Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(9): 25284–25296.
DOI: 10.3934/math.20241234
Received: 09 July 2024
Revised: 19 August 2024
Accepted: 23 August 2024
Published: 29 August 2024

Research article

Non-uniform dependence on periodic initial data for the two-component Fornberg-Whitham system in Besov spaces

Prerona Dutta* **and Barbara Lee Keyfitz**

Department of Mathematics, The Ohio State University, 231 W 18th Ave, Columbus, OH 43210, USA

* **Correspondence:** Email: pdutta@xula.edu.

Abstract: This paper establishes non-uniform continuity of the data-to-solution map in the periodic case for the two-component Fornberg-Whitham system in Besov spaces $B^s_{p,r}(\mathbb{T}) \times B^{s-1}_{p,r}(\mathbb{T})$ for $s > \max\{2 + \frac{1}{p}, \frac{5}{2}\}$. In particular, when $p = 2$ and $r = 2$, this proves the non-uniform dependence on initial data for the system in Sobolev spaces $H^s(\mathbb{T}) \times H^{s-1}(\mathbb{T})$ for $s > \frac{5}{2}$.

Keywords: Fornberg-Whitham system; Besov space; non-uniform dependence
Mathematics Subject Classification: 35Q35, 35B30

1. Introduction

In this paper, we consider the following two-component Fornberg-Whitham (FW) system for a fluid

$$\begin{cases} u_t + uu_x = \left(1 - \partial_x^2\right)^{-1} \partial_x (\rho - u) \\ \rho_t + (\rho u)_x = 0 \\ (u, \rho)(0, x) = (u_0, \rho_0)(x) \end{cases} \tag{1.1}$$

where $x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, $t \in \mathbb{R}^+$. Here, $u = u(x, t)$ is the horizontal velocity of the fluid, and $\rho = \rho(x, t)$ is the height of the fluid surface above a horizontal bottom. This system was first proposed in [5], and local well-posedness and non-uniform dependence on the initial data were established in Sobolev spaces $H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ for $s > \frac{3}{2}$ in [11, 12].

Local well-posedness in Besov spaces $B^s_{p,r}(\mathbb{R}) \times B^{s-1}_{p,r}(\mathbb{R})$ of (1.1) was established in [4] for $s > \max\{2 + \frac{1}{p}, \frac{5}{2}\}$. Besov spaces $B^s_{p,r}$ are a class of functions of interest in the study of nonlinear partial differential equations, as they are based on Sobolev spaces and introduce a measure of generalized Hölder regularity through the index r, along with the Sobolev index of differentiability s and the Lebesgue integrability index p. If s and p are fixed, the spaces $B^s_{p,r}$ grow larger with increasing r. In [4], the FW system was shown to be well-posed in the sense of Hadamard by establishing the existence and uniqueness of the solution to the system (1.1) and then proving continuity of the data-to-solution map when the initial data belong to $B^s_{p,r}(\mathbb{R}) \times B^{s-1}_{p,r}(\mathbb{R})$ for $s > \max\{2 + \frac{1}{p}, \frac{5}{2}\}$.

In this paper, our objective is to prove non-uniform dependence on periodic initial data for the two-component FW system (1.1) in $B^s_{p,r}(\mathbb{T}) \times B^{s-1}_{p,r}(\mathbb{T})$ for $s > \max\{2 + \frac{1}{p}, \frac{5}{2}\}$. We work with periodic initial data, as that simplifies our choice of approximate solutions and the resulting estimates. Setting $\Lambda = 1 - \partial_x^2$, we rewrite (1.1) as

$$\begin{cases} u_t + uu_x = \Lambda^{-1}\partial_x(\rho - u) \\ \rho_t + u\rho_x + \rho u_x = 0 \\ (u, \rho)(0, x) = (u_0, \rho_0)(x) \end{cases} \tag{1.2}$$

where $x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and $t \in \mathbb{R}^+$.

The paper is organized as follows: In Section 2, we recall the standard definitions and properties of Besov spaces, linear transport equations, the operator Λ, and the two-component FW system. In Section 3, we prove non-uniform dependence on initial data for the FW system (1.2) when the initial data belong to $B^s_{p,r}(\mathbb{T}) \times B^{s-1}_{p,r}(\mathbb{T})$ for $s > \max\{2 + \frac{1}{p}, \frac{5}{2}\}$. For this proof, we use a technique previously seen in the study of non-uniform continuity of data-to-solution maps for other nonlinear PDEs, for instance in [6–8, 10, 12]. We construct two sequences of approximate solutions such that the initial data for these sequences converge to each other in $B^s_{p,r}(\mathbb{T}) \times B^{s-1}_{p,r}(\mathbb{T})$. Non-uniform dependence is then established by proving that the approximate and hence the exact solutions remain bounded away from each other for any positive time $t > 0$. This idea was first explored by Kato in [9] to show that the data-to-solution map for Burgers' equation is not Hölder continuous in the H^s norm with $s > 3/2$ for any Hölder exponent.

2. Notation and preliminaries

This section is a review of relevant definitions and results on Besov spaces, linear transport equations, the operator Λ, and the two-component FW system (1.2). We begin by listing some useful notation to be used throughout Section 3.

2.1. Notation

For any $x, y \in \mathbb{R}$,

- $x \lesssim y$ denotes $x \leq \alpha y$ for some constant α.
- $x \approx y$ denotes $x = \beta y$ for some constant β.
- $x \gtrsim y$ denotes $x \geq \gamma y$ for some constant γ.

2.2. Besov spaces

We recall the construction of a dyadic partition of unity from [8]. Consider a smooth bump function χ such that $\operatorname{supp}\chi = [-\frac{4}{3}, \frac{4}{3}]$ and $\chi = 1$ on $[-\frac{3}{4}, \frac{3}{4}]$. For $\xi > 0$, set $\varphi_{-1}(\xi) = \chi$, $\varphi_0(\xi) = \chi\left(\frac{\xi}{2}\right) - \chi(\xi)$ and $\varphi_q(\xi) = \varphi_0(2^{-q}\xi)$. Then, $\operatorname{supp}\varphi_q = [\frac{3}{4} \cdot 2^q, \frac{8}{3} \cdot 2^q]$ and $\sum_{q \geq -1} \varphi_q(\xi) = 1$. Using this partition, a Littlewood-Paley decomposition of any periodic distribution u is defined in [3] as follows:

Definition 2.1 (Littlewood-Paley decomposition). *For any $u \in \mathcal{D}'(\mathbb{T})$ with the Fourier series $u(x) = \sum_{j\in\mathbb{Z}} \hat{u}_j e^{ijx}$ where $\hat{u}_j = \frac{1}{2\pi}\int_0^{2\pi} e^{-ijy}u(y)\,dy$, its Littlewood-Paley decomposition is given by $u = \sum_{q\geq -1} \Delta_q u$, where $\Delta_q u$ are periodic dyadic blocks defined for all $q \in \mathbb{Z}$ as*

$$\Delta_q u = \sum_{j\in\mathbb{Z}} \varphi_q(j)\hat{u}_j e^{ijx} \ .$$

Using this Littlewood-Paley decomposition, Besov spaces on $\mathbb{T}$ are defined in [3] as follows:

Definition 2.2 (Besov spaces). *Let $s \in \mathbb{R}$ and $p,\ r \in [1,\infty]$. Then the Besov spaces of functions are defined as*

$$B^s_{p,r} \equiv B^s_{p,r}(\mathbb{T}) = \{u \in \mathcal{D}'(\mathbb{T}) \,\big|\, \|u\|_{B^s_{p,r}} < \infty\}\ ,$$

where

$$\|u\|_{B^s_{p,r}} = \begin{cases} \left(\sum\limits_{q\geq -1} (2^{sq}\|\Delta_q u\|_{\mathbf{L}^p})^r\right)^{1/r} & \text{if } 1 \leq r < \infty \\ \sup\limits_{q\geq -1} 2^{sq}\|\Delta_q u\|_{\mathbf{L}^p} & \text{if } r = \infty \end{cases} \ .$$

Following are some properties proved in [1, Section 2.8] and [3, Section 1.3] that facilitate the study of nonlinear partial differential equations in Besov spaces.

Lemma 2.3. *Let $s, s_j \in \mathbb{R}$ for $j = 1,2$ and $1 \leq p,r \leq \infty$. Then the following properties hold:*

(1) Topological property: $B^s_{p,r}$ is a Banach space continuously embedded in $\mathcal{D}'(\mathbb{T})$.
(2) Algebraic property: For all $s > 0$, $B^s_{p,r} \cap \mathbf{L}^\infty$ is a Banach algebra.
(3) Interpolation: If $f \in B^{s_1}_{p,r} \cap B^{s_2}_{p,r}$ and $\theta \in [0,1]$, then $f \in B^{\theta s_1 + (1-\theta)s_2}_{p,r}$ and

$$\|f\|_{B^{\theta s_1+(1-\theta)s_2}_{p,r}} \leq \|f\|^{\theta}_{B^{s_1}_{p,r}} \|f\|^{1-\theta}_{B^{s_2}_{p,r}} \ .$$

(4) Embedding: $B^{s_1}_{p,r} \hookrightarrow B^{s_2}_{p,r}$ whenever $s_1 \geq s_2$. In particular, $B^s_{p,r} \hookrightarrow B^{s-1}_{p,r}$ for all $s \in \mathbb{R}$.

Remark on (2) in Lemma 2.3: When $s > \frac{1}{p}$ (or $s \geq \frac{1}{p}$ and $r = 1$), $B^s_{p,r} \hookrightarrow \mathbf{L}^\infty$. We will use the fact that for $0 < s < \frac{1}{p}$, the result is still true as long as the functions are bounded.

2.3. Linear transport equation

Given a linear transport equation, Proposition A.1 in [2] proves the following estimate for its solution size in Besov spaces:

Proposition 2.4. *Consider the linear transport equation*

$$\begin{cases} \partial_t f + v\partial_x f = F \\ f(x,0) = f_0(x) \end{cases} \tag{2.1}$$

where $f_0 \in B^s_{p,r}(\mathbb{T})$, $F \in \mathbf{L}^1((0,T); B^s_{p,r}(\mathbb{T}))$ and v is such that $\partial_x v \in \mathbf{L}^1((0,T); B^{s-1}_{p,r}(\mathbb{T}))$. Suppose $f \in \mathbf{L}^\infty((0,T); B^s_{p,r}(\mathbb{T})) \cap C([0,T]; \mathcal{D}'(\mathbb{T}))$ is a solution to (2.1)*. Let $1 \leq p,r \leq \infty$. If either $s \neq 1 + \frac{1}{p}$, or $s = 1 + \frac{1}{p}$ and $r = 1$, then for a positive constant C that depends on s, p, and r, we have*

$$\|f(t)\|_{B^s_{p,r}} \leq e^{CV(t)}\left(\|f_0\|_{B^s_{p,r}} + C\int_0^t e^{-CV(\tau)}\|F(\tau)\|_{B^s_{p,r}}\,d\tau\right)$$

where

$$V(t) = \int_0^t \|\partial_x v(\tau)\|_{B^{1/p}_{p,r}\cap \mathbf{L}^\infty}\, d\tau \quad \text{if } s < 1 + \frac{1}{p}$$

and

$$V(t) = \int_0^t \|\partial_x v(\tau)\|_{B^{s-1}_{p,r}}\, d\tau \quad \text{otherwise} .$$

For $r < \infty$, $f \in C([0,T], B^s_{p,r}(\mathbb{T}))$, and if $r = \infty$, then $f \in C([0,T], B^{s'}_{p,1}(\mathbb{T}))$ for all $s' < s$.

2.4. The Operator Λ

Let $\Lambda = 1 - \partial_x^2$; then, for any test function g, the Fourier transform of $\Lambda^{-1}g$ is given by $\mathcal{F}\left(\Lambda^{-1}g\right) = \frac{1}{1+\xi^2}\hat{g}(\xi)$. Moreover, for any $s \in \mathbb{R}$, $\Lambda^{-1}\partial_x$ is continuous from $B^{s-1}_{p,r}$ to $B^s_{p,r}$; that is, for all $h \in B^{s-1}_{p,r}$, there exists a constant $\kappa > 0$ depending on s, p, and r such that

$$\|\Lambda^{-1}\partial_x h\|_{B^s_{p,r}} \leq \kappa \|h\|_{B^{s-1}_{p,r}} . \tag{2.2}$$

2.5. The Fornberg-Whitham system

The well-posedness of the two-component FW system (1.2) in Besov spaces was established on the real line in [4] with the following result:

Theorem 2.5. *Let $s > \max\{2 + \frac{1}{p}, \frac{5}{2}\}$, $p \in [1,\infty]$, $r \in [1,\infty]$ and $(u_0, \rho_0) \in B^s_{p,r}(\mathbb{R}) \times B^{s-1}_{p,r}(\mathbb{R})$. Then the system* (1.2) *has a unique solution $(u,\rho) \in C\left([0,T]; B^s_{p,r}(\mathbb{R}) \times B^{s-1}_{p,r}(\mathbb{R})\right)$, where the doubling time T is given by*

$$T = \frac{C}{\left(\|u_0\|_{B^s_{p,r}} + \|\rho_0\|_{B^{s-1}_{p,r}}\right)^2} ,$$

with C being a constant that depends on s, p, and r, and the solution size is estimated as

$$\|(u,\rho)\|_{B^s_{p,r}\times B^{s-1}_{p,r}} \leq 2\left(\|u_0\|_{B^s_{p,r}} + \|\rho_0\|_{B^{s-1}_{p,r}}\right) .$$

Moreover, the data-to-solution map is continuous.

Since we work with $B^s_{p,r}(\mathbb{T}) \times B^{s-1}_{p,r}(\mathbb{T})$ in this paper, we state the following:

Corollary 2.6. *Theorem 2.5 holds when $\mathbb{R}$ is replaced by $\mathbb{T}$.*

Proof. The existence of a solution to (1.2) is proved by altering the mollifier used to prove Theorem 2.5. This adaptation of the mollifier was done for the single Fornberg-Whitham equation in [7, Section 3.1]. Uniqueness and continuous dependence on periodic initial data for the system (1.2) are established by approximation arguments similar to those in [4, Sections 3.2 and 3.3].

□

3. Non-uniform dependence on initial data

In this section, we establish non-uniform dependence on initial data in the periodic case for the two-component FW system (1.2) in Besov spaces.

Theorem 3.1. *Let $s > \max\{2 + \frac{1}{p}, \frac{5}{2}\}$ and $r \in [1, \infty]$. The data-to-solution map $(u_0, \rho_0) \mapsto (u(t), \rho(t))$ of the Cauchy problem* (1.2) *is not uniformly continuous from any bounded subset of $B^s_{p,r}(\mathbb{T}) \times B^{s-1}_{p,r}(\mathbb{T})$ into $C([0,T]; B^s_{p,r}(\mathbb{T})) \times C([0,T]; B^{s-1}_{p,r}(\mathbb{T}))$ where T is given by Theorem 2.5. In particular, there exist two sequences of solutions $\{(u_{\omega,n}, \rho_{\omega,n})\}$ with $\omega = \pm 1$ such that the following hold:*

(i) $\lim_{n\to\infty} \left(\|u_{1,n}(0) - u_{-1,n}(0)\|_{B^s_{p,r}} + \|\rho_{1,n}(0) - \rho_{-1,n}(0)\|_{B^{s-1}_{p,r}} \right) = 0.$

(ii) $\liminf_{n\to\infty} \left(\|u_{1,n} - u_{-1,n}\|_{B^s_{p,r}} + \|\rho_{1,n} - \rho_{-1,n}\|_{B^{s-1}_{p,r}} \right) \gtrsim |\sin t|.$

Proof. For $n \in \mathbb{N}$, we consider two sequences of functions $\{(u^{\omega,n}, \rho^{\omega,n})\}$ with $\omega = \pm 1$, defined by

$$\begin{cases} u^{\omega,n} = \frac{-\omega}{n} + \frac{1}{n^s} \sin(nx + \omega t) \\ \rho^{\omega,n} = \frac{1}{n} + \frac{1}{n^s} \sin(nx + \omega t) \end{cases}.$$

We take initial data

$$\begin{cases} u^0_{\omega,n} = u^{\omega,n}(0) = \frac{-\omega}{n} + \frac{1}{n^s} \sin nx \\ \rho^0_{\omega,n} = \rho^{\omega,n}(0) = \frac{1}{n} + \frac{1}{n^s} \sin nx \end{cases}.$$

Let the solutions to the FW system (1.2) with these initial data be denoted by $(u_{\omega,n}, \rho_{\omega,n})$. At $t = 0$, we have

$$\lim_{n\to\infty} \left(\|u^0_{1,n} - u^0_{-1,n}\|_{B^s_{p,r}} + \|\rho^0_{1,n} - \rho^0_{-1,n}\|_{B^{s-1}_{p,r}} \right) = \lim_{n\to\infty} 2\|n^{-1}\|_{B^s_{p,r}} = 0\,,$$

which proves part *(i)* of Theorem 3.1.

To prove part *(ii)*, first we estimate $\|(u^0_{\omega,n}, \rho^0_{\omega,n})\|_{B^\gamma_{p,r} \times B^{\gamma-1}_{p,r}}$ and $\|(u^{\omega,n}, \rho^{\omega,n})\|_{B^\gamma_{p,r} \times B^{\gamma-1}_{p,r}}$ for any $\gamma > 0$ and $r < \infty$. Using the triangle inequality, we have

$$\|(u^0_{\omega,n}, \rho^0_{\omega,n})\|_{B^\gamma_{p,r} \times B^{\gamma-1}_{p,r}} \le 2\|n^{-1}\|_{B^\gamma_{p,r}} + n^{-s}\|\sin nx\|_{B^\gamma_{p,r}} + n^{1-s}\|\sin nx\|_{B^{\gamma-1}_{p,r}}. \tag{3.1}$$

By Definition 2.2,

$$\|\sin nx\|_{B^\gamma_{p,r}} = \left(\sum_{q \ge -1} 2^{\gamma q r} \|\Delta_q \sin nx\|^r_{\mathbf{L}^p} \right)^{\frac{1}{r}}. \tag{3.2}$$

From Definition 2.1, as shown in the Appendix, we have $\|\Delta_q \sin(nx)\|_{\mathbf{L}^p} = \varphi_q(n)$, where $0 < \varphi_q(n) \le 1$ for all q such that $\frac{1}{\ln(2)} \ln\left(\frac{3}{8}n\right) \le q \le \frac{1}{\ln(2)} \ln\left(\frac{4}{3}n\right)$ and $\varphi_q(n) = 0$ otherwise. Hence, (3.2) implies that for any $\gamma > 0$,

$$\|\sin(nx)\|_{B^\gamma_{p,r}} \le \left(\sum_{q=\frac{1}{\ln(2)}\ln\left(\frac{3}{8}n\right)}^{\frac{1}{\ln(2)}\ln\left(\frac{4}{3}n\right)} (2^q)^{\gamma r} \right)^{\frac{1}{r}}.$$

As $2^q \le \frac{4}{3}n$ for every term in the summation, from the above, we obtain

$$\begin{aligned} \|\sin(nx)\|_{B^\gamma_{p,r}} &\le \left(\sum_{q=\frac{1}{\ln(2)}\ln\left(\frac{3}{8}n\right)}^{\frac{1}{\ln(2)}\ln\left(\frac{4}{3}n\right)} \left(\frac{4}{3}n\right)^{\gamma r} \right)^{\frac{1}{r}} \\ &= \left(\frac{1}{\ln(2)} \left[\ln\left(\frac{4}{3}n\right) - \ln\left(\frac{3}{8}n\right) \right] \right)^{\frac{1}{r}} \left(\frac{4}{3}n\right)^{\gamma} \end{aligned}$$

$$= \left(\frac{1}{\ln(2)}\ln\left(\frac{32}{9}\right)\right)^{\frac{1}{r}}\left(\frac{4}{3}\right)^{\gamma} n^{\gamma} \;=\; C_{\gamma} n^{\gamma}\,. \tag{3.3}$$

Here and in what follows, C_γ is a generic constant that depends only on γ for fixed p and r. Similarly, it follows that for any $\gamma > 0$,

$$\|\cos(nx)\|_{B^{\gamma}_{p,r}} \;\le\; C_{\gamma} n^{\gamma}. \tag{3.4}$$

By Definition 2.1,

$$\Delta_q n^{-1} = \varphi_q(0)\, n^{-1} = \begin{cases} n^{-1} & \text{if } q = -1 \\ 0 & \text{otherwise} \end{cases}\,.$$

So, $\|n^{-1}\|_{B^{\gamma}_{p,r}} = \left(\sum_{q\ge -1} 2^{\gamma q r}\|\Delta_q n^{-1}\|^{r}_{\mathbf{L}^p}\right)^{\frac{1}{r}} = 2^{-\gamma} n^{-1}$. Using this and (3.3), it follows from (3.1) that

$$\begin{aligned}\|(u^{0}_{\omega,n},\rho^{0}_{\omega,n})\|_{B^{\gamma}_{p,r}\times B^{\gamma-1}_{p,r}} &\le 2^{1-\gamma}n^{-1} + C_{\gamma}n^{\gamma}n^{-s} + C_{\gamma}n^{\gamma-1}n^{1-s} \\ &\le C_{\gamma}\max\{n^{-1}, n^{\gamma-s}\}\,. \end{aligned} \tag{3.5}$$

Since $(u^{\omega,n},\rho^{\omega,n})$ is a phase shift of $(u^{0}_{\omega,n},\rho^{0}_{\omega,n})$, we have

$$\|(u^{\omega,n},\rho^{\omega,n})\|_{B^{\gamma}_{p,r}\times B^{\gamma-1}_{p,r}} \;\le\; C_{\gamma}\max\{n^{-1}, n^{\gamma-s}\}\,. \tag{3.6}$$

If $r = \infty$, (3.5) and (3.6) follow immediately from Definition 2.2.

We complete the proof of Theorem 3.1 by establishing *(ii)* for $\{(u^{\omega,n},\rho^{\omega,n})\}$, taking advantage of the following lemma, whose proof follows the proof of Theorem 3.1. Lemma 3.2 establishes that for each n and ω, $(u^{\omega,n},\rho^{\omega,n})$ approximates $(u_{\omega,n},\rho_{\omega,n})$ in $B^{s}_{p,r}(\mathbb{T})\times B^{s-1}_{p,r}(\mathbb{T})$ uniformly on $[0,T]$ for some $T>0$.

Lemma 3.2. *Let $\mathcal{E}^{\omega,n} = (\mathcal{E}^{\omega,n}_1, \mathcal{E}^{\omega,n}_2)$ where $\mathcal{E}^{\omega,n}_1 = u_{\omega,n} - u^{\omega,n}$ and $\mathcal{E}^{\omega,n}_2 = \rho_{\omega,n} - \rho^{\omega,n}$, with $\omega = \pm 1$. Then for all $t \in (0,T)$, where T is given by Theorem 2.5, $\|\mathcal{E}^{\omega,n}(t)\|_{B^{s}_{p,r}\times B^{s-1}_{p,r}} = \|\mathcal{E}^{\omega,n}_1(t)\|_{B^{s}_{p,r}} + \|\mathcal{E}^{\omega,n}_2(t)\|_{B^{s-1}_{p,r}} \to 0$ as $n\to\infty$.*

We show that $(u_{-1,n},\rho_{-1,n})$ and $(u_{1,n},\rho_{1,n})$ stay bounded away from each other for any $t>0$. Since

$$\|u_{1,n} - u_{-1,n}\|_{B^{s}_{p,r}} \;\ge\; \|u^{1,n} - u^{-1,n}\|_{B^{s}_{p,r}} - \|u^{1,n} - u_{1,n}\|_{B^{s}_{p,r}} - \|u^{-1,n} - u_{-1,n}\|_{B^{s}_{p,r}} \tag{3.7}$$

and

$$\|\rho_{1,n} - \rho_{-1,n}\|_{B^{s-1}_{p,r}} \;\ge\; \|\rho^{1,n} - \rho^{-1,n}\|_{B^{s-1}_{p,r}} - \|\rho^{1,n} - \rho_{1,n}\|_{B^{s-1}_{p,r}} - \|\rho^{-1,n} - \rho_{-1,n}\|_{B^{s-1}_{p,r}}\,, \tag{3.8}$$

adding (3.7) and (3.8) we obtain

$$\begin{aligned}\|u_{1,n}-u_{-1,n}\|_{B^{s}_{p,r}}+\|\rho_{1,n}-\rho_{-1,n}\|_{B^{s-1}_{p,r}} &\ge \|u^{1,n}-u^{-1,n}\|_{B^{s}_{p,r}}+\|\rho^{1,n}-\rho^{-1,n}\|_{B^{s-1}_{p,r}}-\|\mathcal{E}^{1,n}(t)\|_{B^{s}_{p,r}\times B^{s-1}_{p,r}}-\|\mathcal{E}^{-1,n}(t)\|_{B^{s}_{p,r}\times B^{s-1}_{p,r}} \\ &\ge n^{-s}\left(\|\sin(nx+t)-\sin(nx-t)\|_{B^{s}_{p,r}} + \|\sin(nx+t)-\sin(nx-t)\|_{B^{s-1}_{p,r}}\right) \\ &\quad - 2\|n^{-1}\|_{B^{s}_{p,r}} - \|\mathcal{E}^{1,n}(t)\|_{B^{s}_{p,r}\times B^{s-1}_{p,r}} - \|\mathcal{E}^{-1,n}(t)\|_{B^{s}_{p,r}\times B^{s-1}_{p,r}} \\ &= 2n^{-s}\left(\|\cos(nx)\|_{B^{s}_{p,r}}|\sin(t)| + \|\cos(nx)\|_{B^{s-1}_{p,r}}|\sin(t)|\right) - 2^{1-\gamma}n^{-1} - \|\mathcal{E}^{1,n}(t)\|_{B^{s}_{p,r}\times B^{s-1}_{p,r}} - \|\mathcal{E}^{-1,n}(t)\|_{B^{s}_{p,r}\times B^{s-1}_{p,r}}\,. \end{aligned} \tag{3.9}$$

By Definition 2.2, if $r = \infty$, we immediately have

$$\|\cos(nx)\|_{B^s_{p,r}} \geq C_s n^s \,, \tag{3.10}$$

where C_s is a constant that depends only on s for a given p. For $1 \leq r < \infty$, there is a similar estimate, whose proof is given in the Appendix. Also, by Lemma 3.2, we have $\|\mathcal{E}^{\omega,n}(t)\|_{B^s_{p,r}\times B^{s-1}_{p,r}} \to 0$ for $\omega = \pm 1$, as $n \to \infty$. Using this and (3.10), it follows from (3.9) that

$$\begin{aligned}
&\liminf_{n\to\infty}\left(\|u_{1,n} - u_{-1,n}\|_{B^s_{p,r}} + \|\rho_{1,n} - \rho_{-1,n}\|_{B^{s-1}_{p,r}}\right)\\
&\qquad\geq\ 2C_s\left(\liminf_{n\to\infty}|\sin(t)| + \liminf_{n\to\infty} n^{-1}|\sin(t)|\right)\ \approx\ |\sin(t)|\ >\ 0\,.
\end{aligned}$$

This proves part *(ii)* of Theorem 3.1 and completes the proof of non-uniform dependence on initial data for the two-component FW system (1.2) in $B^s_{p,r}(\mathbb{T}) \times B^{s-1}_{p,r}(\mathbb{T})$ for $s > \max\{2 + \frac{1}{p}, \frac{5}{2}\}$.

□

Now we prove Lemma 3.2.

Proof. **(Lemma 3.2)** We show that $\|\mathcal{E}^{\omega,n}(t)\|_{B^\gamma_{p,r}\times B^{\gamma-1}_{p,r}} \to 0$ as $n \to \infty$ for any γ with $\max\{s - \frac{3}{2}, 1 + \frac{1}{p}\} < \gamma < s - 1$, and then interpolate between such a γ and a value $\delta > s$. Recall that $\mathcal{E}^{\omega,n}_1 = u_{\omega,n} - u^{\omega,n}$ and $\mathcal{E}^{\omega,n}_2 = \rho_{\omega,n} - \rho^{\omega,n}$. It can be seen that $\mathcal{E}^{\omega,n}_1$ and $\mathcal{E}^{\omega,n}_2$ vanish at $t = 0$ and that they satisfy the equations

$$\begin{cases}\partial_t\mathcal{E}^{\omega,n}_1 + u^{\omega,n}\partial_x\mathcal{E}^{\omega,n}_1 = -\mathcal{E}^{\omega,n}_1\partial_x u_{\omega,n} + \Lambda^{-1}\partial_x(\mathcal{E}^{\omega,n}_2 - \mathcal{E}^{\omega,n}_1) - R_1\\ \partial_t\mathcal{E}^{\omega,n}_2 + u_{\omega,n}\partial_x\mathcal{E}^{\omega,n}_2 = -\mathcal{E}^{\omega,n}_2\partial_x u_{\omega,n} - \rho^{\omega,n}\partial_x\mathcal{E}^{\omega,n}_1 - \mathcal{E}^{\omega,n}_1\partial_x\rho^{\omega,n} - R_2\end{cases}. \tag{3.11}$$

Here, R_1 and R_2 are the approximate solutions for the FW system, that is,

$$\begin{cases}R_1 = \partial_t u^{\omega,n} + u^{\omega,n}\partial_x u^{\omega,n} - \Lambda^{-1}\partial_x(\rho^{\omega,n} - u^{\omega,n})\\ R_2 = \partial_t\rho^{\omega,n} + \partial_x(\rho^{\omega,n}u^{\omega,n})\end{cases}.$$

- Estimate for $\|R_1\|_{B^\gamma_{p,r}}$: Using the definitions of $u^{\omega,n}$ and $\rho^{\omega,n}$, we have

$$R_1 = \partial_t u^{\omega,n} + u^{\omega,n}\partial_x u^{\omega,n} - \Lambda^{-1}\partial_x(\rho^{\omega,n} - u^{\omega,n}) = \frac{1}{2n^{2s-1}}\sin(2(nx+\omega t))\,.$$

  Then by (3.3),

$$\|R_1\|_{B^\gamma_{p,r}} \leq C_\gamma n^{\gamma-2s+1}\,.$$

- Estimate for $\|R_2\|_{B^{\gamma-1}_{p,r}}$: Using the definitions of $u^{\omega,n}$ and $\rho^{\omega,n}$,

$$R_2 = \partial_t\rho^{\omega,n} + \partial_x(\rho^{\omega,n}u^{\omega,n}) = \frac{1}{n^s}\cos(nx + \omega t) + \frac{1}{n^{2s-1}}\sin(2(nx+\omega t))\,.$$

  So from (3.3) and (3.4), it follows that

$$\|R_2\|_{B^{\gamma-1}_{p,r}} \leq C_\gamma\left(n^{-s}n^{\gamma-1} + n^{1-2s}n^{\gamma-1}\right) \leq C_\gamma n^{\gamma-s-1}\,.$$

Therefore,

$$\|R_1\|_{B^{\gamma}_{p,r}} + \|R_2\|_{B^{\gamma-1}_{p,r}} \lesssim n^{\gamma-s-1} . \tag{3.12}$$

Since $\mathcal{E}_1^{\omega,n}(t)$ and $\mathcal{E}_2^{\omega,n}(t)$ satisfy the linear transport equations (3.11), to estimate the error $\|\mathcal{E}^{\omega,n}(t)\|_{B^{\gamma}_{p,r}\times B^{\gamma-1}_{p,r}}$, we apply Proposition 2.4 to obtain

$$\|\mathcal{E}_1^{\omega,n}(t)\|_{B^{\gamma}_{p,r}} \leq K_1 e^{K_1 V_1(t)} \int_0^t e^{-K_1 V_1(\tau)} \|F_1(\tau)\|_{B^{\gamma}_{p,r}}\, d\tau \tag{3.13}$$

and

$$\|\mathcal{E}_2^{\omega,n}(t)\|_{B^{\gamma-1}_{p,r}} \leq K_2 e^{K_2 V_2(t)} \int_0^t e^{-K_2 V_2(\tau)} \|F_2(\tau)\|_{B^{\gamma-1}_{p,r}}\, d\tau \tag{3.14}$$

where K_1, K_2 are positive constants depending on γ and

$$F_1(t) = -\mathcal{E}_1^{\omega,n} \partial_x u_{\omega,n} + \Lambda^{-1}\partial_x(\mathcal{E}_2^{\omega,n} - \mathcal{E}_1^{\omega,n}) - R_1 , \tag{3.15}$$

$$F_2(t) = -\mathcal{E}_2^{\omega,n} \partial_x u_{\omega,n} - \rho^{\omega,n}\partial_x \mathcal{E}_1^{\omega,n} - \mathcal{E}_1^{\omega,n}\partial_x \rho^{\omega,n} - R_2 . \tag{3.16}$$

$$V_1(t) = \int_0^t \|\partial_x u^{\omega,n}(\tau)\|_{B^{\gamma-1}_{p,r}}\, d\tau ,$$

$$V_2(t) = \begin{cases} \int_0^t \|\partial_x u_{\omega,n}(\tau)\|_{B^{1/p}_{p,r}\cap \mathbf{L}^\infty}\, d\tau & \text{if } \gamma < 2 + \frac{1}{p} \\ \int_0^t \|\partial_x u_{\omega,n}(\tau)\|_{B^{\gamma-2}_{p,r}}\, d\tau & \text{otherwise} \end{cases} .$$

Since $\max\{s - \frac{3}{2}, 1 + \frac{1}{p}\} < \gamma < s - 1$, we have

$$V_1(t) \lesssim n^{\gamma-s} t \leq n^{-1} t \quad \text{and} \quad V_2(t) \leq C \int_0^t \|u_{\omega,n}(\tau)\|_{B^{\gamma}_{p,r}}\, d\tau \tag{3.17}$$

for some constant C that depends on γ, p, and r. By Theorem 2.5 and Eq (3.5), it follows that

$$V_2(t) \leq 2C \int_0^t \left\| \left(u^0_{\omega,n}, \rho^0_{\omega,n}\right) \right\|_{B^{\gamma}_{p,r}\times B^{\gamma-1}_{p,r}}\, d\tau \lesssim n^{-1} t . \tag{3.18}$$

Let $K = \max\{K_1, K_2\}$. Using (3.17) and (3.18), we combine (3.13) and (3.14) to obtain

$$\|\mathcal{E}_1^{\omega,n}(t)\|_{B^{\gamma}_{p,r}} + \|\mathcal{E}_2^{\omega,n}(t)\|_{B^{\gamma-1}_{p,r}} \lesssim \int_0^t e^{K(t-\tau)/n} \left(\|F_1(\tau)\|_{B^{\gamma}_{p,r}} + \|F_2(\tau)\|_{B^{\gamma-1}_{p,r}} \right) d\tau . \tag{3.19}$$

- Estimate for $\|F_1(\tau)\|_{B^{\gamma}_{p,r}}$: From (3.15), as $B^{\gamma}_{p,r}$ is a Banach algebra, we have

$$\begin{aligned} \|F_1\|_{B^{\gamma}_{p,r}} &\leq \|\mathcal{E}_1^{\omega,n}\|_{B^{\gamma}_{p,r}} \|\partial_x u_{\omega,n}\|_{B^{\gamma}_{p,r}} + \|\Lambda^{-1}\partial_x(\mathcal{E}_2^{\omega,n} - \mathcal{E}_1^{\omega,n})\|_{B^{\gamma}_{p,r}} + \|R_1\|_{B^{\gamma}_{p,r}} \\ &\leq \|\mathcal{E}_1^{\omega,n}\|_{B^{\gamma}_{p,r}} \|u_{\omega,n}\|_{B^{\gamma+1}_{p,r}} + \|\Lambda^{-1}\partial_x(\mathcal{E}_2^{\omega,n} - \mathcal{E}_1^{\omega,n})\|_{B^{\gamma}_{p,r}} + \|R_1\|_{B^{\gamma}_{p,r}} . \end{aligned} \tag{3.20}$$

From (2.2),

$$\|\Lambda^{-1}\partial_x(\mathcal{E}_2^{\omega,n} - \mathcal{E}_1^{\omega,n})\|_{B^{\gamma}_{p,r}} \leq \kappa \|\mathcal{E}_2^{\omega,n} - \mathcal{E}_1^{\omega,n}\|_{B^{\gamma-1}_{p,r}} \leq M \left(\|\mathcal{E}_1^{\omega,n}\|_{B^{\gamma}_{p,r}} + \|\mathcal{E}_2^{\omega,n}\|_{B^{\gamma-1}_{p,r}} \right) \tag{3.21}$$

where M is a constant depending on γ, p, and r. By Theorem 2.5, we have

$$\|u_{\omega,n}\|_{B^{\gamma+1}_{p,r}} \leq 2\left\| \left(u^0_{\omega,n}, \rho^0_{\omega,n}\right) \right\|_{B^{\gamma+1}_{p,r}\times B^{\gamma}_{p,r}} ,$$

so by (3.5), $\|u_{\omega,n}\|_{B^{\gamma+1}_{p,r}} \leq 2C_\gamma \max\{n^{-1}, n^{\gamma+1-s}\}$. As $\gamma > \max\{s-\frac{3}{2}, 1+\frac{1}{p}\}$,

$$\|u_{\omega,n}\|_{B^{\gamma+1}_{p,r}} \lesssim n^{\gamma+1-s} . \tag{3.22}$$

Using (3.21) and (3.22), from (3.20), we obtain

$$\|F_1(\tau)\|_{B^{\gamma}_{p,r}} \lesssim \left(M + n^{\gamma+1-s}\right)\|\mathcal{E}^{\omega,n}_1(\tau)\|_{B^{\gamma}_{p,r}} + M\|\mathcal{E}^{\omega,n}_2(\tau)\|_{B^{\gamma-1}_{p,r}} + \|R_1(\tau)\|_{B^{\gamma}_{p,r}} . \tag{3.23}$$

- Estimate for $\|F_2(\tau)\|_{B^{\gamma-1}_{p,r}}$: We may use the algebra property, item (2) of Lemma 2.3, for $B^{\gamma-1}_{p,r}$ since $\gamma - 1 > \max\{s-\frac{5}{2}, \frac{1}{p}\} > 0$ and the functions we are dealing with are bounded. Then, from (3.16),

$$\begin{aligned}
\|F_2\|_{B^{\gamma-1}_{p,r}} &\leq \|\mathcal{E}^{\omega,n}_2\|_{B^{\gamma-1}_{p,r}}\|\partial_x u_{\omega,n}\|_{B^{\gamma-1}_{p,r}} + \|\rho^{\omega,n}\|_{B^{\gamma-1}_{p,r}}\|\partial_x \mathcal{E}^{\omega,n}_1\|_{B^{\gamma-1}_{p,r}} \\
&\quad + \|\partial_x \rho^{\omega,n}\|_{B^{\gamma-1}_{p,r}}\|\mathcal{E}^{\omega,n}_1\|_{B^{\gamma-1}_{p,r}} + \|R_2\|_{B^{\gamma-1}_{p,r}} \\
&\lesssim n^{-1}\|\mathcal{E}^{\omega,n}_1\|_{B^{\gamma}_{p,r}} + \|\mathcal{E}^{\omega,n}_2\|_{B^{\gamma-1}_{p,r}}\|u_{\omega,n}\|_{B^{\gamma}_{p,r}} + \|R_2\|_{B^{\gamma-1}_{p,r}} .
\end{aligned} \tag{3.24}$$

By Corollary 2.6, $\|u_{\omega,n}\|_{B^{\gamma}_{p,r}} \leq 2\left\| \left(u^0_{\omega,n}, \rho^0_{\omega,n}\right) \right\|_{B^{\gamma}_{p,r}\times B^{\gamma-1}_{p,r}}$, which implies

$$\|u_{\omega,n}\|_{B^{\gamma}_{p,r}} \leq 2C_\gamma \max\{n^{-1}, n^{\gamma-s}\}$$

by (3.5). As $\gamma < s-1$, $\|u_{\omega,n}\|_{B^{\gamma}_{p,r}} \lesssim n^{-1}$. Using this in (3.24) yields

$$\|F_2(\tau)\|_{B^{\gamma-1}_{p,r}} \lesssim n^{-1}\|\mathcal{E}^{\omega,n}_1(\tau)\|_{B^{\gamma}_{p,r}} + n^{-1}\|\mathcal{E}^{\omega,n}_2(\tau)\|_{B^{\gamma-1}_{p,r}} + \|R_2(\tau)\|_{B^{\gamma-1}_{p,r}} . \tag{3.25}$$

Adding (3.23) and (3.25) gives

$$\begin{aligned}
\|F_1(\tau)\|_{B^{\gamma}_{p,r}} + \|F_2(\tau)\|_{B^{\gamma-1}_{p,r}} &\lesssim (M + n^{\gamma+1-s})\left(\|\mathcal{E}^{\omega,n}_1(\tau)\|_{B^{\gamma}_{p,r}} + \|\mathcal{E}^{\omega,n}_2(\tau)\|_{B^{\gamma-1}_{p,r}}\right) \\
&\quad + \|R_1(\tau)\|_{B^{\gamma}_{p,r}} + \|R_2(\tau)\|_{B^{\gamma-1}_{p,r}} .
\end{aligned} \tag{3.26}$$

Substituting (3.26) into (3.19), we obtain

$$\|\mathcal{E}^{\omega,n}(t)\|_{B^{\gamma}_{p,r}\times B^{\gamma-1}_{p,r}} \lesssim f(t) + \int_0^t g(\tau)\|\mathcal{E}^{\omega,n}(\tau)\|_{B^{\gamma}_{p,r}\times B^{\gamma-1}_{p,r}}\, d\tau \tag{3.27}$$

where

$$f(t) \approx \int_0^t e^{K(t-\tau)/n}\left(\|R_1(\tau)\|_{B^{\gamma}_{p,r}} + \|R_2(\tau)\|_{B^{\gamma-1}_{p,r}}\right) d\tau \tag{3.28}$$

and

$$g(\tau) \approx (M + n^{\gamma+1-s})e^{K(t-\tau)/n} \leq (M+1)e^{K(t-\tau)/n} . \tag{3.29}$$

Using Grönwall's inequality, from (3.27) we obtain

$$\|\mathcal{E}^{\omega,n}(t)\|_{B^{\gamma}_{p,r}\times B^{\gamma-1}_{p,r}} \lesssim f(t) + \int_0^t g(\tau)f(\tau)e^{\int_\tau^t g(z)\, dz}\, d\tau . \tag{3.30}$$

Using (3.12) along with (3.28) and (3.29), from (3.30), we obtain

$$\|\mathcal{E}^{\omega,n}(t)\|_{B^{\gamma}_{p,r}\times B^{\gamma-1}_{p,r}} \lesssim n^{\gamma-s-1}\,, \tag{3.31}$$

which means that $\|\mathcal{E}^{\omega,n}(t)\|_{B^{\gamma}_{p,r}\times B^{\gamma-1}_{p,r}} \to 0$ as $n \to \infty$ for any $\max\{s-\frac{3}{2}, 1+\frac{1}{p}\} < \gamma < s-1$.

On the other hand, if $\delta \in (s, s+1)$, then noting that the solution with the given data is in $B^{\delta}_{p,r} \times B^{\delta-1}_{p,r}$ for any δ we have, for $0 < t < T$ (from Theorem 2.5)

$$\begin{aligned}\|\mathcal{E}^{\omega,n}(t)\|_{B^{\delta}_{p,r}\times B^{\delta-1}_{p,r}} &\leq \|(u_{\omega,n},\rho_{\omega,n})\|_{B^{\delta}_{p,r}\times B^{\delta-1}_{p,r}} + \|(u^{\omega,n},\rho^{\omega,n})\|_{B^{\delta}_{p,r}\times B^{\delta-1}_{p,r}} \\ &\leq 2\|(u^{0}_{\omega,n},\rho^{0}_{\omega,n})\|_{B^{\delta}_{p,r}\times B^{\delta-1}_{p,r}} + \|(u^{\omega,n},\rho^{\omega,n})\|_{B^{\delta}_{p,r}\times B^{\delta-1}_{p,r}}\,,\end{aligned} \tag{3.32}$$

where we have used the solution size estimate in Theorem 2.5. Now, for $\delta < s+1$, Eqs (3.5) and (3.6) imply that $\|(u^{0}_{\omega,n},\rho^{0}_{\omega,n})\|_{B^{\delta}_{p,r}\times B^{\delta-1}_{p,r}} \leq C_{\delta} n^{\delta-s}$ and $\|(u^{\omega,n},\rho^{\omega,n})\|_{B^{\delta}_{p,r}\times B^{\delta-1}_{p,r}} \leq C_{\delta} n^{\delta-s}$, where C_{δ} denotes a constant that depends only on δ, for a given p and r. So (3.32) yields

$$\|\mathcal{E}^{\omega,n}(t)\|_{B^{\delta}_{p,r}\times B^{\delta-1}_{p,r}} \lesssim n^{\delta-s}\,. \tag{3.33}$$

We use the interpolation property, item (3) from Lemma 2.3, with $\theta = \frac{\delta-s}{\delta-\gamma}$, to obtain

$$\|\mathcal{E}^{\omega,n}(t)\|_{B^{s}_{p,r}\times B^{s-1}_{p,r}} \leq \|\mathcal{E}^{\omega,n}(t)\|^{\theta}_{B^{\gamma}_{p,r}\times B^{\gamma-1}_{p,r}} \|\mathcal{E}^{\omega,n}(t)\|^{1-\theta}_{B^{\delta}_{p,r}\times B^{\delta-1}_{p,r}}\,. \tag{3.34}$$

From (3.34), using (3.31) and (3.33), we obtain

$$\|\mathcal{E}^{\omega,n}(t)\|_{B^{s}_{p,r}\times B^{s-1}_{p,r}} \lesssim \left(n^{\gamma-s-1}\right)^{\frac{\delta-s}{\delta-\gamma}} \left(n^{\delta-s}\right)^{\frac{s-\gamma}{\delta-\gamma}} = n^{-\theta}\,. \tag{3.35}$$

As $\theta \in (0,1)$, (3.35) implies that $\|\mathcal{E}^{\omega,n}(t)\|_{B^{s}_{p,r}\times B^{s-1}_{p,r}} \to 0$ as $n \to \infty$ for any $s > \max\{2+\frac{1}{p}, \frac{5}{2}\}$. This completes the proof of Lemma 3.2.

□

When $p = r = 2$, $B^{s}_{2,2}$ and H^{s} are equivalent by [2, Proposition 1.2], and so we obtain the following corollary:

Corollary 3.3. *The data-to-solution map for the two-component FW system* (1.2) *is not uniformly continuous from any bounded subset of* $H^{s}(\mathbb{T}) \times H^{s-1}(\mathbb{T})$ *into* $C([0,T]; H^{s}(\mathbb{T})) \times C([0,T]; H^{s-1}(\mathbb{T}))$ *for* $s > \frac{5}{2}$.

4. Conclusions

In this paper, we considered the two-component Fornberg-Whitham (FW) system (1.2) and used a sequential approach to prove that its data-to-solution map is not uniformly continuous for periodic initial data belonging to Besov spaces $B^{s}_{p,r}(\mathbb{T}) \times B^{s-1}_{p,r}(\mathbb{T})$ where $s > \max\{2+\frac{1}{p}, \frac{5}{2}\}$. As a corollary, this establishes non-uniform dependence on periodic initial data for the FW system (1.2) in Sobolev spaces $H^{s}(\mathbb{T}) \times H^{s-1}(\mathbb{T})$ for $s > \frac{5}{2}$.

Appendix

In this appendix, we provide a lower bound on $\|\cos(nx)\|_{B^s_{p,r}}$ for any $s > 0$ and $1 \leq r < \infty$. By Definition 2.2,

$$\|\cos(nx)\|_{B^s_{p,r}} = \left(\sum_{q\geq -1} 2^{sqr} \|\Delta_q \cos nx\|^r_{\mathbf{L}^p} \right)^{\frac{1}{r}} . \tag{4.1}$$

By Definition 2.1, $\Delta_q \cos(nx) = \varphi_q(n)e^{inx}$. Therefore, $\|\Delta_q \cos(nx)\|_{\mathbf{L}^p} = \varphi_q(n)$, where $0 < \varphi_q(n) \leq 1$ for all q such that $\frac{1}{\ln(2)} \ln\left(\frac{3}{8}n\right) \leq q \leq \frac{1}{\ln(2)} \ln\left(\frac{4}{3}n\right)$ and $\varphi_q(n) = 0$ otherwise, (4.1) implies that

$$\|\cos(nx)\|_{B^s_{p,r}} = \left(\sum_{q=\frac{1}{\ln(2)}\ln\left(\frac{3}{8}n\right)}^{\frac{1}{\ln(2)}\ln\left(\frac{4}{3}n\right)} (2^q)^{sr} \varphi^r_q(n) \right)^{\frac{1}{r}} .$$

Since $2^q \geq \frac{3}{8}n$ for all terms in the summation, from the above we have

$$\|\cos(nx)\|_{B^s_{p,r}} \geq \left(\frac{3}{8}\right)^s n^s \left(\sum_{q=\frac{1}{\ln(2)}\ln\left(\frac{3}{8}n\right)}^{\frac{1}{\ln(2)}\ln\left(\frac{4}{3}n\right)} \varphi^r_q(n) \right)^{\frac{1}{r}} . \tag{4.2}$$

Recall that $\varphi_0(\xi) = \chi\left(\frac{\xi}{2}\right) - \chi(\xi)$ and $\varphi_q(\xi) = \varphi_0(2^{-q}\xi)$ for any $q > -1$, where supp $\chi = [-\frac{4}{3}, \frac{4}{3}]$ and $\chi = 1$ on $[-\frac{3}{4}, \frac{3}{4}]$. This means that supp $\varphi_q = [\frac{3}{4} \cdot 2^q, \frac{8}{3} \cdot 2^q]$ for any $q \geq 1$ and furthermore, $\varphi_q = 1$ on the interval $[\frac{4}{3} \cdot 2^q, \frac{3}{2} \cdot 2^q]$. In other words, $\varphi_q(n) = 1$ for $\frac{1}{\ln(2)} \ln\left(\frac{2}{3}n\right) \leq q \leq \frac{1}{\ln(2)} \ln\left(\frac{3}{4}n\right)$. Therefore, from (4.2) we have

$$\begin{aligned}
\|\cos(nx)\|_{B^s_{p,r}} &\geq \left(\frac{3}{8}\right)^s n^s \left(\sum_{q=\frac{1}{\ln(2)}\ln\left(\frac{2}{3}n\right)}^{\frac{1}{\ln(2)}\ln\left(\frac{3}{4}n\right)} 1 \right)^{\frac{1}{r}} \\
&= \left(\frac{3}{8}\right)^s n^s \left(\frac{1}{\ln(2)} \left[\ln\left(\frac{3}{4}n\right) - \ln\left(\frac{2}{3}n\right) \right] \right)^{\frac{1}{r}} \\
&= \left(\frac{1}{\ln(2)} \ln\left(\frac{9}{8}\right) \right)^{\frac{1}{r}} \left(\frac{3}{8}\right)^s n^s = C_s n^s,
\end{aligned}$$

where C_s is a constant that depends only on s, for a given p and r. The same estimate holds for $\|\sin(nx)\|_{B^s_{p,r}}$ as well.

Author contributions

All authors contributed equally towards conceptualization, formal analysis, investigation and methodology in this project; Writing of the original draft was done by Prerona Dutta; thereafter all authors together completed the review and editing process. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

We thank John Holmes, at The Ohio State University, for his valuable suggestions on this project. We would also like to thank the anonymous referees for their comments which greatly helped in improving the paper overall.

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. H. Bahouri, J. Chemin, R. Danchin, *Fourier analysis and nonlinear partial differential equations*, Springer-Verlag Berlin Heidelberg, 2011.
2. R. Danchin, A few remarks on the Camassa-Holm equation, *Differ. Integral Equ.*, **14** (2001), 953–988.
3. R. Danchin, Fourier analysis methods for PDE's, *Lect. Notes*, 2005. Available from: `https://perso.math.u-pem.fr/danchin.raphael/cours/courschine.pdf.`
4. P. Dutta, Well-posedness of the two-component Fornberg-Whitham system in Besov spaces, *La Mat.*, **3** (2024), 704–720. https://doi.org/10.1007/s44007-024-00103-3
5. X. Fan, L. Tian, S. Yang, J. Yin, Bifurcations of traveling wave solutions for a two-component Fornberg-Whitham equation, *Commun. Nonlinear Sci. Numer. Simulat.*, **16** (2011), 3956–3963. https://doi.org/10.1016/j.cnsns.2011.02.010
6. A. A. Himonas, G. Misiolek, Non-uniform dependence on initial data of solutions to the Euler equations of hydrodynamics, *Commun. Math. Phys.*, **296** (2010), 285–301. https://doi.org/10.1007/s00220-010-0991-1
7. J. Holmes, R. C. Thompson, Well-posedness and continuity properties of the Fornberg-Whitham equation in Besov spaces, *J. Differ. Equations*, **263** (2017), 4355–4381. https://doi.org/10.1016/j.jde.2017.05.019
8. J. Holmes, F. Tiglay, Non-uniform dependence of the data-to-solution map for the Hunter-Saxton equation in Besov spaces, *J. Evol. Equ.*, **18** (2018), 1173–1187. https://doi.org/10.1007/s00028-018-0436-4
9. T. Kato, The Cauchy problem for quasi-linear symmetric hyperbolic systems, *Arch. Ration. Mech. Anal.*, **58** (1975), 181–205. https://doi.org/10.1007/BF00280740
10. B. L. Keyfitz, F. Tiglay, Non-uniform dependence on initial data for compressible gas dynamics: The periodic Cauchy problem, *J. Differ. Equations*, **263** (2017), 6494–6511. https://doi.org/10.1016/j.jde.2017.07.020

11. F. Xu, Y. Zhang, F. Li, The well-posedness, blow-up, and traveling waves for a two-component Fornberg-Whitham system, *J. Math. Phys.*, **62** (2021), 1–17. https://doi.org/10.1063/5.0030436

12. Y. Yu, J. Li, Non-uniform dependence of the data-to-solution map for the two-component Fornberg-Whitham system, *Ann. Mat. Pur. Appl.*, **202** (2022), 59–76. https://doi.org/10.1007/s10231-022-01232-8

AIMS Press

© 2024 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (https://creativecommons.org/licenses/by/4.0)