = \mathbb{P}_1\Theta(\mathbb{P}_1)\mathbb{P}_1 + \mathbb{P}_1\Theta(\mathbb{P}_1)\mathbb{P}_2 + \mathbb{P}_2\Theta(\mathbb{P}_1)\mathbb{P}_1 + \mathbb{P}_2\Theta(\mathbb{P}_1)\mathbb{P}_2.$$

Now,

$$\begin{aligned}\Omega(\mathbb{P}_1) &= \Theta(\mathbb{P}_1) - \mathbb{P}_1\Theta(\mathbb{P}_1)\mathbb{P}_2 - \mathbb{P}_2\Theta(\mathbb{P}_1)\mathbb{P}_1\\ &= \mathbb{P}_1\Theta(\mathbb{P}_1)\mathbb{P}_1 + \mathbb{P}_1\Theta(\mathbb{P}_1)\mathbb{P}_2 + \mathbb{P}_2\Theta(\mathbb{P}_1)\mathbb{P}_1 + \mathbb{P}_2\Theta(\mathbb{P}_1)\mathbb{P}_2 - \mathbb{P}_1\Theta(\mathbb{P}_1)\mathbb{P}_2 - \mathbb{P}_2\Theta(\mathbb{P}_1)\mathbb{P}_1\\ &= \mathbb{P}_1\Theta(\mathbb{P}_1)\mathbb{P}_1 + \mathbb{P}_2\Theta(\mathbb{P}_1)\mathbb{P}_2.\end{aligned}$$

Therefore, by using Lemma 3.4, we obtain $\Omega(\mathbb{P}_1) \in \mathcal{Z}(\mathcal{S})$.
(3) Since $\mathbb{P}_2\mathbb{P}_1\mathbb{P}_1\ldots\mathbb{P}_n = 0$ and $\Omega(\mathbb{P}_1) \in \mathcal{Z}(\mathcal{S})$, then

$$0 = \Omega(P_n(\mathbb{P}_2, \mathbb{P}_1, \ldots, \mathbb{P}_1)) = P_n(\Omega(\mathbb{P}_2), \mathbb{P}_1, \ldots, \mathbb{P}_1) = (-1)^{n+1}\mathbb{P}_1\Omega(\mathbb{P}_2)\mathbb{P}_2 + \mathbb{P}_2\Omega(\mathbb{P}_2)\mathbb{P}_1.$$

By using pre-and-post multiplication by $\mathbb{P}_1$ in the above, we obtain $\mathbb{P}_1\Omega(\mathbb{P}_2)\mathbb{P}_2 = 0$. Similarly, $\mathbb{P}_2\Omega(\mathbb{P}_2)\mathbb{P}_1 = 0$. Now, from the definition of Ω, we have

$$\Omega(\mathbb{P}_2) = \Theta(\mathbb{P}_1) + \mathbb{P}_2\Theta(\mathbb{P}_1)\mathbb{P}_1 + \mathbb{P}_1\Theta(\mathbb{P}_1)\mathbb{P}_2.$$

By using pre-and-post multiplication by $\mathbb{P}_1$ in the above, we obtain $\mathbb{P}_1\Omega(\mathbb{P}_2)\mathbb{P}_1 = \mathbb{P}_1\Theta(\mathbb{P}_2)\mathbb{P}_1$. Similarly, by using pre-and-post multiplication by $\mathbb{P}_2$ in the above equation, we have $\mathbb{P}_2\Omega(\mathbb{P}_2)\mathbb{P}_2 = \mathbb{P}_2\Theta(\mathbb{P}_2)\mathbb{P}_2$. By Peirce-decomposition and using Lemma 3.4, we obtain

$$\Omega(\mathbb{P}_2) = \mathbb{P}_1\Omega(\mathbb{P}_2)\mathbb{P}_1 + \mathbb{P}_2\Omega(\mathbb{P}_2)\mathbb{P}_2 = \mathbb{P}_1\Theta(\mathbb{P}_2)\mathbb{P}_1 + \mathbb{P}_2\Theta(\mathbb{P}_2)\mathbb{P}_2 \in \mathcal{Z}(\mathcal{S}).$$

This completes the proof. □

Lemma 3.6. $\Omega(\mathcal{S}_{ij}) \subseteq \mathcal{S}_{ij}$ $(1 \leq i \neq j \leq 2)$.

Proof. It is sufficient to prove for $i = 1$ and $j = 2$ first. For other values, the proof will follow a similar fashion. We have $\mathcal{D}_{12}\mathbb{P}_1\mathbb{P}_1\mathbb{P}_2 \ldots \mathbb{P}_2 = 0$. Also, by using Lemma 3.5, we have

$$\begin{aligned} \Omega(\mathcal{D}_{12}) &= \Omega(P_n(\mathcal{D}_{12}, \mathbb{P}_1\mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_2)) = P_n(\Omega(\mathcal{D}_{12}), \mathbb{P}_1, \mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_2) \\ &= \mathbb{P}_1\Omega(\mathcal{D}_{12})\mathbb{P}_2 + (-1)^{n+1}\mathbb{P}_2\Omega(\mathcal{D}_{12})\mathbb{P}_1. \end{aligned}$$

By pre-and-post multiplication by $\mathbb{P}_1$ in the above equation, we obtain $\mathbb{P}_1\Omega(\mathcal{D}_{12})\mathbb{P}_1 = 0$. Similarly, $\mathbb{P}_2\Omega(\mathcal{D}_{12})\mathbb{P}_2 = 0$.

Now, if n is even, then $\Omega(\mathcal{D}_{12}) = \mathbb{P}_1\Omega(\mathcal{D}_{12})\mathbb{P}_2 - \mathbb{P}_2\Omega(\mathcal{D}_{12})\mathbb{P}_1$. By multiplying $\mathbb{P}_2$ from left and $\mathbb{P}_1$ from right, we obtain $\mathbb{P}_2\Omega(\mathcal{D}_{12})\mathbb{P}_1 = 0$. Also, by multiplying $\mathbb{P}_1$ from left and right, we obtain $\mathbb{P}_1\Omega(\mathcal{D}_{12})\mathbb{P}_1 = 0$. Similarly, $\mathbb{P}_2\Omega(\mathcal{D}_{12})\mathbb{P}_2 = 0$. Hence, $\Omega(\mathcal{D}_{12}) \in \mathcal{S}_{12}$. Now, we assume that n is odd. Since $\mathcal{D}_{12}\mathcal{E}_{12}\mathcal{F}_{12}\mathbb{P}_2 \ldots \mathbb{P}_2 = 0$ for any $\mathcal{D}_{12}, \mathcal{E}_{12}, \mathcal{F}_{12} \in \mathcal{S}_{12}$, we have

$$\begin{aligned} 0 &= \Omega(P_n(\mathcal{D}_{12}, \mathcal{E}_{12}, \mathcal{F}_{21}, \mathbb{P}_2, \ldots, \mathbb{P}_2)) \\ &= P_n(\Omega(\mathcal{D}_{12}), \mathcal{E}_{12}, \mathcal{F}_{21}, \mathbb{P}_2, \ldots, \mathbb{P}_2) + P_n(\mathcal{D}_{12}, \Omega(\mathcal{E}_{12}), \mathcal{F}_{21}, \mathbb{P}_2, \ldots, \mathbb{P}_2) \\ &= P_{n-1}([\Omega(\mathcal{D}_{12}), \mathcal{E}_{12}], \mathcal{F}_{21}, \mathbb{P}_2, \ldots, \mathbb{P}_2) + P_{n-1}([\mathcal{D}_{12}, \Omega(\mathcal{E}_{12})], \mathcal{F}_{21}, \mathbb{P}_2, \ldots, \mathbb{P}_2) \\ &= P_{n-1}([\Omega(\mathcal{D}_{12}), \mathcal{E}_{12}], \mathcal{F}_{21}, \mathbb{P}_2, \ldots, \mathbb{P}_2 + [\mathcal{D}_{12}, \Omega(\mathcal{E}_{12})], \mathcal{F}_{21}, \mathbb{P}_2, \ldots, \mathbb{P}_2) \\ &= P_{n-1}(l, \mathcal{F}_{21}, \mathbb{P}_2, \ldots, \mathbb{P}_2) \\ &= P_{n-2}(l, \mathcal{F}_{21}, \mathbb{P}_2, \ldots, \mathbb{P}_2) \\ &= [l, \mathcal{F}_{21}], \end{aligned}$$

and

$$\begin{aligned} 0 &= \Omega(P_n(\mathcal{D}_{12}, \mathcal{E}_{12}, \mathcal{F}_{12}, \mathbb{P}_1, \ldots, \mathbb{P}_1)) \\ &= P_n(\Omega(\mathcal{D}_{12}), \mathcal{E}_{12}, \mathcal{F}_{12}, \mathbb{P}_1, \ldots, \mathbb{P}_1) + P_n(\mathcal{D}_{12}, \Omega(\mathcal{E}_{12}), \mathcal{F}_{12}, \mathbb{P}_1, \ldots, \mathbb{P}_1) \\ &= P_{n-1}([\Omega(\mathcal{D}_{12}), \mathcal{E}_{12}], \mathcal{F}_{12}, \mathbb{P}_1, \ldots, \mathbb{P}_1) + P_{n-1}([\mathcal{D}_{12}, \Omega(\mathcal{E}_{12})], \mathcal{F}_{12}, \mathbb{P}_1, \ldots, \mathbb{P}_1) \\ &= P_{n-1}([\Omega(\mathcal{D}_{12}), \mathcal{E}_{12}], \mathcal{F}_{12}, \mathbb{P}_1, \ldots, \mathbb{P}_1 + [\mathcal{D}_{12}, \Omega(\mathcal{E}_{12})], \mathcal{F}_{12}, \mathbb{P}_1, \ldots, \mathbb{P}_1) \\ &= P_{n-1}(l, \mathcal{F}_{12}, \mathbb{P}_1, \ldots, \mathbb{P}_1) \\ &= P_{n-2}(l, \mathcal{F}_{12}, \mathbb{P}_1, \ldots, \mathbb{P}_1) = [l, \mathcal{F}_{12}], \end{aligned}$$

where $l = [\Omega(\mathcal{D}_{12}), \mathcal{E}_{12}] + [\mathcal{D}_{12}\Omega(\mathcal{E}_{12})] \in \mathcal{S}_{11} + \mathcal{S}_{22}$. Now, $[l, \mathcal{F}_{21}] = 0$. That means $l\mathcal{F}_{21} - \mathcal{F}_{21}l = 0$. Thus, $\mathbb{P}_2 l\mathbb{P}_2\mathcal{F}_{21} = \mathcal{F}_{21}\mathbb{P}_1 l\mathbb{P}_1 = 0$ for any $\mathcal{F}_{21} \in \mathcal{S}_{12}$. Hence, by Lemma 2.1, $\mathbb{P}_1 l\mathbb{P}_1 + \mathbb{P}_2 l\mathbb{P}_2 \in \mathcal{Z}(\mathcal{S})$. Now,

$$\begin{aligned} l &= [\Omega(\mathcal{D}_{12}), \mathcal{E}_{12}] + [\mathcal{D}_{12}, \Omega(\mathcal{E}_{12})] \\ &= [\Omega(\mathcal{D}_{12}), \mathcal{E}_{12}] - P_n(\mathcal{D}_{12}, \mathbb{P}_1, \ldots, \mathbb{P}_1, \Omega(\mathcal{E}_{12})) \\ &= [\Omega(\mathcal{D}_{12}), \mathcal{E}_{12}] - \Omega(P_n(\mathcal{D}_{12}, \mathbb{P}_1, \ldots, \mathbb{P}_1, \mathcal{E}_{12})) + P_n(\Omega(\mathcal{D}_{12}), \mathbb{P}_1, \ldots, \mathbb{P}_1, \mathcal{E}_{12}) \\ &= [\Omega(\mathcal{D}_{12}), \mathcal{E}_{12}] + [\mathbb{P}_2\Omega(\mathcal{D}_{12})\mathbb{P}_1, \mathcal{E}_{12}]. \end{aligned}$$

Thus, $[2\mathbb{P}_2\Omega(\mathcal{D}_{12})\mathbb{P}_1, \mathcal{E}_{12}] = l \in \mathcal{Z}(\mathcal{S})$. That is, $[\mathbb{P}_2\Omega(\mathcal{D}_{12})\mathbb{P}_1, \mathcal{E}_{12}] = l \in \mathcal{Z}(\mathcal{S})$. Hence, $\mathbb{P}_2\Omega(\mathcal{D}_{12})\mathbb{P}_1\mathcal{E}_{12} = 0$ for any $\mathcal{E}_{12} \in \mathcal{S}_{12}$. By using primeness of $\mathcal{S}$, we get $\mathbb{P}_2\Omega(\mathcal{D}_{12})\mathbb{P}_1 = 0$. Hence, $\Omega(\mathcal{S}_{12}) \subseteq \mathcal{S}_{12}$. □

Lemma 3.7. *For any $\mathcal{D}_{ii} \in \mathcal{S}_{ii}$ for $i = 1, 2$, there is always a map $\eta_i : \mathcal{S}_{ii} \to \mathcal{Z}(\mathcal{S})$ so that $\Omega(\mathcal{D}_{ii}) - f_i(\mathcal{D}_{ii}) \in \mathcal{S}_{ii}$.*

Proof. Since $\mathcal{D}_{11}\mathbb{P}_2\mathbb{P}_2 \ldots \mathbb{P}_2 = 0$ and using Lemma 3.4, we obtain

$$0 = \Omega(P_n(\mathcal{D}_{11}, \mathbb{P}_2, \ldots, \mathbb{P}_2)) = P_n(\Omega(\mathcal{D}_{11}), \mathbb{P}_2, \ldots, \mathbb{P}_2) = \mathbb{P}_1\Omega(\mathcal{D}_{11})\mathbb{P}_2 + (-1)^{n-1}\mathbb{P}_2\Omega(\mathcal{D}_{11})\mathbb{P}_1.$$

Multiplying $\mathbb{P}_1$ from left, we obtain $\mathbb{P}_1\Omega(\mathcal{D}_{11})\mathbb{P}_2 = 0$. Similarly, multiplying $\mathbb{P}_2$ from left, $\mathbb{P}_2\Omega(\mathcal{D}_{11})\mathbb{P}_1 = 0$. Hence, $\Omega(\mathcal{D}_{11}) \in \mathcal{S}_{11} + \mathcal{S}_{22}$. Similarly, $\Omega(\mathcal{D}_{22}) \in \mathcal{S}_{11} + \mathcal{S}_{22}$.

Since $\mathcal{D}_{11}\mathcal{E}_{22}\mathcal{X}_{12}\mathbb{P}_2 \ldots \mathbb{P}_2 = 0$ and for any $\mathcal{E}_{22} \in \mathcal{S}_{22}, \mathcal{X}_{12} \in \mathcal{S}_{12}$, we have

$$\begin{aligned} 0 &= \Omega(P_n(\mathcal{D}_{11}, \mathcal{E}_{22}, \mathcal{X}_{12}, \mathbb{P}_1, \ldots, \mathbb{P}_1)) \\ &= P_n(\Omega(\mathcal{D}_{11}), \mathcal{E}_{22}, \mathcal{X}_{12}, \mathbb{P}_1, \ldots, \mathbb{P}_1) + P_n(\mathcal{D}_{11}, \Omega(\mathcal{E}_{22}), \mathcal{X}_{12}, \mathbb{P}_1, \ldots, \mathbb{P}_1) \\ &= P_{n-1}([\Omega(\mathcal{D}_{11}), \mathcal{E}_{22}], \mathcal{X}_{12}, \mathbb{P}_1, \ldots, \mathbb{P}_1) + P_{n-1}([\mathcal{D}_{11}, \Omega(\mathcal{E}_{22})], \mathcal{X}_{12}, \mathbb{P}_1, \ldots, \mathbb{P}_1) \\ &= P_{n-1}([\Omega(\mathcal{D}_{11}), \mathcal{E}_{22}], \mathcal{X}_{12}, \mathbb{P}_1, \ldots, \mathbb{P}_1 + [\mathcal{D}_{11}, \Omega(\mathcal{E}_{22})], \mathcal{X}_{12}, \mathbb{P}_1, \ldots, \mathbb{P}_1) \\ &= P_{n-1}(m, \mathcal{X}_{12}, \mathbb{P}_1, \ldots, \mathbb{P}_1) \\ &= P_{n-2}(m, \mathcal{X}_{12}, \mathbb{P}_1, \ldots, \mathbb{P}_1) \\ &= [m, \mathcal{X}_{12}], \end{aligned}$$

and

$$\begin{aligned} 0 &= \Omega(P_n(\mathcal{D}_{11}, \mathcal{E}_{22}, \mathcal{X}_{21}, \mathbb{P}_2, \ldots, \mathbb{P}_2)) \\ &= P_n(\Omega(\mathcal{D}_{11}), \mathcal{E}_{22}, \mathcal{X}_{21}, \mathbb{P}_2, \ldots, \mathbb{P}_2) + P_n(\mathcal{D}_{11}, \Omega(\mathcal{E}_{22}), \mathcal{X}_{21}, \mathbb{P}_2, \ldots, \mathbb{P}_2) \\ &= P_{n-1}([\Omega(\mathcal{D}_{11}), \mathcal{E}_{22}], \mathcal{X}_{21}, \mathbb{P}_2, \ldots, \mathbb{P}_2) + P_{n-1}([\mathcal{D}_{11}, \Omega(\mathcal{E}_{22})], \mathcal{X}_{21}, \mathbb{P}_2, \ldots, \mathbb{P}_2) \\ &= P_{n-1}([\Omega(\mathcal{D}_{11}), \mathcal{E}_{22}], \mathcal{X}_{21}, \mathbb{P}_2, \ldots, \mathbb{P}_2 + [\mathcal{D}_{11}, \Omega(\mathcal{E}_{22})], \mathcal{X}_{12}, \mathbb{P}_2, \ldots, \mathbb{P}_2) \\ &= P_{n-1}(m, \mathcal{X}_{21}, \mathbb{P}_2, \ldots, \mathbb{P}_2) \\ &= P_{n-2}(m, \mathcal{X}_{21}, \mathbb{P}_2, \ldots, \mathbb{P}_2) \\ &= [m, \mathcal{X}_{21}], \end{aligned}$$

where $m = [\Omega(\mathcal{D}_{11}), \mathcal{E}_{22}] + [\mathcal{D}_{11}, \Omega(\mathcal{E}_{22})] \in \mathcal{S}_{11} + \mathcal{S}_{22}$. Now, $[m, \mathcal{X}_{12}] = [m, \mathcal{X}_{21}] = 0$ and using Lemma 2.1, we have $\mathbb{P}_1 m\mathbb{P}_1 + \mathbb{P}_2 m\mathbb{P}_2 \in \mathcal{Z}(\mathcal{S})$. Therefore, $m = [\Omega(\mathcal{D}_{11}), \mathcal{E}_{22}] + [\mathcal{D}_{11}, \Omega(\mathcal{E}_{22})] \in \mathcal{Z}(\mathcal{S})$.

Multiplying both sides by $\mathbb{P}_2$ yields $[\mathbb{P}_2\Omega(\mathcal{D}_{11})\mathbb{P}_2, \mathcal{E}_{22}] \in \mathcal{Z}(\mathcal{S})\mathbb{P}_2$. By the Kleinecke-Shirokov theorem, we obtain

$$[\mathbb{P}_2\Omega\left(\mathcal{D}_{11}\right)\mathbb{P}_2, \mathcal{E}_{22}] = 0,$$

for any $\mathcal{E}_{22} \in \mathcal{Z}(\mathcal{S}_{22})$. That means $\mathbb{P}_2\Omega(\mathcal{D}_{11})\mathbb{P}_2 \in \mathcal{Z}(\mathcal{S}_{22})$. Therefore, $\mathbb{P}_2\Omega(\mathcal{D}_{11})\mathbb{P}_2 = z$ for some $z \in \mathcal{Z}(\mathcal{S}_{22})$. Multiplying $\mathbb{P}_2$ from both sides, we get $\mathbb{P}_2\Omega(\mathcal{D}_{11})\mathbb{P}_2 = z\mathbb{P}_2$. Now, define a map $\eta_1 : \mathcal{S}_{11} \to \mathcal{Z}(\mathcal{S})$ such that $\eta_1(\mathcal{D}_{11})\mathbb{P}_2 = \mathbb{P}_2\Omega(\mathcal{D}_{11})\mathbb{P}_2$. Hence,

$$\begin{aligned}
\Omega(\mathcal{D}_{11}) - \eta_1(\mathcal{D}_{11}) &= \mathbb{P}_1\Omega(\mathcal{D}_{11})\mathbb{P}_1 + \mathbb{P}_2\Omega(\mathcal{D}_{22})\mathbb{P}_2 - \eta_1(\mathcal{D}_{11}) \\
&= \mathbb{P}_1\Omega(\mathcal{D}_{11})\mathbb{P}_1 + \eta_1(\mathcal{D}_{11})\mathbb{P}_2 - \eta_1(\mathcal{D}_{11}) \\
&= \mathbb{P}_1\Omega(\mathcal{D}_{11})\mathbb{P}_1 - \eta_1(\mathcal{D}_{11})\mathbb{P}_1.
\end{aligned}$$

Therefore, $\mathbb{P}_1\Omega(\mathcal{D}_{11})\mathbb{P}_1 - \eta_1(\mathcal{D}_{11})\mathbb{P}_1 \in \mathcal{S}_{11}$. Hence, $\Omega(\mathcal{D}_{11}) - \eta_1(\mathcal{D}_{11}) \in \mathcal{S}_{11}$. Similarly, we can show the result for $i = 2$. □

For any $\mathcal{D} = \mathcal{D}_{11} + \mathcal{D}_{12} + \mathcal{D}_{21} + \mathcal{D}_{22} \in \mathcal{S}$, define a mappings $\psi : \mathcal{S} \to \mathcal{S}$ and $\tau : \mathcal{S} \to \mathcal{Z}(\mathcal{S})$ satisfying $\psi(\mathcal{D}) = \sum_{i,j=1}^{2} \phi\left(\mathcal{D}_{ij}\right) - f_1\left(\mathcal{D}_{11}\right) - f_2\left(\mathcal{D}_{22}\right)$ and $\tau(\mathcal{D}) = \Omega(\mathcal{D}) - \psi(\mathcal{D})$ for any $\mathcal{D} \in \mathcal{S}$. It is easy to verify that ψ has the following properties:

Lemma 3.8. *For any $\mathcal{D}_{ij} \in \mathcal{S}_{ij}$, we have*

(1) $\psi\left(\mathcal{D}_{ij}\right) = \Omega\left(\mathcal{D}_{ij}\right) \in \mathcal{S}_{ij}, 1 \leq i \neq j \leq 2.$
(2) $\psi\left(\mathcal{D}_{ii}\right) \in \mathcal{S}_{ii}, i = 1, 2.$
(3) $\psi\left(\mathcal{D}_{11} + \mathcal{D}_{12} + \mathcal{D}_{21} + \mathcal{D}_{22}\right) = \psi\left(\mathcal{D}_{11}\right) + \psi\left(\mathcal{D}_{12}\right) + \psi\left(\mathcal{D}_{21}\right) + \psi\left(\mathcal{D}_{22}\right).$
(4) ψ is an additive map on $\mathcal{S}_{ij}, 1 \leq i \neq j \leq 2$.

Lemma 3.9. *For $\mathcal{D}_{ij} \in \mathcal{S}_{ij}, (1 \leq i \neq j \leq 2)$, we have*

(1) $\psi\left(\mathcal{D}_{ii}\mathcal{D}_{ij}\right) = \psi\left(\mathcal{D}_{ii}\right)\mathcal{D}_{ij} + \mathcal{D}_{ii}\psi\left(\mathcal{D}_{ij}\right).$
(2) $\psi\left(\mathcal{D}_{ij}\mathcal{D}_{jj}\right) = \psi\left(\mathcal{D}_{ij}\right)\mathcal{D}_{jj} + \mathcal{D}_{ij}\psi\left(\mathcal{D}_{jj}\right).$

Proof. (1) Since $\mathcal{D}_{ii}\mathcal{D}_{ij}\mathbb{P}_i \ldots \mathbb{P}_i = 0, (i \neq j)$, we have

$$\begin{aligned}
\psi\left(\mathcal{D}_{ii}\mathcal{D}_{ij}\right) &= \phi\left(\mathcal{D}_{ii}\mathcal{D}_{ij}\right) \\
&= \phi\left(P_n\left(\mathcal{D}_{ij}, \mathcal{D}_{ii}, \mathbb{P}_i, \mathbb{P}_i, \ldots, \mathbb{P}_i\right)\right) \\
&= P_n\left(\phi\left(\mathcal{D}_{ij}\right), \mathcal{D}_{ii}, \mathbb{P}_i, \ldots, \mathbb{P}_i\right) + P_n\left(\mathcal{D}_{ij}, \phi\left(\mathcal{D}_{ii}\right), \mathbb{P}_i, \ldots, \mathbb{P}_i\right) \\
&= P_n\left(\psi\left(\mathcal{D}_{ij}\right), \mathcal{D}_{ii}, \mathbb{P}_i, \ldots, \mathbb{P}_i\right) + P_n\left(\mathcal{D}_{ij}, \psi\left(\mathcal{D}_{ii}\right), \mathbb{P}_i, \ldots, \mathbb{P}_i\right) \\
&= \mathcal{D}_{ii}\psi\left(\mathcal{D}_{ij}\right) + \psi\left(\mathcal{D}_{ii}\right)\mathcal{D}_{ij}.
\end{aligned}$$

(2) By using the same approach as in (1), one can conclude that

$$\psi\left(\mathcal{D}_{ij}\mathcal{D}_{jj}\right) = \psi\left(\mathcal{D}_{ij}\right)\mathcal{D}_{jj} + \mathcal{D}_{ij}\psi\left(\mathcal{D}_{jj}\right).$$

□

Lemma 3.10. *$\psi(\mathcal{D}_{ii}\mathcal{E}_{ii}) = \mathcal{D}_{ii}\psi(\mathcal{E}_{ii}) + \psi(\mathcal{D}_{ii})\mathcal{E}_{ii}$ for any $\mathcal{D}_{ii}, \mathcal{E}_{ii} \in \mathcal{S}_{ii}$.*

Proof. For any $\mathcal{X}_{ij} \in \mathcal{S}_{ij}$ and using Lemma 3.9, we have

$$\begin{aligned}\mathcal{D}_{ii}\mathcal{E}_{ii}\psi\left(\mathcal{X}_{ij}\right) + \psi(\mathcal{D}_{ii}\mathcal{E}_{ii})\mathcal{X}_{ij} &= \psi\left(\mathcal{D}_{ii}\mathcal{E}_{ii}\mathcal{X}_{ij}\right)\\ &= \mathcal{D}_{ii}\psi\left(\mathcal{E}_{ii}\mathcal{X}_{ij}\right) + \psi(\mathcal{D}_{ii})\mathcal{E}_{ii}\mathcal{X}_{ij}\\ &= \mathcal{D}_{ii}\mathcal{E}_{ii}\psi\left(\mathcal{X}_{ij}\right) + \mathcal{D}_{ii}\psi(\mathcal{E}_{ii})\mathcal{X}_{ij} + \psi(\mathcal{D}_{ii})\mathcal{E}_{ii}\mathcal{X}_{ij}.\end{aligned}$$

Therefore, $(\psi(\mathcal{D}_{ii}\mathcal{E}_{ii}) - \mathcal{D}_{ii}\psi(\mathcal{E}_{ii}) - \psi(\mathcal{D}_{ii})\mathcal{E}_{ii})\mathcal{X}_{ij} = 0$. By using primeness of $\mathcal{S}$, we get $\psi(\mathcal{D}_{ii}\mathcal{E}_{ii}) - \mathcal{D}_{ii}\psi(\mathcal{E}_{ii}) - \psi(\mathcal{D}_{ii})\mathcal{E}_{ii}) = 0$. Hence, $\psi(\mathcal{D}_{ii}\mathcal{E}_{ii}) = \mathcal{D}_{ii}\psi(\mathcal{E}_{ii}) + \psi(\mathcal{D}_{ii})\mathcal{E}_{ii}$. □

Lemma 3.11. *ψ is additive.*

Proof. By using Lemma 3.8, we can see thet ψ is additive on $\mathcal{S}_{ij}$. For any $\mathcal{D}_{11}, \mathcal{E}_{11} \in \mathcal{S}_{11}$ and $\mathcal{F}_{12} \in \mathcal{S}_{12}$, we have

$$\psi((\mathcal{D}_{11} + \mathcal{E}_{11})\mathcal{F}_{12}) = \psi(\mathcal{D}_{11}\mathcal{F}_{12}) + \psi(\mathcal{E}_{11}\mathcal{F}_{12}) = \psi(\mathcal{D}_{11})\mathcal{F}_{12} + \mathcal{D}_{11}\psi(\mathcal{F}_{12}) + \psi(\mathcal{E}_{11})\mathcal{D}_{12} + \mathcal{E}_{11}\psi(\mathcal{F}_{12}).$$

Alternatively, we can write

$$\psi((\mathcal{D}_{11} + \mathcal{E}_{11})\mathcal{F}_{12}) = \psi(\mathcal{D}_{11} + \mathcal{E}_{11})\mathcal{F}_{12} + (\mathcal{D}_{11} + \mathcal{E}_{11})\psi(\mathcal{F}_{12}).$$

From the last two expressions, we obtain

$$(\psi(\mathcal{D}_{11} + \mathcal{E}_{11}) - \psi(\mathcal{D}_{11}) - \psi(\mathcal{E}_{11}))\mathcal{F}_{12} = 0.$$

By using primeness of $\mathcal{S}$, we obtain

$$\psi(\mathcal{D}_{11} + \mathcal{E}_{11}) = \psi(\mathcal{D}_{11}) + \psi(\mathcal{E}_{11}).$$

In the similar way

$$\psi(\mathcal{D}_{22} + \mathcal{E}_{22}) = \psi(\mathcal{D}_{22}) + \psi(\mathcal{E}_{22}).$$

Thus, for any $\mathcal{D} = \mathcal{D}_{11} + \mathcal{D}_{12} + \mathcal{D}_{21} + \mathcal{D}_{22}$ and $\mathcal{E} = \mathcal{E}_{11} + \mathcal{E}_{12} + \mathcal{E}_{21} + \mathcal{E}_{22}$, we have

$$\begin{aligned}\psi(\mathcal{D} + \mathcal{E}) &= \psi(\mathcal{D}_{11} + \mathcal{D}_{12} + \mathcal{D}_{21} + \mathcal{D}_{22} + \mathcal{E}_{11} + \mathcal{E}_{12} + \mathcal{E}_{21} + \mathcal{E}_{22})\\ &= \psi(\mathcal{D}_{11} + \mathcal{E}_{11}) + \psi(\mathcal{D}_{12} + \mathcal{E}_{12}) + \psi(\mathcal{D}_{21} + \mathcal{E}_{21}) + \psi(\mathcal{D}_{22} + \mathcal{E}_{22})\\ &= \psi(\mathcal{D}_{11} + \mathcal{D}_{12} + \mathcal{D}_{21} + \mathcal{D}_{22}) + \psi(\mathcal{E}_{11} + \mathcal{E}_{12} + \mathcal{E}_{21} + \mathcal{E}_{22})\\ &= \psi(\mathcal{D}) + \psi(\mathcal{E}).\end{aligned}$$

This completes the proof. □

Lemma 3.12. *For any $\mathcal{D}_{ij} \in \mathcal{S}_{ij}$ with $\mathcal{E}_{ji} \in \mathcal{S}_{ji}, (1 \leq i \neq j \leq 2,)$ the following holds true:*

$$\psi\left(\mathcal{D}_{ij}\mathcal{E}_{ji}\right) = \psi\left(\mathcal{D}_{ij}\right)\mathcal{E}_{ji} + \mathcal{D}_{ij}\psi\left(\mathcal{E}_{ji}\right).$$

Proof. Since $\mathcal{D}_{21}\mathcal{E}_{12}\mathcal{X}_{12}(-\mathbb{P}_1)\ldots(-\mathbb{P}_1) = 0$, we have

$$\begin{aligned}
& \Omega(P_n(\mathcal{D}_{21}, \mathcal{E}_{12}, \mathcal{X}_{12}, (-\mathbb{P}_1), \ldots, (-\mathbb{P}_1))) \\
= \ & \Omega(-\mathcal{D}_{12}\mathcal{E}_{21}\mathcal{X}_{12} - \mathcal{X}_{12}\mathcal{D}_{12}\mathcal{E}_{12}) \\
= \ & \Omega(P_n(\mathcal{D}_{21}, \mathcal{E}_{12}, \mathcal{X}_{12}, (-\mathbb{P}_1), \ldots, (-\mathbb{P}_1)) \\
= \ & \Omega(-\mathcal{E}_{12}\mathcal{D}_{21}\mathcal{X}_{12}) - \Omega(\mathcal{X}_{12}\mathcal{D}_{21}\mathcal{E}_{12}) \\
= \ & -\psi(\mathcal{E}_{12}\mathcal{D}_{21}\mathcal{X}_{12}) - \psi(\mathcal{X}_{12}\mathcal{D}_{21}\mathcal{E}_{12}) \\
= \ & -\psi(\mathcal{E}_{12}\mathcal{D}_{21})\mathcal{X}_{12} - \mathcal{E}_{12}\mathcal{D}_{21}\psi(\mathcal{X}_{12}) - \psi(\mathcal{X}_{12})\mathcal{D}_{21}\mathcal{E}_{12} - \mathcal{X}_{12}\psi(\mathcal{D}_{21}\mathcal{E}_{12}).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
& \Omega(P_n(\mathcal{D}_{21}, \mathcal{E}_{12}, \mathcal{X}_{12}, (-\mathbb{P}_1), \ldots, (-\mathbb{P}_1))) \\
= \ & P_n(\Omega(\mathcal{D}_{21}), \mathcal{E}_{12}, \mathcal{X}_{12}, (-\mathbb{P}_1), \ldots, (-\mathbb{P}_1)) \\
& +P_n(\mathcal{D}_{21}, \Omega(\mathcal{E}_{12}), \mathcal{X}_{12}, (-\mathbb{P}_1), \ldots, (-\mathbb{P}_1)) \\
& +P_n(\mathcal{D}_{21}, \mathcal{E}_{12}, \Omega(\mathcal{X}_{12}), (-\mathbb{P}_1), \ldots, (-\mathbb{P}_1)) \\
= \ & P_n(\psi(\mathcal{D}_{21}), \mathcal{E}_{12}, \mathcal{X}_{12}, (-\mathbb{P}_1), \ldots, (-\mathbb{P}_1)) \\
& +P_n(\mathcal{D}_{21}, \psi(\mathcal{E}_{12}), \mathcal{X}_{12}, (-\mathbb{P}_1), \ldots, (-\mathbb{P}_1)) \\
& +P_n(\mathcal{D}_{21}, \mathcal{E}_{12}, \psi(\mathcal{X}_{12}), (-\mathbb{P}_1), \ldots, (-\mathbb{P}_1)) \\
= \ & -\mathcal{E}_{12}\psi(\mathcal{D}_{21})\mathcal{X}_{12} - \mathcal{X}_{12}\Psi(\mathcal{D}_{21})\mathcal{E}_{12} \\
& -\psi(\mathcal{E}_{12})\mathcal{D}_{21}\mathcal{X}_{12} - \mathcal{X}_{12}\mathcal{D}_{21}\psi(\mathcal{E}_{12}) \\
& -\mathcal{E}_{12}\mathcal{D}_{21}\psi(\mathcal{X}_{12}) - \psi(\mathcal{X}_{12})\mathcal{D}_{21}\mathcal{E}_{12}.
\end{aligned}$$

From the above two equations, we obtain

$$(\psi(\mathcal{E}_{12}\mathcal{D}_{21}) + \mathcal{E}_{12}\psi(\mathcal{D}_{21}) + \psi(\mathcal{E}_{12})\mathcal{D}_{21})\mathcal{X}_{12} = \mathcal{X}_{12}(\psi(\mathcal{D}_{21}\mathcal{E}_{12}) + \mathcal{D}_{21}\psi(\mathcal{E}_{12}) + \psi(\mathcal{D}_{21})\mathcal{E}_{12}). \tag{3.2}$$

Also, since $\mathcal{E}_{12}\mathcal{D}_{21}\mathcal{X}_{21}(-\mathbb{P}_2)\ldots(-\mathbb{P}_2) = 0$, we have

$$\begin{aligned}
& \Omega(P_n(\mathcal{E}_{12}, \mathcal{D}_{21}, \mathcal{X}_{21}, (-\mathbb{P}_2), \ldots, (-\mathbb{P}_2)) \\
= \ & \Omega(-\mathcal{D}_{21}\mathcal{E}_{12}\mathcal{X}_{21} - \mathcal{X}_{21}\mathcal{E}_{12}\mathcal{D}_{21}) \\
= \ & \Omega(-\mathcal{D}_{21}\mathcal{E}_{12}\mathcal{X}_{21}) - \Omega(\mathcal{X}_{12}\mathcal{E}_{12}\mathcal{D}_{21}) \\
= \ & -\psi(\mathcal{D}_{21}\mathcal{E}_{12}\mathcal{X}_{21}) - \psi(\mathcal{X}_{12}\mathcal{E}_{12}\mathcal{D}_{21}) \\
= \ & -\psi(\mathcal{D}_{21}\mathcal{E}_{12})\mathcal{X}_{21} - \mathcal{D}_{21}\mathcal{E}_{12}\psi(\mathcal{X}_{21}) - \psi(\mathcal{X}_{21})\mathcal{E}_{12}\mathcal{D}_{21} - \mathcal{X}_{21}\psi(\mathcal{E}_{12}\mathcal{D}_{21}).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
& \Omega(P_n(\mathcal{E}_{12}, \mathcal{D}_{21}, \mathcal{X}_{21}, (-\mathbb{P}_2), \ldots, (-\mathbb{P}_2))) \\
= \ & P_n(\Omega(\mathcal{E}_{12}), \mathcal{D}_{21}, \mathcal{X}_{21}, (-\mathbb{P}_2), \ldots, (-\mathbb{P}_2)) \\
& +P_n(\mathcal{E}_{12}, \Omega(\mathcal{D}_{21}), \mathcal{X}_{21}, (-\mathbb{P}_2), \ldots, (-\mathbb{P}_2)) \\
& +P_n(\mathcal{E}_{12}, \mathcal{D}_{21}, \Omega(\mathcal{X}_{21}), (-\mathbb{P}_2), \ldots, (-\mathbb{P}_2)) \\
= \ & P_n(\psi(\mathcal{E}_{12}), \mathcal{D}_{21}, \mathcal{X}_{21}, (-\mathbb{P}_2)\ldots(-\mathbb{P}_2))
\end{aligned}$$

$$
\begin{aligned}
& +P_n(\mathcal{E}_{12}, \psi(\mathcal{D}_{21}), \mathcal{X}_{21}, (-\mathbb{P}_2), \ldots, (-\mathbb{P}_2)) \\
& +P_n(\mathcal{E}_{12}, \mathcal{D}_{21}, \psi(\mathcal{X}_{21}), (-\mathbb{P}_2), \ldots, (-\mathbb{P}_2)) \\
= \; & -\mathcal{D}_{21}\psi(\mathcal{E}_{12})\mathcal{X}_{21} - \mathcal{X}_{21}\Psi(\mathcal{E}_{12})\mathcal{D}_{21} \\
& -\psi(\mathcal{D}_{21})\mathcal{E}_{12}\mathcal{X}_{21} - \mathcal{X}_{21}\mathcal{E}_{12}\psi(\mathcal{D}_{21}) \\
& -\mathcal{D}_{21}\mathcal{E}_{12}\psi(\mathcal{X}_{21}) - \psi(\mathcal{X}_{21})\mathcal{E}_{12}\mathcal{D}_{21}.
\end{aligned}
$$

By comparing the above two equations, we obtain

$$
(-\psi(\mathcal{D}_{21}\mathcal{E}_{12}) + \mathcal{D}_{21}\psi(\mathcal{E}_{12}) + \psi(\mathcal{D}_{21})\mathcal{E}_{12})\mathcal{X}_{21} = \mathcal{X}_{21}(\psi(\mathcal{E}_{12}\mathcal{D}_{21}) - \mathcal{E}_{12}\psi(\mathcal{D}_{21}) - \psi(\mathcal{E}_{12})\mathcal{D}_{21}). \tag{3.3}
$$

By using Eqs (3.2) and (3.3), we obtain

$$
\psi(\mathcal{D}_{12}\mathcal{E}_{21}) - \mathcal{D}_{12}\psi(\mathcal{E}_{21}) - \psi(\mathcal{D}_{12})\mathcal{E}_{21} \in \mathcal{Z}(\mathcal{S}).
$$

Assume that $\psi(\mathcal{D}_{12}\mathcal{E}_{21}) - \mathcal{D}_{12}\psi(\mathcal{E}_{21}) - \psi(\mathcal{D}_{12})\mathcal{E}_{21} = \alpha(\mathcal{D}_{12}, \mathcal{E}_{21})$ for any $\alpha(\mathcal{D}_{12}, \mathcal{E}_{21}) \in \mathcal{Z}(\mathcal{S}_{11})$. Now

$$
\begin{aligned}
\alpha(\mathcal{X}_{11}\mathcal{D}_{12}, \mathcal{E}_{21}) &= \psi(\mathcal{X}_{11}\mathcal{D}_{12}\mathcal{E}_{21}) - \mathcal{X}_{11}\mathcal{D}_{12}\psi(\mathcal{E}_{21}) - \psi(\mathcal{X}_{11}\mathcal{D}_{12})\mathcal{E}_{21} \\
&= \psi(\mathcal{X}_{11})\mathcal{D}_{12}\mathcal{E}_{21} + \mathcal{X}_{11}\psi(\mathcal{D}_{12}\mathcal{E}_{21}) - \mathcal{X}_{11}\mathcal{D}_{12}\psi(\mathcal{E}_{21}) \\
&\quad -\psi(\mathcal{X}_{11})\mathcal{D}_{12}\mathcal{E}_{21} - \mathcal{X}_{11}\psi(\mathcal{D}_{12})\mathcal{E}_{21} \\
&= \mathcal{X}_{11}(\psi(\mathcal{D}_{12}\mathcal{E}_{21}) - \psi(\mathcal{D}_{12})\mathcal{E}_{21} - \mathcal{D}_{12}\psi(\mathcal{E}_{21})).
\end{aligned}
$$

Also, $\mathcal{S}_{11}\alpha(\mathcal{D}_{12}, \mathcal{E}_{21})$ is a central ideal in $\mathcal{S}_{11}$. As $\mathcal{S}_{11}$ contains no nonzero central ideal. Therefore, $\alpha(\mathcal{D}_{12}, \mathcal{E}_{21}) = 0$. Hence,

$$
\psi(\mathcal{D}_{12}\mathcal{E}_{21}) = \mathcal{D}_{12}\psi(\mathcal{E}_{21}) + \psi(\mathcal{D}_{12})\mathcal{E}_{21}.
$$

By using the same approach, one can also prove the result for $i = 2, j = 1$. □

Proof of Theorem 3.1. It follows by Lemmas 3.9–3.12 that ψ is an additive derivation. For $\mathcal{D} \in \mathcal{S}$, we have

$$
\Theta(\mathcal{D}) = \Omega(\mathcal{D}) + [\mathcal{D}, M] = \psi(\mathcal{D}) + \tau(\mathcal{D}) + [\mathcal{D}, M] = d(\mathcal{D}) + \tau(\mathcal{D}),
$$

where $M = \mathbb{P}_1\Theta(\mathbb{P}_1)\mathbb{P}_2 - \mathbb{P}_2\Theta(\mathbb{P}_1)\mathbb{P}_1$ and $d(\mathcal{D})$ is a derivation. It only remains to prove τ vanishes at $(n-1)^{th}$ commutator, i.e., we show that $\tau(P_n(\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n)) = 0$ for all $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n \in \mathcal{S}$. By using Lemma 3.8, we obtain

$$
\begin{aligned}
& \tau(P_n(\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n)) \\
= \; & \Omega(P_n(\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n)) - \psi(P_n(\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n)) \\
= \; & \sum_{i=1}^{n} P_n(\mathcal{D}_1, \ldots, \mathcal{D}_{i-1}, \Omega(\mathcal{D}_i), \mathcal{D}_{i+1}, \ldots, \mathcal{D}_n) - \sum_{i=1}^{n} P_n(\mathcal{D}_1, \ldots, \mathcal{D}_{i-1}, \Psi(\mathcal{D}_i), \mathcal{D}_{i+1}, \ldots, \mathcal{D}_n) \\
= \; & \sum_{i=1}^{n} P_n(\mathcal{D}_1, \ldots, \mathcal{D}_{i-1}, \psi(\mathcal{D}_i), \mathcal{D}_{i+1}, \ldots, \mathcal{D}_n) - \sum_{i=1}^{n} P_n(\mathcal{D}_1, \ldots, \mathcal{D}_{i-1}, \Psi(\mathcal{D}_i), \mathcal{D}_{i+1}, \ldots, \mathcal{D}_n) \\
= \; & 0.
\end{aligned}
$$

This completes the proof of Theorem 3.1 here. □

Author contributions

All authors are contributed equally.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The first author gratefully acknowledges the funding of the Deanship of Graduate Studies and Scientific Research, Jazan University, Saudi Arabia, through project number: RG24-S053.

Conflict of interest

The authors declare no conflict of interest.

References

1. M. Ashraf, B. Wani, F. Wei, Multiplicative $*$-Lie triple higher derivations of standard operator algebras, *Quaest. Math.*, **42** (2019), 857–884. http://dx.doi.org/10.2989/16073606.2018.1502213
2. M. Ashraf, A. Jabeen, Characterizations of additive ζ-Lie derivations on unital algebras, *Ukr. Math. J.*, **73** (2021), 532–546. http://dx.doi.org/10.1007/s11253-021-01941-y
3. K. Fallahi, H. Ghahramani, Anti-derivable linear maps at zero on standard operator algebras, *Acta Math. Hungar.*, **167** (2022), 287–294. http://dx.doi.org/10.1007/s10474-022-01243-0
4. P. Ji, W. Qi, Characterizations of Lie derivations of triangular algebras, *Linear Algebra Appl.*, **435** (2011), 1137–1146. http://dx.doi.org/10.1016/j.laa.2011.02.048
5. P. Ji, W. Qi, X. Sun, Characterizations of Lie derivations of factor von Neumann algebras, *Linear Multilinear A.*, **61** (2013), 417–428. http://dx.doi.org/10.1080/03081087.2012.689982
6. L. Liu, Lie triple derivations on factor von Neumann algebras, *Bull. Korean Math. Soc.*, **52** (2015), 581–591. http://dx.doi.org/10.4134/BKMS.2015.52.2.581
7. F. Lu, W. Jing, Characterizations of Lie derivations of B(X), *Linear Algebra Appl.*, **432** (2010), 89–99. http://dx.doi.org/10.1016/j.laa.2009.07.026
8. X. Qi, Characterization of (generalized) Lie derivations on J-subspace lattice algebras by local action, *Aequat. Math.*, **87** (2014), 53–69. http://dx.doi.org/10.1007/s00010-012-0177-3
9. X. Qi, J. Hou, Characterization of Lie derivations on von Neumann algebras, *Linear Algebra Appl.*, **438** (2013), 533–548. http://dx.doi.org/10.1016/j.laa.2012.08.019
10. N. Rehman, J. Nisar, M. Nazim, Nonlinear mixed Jordan triple*-derivations on standard operator algebras, *Filomat*, **37** (2023), 3143–3151. http://dx.doi.org/10.2298/FIL2310143R
11. X. Tang, M. Xiao, P. Wang, Local properties of Virasoro-like algebra, *J. Geom. Phys.*, **186** (2023), 104772. http://dx.doi.org/10.1016/j.geomphys.2023.104772

12. Y. Wang, Lie n-derivations of unital algebras with idempotents, *Linear Algebra Appl.*, **458** (2014), 512–525. http://dx.doi.org/10.1016/j.laa.2014.06.029
13. Y. Wang, Y. Wang, Multiplicative Lie n-derivations of generalized matrix algebras, *Linear Algebra Appl.*, **438** (2013), 2599–2616. http://dx.doi.org/10.1016/j.laa.2012.10.052
14. C. Xia, X. Dong, D. Wang, C. Zhang, Local properties of Schrödinger-Virasoro type Lie algebras and a full diamond thin Lie algebra, *Commun. Algebra*, **52** (2024), 4650-4665. http://dx.doi.org/10.1080/00927872.2024.2356244
15. X. Zhao, Nonlinear Lie triple derivations by local actions on triangular algebras, *J. Algebra Appl.*, **22** (2023), 2350059. http://dx.doi.org/10.1142/S0219498823500597
16. X. Zhao, H. Hao, Non-global nonlinear Lie triple derivable maps on finite von Neumann algebras, *Bull. Iran. Math. Soc.*, **47** (2021), 307–322. http://dx.doi.org/10.1007/s41980-020-00493-4
17. Y. Zhong, X. Tang, n-derivations of the extended Schrödinger-Virasoro Lie algebra, *Algebr. Colloq.*, **28** (2021), 105–118. http://dx.doi.org/10.1142/S1005386721000109