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Research article

Global stability of traveling fronts of a diffusion system with the Belousov-Zhabotinskii reaction

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Abstract: This paper studied the asymptotic stability of traveling fronts of the Belousov-Zhabotinskii (BZ for short) system. Under the condition that the initial perturbation decays as $|x| \to \infty$, we came to the conclusion that the traveling fronts were globally exponentially stable. The main method was the super and sub-solutions combined with a squeezing technique.

Keywords: global stability; Belousov-Zhabotinskii; traveling fronts; super and sub solutions; squeezing technique
Mathematics Subject Classification: 35K40, 35K57, 35C07

1. Introduction

This paper aims to study the stability of the following diffusion equation with the Belousov-Zhabotinskii (BZ) reaction:

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + u(x,t)(1 - u(x,t) - rv(x,t)), \\ v_t(x,t) = \Delta v(x,t) - bu(x,t)v(x,t), \end{cases} \tag{1.1}$$

where r, b are positive parameters and u, v corresponds to the concentration of bromous acid and bromide ion, respectively. The BZ chemical reaction is a famous oscillation reaction discovered by Belousov. About ten years later, Zaikin and Zhabotinskii observed traveling wave phenomena in such a chemical activity [32], and then Field and his coworkers [4, 5] established a model to describe chemical wave in the BZ reaction. Later, based on experimental and numerical results, Murray [21, 22] nondimensionalized the model to be system (1.1).

Since the traveling wave solution of (1.1) is found to be an appropriate mathematical tool to describe the chemical wave observed in the BZ reaction, it has attracted a lot of attention, for example, see the recent works [3, 6, 7, 15, 26, 27, 33] and references therein for the study of traveling wave solutions of (1.1).

It is natural to ask whether the aforementioned traveling wave solutions of (1.1) are stable or not, since the stability of traveling waves is also very important (but more difficult) to reaction diffusion equations. There are several methods to prove the stability of traveling wave solutions, among which three methods are frequently-used; see [8, 24, 31] for the spectral analysis method, see [17–20] for the weighted energy method, and see [1,25,29] for the squeezing technique combined with the comparison principle.

For scalar equations $u_t = \Delta u + f(u), x \in \mathbb{R},\ t > 0$, the stability of traveling waves has been well studied; see [8, 11, 13, 23, 30, 31] and the references therein. For the systems of reaction diffusion equations, Kapitula [10] considered a semi-linear parabolic system. Using semigroup theory, Kapitula proved that the wave fronts are stable in polynomially weighted L^∞ spaces, and the convergence speed is given by the detail semigroup estimate. By detail spectral analysis, Sattinger [23] proved that the traveling wave fronts were stable to perturbations in some exponentially weighted L^∞ spaces. Kan-on and Fang [9] obtained the asymptotic stability of monotone traveling waves for a competition-diffusion system by using spectral analysis. Kessler and Levine [12] investigated linear stability as well as nonlinear stability of the traveling wave solutions in a piecewise linear Oregonator model arising in the BZ reaction, but they did not estimate the convergence speed of the traveling waves. Lv and Wang [16] studied the asymptotic stability of a cooperative Lotka-Volterra system by the weighted energy method and obtained the time decay rates, which is also valid for the BZ system. However, their results depend mainly on the condition that the initial perturbations are in a weighted H^1 space. Recently, Du et al. [2] and Wang et al. [28] also considered the stability of the BZ system with delay, but their stability results are based on the weighted energy method or the weighted spaces.

With all the above in mind, the purpose of this article is to study the asymptotic stability of traveling fronts of system (1.1) by the squeezing technique combined with the comparison principle. Note that in system (1.1), r is a key parameter that decides the characteristic of the BZ system. Precisely, the BZ system is mono-stable if $r \in (0, 1]$, while it is bistable if $r > 1$.

In the current paper, we always assume that $r > 1$ and $b > 0$, i.e., we study the bistable case. Let $u_1(x,t) = u(x,t), u_2(x,t) = 1 - v(x,t)$, $\mathbf{u} = (u_1, u_2)$ and $\mathbf{F}(\mathbf{u}) = (f_1(\mathbf{u}), f_2(\mathbf{u})) = (u_1(1 - r - u_1 + ru_2), bu_1(1 - u_2))$, then system (1.1) can be rewritten as

$$\mathbf{u}_t(x,t) = \Delta \mathbf{u}(x,t) + \mathbf{F}(\mathbf{u}(x,t)), \quad x \in \mathbb{R},\ t > 0. \tag{1.2}$$

We should emphasize that the 'bistable' case here is not standard. Actually, it is easy to see that $(0, u_2)$ is a steady state of (1.2) for any $u_2 \in \mathbb{R}$, which means that (1.2) is degenerate at the equilibrium (0,0).

It is known from [7, 26] that system (1.2) admits a unique (up to translation) positive traveling front $(\mathbf{U}(\xi), c)$, $\mathbf{U}(\xi) = (U_1(\xi), U_2(\xi))$, satisfying $U_1(\xi) < U_2(\xi)$ and

$$\begin{cases} U_1''(\xi) - cU_1'(\xi) + U_1(\xi)(1 - r - U_1(\xi) + rU_2(\xi)) = 0, \\ U_2''(\xi) - cU_2'(\xi) + bU_1(\xi)(1 - U_2(\xi)) = 0, \\ 0 < U_i(\xi) < 1, U_i(-\infty) = 0, U_i(+\infty) = 1, i = 1, 2. \end{cases} \tag{1.3}$$

The main result of this paper is stated as follows.

Theorem 1.1. *Assume $b > 0$ and $r > 1$. If the initial value $\mathbf{u}_0(x) \in [\mathbf{0}, \mathbf{1}]$ satisfies*

$$\limsup_{x\to+\infty} \left|u_{0,i}(x) - U_i(x)\right| = 0 \quad \text{and} \quad \limsup_{x\to-\infty} \left|u_{0,i}(x) - U_i(x)\right| e^{-\alpha_i\lambda_2 x} = 0, \ \ i = 1, 2, \tag{1.4}$$

where $\lambda_2 = c$, then the solution $\mathbf{u}(x, t; \mathbf{u}_0)$ of (1.2) with $\mathbf{u}(x, 0; \mathbf{u}_0) = \mathbf{u}_0(x)$ satisfies

$$\frac{|u_i(x, t; \mathbf{u}_0) - U_i(x + ct + \xi_0)|}{U_2^{\alpha_i}(x + ct + \xi_0)} \le Ce^{-kt}, \quad x \in \mathbb{R}, \ t > 0, \ \ i = 1, 2, \tag{1.5}$$

where $0 < \alpha_2 < \alpha_1 < 1$, and $C, k > 0, \xi_0 \in \mathbb{R}$ are some constants.

Remark: This result implies that the traveling front $\mathbf{U}(x + ct)$ is asymptotically stable under initial perturbations that decay as $|x| \to \infty$, but can be possibly large in any finite intervals. The convergence rate is exponential.

To use the comparison argument, we modify system (1.2). Let $\tilde{\mathbf{F}}(\mathbf{u}) = \mathbf{F}(\mathbf{u}) + \mathbf{G}(\mathbf{u})$, where $\mathbf{G}(\mathbf{u}) = (g_1(\mathbf{u}), g_2(\mathbf{u}))$ with

$$g_1(\mathbf{u}) = ru_2 \max\{0, -u_1\}, \qquad g_2(\mathbf{u}) = b(u_1 - 1) \max\{0, u_2 - 1\}.$$

It is easy to check that

$$\tilde{f}_i(\mathbf{u}) = f_i(\mathbf{u}), \ \partial_{u_j}\tilde{f}_i(\mathbf{u}) = \partial_{u_j} f_i(\mathbf{u}) \quad \text{if } \mathbf{0} \le u_i \le \mathbf{1}, \ i = 1, 2 \text{ with } j \ne i,$$

and

$$\partial_{u_j}\tilde{f}_i(\mathbf{u}) \ge 0 \quad \text{if } (u_1, u_2) \in \mathbb{R}^2 \ \text{ and } \ j \ne i.$$

Thus, the comparison principle works for the following Cauchy problem:

$$\begin{cases} \tilde{\mathbf{v}}_t(x, t) = \tilde{\mathbf{v}}_{xx}(x, t) + \tilde{\mathbf{F}}(\tilde{\mathbf{v}}(x, t)), & x \in \mathbb{R}, t > 0, \\ \tilde{\mathbf{v}}(x, 0) = \mathbf{v}_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.6}$$

That is, if we denote the solution of (1.6) by $\tilde{\mathbf{v}}(x, t; \mathbf{v}_0)$, then $\tilde{\mathbf{v}}(x, t; \mathbf{v}_0^1) \le \tilde{\mathbf{v}}(x, t; \mathbf{v}_0^2)$ if $\mathbf{v}_0^1(x) \le \mathbf{v}_0^2(x)$. It is also easy to see that $[\mathbf{0}, \mathbf{1}]$ is the invariant interval for the solution of (1.6), namely, if $\mathbf{v}_0(x) \in [\mathbf{0}, \mathbf{1}]$, then $\tilde{\mathbf{v}}(x, t; \mathbf{v}_0) \in [\mathbf{0}, \mathbf{1}]$. Thus, for $\mathbf{v}_0(x) \in [\mathbf{0}, \mathbf{1}]$, the solution $\tilde{\mathbf{v}}(x, t; \mathbf{v}_0)$ of (1.6) is also the solution of (1.2) with the same initial data, i.e., $\tilde{\mathbf{v}}(x, z, t; \mathbf{v}_0) \equiv \mathbf{v}(x, z, t; \mathbf{v}_0)$, where $\mathbf{v}(x, z, t; \mathbf{v}_0)$ denotes the solution of (1.2) with initial data $\mathbf{v}_0$.

The outline of this paper is as follows. In Section 2, we give some notations and known results. In Section 3, we construct some super and sub-solutions for later use. The final section is devoted to the proof of the main theorem.

2. Preliminary

First we introduce some notations.

For vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, we define their order relationships. We use $\mathbf{x} < \mathbf{y}$ to mean $x_i < y_i, i = 1, 2$, and $\mathbf{x} \le \mathbf{y}$ to mean $x_i \le y_i, i = 1, 2$. The interval is $[\mathbf{x}_1, \mathbf{x}_2] := \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x}_1 \le \mathbf{x} \le \mathbf{x}_2\}$. Particularly, we denote $\mathbf{0} = (0, 0)$ and $\mathbf{1} = (1, 1)$.

Now, we list the asymptotic behaviors of the wavefront profile $\mathbf{U}(\xi)$ at the space infinity; see [26, Lemmas 13 and 14].

$$U_1(\xi+\xi_1)=e^{\lambda_1\xi}+O(e^{(2\lambda_2-\sigma)\xi}),\ U_2(\xi+\xi_1)=Ae^{\lambda_2\xi}+O(e^{(\lambda_1-\sigma)\xi}),\quad \xi\to-\infty,$$
$$U_1'(\xi)=\lambda_1 e^{\lambda_1\xi}+O(e^{(2\lambda_2-\sigma)\xi}),\ U_2'(\xi)=A\lambda_2e^{\lambda_2\xi}+O(e^{(\lambda_1-\sigma)\xi}),\ \ \xi\to-\infty,$$

where $\lambda_2=c$ and λ_1 is the positive root of the equation $x^2-cx+1-r=0$, and

$$(U_1(\xi+\xi_0),U_1'(\xi+\xi_0))=\begin{cases}-Ae^{\tilde{\zeta}_1\xi}(1,\tilde{\zeta}_1)+\left(1-\dfrac{re^{\zeta_1\xi}}{b-1},-\dfrac{r\zeta_1e^{\zeta_1\xi}}{b-1}\right)+O\left(e^{(\zeta_1-\sigma)\xi}\right),\ b\neq1,\\ \left(1-\dfrac{r(\xi+d)e^{\zeta_1\xi}}{c-2\zeta_1},-\dfrac{r(\xi+d_1)\zeta_1e^{\zeta_1\xi}}{c-2\zeta_1}\right)+O\left(e^{(\zeta_1-\sigma)\xi}\right),\ b=1,\end{cases}$$
$$(U_2(\xi+\xi_0),U_2'(\xi+\xi_0))=\left(1-e^{\zeta_1\xi},-\zeta_1e^{\zeta_1\xi}\right)+O\left(e^{(\zeta_1-\sigma)\xi}\right),\ \xi\to+\infty,$$

where $\tilde{\zeta}_1=\frac{1}{2}(c-\sqrt{c^2+4})$ and $\zeta_1=\frac{1}{2}(c-\sqrt{c^2+4b})$. In the above, $A>0$, $\sigma>0$, ξ_0, ξ_1, d, and d_1 are appropriate constants. Since $\lambda_1>\lambda_2$, it is easy to see that

$$\lim_{\xi\to-\infty}\frac{U_2'(\xi)}{U_2(\xi)}=\lambda_2=c.$$

We can find two positive constants L_1,L_2 such that

$$L_1e^{\max\{\lambda_1,2\lambda_2\}\xi}<U_1(\xi),U_1'(\xi)<L_2e^{\min\{\lambda_1,2\lambda_2\}\xi},\quad \xi<0. \tag{2.1}$$

$$L_1e^{\lambda_1\xi}<U_2(\xi),U_2'(\xi)<L_2e^{\lambda_2\xi},\quad \xi<0. \tag{2.2}$$

In the following of this section, we give some notations. Denote $f_{ij}(\mathbf{u})=\frac{\partial f_i(\mathbf{u})}{\partial u_j}$, and let

$$D\mathbf{F}(\mathbf{u})=(f_{ij}(\mathbf{u}))_{2\times2}=\begin{pmatrix}1-r-2u_1+ru_2 & ru_1\\ b(1-u_2) & -bu_1\end{pmatrix}.$$

We introduce a vector $\mathbf{q}:=(q_1,q_2)\in(\mathbf{0},\mathbf{1})$ and denote its transpose by $\mathbf{q}^T$. Notice the fact $r>1$, then we can find an appropriate $\mathbf{q}$ with $q_1>rq_2$ such that

$$D\mathbf{F}(\mathbf{1})\cdot\mathbf{q}^T=(-q_1+rq_2,-bq_2)<\mathbf{0}.$$

Since $D\mathbf{F}(\mathbf{u})$ is continuous in $\mathbf{u}$, we can fix a small number $\varepsilon\in(0,1)$ such that

$$D\mathbf{F}(\mathbf{u})\cdot\mathbf{q}^T\leq\frac{1}{2}D\mathbf{F}(\mathbf{1})\cdot\mathbf{q}^T \tag{2.3}$$

for any $\mathbf{u}\in[(1-\varepsilon)\mathbf{1},(1+\varepsilon)\mathbf{1}]$.

Throughout this paper, we always denote

$$N_1:=\sup_{x\in\mathbb{R}}\left|\frac{U_2'(x)}{U_2(x)}\right|,\quad N_2:=\sup_{x\in\mathbb{R}}\left|\frac{U_2''(x)}{U_2(x)}\right|\quad M_0:=\sup_{1\leq i,j\leq2}\sup_{\mathbf{u}\in[-\mathbf{1},\mathbf{2}]}|f_{ij}(\mathbf{u})|,$$

and

$$\Pi_1(x):=x^2-cx+1-r,\quad \Pi_2(x):=x^2-cx.$$

Obviously, $\Pi_i(\alpha_i\lambda_2)<0$ for $i=1,2$; see Theorem 1.1 for α_i.

3. Super-sub-solutions and some technical lemmas

In this section, we establish the super and sub-solutions as well as some technical lemmas, which will be used in the proof of the main result.

Define the operator $\tilde{\mathcal{L}}$ by

$$\tilde{\mathcal{L}}[\mathbf{v}] := \mathbf{v}_t - \mathbf{v}_{xx} - \tilde{\mathbf{F}}(\mathbf{v}).$$

Lemma 3.1. *Let $r > 1$ and $b > 0$. Denote $\beta_0 := \min\left\{\frac{\min_{i=1,2}\{-\frac{1}{4}\Pi_i(\alpha_i\lambda_2)\}}{N_1+1}, \frac{q_1-rq_2}{16q_1}, \frac{b}{8}\right\}$, then for each fixed $\beta \in (0,\beta_0]$, there exist a positive number $\rho = \rho(\beta)$ and a $\delta = \delta(\rho)$ such that the functions defined by*

$$\mathbf{v}^{\pm}(x,t) = \mathbf{U}(x + ct + \xi \pm \rho\delta(1 - e^{-\beta t})) \pm \delta e^{-\beta t}\boldsymbol{\mathcal{U}}^{\boldsymbol{\alpha}}(x + ct + \xi \pm \rho\delta(1 - e^{-\beta t}))$$

are a super-solution and a sub-solution to (1.6) *on $t \in [0,\infty)$, respectively, where*

$$\boldsymbol{\mathcal{U}}^{\boldsymbol{\alpha}}(\xi) := \left(q_1 U_2^{\alpha_1}(\xi), q_2 U_2^{\alpha_2}(\xi)\right).$$

Proof. First, we prove that $\mathbf{v}^+$ is a super-solution, i.e.,

$$\tilde{\mathcal{L}}[\mathbf{v}^+]_i = \partial_t v_i^+(x,t) - \partial_{xx} v_i^+(x,t) - \tilde{f}_i(\mathbf{v}^+(x,t)) \geq 0, \quad i = 1,2.$$

Let $\eta^+ = x + ct + \xi + \rho\delta(1 - e^{-\beta t})$. A direct computation gives that

$$\eta_t^+ = c + \rho\delta\beta e^{-\beta t}, \quad \eta_x^+ = 1, \quad \eta_{xx}^+ = 0.$$

For any $t \geq 0$, we have

$$\begin{aligned}
\tilde{\mathcal{L}}[\mathbf{v}^+]_i &= U_i'(\eta^+)\eta_t^+ - \delta\beta e^{-\beta t} q_i U_2^{\alpha_i}(\eta^+) + \delta e^{-\beta t}\alpha_i q_i U_2^{\alpha_i - 1}(\eta^+)U_2'(\eta^+)\eta_t^+ \\
&\quad - U_i''(\eta^+) - \delta e^{-\beta t} q_i U_2^{\alpha_i}(\eta^+)\left[\alpha_i(\alpha_i - 1)\left(\frac{U_2'(\eta^+)}{U_2(\eta^+)}\right)^2 + \alpha_i\frac{U_2''(\eta^+)}{U_2(\eta^+)}\right] \\
&\quad - f_i(\mathbf{U}(\eta^+)) + f_i(\mathbf{U}(\eta^+)) - \tilde{f}_i(\mathbf{v}^+(x,t)) \\
&= \rho\delta\beta e^{-\beta t} U_i'(\eta^+) + \delta e^{-\beta t} q_i U_2^{\alpha_i}(\eta^+)\Bigg[-\beta + \alpha_i\left(c + \rho\delta\beta e^{-\beta t}\right)\frac{U_2'(\eta^+)}{U_2(\eta^+)} \\
&\quad - \alpha_i(\alpha_i - 1)\left(\frac{U_2'(\eta^+)}{U_2(\eta^+)}\right)^2 - \alpha_i\frac{U_2''(\eta^+)}{U_2(\eta^+)}\Bigg] + f_i(\mathbf{U}(\eta^+)) - \tilde{f}_i(\mathbf{v}^+(x,t)) \\
&\geq \rho\delta\beta e^{-\beta t} U_i'(\eta^+) + \delta e^{-\beta t} q_i U_2^{\alpha_i}(\eta^+)\Bigg[-\beta + \alpha_i c\frac{U_2'(\eta^+)}{U_2(\eta^+)} - \alpha_i^2\left(\frac{U_2'(\eta^+)}{U_2(\eta^+)}\right)^2 \\
&\quad + \alpha_i\left(\frac{U_2'(\eta^+)}{U_2(\eta^+)}\right)^2 - \alpha_i\frac{U_2''(\eta^+)}{U_2(\eta^+)}\Bigg] + f_i(\mathbf{U}(\eta^+)) - \tilde{f}_i(\mathbf{v}^+(x,t)),
\end{aligned}$$

since $\rho\delta\beta e^{-\beta t} U_i'(\eta^+) > 0$. Then, we consider three cases.

Case 1. $\eta^+ < -X$ for $X > 0$ large enough.

By the fact $\lim_{x\to-\infty} U_2'(x) = 0$, there exists a constant $X_1 > 0$ such that $U_2(\eta^+) \leq \frac{1}{2}$ for any $\eta^+ \leq -X_1$, and following this we have

$$0 < v_2^+(x,t) \leq \frac{1}{2} + \frac{1}{2}\delta < 1, \quad \forall\, \eta^+ \leq -X_1.$$

This implies that $\tilde{f}_i(\mathbf{v}^+(x,t)) = f_i(\mathbf{v}^+(x,t))$ for $\eta^+ \le -X_1$ and $i = 1,2$.

Moreover, since $\lambda_2 = \lim\limits_{x\to-\infty} \frac{U_2'(x)}{U_2(x)}$ and $\lambda_2^2 = \lim\limits_{x\to-\infty} \frac{U_2''(x)}{U_2(x)}$, we have

$$-\left(\frac{U_2'(x)}{U_2(x)}\right)^2 + \frac{U_2''(x)}{U_2(x)} \to 0,$$

$$\left(\alpha_1 \frac{U_2'(x)}{U_2(x)}\right)^2 - c\alpha_1 \frac{U_2'(x)}{U_2(x)} + 1 - r \to \Pi_1(\alpha_1\lambda_2) < 0,$$

and

$$\left(\alpha_2 \frac{U_2'(x)}{U_2(x)}\right)^2 - c\alpha_2 \frac{U_2'(x)}{U_2(x)} \to \Pi_2(\alpha_2\lambda_2) < 0$$

as $x \to -\infty$. Thus, there exists $X_2 > 0$ large enough such that

$$\left|-\left(\frac{U_2'(x)}{U_2(x)}\right)^2 + \frac{U_2''(x)}{U_2(x)}\right| < -\frac{1}{8}\Pi_i(\alpha_i\lambda_2), \tag{3.1}$$

$$\left(\alpha_1 \frac{U_2'(x)}{U_2(x)}\right)^2 - c\alpha_1 \frac{U_2'(x)}{U_2(x)} + 1 - r < \frac{1}{2}\Pi_1(\alpha_1\lambda_2), \tag{3.2}$$

and

$$\left(\alpha_2 \frac{U_2'(x)}{U_2(x)}\right)^2 - c\alpha_2 \frac{U_2'(x)}{U_2(x)} < \frac{1}{2}\Pi_2(\alpha_2\lambda_2) \tag{3.3}$$

for any $x < -X_2$.

For the reaction term, we have

$$f_i(\mathbf{v}^+(x,t)) - f_i(\mathbf{U}(\eta^+)) = \left(\sum_{j=1}^{2} f_{ij}\left(\eta_{\tau_i}\right) q_j U_2^{\alpha_j}(\eta^+)\right)\delta e^{-\beta t},$$

where $\eta_{\tau_i} := \mathbf{U}(\eta^+) + \tau_i\delta e^{-\beta t}\boldsymbol{\mathcal{U}}^{\boldsymbol{\alpha}}(\eta^+)$ with $\tau_i \in (0,1)$, $i = 1,2$. If $i = 1$,

$$\begin{aligned}
&\frac{1}{q_1 U_2^{\alpha_1}(\eta^+)} \sum_{j=1}^{2} f_{1j}(\eta_{\tau_1}) q_j U_2^{\alpha_j}(\eta^+) \\
&= f_{11}(\eta_{\tau_1}) + r\left(U_1(\eta^+) + \tau_1\delta e^{-\beta t} q_1 U_2^{\alpha_1}(\eta^+)\right)\frac{q_2 U_2^{\alpha_2}(\eta^+)}{q_1 U_2^{\alpha_1}(\eta^+)} \\
&\le f_{11}(\eta_{\tau_1}) + r(1+\delta)U_2^{\alpha_2}(\eta^+) \\
&\to 1 - r
\end{aligned}$$

as $\eta^+ \to -\infty$. If $i = 2$, by the fact that $\lim_{x\to-\infty} U_2^{\alpha_1-\alpha_2}(x) = 0$, we have

$$\frac{1}{q_1 U_2^{\alpha_1}(\eta^+)} \sum_{j=1}^{2} f_{2j}(\eta_{\tau_2}) q_j U_2^{\alpha_j}(\eta^+) = f_{21}(\eta_{\tau_2})\frac{q_1}{q_2} U_2^{\alpha_1-\alpha_2}(\eta^+) + f_{21}(\eta_{\tau_2}) \to 0$$

as $\eta^+ \to -\infty$. It follows that there exists $X_3 > 0$ large enough such that

$$\frac{1}{q_1 U_2^{\alpha_1}(\eta^+)} \sum_{j=1}^{2} f_{1j}(\eta_{\tau_1}) q_j U_2^{\alpha_j}(\eta^+) - (1-r) \leq -\frac{1}{8}\Pi_1(\alpha_1\lambda_2), \tag{3.4}$$

and

$$\frac{1}{q_2 U_2^{\alpha_2}(\eta^+)} \sum_{j=1}^{2} f_{2j}(\eta_{\tau_2}) q_j U_2^{\alpha_j}(\eta^+) \leq -\frac{1}{8}\Pi_2(\alpha_2\lambda_2) \tag{3.5}$$

for any $\eta^+ < -X_3$.

Take $X = \max\{X_1, X_2, X_3\}$. Then, for any $\eta^+ < -X$, (3.1)–(3.5) yield that

$$\begin{aligned} \tilde{\mathcal{L}}[\mathbf{v}^+]_i \ &\geq\ \delta e^{-\beta t} q_i U_2^{\alpha_i}(\eta^+) \Bigg\{ -\beta - \alpha_i \left| -\left(\frac{U_2'(\eta^+)}{U_2(\eta^+)}\right)^2 + \frac{U_2''(\eta^+)}{U_2(\eta^+)} \right| \\ &\quad -\left(\alpha_i \frac{U_2'(\eta^+)}{U_2(\eta^+)}\right)^2 + c\alpha_i \frac{U_2'(\eta^+)}{U_2(\eta^+)} \Bigg\} + f_i(\mathbf{U}(\eta^+)) - f_i(\mathbf{v}^+(x,t)) \\ &\geq\ \delta e^{-\beta t} q_i U_2^{\alpha_i}(\eta^+) \left\{ -\beta + \frac{1}{8}\Pi_i(\alpha_i\lambda_2) - \frac{1}{2}\Pi_i(\alpha_i\lambda_2) + \frac{1}{8}\Pi_i(\alpha_i\lambda_2) \right\} \\ &\geq\ 0, \end{aligned}$$

provided that $\beta \leq \min_{i=1,2}\left\{-\frac{1}{4}\Pi_i(\alpha_i\lambda_2)\right\}$.

Case 2. $\eta^+ > X'$ for $X' > 0$ large enough.

In this case, it is not difficult to see that

$$\tilde{\mathcal{L}}[\mathbf{v}^+]_i \geq \delta e^{-\beta t} q_i U_2^{\alpha_i}(\eta^+) \left(-\beta - \alpha_i \frac{U_2''(\eta^+)}{U_2(\eta^+)} \right) + f_i(\mathbf{U}(\eta^+)) - \tilde{f}_i(\mathbf{v}^+(x,t)).$$

Recall the definition of $\tilde{f}_i$. We know

$$\tilde{f}_i(\mathbf{v}^+(x,z)) - f_i(\mathbf{U}(\eta^+)) \leq \left(\sum_{j=1}^{2} f_{ij}\left(\eta_{\tau_i}\right) q_j U_2^{\alpha_j}(\eta^+) \right) \delta e^{-\beta t} + b\delta^2 e^{-2\beta t} U_2^{\alpha_i}(\eta^+) q_1 q_2,$$

where $\eta_{\tau_i} := \mathbf{U}(\eta^+) + \tau_i \delta e^{-\beta t} \boldsymbol{\mathcal{U}}^{\boldsymbol{\alpha}}(\eta^+)$ with $\tau_i \in (0,1), i = 1,2$. Since $\lim_{x\to\infty} U_2(x) = 1$, there exists $X_1' > 0$ large enough such that

$$0 < U_2^{\alpha_2-\alpha_1}(\eta^+) - 1 \leq \frac{q_1 - rq_2}{4q_1}, \quad \forall \eta^+ \geq X_1'. \tag{3.6}$$

Moreover, there exists $X_2' > 0$ large enough such that

$$(1-\varepsilon)\mathbf{1} < \eta_{\tau_i} < (1+\varepsilon)\mathbf{1}$$

for any $\eta^+ > X_2'$, provided that $\delta \leq \varepsilon$. If $i = 1$, it follows from (2.3) that for any $\eta^+ > X_2'$, we have

$$\frac{1}{q_1 U_2^{\alpha_1}(\eta^+)} \sum_{j=1}^{2} f_{1j}(\eta_{\tau_1}) q_j U_2^{\alpha_j}(\eta^+)$$

$$
\begin{aligned}
&= \frac{1}{q_1}\left[f_{11}(\eta_{\tau_1})q_1 + f_{12}(\eta_{\tau_1})q_2\right] + f_{12}(\eta_{\tau_1})\frac{q_2}{q_1}\left(1 - U_2^{\alpha_2-\alpha_1}(\eta^+)\right)\\
&\le \frac{-q_1 + rq_2}{2q_1} + \frac{q_1 - rq_2}{4q_1} = -\frac{q_1 - rq_2}{4q_1}.
\end{aligned}
$$

If $i = 2$, it follows (2.3) and (3.6) that for any $\eta^+ > \max\{X'_1, X'_2\}$, we have

$$
\begin{aligned}
&\frac{1}{q_2 U_2^{\alpha_2}(\eta^+)} \sum_{j=1}^{2} f_{2j}(\eta_{\tau_1}) q_j U_2^{\alpha_j}(\eta^+)\\
&= \frac{1}{q_2}\left[f_{11}(\eta_{\tau_1})q_1 + f_{12}(\eta_{\tau_1})q_2\right] + f_{21}(\eta_{\tau_2})\frac{q_1}{q_2}\left(U_2^{\alpha_1-\alpha_2}(\eta^+) - 1\right)\\
&\le -\frac{b}{2}.
\end{aligned}
$$

Recalling the asymptotic behaviors of U_2, we can choose a positive X'_3 large enough such that for any $\eta^+ > X'_3$, it holds

$$
\left|\frac{U_2''(\eta^+)}{U_2(\eta^+)}\right| \le \frac{1}{2}\min\left\{\frac{q_1 - rq_2}{4q_1}, \frac{b}{2}\right\}.
$$

Take $X' = \max\{X'_1, X'_2, X'_3\}$. Then, for any $\eta^+ > X'$, we have

$$
\begin{aligned}
\tilde{\mathcal{L}}[\mathbf{v}^+]_i &\ge \delta e^{-\beta t} q_i U_2^{\alpha_i}(\eta^+)\left\{-\beta - \frac{1}{2}\min\left\{\frac{q_1 - rq_2}{4q_1}, \frac{b}{2}\right\}\right.\\
&\quad \left. + \min\left\{\frac{q_1 - rq_2}{4q_1}, \frac{b}{2}\right\} - b\delta\right\}\\
&\ge 0,
\end{aligned}
$$

provided that

$$
\beta \le \frac{1}{4}\min\left\{\frac{q_1 - rq_2}{4q_1}, \frac{b}{2}\right\}, \quad \delta \le \frac{1}{4b}\min\left\{\frac{q_1 - rq_2}{4q_1}, \frac{b}{2}\right\}.
$$

Case 3. $-X \le \eta^+ \le X'$.

Let $u_* := \min_{i=1,2} \min_{-X \le x \le X'} U'_i(x)$. It is easy to see that

$$
\begin{aligned}
\tilde{f}_i(\mathbf{v}^+(x, z)) - f_i(\mathbf{U}(\eta^+)) &\le \left(\sum_{j=1}^{2} f_{ij}(\eta_{\tau_i})\, q_j U_2^{\alpha_j}(\eta^+)\right)\delta e^{-\beta t} + b\delta^2 e^{-2\beta t} U_2^{\alpha_i}(\eta^+) q_1 q_2\\
&\le (2M_0 + b)\delta e^{-\beta t}.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
\tilde{\mathcal{L}}[\mathbf{v}^+]_i &\ge \delta e^{-\beta t}\left(\rho\beta U'_i(\eta^+) - q_i\beta - q_i\left|\frac{U_2''(\eta^+)}{U_2(\eta^+)}\right|\right) + f_i(\mathbf{U}(\eta^+)) - \tilde{f}_i(\mathbf{v}^+(x, t))\\
&\ge \delta e^{-\beta t}(\rho\beta u_* - 1 - N_2 - 2M_0 - b)\\
&\ge 0,
\end{aligned}
$$

provided that $\rho > \frac{1+N_2+2M_0+b}{\beta u_*}$.

Combing the three cases above, we have proved that $\mathbf{v}^+(x,t)$ is a super-solution. In the following, we prove that $\mathbf{v}^-(x,t)$ is a sub-solution.

Let $\eta^- = x + ct + \xi - \rho\delta(1 - e^{-\beta t})$. Similarly, we have

$$\begin{aligned}\tilde{\mathcal{L}}[\mathbf{v}^-]_i &= -\rho\delta\beta e^{-\beta t}U_i'(\eta^-) + \delta e^{-\beta t}q_iU_2^{\alpha_i}(\eta^-)\Bigg[\beta - \alpha_i\left(c - \rho\delta\beta e^{-\beta t}\right)\frac{U_2'(\eta^-)}{U_2(\eta^-)} \\ &\quad +\alpha_i(\alpha_i-1)\left(\frac{U_2'(\eta^-)}{U_2(\eta^-)}\right)^2 - \alpha_i\frac{U_2''(\eta^-)}{U_2(\eta^-)}\Bigg] + f_i(\mathbf{U}(\eta^-)) - \tilde{f}_i(\mathbf{v}^-(x,t)).\end{aligned}$$

We also divide the whole interval into three parts.

Case 1. $\eta^- < -Y$ for some $Y > 0$ large enough.

Recalling the definition of $\tilde{f}_i$, we have $\tilde{f}_2(\mathbf{v}^-(x,t)) = f_2(\mathbf{v}^-(x,t))$ and

$$\begin{aligned}&\frac{1}{q_1U_2^{\alpha_1}(\eta^-)}\left(f_1(\mathbf{U}(\eta^-)) - \tilde{f}_1(\mathbf{v}^-(x,t))\right) \\ &= \frac{1}{q_1U_2^{\alpha_1}(\eta^-)}\left(\sum_{j=1}^{2} f_{1j}(\eta_{\tau_1})q_jU_2^{\alpha_j}(\eta^-) - rv_2^-(\eta^-)\max\{0, -v_1^-(\eta^-)\}\right) \\ &\le f_{11}(\eta_{\tau_1}) + r\left(U_1(\eta^-) + \tau_1\delta e^{-\beta t}q_1U_2^{\alpha_1}(\eta^-)\right)\frac{q_2U_2^{\alpha_2}(\eta^-)}{q_1U_2^{\alpha_1}(\eta^-)} + \frac{r|v_1^-(\eta^-)||v_2^-(\eta^-)|}{q_1U_2^{\alpha_1}(\eta^-)} \\ &\le f_{11}(\eta_{\tau_1}) + r(1+\delta q_1)U_2^{\alpha_2}(\eta^-) + \frac{r|v_1^-(\eta^-)||v_2^-(\eta^-)|}{q_1U_2^{\alpha_1}(\eta^-)} \\ &\to 1 - r\end{aligned}$$

as $\eta^- \to -\infty$, since

$$\lim_{x\to-\infty}\frac{|v_1^-(\eta^-)||v_2^-(\eta^-)|}{q_1U_2^{\alpha_1}(\eta^-)} = 0.$$

Thus, there exists $Y_1 > 0$ large enough such that

$$\frac{1}{q_1U_2^{\alpha_1}(\eta^-)}\left(f_1(\mathbf{U}(\eta^-)) - \tilde{f}_1(\mathbf{v}^-(x,t))\right) - (1-r) \le -\frac{1}{8}\Pi_1(\alpha_1\lambda_2). \tag{3.7}$$

Take $Y = \max\{Y_1, X\}$. It then follows from (3.1)–(3.3), (3.5), and (3.7) that

$$\begin{aligned}\tilde{\mathcal{L}}[\mathbf{v}^-]_i &= -\rho\delta\beta e^{-\beta t}U_i'(\eta^-) + \delta e^{-\beta t}q_iU_2^{\alpha_i}(\eta^-)\Bigg\{\beta + \rho\delta\beta e^{-\beta t}\frac{U_2'(\eta^-)}{U_2(\eta^-)} \\ &\quad + \left(\alpha_i\frac{U_2'(\eta^-)}{U_2(\eta^-)}\right)^2 - c\alpha_i\frac{U_2'(\eta^-)}{U_2(\eta^-)} + \alpha_i\Bigg[-\left(\frac{U_2'(\eta^-)}{U_2(\eta^-)}\right)^2 + \frac{U_2''(\eta^-)}{U_2(\eta^-)}\Bigg]\Bigg\} \\ &\quad + f_i(\mathbf{U}(\eta^-)) - \tilde{f}_i(\mathbf{v}^-(x,t)) \\ &\le \delta e^{-\beta t}q_iU_2^{\alpha_i}(\eta^-)\left(\beta + \rho\beta N_1 + \frac{1}{2}\Pi_i(\alpha_i\lambda_2) - \frac{1}{8}\Pi_i(\alpha_i\lambda_2) - \frac{1}{8}\Pi_i(\alpha_i\lambda_2)\right) \\ &\le 0,\end{aligned}$$

provided that $\delta < 1/\rho$ and $\beta + \beta N_1 \le \min_{i=1,2}\left\{-\frac{1}{4}\Pi_i(\alpha_i\lambda_2)\right\}$.

Case 2. $\eta^- > Y'$ for some $Y' > 0$ large enough.

Since $\lim_{x\to\infty} \frac{U_2'(x)}{U_2(x)} = 0, \lim_{x\to\infty} \frac{U_2''(x)}{U_2(x)} = 0$, there exists $Y_1' > 0$ large enough such that

$$\frac{U_2'(\eta^-)}{U_2(\eta^-)} + \left|\frac{U_2''(\eta^-)}{U_2(\eta^-)}\right| \le \frac{1}{2}\min\left\{\frac{q_1 - rq_2}{2q_1}, \frac{b}{2}\right\}. \tag{3.8}$$

By the definition of $\mathbf{v}^-$, there exists a positive Y_1' large enough such that $v_1(\eta^-) > 0$ for all $\eta^- > Y_1'$. This implies that $\tilde{f}_1(\mathbf{v}^-) = f_1(\mathbf{v}^-)$. Take $Y' = \max\{Y_1', X'\}$, and we have

$$\begin{aligned}\tilde{\mathcal{L}}[\mathbf{v}^-]_i &\le \delta e^{-\beta t} q_i U_2^{\alpha_i}(\eta^-)\left(\beta + \frac{U_2'(\eta^-)}{U_2(\eta^-)} + \left|\frac{U_2''(\eta^-)}{U_2(\eta^-)}\right|\right) + f_i(\mathbf{U}(\eta^-)) - f_i(\mathbf{v}^-(x,t)) \\ &\le \delta e^{-\beta t} q_i U_2^{\alpha_i}(\eta^-)\left\{\beta + \frac{1}{2}\min\left\{\frac{q_1 - rq_2}{2q_1}, \frac{b}{2}\right\} - \min\left\{\frac{q_1 - rq_2}{2q_1}, \frac{b}{2}\right\}\right\} \\ &\le 0,\end{aligned}$$

provided that $\beta \le \frac{1}{4}\min\left\{\frac{q_1 - rq_2}{q_1}, b\right\}$.

Case 3. $Y \le \eta^- \le Y'$.

For simplicity, we still define $u_* := \min_{i=1,2} \min_{-Y\le x\le Y'} U_i'(x)$. It is easy to see that

$$|g_1(\mathbf{v}^-(x,t))| = |rv_2^-(x,t)\max\{0, -v_1^-(x,t)\}| \le r\delta e^{-\beta t}.$$

Thus, we have

$$\begin{aligned}\tilde{\mathcal{L}}[\mathbf{v}^-]_i &\le \delta e^{-\beta t}\left(-\rho\beta U_i'(\eta^-) + \rho\delta\beta e^{-\beta t}\frac{U_2'(\eta^-)}{U_2(\eta^-)} - \alpha_i \frac{U_2''(\eta^-)}{U_2(\eta^-)}\right) \\ &\quad + f_i(\mathbf{U}(\eta^-)) - \tilde{f}_i(\mathbf{v}^-(x,t)) \\ &\le \delta e^{-\beta t}(-\rho\beta u_* + N_1 + N_2 + 2M_0 + r) \\ &\le 0,\end{aligned}$$

provided that $\rho\beta u_* \ge N_1 + N_2 + 2M_0 + r$. Finally, let

$$\rho \ge \max\left\{\frac{1}{\beta}\cdot\frac{\max\{1, N_1\} + N_2 + 2M_0 + \max\{b, r\}}{u_*}, 1\right\},$$

and $0 < \delta < \delta_0 := \min\left\{\frac{q_1 - rq_2}{16bq_1}, \frac{1}{8}, \varepsilon, \frac{1}{\rho}\right\}$. The proof is completed. □

We introduce an auxiliary lemma below before moving on.

Lemma 3.2. *For any pair of super-solution and sub-solution $\boldsymbol{\omega}^\pm(x,t) \in [\mathbf{-1}, \mathbf{2}]$ with $\boldsymbol{\omega}^+(x,0) \ge \boldsymbol{\omega}^-(x,0)$, there holds $\boldsymbol{\omega}^+(x,t) \ge \boldsymbol{\omega}^-(x,t)$ for $t \ge 0$ and $x \in \mathbb{R}$. Furthermore, one has*

$$\omega_i^+(x,t) - \omega_i^-(x,t) \ge \theta(J,t)\int_z^{z+1}(\omega_i^+(y,0) - \omega_i^-(y,0))dy, \quad i = 1,2, \tag{3.9}$$

for any $x \in \mathbb{R}$ with $|x - z| \le J$, where $\theta(J,t) = \frac{1}{\sqrt{4\pi t}}\exp\left\{-m_0 t - \frac{(J+1)^2}{4t}\right\}$ with m_0 as a positive constant.

Proof. The comparison principle for parabolic systems implies $\boldsymbol{\omega}^+(x,t) \geq \boldsymbol{\omega}^-(x,t)$ directly. Let $\boldsymbol{\omega} := \boldsymbol{\omega}^+ - \boldsymbol{\omega}^-$. Then, $\boldsymbol{\omega}$ satisfies

$$\begin{aligned} \partial_t \boldsymbol{\omega} &\geq \partial_{xx}\boldsymbol{\omega} + \tilde{F}(\boldsymbol{\omega}^+) - \tilde{F}(\boldsymbol{\omega}^-) \\ &\geq \partial_{xx}\boldsymbol{\omega} + D\tilde{F}(\eta_{\tau_i}) \cdot \boldsymbol{\omega}, \end{aligned}$$

where $\eta_{\tau_i} = \tau_i \omega^+ + (1-\tau_i)\omega^-$ with $\tau_i \in (0,1)$. In view of $\tilde{f}_{ij}(\eta_{\tau_i}) \geq 0$ for $i \neq j$, we have

$$\begin{aligned} \partial_t \omega_i &\geq \partial_{xx}\omega_i + \tilde{f}_{ii}(\eta_{\tau_i})\omega_i \\ &\geq \partial_{xx}\omega_i - m_0\omega_i, \end{aligned}$$

where $m_0 := \sup_{\mathbf{u}\in[-\mathbf{1},\mathbf{2}], i=1,2} |f_{ii}(\mathbf{u})|$. Thus, we obtain

$$\omega_i(x,t) \quad \geq \quad e^{-m_0 t} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} \exp\left\{-\frac{(x-y)^2}{4t}\right\} \omega_i(y,0)dy,$$

and following this we get (3.9). □

Lemma 3.3. *Assume $r > 1$ and $b > 0$. Let $\tilde{\mathbf{v}}(x,t) \in [\mathbf{0},\mathbf{1}]$ be a solution of* (1.6)*. If $\tilde{\mathbf{v}}(x,t)$ satisfies*

$$\mathbf{U}(x+cT+\xi) - \delta \boldsymbol{\mathcal{U}}^{\alpha}(x+cT+\xi) \leq \tilde{\mathbf{v}}(x,T) \leq \mathbf{U}(x+cT+\xi+h) + \delta \boldsymbol{\mathcal{U}}^{\alpha}(x+cT+\xi+h)$$

for some $T \geq 0$, where $\xi \in \mathbb{R}, h > 0$ are some constants, then for any $t \geq T+1$, there exists a positive number ϵ^* such that

$$\begin{aligned} &\mathbf{U}(x+ct+\hat{\xi}(t)) - \hat{\delta}(t)\boldsymbol{\mathcal{U}}^{\alpha}(x+ct+\hat{\xi}(t)) \\ &\leq \tilde{\mathbf{v}}(x,t) \\ &\leq \mathbf{U}(x+ct+\hat{\xi}(t)+\hat{h}(t)) + \hat{\delta}(t)\boldsymbol{\mathcal{U}}^{\alpha}(x+ct+\hat{\xi}(t)+\hat{h}(t)), \end{aligned}$$

where $\hat{\xi}(t), \hat{\delta}(t)$, and $\hat{h}(t)$ satisfy

$$\xi - \rho\delta + \epsilon^* \min\{1,h\} \leq \hat{\xi}(t) \leq \xi + 2\rho\epsilon^* \min\{1,h\},$$

$$\hat{\delta}(t) = (\epsilon^* \min\{1,h\} + \delta e^{-\beta})e^{-\beta[t-(T+1)]},$$

and

$$0 < \hat{h}(t) \leq h + 2\rho\delta - \rho\epsilon^* \min\{1,h\}.$$

Here, δ and ρ are defined as in Lemma 3.1.

Proof. Obviously, $\mathbf{v}(x,t) = \tilde{\mathbf{v}}(x,t+T)$ is also a solution to (1.6) with $\mathbf{v}(x,0) = \tilde{\mathbf{v}}(x,T)$. It then follows from Lemma 3.1 and the comparison principle that

$$\begin{aligned} &\mathbf{U}\left(x+c(T+t)+\xi-\rho\delta(1-e^{-\beta t})\right) - \delta e^{-\beta t}\boldsymbol{\mathcal{U}}^{\alpha}\left(x+c(T+t)+\xi-\rho\delta(1-e^{-\beta t})\right) \\ &\leq \tilde{\mathbf{v}}(x,t+T) \leq \mathbf{U}\left(x+c(T+t)+\xi+h+\rho\delta(1-e^{-\beta t})\right) \\ &\qquad + \delta e^{-\beta t}\boldsymbol{\mathcal{U}}^{\alpha}\left(x+c(T+t)+\xi+h+\rho\delta(1-e^{-\beta t})\right) \end{aligned}$$

for all $x \in \mathbb{R},\ t \geq 0$.

Let $\epsilon_1 = \frac{1}{2}\min_{i=1,2}\{U_i'(\eta) : |\eta - cT - \xi| \le 2\} > 0$ and $\bar{h} = \min\{1, h\}$. By the mean value theorem, we obtain

$$\int_0^1 \Big(U_i(y + cT + \xi + \bar{h}) - U_i(y + cT + \xi)\Big)\, dy \ge 2\epsilon_1\bar{h}.$$

Then, at least one of the following is true:

$$\text{(i)} \int_0^1 (\tilde{v}_i(y, T) - U_i(y + cT + \xi))\, dy \ge \epsilon_1\bar{h},$$
$$\text{(ii)} \int_0^1 (U_i(y + cT + \xi + h) - \tilde{v}_i(y, T))\, dy \ge \epsilon_1\bar{h}.$$

We consider case (i) only, and the case (ii) is similar. Since $\alpha_i < 1$, we have

$$\lim_{x\to-\infty} \frac{U_i'(x)}{U_i^{\alpha_i}(x)} = 0, \quad i = 1, 2.$$

Thus, there exists a $M_3 > 0$ such that

$$\frac{U_i'(x)}{U_i^{\alpha_i}(x)} \le \frac{1}{2\rho}, \quad \text{for } |x| \ge M_3,\ i = 1, 2. \tag{3.10}$$

Let $J_1 = M_3 + c + 2, z_0 = -cT - \xi$ and $J_2 = J_1 + c + 3$. Now, we divide $\mathbb{R}$ into two intervals:

$$\Omega_1 = \{x \in \mathbb{R} \,|\, |x - z_0| \le J_1\} \quad \text{and} \quad \Omega_2 = \mathbb{R} \setminus \Omega_1.$$

If $x \in \Omega_1$, letting $t = 1$ and $\bar{\theta} = \theta(J_1 + |z_0|, 1)$, then by (3.9) we have

$$\begin{aligned}
&\tilde{v}_i(x, T + 1)\\
&\ge U_i(x+c(T + 1)+\xi-\rho\delta(1-e^{-\beta}))-\delta e^{-\beta}U_2^{\alpha_i}(x+c(T+1)+\xi-\rho\delta(1-e^{-\beta}))\\
&\quad +\bar{\theta}\int_0^1 (\tilde{v}_i(y, T) - U_i(y + cT + \xi))\, dy\\
&\ge U_i(x - z_0 + c - \rho\delta(1 - e^{-\beta})) - \delta e^{-\beta}U_2^{\alpha_i}(x - z_0 + c - \rho\delta(1 - e^{-\beta})) + \bar{\theta}\epsilon_1\bar{h}.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&U_i(x - z_0 + c + 2\rho\epsilon^*\bar{h} - \rho\delta(1 - e^{-\beta})) - U_i(x - z_0 + c - \rho\delta(1 - e^{-\beta}))\\
&= 2\rho\epsilon^*\bar{h}U_i'(\eta_1) \le \bar{\theta}\epsilon_1\bar{h},
\end{aligned}$$

where $\eta_1 = x - z_0 + c - \rho\delta(1 - e^{-\beta}) + \theta_i \cdot 2\rho\epsilon^*\bar{h}$ with $\theta_i \in (0, 1)$ and

$$\epsilon^* < \min\left\{\frac{1}{2\rho}, \min_{i=1,2}\min_{|\eta|\le J_2} \frac{\bar{\theta}\epsilon_1}{2\rho U_i'(\eta)}\right\}.$$

Clearly, $|\eta_1| \le |x - z_0| + c + \rho\delta(1 - e^{-\beta}) + 2\epsilon^*\bar{h} \le J_2$, and, thus,

$$\begin{aligned}
\tilde{v}_i(x, T + 1) \ \ge\ & U_i(x - z_0 + c + 2\rho\epsilon^*\bar{h} - \rho\delta(1 - e^{-\beta}))\\
& -\delta e^{-\beta}U_2^{\alpha_i}(x - z_0 + c + 2\rho\epsilon^*\bar{h} - \rho\delta(1 - e^{-\beta})).
\end{aligned}$$

On the other hand, it is obvious that

$$\begin{aligned}|\eta_1| &\geq |x-z_0|-(c+\rho\delta(1-e^{-\beta})+2\epsilon^*\bar{h})\\ &\geq J_1-(c+\rho\delta(1-e^{-\beta})+2\epsilon^*\bar{h})\geq M_3.\end{aligned}$$

Thus, by (3.10) we have

$$\begin{aligned}&U_i(x-z_0+c-\rho\delta(1-e^{-\beta}))-U_i(x-z_0+c+2\rho\epsilon^*\bar{h}-\rho\delta(1-e^{-\beta}))\\ &=-2\rho\epsilon^*\bar{h}\cdot U_i'(\eta_1)\geq -U_i^{\alpha_i}(\eta_1)\epsilon^*\bar{h}\geq -U_2^{\alpha_i}(\eta_1)\epsilon^*\bar{h}.\end{aligned}$$

Then, for $x\in\Omega_2$, we have

$$\begin{aligned}\tilde{v}_i(x,T+1) &\geq U_i(x-z_0+c+2\rho\epsilon^*\bar{h}-\rho\delta(1-e^{-\beta}))-U_2^{\alpha_i}(\eta_1)\epsilon^*\bar{h}\\ &\quad -\delta e^{-\beta}U_2^{\alpha_i}(x-z_0+c+2\rho\epsilon^*\bar{h}-\rho\delta(1-e^{-\beta}))\\ &\geq U_i(x-z_0+c+2\rho\epsilon^*\bar{h}-\rho\delta(1-e^{-\beta}))\\ &\quad -(\epsilon^*\bar{h}+\delta e^{-\beta})U_2^{\alpha_i}(x-z_0+c+2\rho\epsilon^*\bar{h}-\rho\delta(1-e^{-\beta})).\end{aligned}$$

Combining the above two cases, we know that

$$\begin{aligned}\tilde{v}_i(x,T+1) &\geq U_i(x-z_0+c+2\rho\epsilon^*\bar{h}-\rho\delta(1-e^{-\beta}))\\ &\quad -(\epsilon^*\bar{h}+\delta e^{-\beta})U_2^{\alpha_i}(x-z_0+c+2\rho\epsilon^*\bar{h}-\rho\delta(1-e^{-\beta}))\end{aligned}$$

holds for all $x\in\mathbb{R}$ and $i=1,2$. Denote $\eta_2=2\rho\epsilon^*\bar{h}-\rho\delta(1-e^{-\beta})$. Then, the comparison principle implies that

$$\begin{aligned}\tilde{v}_i(x,t) &\geq U_i\left(x+ct+\xi+\eta_2-\rho(\epsilon^*\bar{h}+\delta e^{-\beta})\left(1-e^{-\beta[t-(T+1)]}\right)\right)\\ &\quad -(\epsilon^*\bar{h}+\delta e^{-\beta})e^{-\beta[t-(T+1)]}U_2^{\alpha_i}\left(x+ct+\xi+\eta_2-\rho(\epsilon^*\bar{h}+\delta e^{-\beta})\left(1-e^{-\beta[t-(T+1)]}\right)\right)\\ &= U_i\left(x+ct+\hat{\xi}(t)\right)-\hat{\delta}(t)U_2^{\alpha_i}\left(x+ct+\hat{\xi}(t)\right)\end{aligned}$$

for all $t\geq T+1$, where

$$\hat{\xi}(t)=\xi+2\rho\epsilon^*\bar{h}-\rho\delta(1-e^{-\beta})-\rho(\epsilon^*\bar{h}+\delta e^{-\beta})\left(1-e^{-\beta[t-(T+1)]}\right),$$

and

$$\hat{\delta}(t)=(\epsilon^*\bar{h}+\delta e^{-\beta})e^{-\beta[t-(T+1)]}.$$

A direct computation gives that

$$\xi+\rho\epsilon^*\bar{h}-\rho\delta\leq\hat{\xi}(t)\leq\xi+2\rho\epsilon^*\bar{h},\quad\forall t\geq T+1.$$

Similarly, for any $t\geq T+1$, we have

$$\begin{aligned}\tilde{v}_i(x,t) &\leq U_i\left(x+ct+\xi+h+\rho\delta(1-e^{-\beta(t-T)})\right)\\ &\quad +\delta e^{-\beta(t-T)}U_2^{\alpha_i}\left(x+ct+\xi+h+\rho\delta(1-e^{-\beta(t-T)}\right)\\ &\leq U_i\left(x+ct+\hat{\xi}(t)+\hat{h}(t)\right)+\hat{\delta}(t)U_2^{\alpha_i}\left(x+ct+\hat{\xi}(t)+\hat{h}(t)\right),\end{aligned}$$

where

$$\hat{h}(t)=h+2\rho\delta\left(1-e^{-\beta(t-T)}\right)-\rho\epsilon^*\bar{h}-\rho\epsilon^*\bar{h}e^{-\beta[t-(T+1)]}.$$

It is easy to verify that

$$0<h-2\rho\epsilon^*\bar{h}+2\rho\delta(1-e^{-\beta})\leq\hat{h}(t)\leq h-\rho\epsilon^*\bar{h}+2\rho\delta.$$

The proof is completed. □

4. Asymptotic stability of traveling fronts

We prove the main result in this section.

Lemma 4.1. *Let* $\mathbf{u}_0(x) \in C(\mathbb{R}, [\mathbf{0}, \mathbf{1}])$ *satisfy* (1.4). *Then, for some constants* z_0, z_1, z_2, *the solution* $\tilde{\mathbf{v}}(x, t; \mathbf{u}_0)$ *of* (1.6) *satisfies*

$$U_i(x + ct - z_1) - q_i(x, t) \leq \tilde{v}_i(x, t; \mathbf{u}_0) \leq U_i(x + ct - z_2) + q_i(x, t) \tag{4.1}$$

for all $x \in \mathbb{R},\ t > 0$, *where* $q_i(x, t) = q_{0,i} e^{-\epsilon t} \min\{e^{\alpha_i \lambda_2 (x+ct-z_0)}, 1\}$ *with* $q_{0,i} = q_0 q_i, i = 1, 2,$ *and*

$$0 < \epsilon \leq \frac{1}{2} \min\left\{\frac{q_1 - rq_2}{2q_1}, \frac{b}{2}\right\},\ \ 0 < q_0 \leq \min\left\{\frac{1}{2b} \min\left\{\frac{q_1 - rq_2}{2q_1}, \frac{b}{2}\right\}, \frac{\varepsilon}{q_2}\right\}.$$

Proof. Let $z = x + ct$, and define

$$\omega_i^{\pm}(x, t) := U_i(z \pm \eta(t)) \pm q_i^{\pm}(x, t), \quad i = 1, 2,$$

where $q_i^{\pm}(x, t) = q_{0,i} e^{-\epsilon t} \min\{e^{\alpha_i \lambda_2 (z - z_0 \pm \eta(0))}, 1\}$, $\epsilon > 0$ will be chosen later, and $\eta(t)$ is bounded and to be chosen so that $\eta'(t) > 0$. To prove (4.1), we need only to prove that $\boldsymbol{\omega}^{\pm}(x, t) = (\omega_1^{\pm}(x, t), \omega_2^{\pm}(x, t))$ are a pair of super and sub-solutions by the comparison principle. Precisely, we need to prove that

$$\tilde{\mathcal{L}}[\boldsymbol{\omega}^{\pm}] = \boldsymbol{\omega}_t^{\pm} - \boldsymbol{\omega}_{xx}^{\pm} - \tilde{\mathbf{F}}(\boldsymbol{\omega}^{\pm}) \geq (\leq) \mathbf{0}.$$

We give the proof for $\boldsymbol{\omega}^{+}(x, t)$ first. A direct calculation gives that

$$\begin{aligned}
&\tilde{\mathcal{L}}[\boldsymbol{\omega}^+]_i \\
&= (\eta'(t) + c) U_i'(z + \eta(t)) + \partial_t q_i^+(x, t) - U_i''(z + \eta(t)) - \partial_{xx} q_i^+(x, t) - \tilde{f}_i(\boldsymbol{\omega}^+(x, t)) \\
&= \eta'(t) U_i'(z + \eta(t)) + \partial_t q_i^+(x, t) - \partial_{xx} q_i^+(x, t) + f_i(\mathbf{U}(z + \eta(t))) - \tilde{f}_i(\boldsymbol{\omega}^+(x, t)),
\end{aligned}$$

and

$$\partial_t q_i^+(x, t) - \partial_{xx} q_i^+(x, t) = \begin{cases} -\epsilon q_i^+(x, t) + [-(\alpha_i \lambda)^2 + c(\alpha_i \lambda)] q_i^+(x, t), & \text{if } z < z_0 - \eta(0), \\ -\epsilon q_i^+(x, t), & \text{if } z \geq z_0 - \eta(0). \end{cases}$$

Take $\delta \in (0, \varepsilon]$ small enough, and consider three cases.

Case 1. $U_i(z + \eta(t)) \in [\delta, 1 - \delta], i = 1, 2.$

Define $C_\delta := \min_{i=1,2} \min_{U_i(x) \in [\delta, 1-\delta]} U_i'(x)$. Then, we have

$$\begin{aligned}
\tilde{\mathcal{L}}[\boldsymbol{\omega}^+]_i &= \eta'(t) U_i'(z + \eta(t)) - \epsilon q_i^+(x, t) + [-(\alpha_i \lambda)^2 + c(\alpha_i \lambda)] q_i^+(x, t) \\
&\quad + f_i(\mathbf{U}(z + \eta(t))) - \tilde{f}_i(\boldsymbol{\omega}^+(x, t)) \\
&\geq \eta'(t) C_\delta - \epsilon q_{0,i} e^{-\epsilon t} - M_0 (q_{0,1} + q_{0,2}) e^{-\epsilon t} - b q_{0,1} q_{0,2} e^{-2\epsilon t} \\
&\geq 0,
\end{aligned}$$

provided that $\eta'(t) C_\delta \geq (\epsilon + 2M_0 + \max\{b, r\}) e^{-\epsilon t}$.

Case 2. $U_i(z + \eta(t)) \geq 1 - \delta.$

By the monotonicity of $U_i(\cdot)$, we know $z + \eta(t) \geq U_i^{-1}(1 - \delta)$. It follows that there exists a constant z_0 such that $z \geq z_0 - \eta(0)$. Thus, we have

$$\begin{aligned}\tilde{\mathcal{L}}[\boldsymbol{\omega}^+]_i &= \eta'(t)U_i'(z + \eta(t)) - \epsilon q_i^+(x,t) + f_i(\mathbf{U}(z + \eta(t))) - \tilde{f}_i(\boldsymbol{\omega}^+(x,t)) \\ &\geq \eta'(t)U_i'(z + \eta(t)) - \epsilon q_i^+(x,t) \\ &\quad - \sum_{j=1}^{2} f_{ij}\left(\mathbf{U}(z + \eta(t)) + \tau_i \mathbf{q}^+(x,t)\right) q_j^+(x,t) - b q_1^+(x,t) q_2^+(x,t).\end{aligned}$$

It is obvious that

$$1 - \varepsilon \leq U_i(z + \eta(t)) + \tau_i q_i^+(x,t) \leq 1 + \varepsilon.$$

Recalling (2.3), we have

$$\begin{aligned}\tilde{\mathcal{L}}[\boldsymbol{\omega}^+]_i &\geq -\epsilon q_i^+(x,t) + q_0 e^{-\epsilon t} \min\left\{\frac{q_1 - r q_2}{2}, \frac{b q_2}{2}\right\} - b q_0^2 q_1 q_2 e^{-2\epsilon t} \\ &\geq 0,\end{aligned}$$

provided that $\epsilon \leq \frac{1}{2}\min\left\{\frac{q_1 - r q_2}{2q_1}, \frac{b}{2}\right\}$ and $q_0 \leq \frac{1}{2b}\min\left\{\frac{q_1 - r q_2}{2q_1}, \frac{b}{2}\right\}$.

Case 3. $U_i(z + \eta(t)) < \delta$.

Obviously, $z \leq z_0 - \eta(0)$ in this case for the same z_0 defined in Case 2. Then, we have

$$\begin{aligned}\tilde{\mathcal{L}}[\boldsymbol{\omega}^+]_i &\geq -\epsilon q_i^+(x,t) + [-(\alpha_i\lambda)^2 + c(\alpha_i\lambda)] q_i^+(x,t) \\ &\quad - \sum_{j=1}^{2} f_{ij}\left(\mathbf{U}(z + \eta(t)) + \tau_i \mathbf{q}^+(x,t)\right) q_j^+(x,t).\end{aligned}$$

Recalling (2.1), we have

$$\frac{f_{12}\left(\mathbf{U}(z + \eta(t)) + \tau_i \mathbf{q}^+(x,t)\right)}{q_1^+(x,t)} q_2^+(x,t) = \frac{r\left(U_1(z + \eta(t)) + \tau_1 q_1^+(x,t)\right)}{q_1^+(x,t)} q_2^+(x,t) \to 0$$

as $z \to -\infty$. Thus, if $i = 1$, it follows that

$$\frac{1}{q_1^+(x,t)} \sum_{j=1}^{2} f_{1j}\left(\mathbf{U}(z + \eta(t)) + \tau_1 \mathbf{q}^+\right) q_j^+(x,t) \to 1 - r, \quad \text{as } z \to -\infty.$$

Moreover,

$$\frac{q_1^+(x,t)}{q_2^+(x,t)} = \frac{q_{0,1}}{q_{0,2}} e^{(\alpha_1 - \alpha_2)\lambda_2(z - z_0 + \eta(0))} \to 0, \quad \text{as } z \to -\infty.$$

Thus, if $i = 2$,

$$\frac{1}{q_2^+(x,t)} \sum_{j=1}^{2} f_{2j}\left(\mathbf{U}(z + \eta(t)) + \tau_1 \mathbf{q}^+\right) q_j^+(x,t) \to 0, \quad \text{as } z \to -\infty.$$

Combing the facts above and taking $\delta > 0$ small enough, we have

$$\mathcal{L}[\omega^+]_i \quad \geq \quad q_i^+(x,t)\left(-\epsilon - \Pi_i(\alpha_i\lambda_2) + \frac{1}{2}\Pi_i(\alpha_i\lambda_2)\right) \geq 0,$$

provided that $\epsilon \leq \min_{i=1,2}\left\{-\frac{1}{2}\Pi_i(\alpha_i\lambda)\right\}$.

Now we need to prove that $\mathbf{u}_0(x) \leq \omega^+(x,0)$ for all $x \in \mathbb{R}$. In fact, it follows from (1.4) for any $q_{0,i} > 0$ that there exists a positive constant M large enough such that for $x < -M$, it holds

$$U_i(x) + q_{0,i}e^{\alpha_i\lambda_2 x} \geq u_{0,i}(x), \quad i = 1,2.$$

Using (1.4) again, we can find a positive constant M' large enough such that for $x > M'$, it holds

$$U_i(x) + q_{0,i} \geq 1 \geq u_{0,i}(x), \quad i = 1,2.$$

For the case $-M \leq X \leq M'$, we can choose $\eta(0)$ sufficiently large to guarantee $\omega_i^+(x,0) \geq u_{0,i}(x)$. Therefore, $\omega^+(x,t)$ is a super-solution to (1.6).

In the following, we prove that $\omega^-(x,t)$ is a sub-solution. We have

$$\tilde{\mathcal{L}}[\omega^-]_i = -\eta'(t)U_i'(z-\eta(t)) - \partial_t q_i^-(x,t) + \partial_{xx}q_i^-(x,t) + f_i(\mathbf{U}(z-\eta(t))) - \tilde{f}_i(\omega^-(x,t))$$

and

$$-\partial_t q_i^-(x,t) + \partial_{xx}q_i^-(x,t) = \begin{cases} \epsilon q_i^-(x,t) + [(\alpha_i\lambda)^2 - c(\alpha_i\lambda)]q_i^-(x,t), & \text{if } z < z_0 + \eta(0), \\ \epsilon q_i^-(x,t), & \text{if } z \geq z_0 + \eta(0). \end{cases}$$

Similar to the proof for $\omega^+(x,t)$, we consider three cases.

Case 1. $U_i(z-\eta(t)) \in [\delta, 1-\delta], i = 1,2$.

Then, we have

$$\begin{aligned} \tilde{\mathcal{L}}[\omega^-]_i &\leq -\eta'(t)U_i'(z-\eta(t)) + \epsilon q_i^-(x,t) + f_i(\mathbf{U}(z-\eta(t))) - \tilde{f}_i(\omega^-(x,t)) \\ &\leq -\eta'(t)C_\delta + \epsilon q_{0,i}e^{-\epsilon t} + M_0(q_{0,1}+q_{0,2})e^{-\epsilon t} + rq_{0,1}e^{-\epsilon t} \\ &\leq 0. \end{aligned}$$

Case 2. $U_i(z-\eta(t)) > 1-\delta, i = 1,2$.

Then, there exists a z_0 such that $z \geq z_0 + \eta(0)$. It follows that $q_i^-(x,t) = q_{0,i}e^{-\epsilon t}$ and

$$\tilde{\mathcal{L}}[\omega^-]_i \leq \epsilon q_{0,i}e^{-\epsilon t} + q_0 e^{-\epsilon t}\sum_{j=1}^{2} f_{ij}\left(\mathbf{U}(z-\eta(t)) - \tau_i\mathbf{q}^-(x,t)\right)q_j \leq 0$$

for $\epsilon \leq \frac{1}{2}\min\left\{\frac{q_1 - rq_2}{2q_1}, \frac{b}{2}\right\}$.

Case 3. $U_i(z-\eta(t)) < \delta, i = 1,2$.

We have $z \leq z_0 + \eta(0)$, $q_i^-(x,t) = q_{0,i}e^{-\epsilon t}e^{\alpha_i\lambda_2(z-z_0-\eta(0))}$ and

$$\tilde{\mathcal{L}}[\omega^-]_i \quad \leq \quad \epsilon q_i^-(x,t) + [(\alpha_i\lambda)^2 - c(\alpha_i\lambda)]q_i^-(x,t)$$

$$+\sum_{j=1}^{2} f_{ij}\left(\mathbf{U}(z-\eta(t))-\tau_i\mathbf{q}^-(x,t)\right)q_j^-(x,t)-g_i(\boldsymbol{\omega}^-).$$

Then, a similar discussion as in Lemma 3.1 for $\mathbf{v}^-$ yields that

$$\tilde{\mathcal{L}}[\boldsymbol{\omega}^-]_i \quad \le \quad q_i^-(x,t)\left(\epsilon+\Pi_i(\alpha_i\lambda_2)-\frac{1}{2}\Pi_i(\alpha_i\lambda_2)\right)\le 0$$

for $\epsilon \le \min_{i=1,2}\left\{-\frac{1}{2}\Pi_i(\alpha_i\lambda_2)\right\}$. To complete the proof, we need only to prove that $\mathbf{u}_0(x)\ge\boldsymbol{\omega}^-(x,0)$. In fact, by (1.4), there exists two positive numbers $\tilde{M}$ and $\tilde{M}'$ large enough such that

$$\omega_i^-(x,0)\le 1-q_{0,i}\le u_{0,i}(x),\quad \forall x>\tilde{M}',$$

and

$$U_i(x)-q_{0,i}e^{\alpha_i\lambda_2 x}\le u_{0,i}(x),\quad \forall x<-\tilde{M}.$$

For $-\tilde{M}\le x\le\tilde{M}'$, we can choose $\eta(0)$ large enough such that $\omega_i^-(x,0)\le u_{0,i}(x)$. Thus, $\boldsymbol{\omega}^-(x,t)$ is a sub-solution. The proof is completed. □

Lemma 4.2. *Assume the assumptions of Lemma 4.1 hold. Then, for any $\delta\in(0,1)$, there exist $T=T(\delta)\ge 0$, ξ, and $h\in\mathbb{R}$ such that*

$$U_i(x+ct+\xi)-\delta U_2^{\alpha_i}(x+ct+\xi)\le\tilde{v}_i(x,t,\mathbf{u}_0)\le U_i(x+ct+\xi+h)+\delta U_2^{\alpha_i}(x+ct+\xi+h)\qquad(4.2)$$

for $x\in\mathbb{R}$ and $t\ge T$, $i=1,2$.

Proof. Recall the asymptotic behaviors of $\mathbf{U}$. We know $U_2<1$ and $U_2^{\alpha_i}(x)\sim A^{\alpha_i}e^{\alpha_i\lambda_2 x}$ as $x\to-\infty$. In view of Lemma 4.1, there exists a constant $M>0$ such that

$$U_i(x+ct-z_1)-Me^{-\epsilon t}U_2^{\alpha_i}(x+ct-z_1)\le\tilde{v}_i(x,t,\mathbf{u}_0)\le U_i(x+ct-z_2)+Me^{-\epsilon t}U_2^{\alpha_i}(x+ct-z_2)$$

for $x\in\mathbb{R}, t>0$. For any $\delta\in(0,1)$, we can choose an appropriate $T\ge 0$ such that $Me^{-\epsilon T}\le\delta$. Moreover, let $\xi=-z_1$ and $h=z_1-z_2$. Then, (4.2) follows. □

Now, we prove the main result.

Proof of Theorem 1.1. Let β,ρ, and δ be as in Lemma 3.1. Let ϵ^* be defined in Lemma 3.3. Further, we choose a $\delta^*=\frac{1}{4}\epsilon^*<1$, then it follows that $0<k^*:=\rho(\epsilon^*-2\delta^*)<\frac{1}{4}$. Fix a $t^*\ge 1$ such that

$$e^{-\beta(t^*-1)}(4+e^{-\beta})<\frac{1}{2}<1-k^*.$$

First, we prove two claims.

Claim I. There exist $T^*>0$ and ξ^* such that

$$U_i(x+cT^*+\xi^*)-\delta^*U_2^{\alpha_i}(x+cT^*+\xi^*)\le u_i(x,t)\le U_i(x+cT^*+\xi^*+1)+\delta^*U_2^{\alpha_i}(x+cT^*+\xi^*+1)\quad(4.3)$$

for $x\in\mathbb{R}$.

Indeed, by Lemma 4.2, there exist $T = T(\delta^*) \geq 0$, ξ, and h such that

$$U_i(x + cT + \xi) - \delta^* U_2^{\alpha_i}(x + cT + \xi) \leq u_i(x, t, \mathbf{u}_0) \leq U_i(x + cT + \xi + h) + \delta^* U_2^{\alpha_i}(x + cT + \xi + h) \quad (4.4)$$

for $x \in \mathbb{R}$. If $h \leq 1$, then by the monotonicity of $\mathbf{U}(\cdot)$, Claim I holds. If $h > 1$, we can choose an integer $N \geq 1$ such that $0 < h - Nk^* \leq k^* < 1$. Then, (4.4) and Lemma 3.3 together with the choice of k^* and t^* imply that

$$\begin{aligned} &\mathbf{U}\left(x + c(t^* + T) + \hat{\xi}(t^* + T)\right) - \hat{\delta}(t^* + T)\mathcal{U}^\alpha\left(x + c(t^* + T) + \hat{\xi}(t^* + T)\right) \\ &\leq \mathbf{u}(x, t^* + T) \leq \mathbf{U}\left(x + c(t^* + T) + \hat{\xi}(t^* + T) + \hat{h}(t^* + T)\right) \\ &\qquad + \hat{\delta}(t^* + T)\mathcal{U}^\alpha\left(x + c(t^* + T) + \hat{\xi}(t^* + T) + \hat{h}(t^* + T)\right), \end{aligned} \quad (4.5)$$

where

$$\begin{aligned} \hat{\delta}(t^* + T) &= (\epsilon^* + \delta^* e^{-\beta})e^{-\beta(t^*-1)} \leq \delta^*(1 - k^*), \\ \xi - \rho\delta^* + \epsilon^* &\leq \hat{\xi}(t^* + T) \leq \xi + 2\epsilon^*, \end{aligned}$$

and

$$0 < \hat{h}(t^* + T) \leq h - \rho(\epsilon^* - 2\delta^*) = h - k^*.$$

Applying Lemma 3.3 again, we conclude that (4.5), with $t^* + T$ replaced by $Nt^* + T$, holds for some $\hat{\xi} \in \mathbb{R}, 0 < \hat{\delta} \leq \delta^*(1 - k^*)^N$ and $0 \leq \hat{h} \leq h - Nk^* \leq 1$. Let $T^* = Nt^* + T, \xi^* = \hat{\xi}$. By the monotonicity of $\mathbf{U}(\cdot)$, (4.3) holds.

Claim II. Define $p = \rho(2\epsilon^* + \delta^*), T_m = T^* + mt^*, \delta_m^* = (1 - k^*)^m\delta^*$, and $h_m = (1 - k^*)^m, m \geq 0$. Then, there exists a sequence $\{\hat{\xi}_m\}_{m=0}^\infty$ with $\hat{\xi}_0 = \xi^*$ such that

$$|\hat{\xi}_{m+1} - \hat{\xi}_m| \leq ph_m, \; m \geq 0,$$

and

$$\begin{aligned} &\mathbf{U}\left(x + cT_m + \hat{\xi}_m\right) - \delta_m^*\mathcal{U}^\alpha\left(x + cT_m + \hat{\xi}_m\right) \\ &\leq \mathbf{u}(x, T_m, \mathbf{u}_0) \leq \mathbf{U}\left(x + cT_m + \hat{\xi}_m + h_m\right) + \delta_m^*\mathcal{U}^\alpha\left(x + cT_m + \hat{\xi}_m + h_m\right). \end{aligned} \quad (4.6)$$

We prove Claim II by mathematical induction. Clearly, Claim I implies that (4.6) holds for $m = 0$. Suppose that (4.6) holds for $m = l \geq 0$. Now, we are going to prove that (4.6) holds for $m = l + 1$. Let $T = T_l, \xi = \hat{\xi}_l, h = h_l, \delta = \delta_l^*$ and $t = T_l + t^* = T_{l+1} \geq T_l + 1$, then Lemma 3.3 yields that

$$\begin{aligned} &\mathbf{U}\left(x + cT_l + \hat{\xi}\right) - \hat{\delta}\mathcal{U}^\alpha\left(x + cT_l + \hat{\xi}\right) \\ &\leq \mathbf{u}(x, T_{l+1}, \mathbf{u}_0) \leq \mathbf{U}\left(x + cT_l + \hat{\xi} + \hat{h}\right) + \hat{\delta}\mathcal{U}^\alpha\left(x + cT_l + \hat{\xi} + \hat{h}(t)\right), \end{aligned}$$

where

$$\begin{aligned} \hat{\delta} &= (\epsilon^* h_l + \delta_l^* e^{-\beta})e^{-\beta(T_{l+1} - T_l - 1)} = (\epsilon^* + \delta^* e^{-\beta})e^{-\beta(t^*-1)}(1 - k^*)^l \\ &= \delta^*(4 + e^{-\beta})e^{-\beta(t^*-1)}(1 - k^*)^l \leq \delta^*(1 - k^*)^{l+1} = \delta_{l+1}^*, \\ \hat{\xi}_l - \rho\delta_l^* &+ \rho\epsilon^* h_l \leq \hat{\xi} \leq \hat{\xi}_l + 2\rho\epsilon^* h_l, \end{aligned}$$

and

$$\begin{aligned}\hat{h} &= h_l + 2\rho\delta_l^* - \rho\epsilon h_l - \rho(\epsilon^* h_l + 2\delta_l e^{-\beta})e^{-\beta(t^*-1)} \leq h_l - \rho(\epsilon^* h_l - 2\delta_l^*) \\ &= h_l(1-\rho(\epsilon^* - 2\delta^*)) = (1-k^*)^{l+1} = h_{l+1}.\end{aligned}$$

Choose $\hat{\xi}_{l+1} = \hat{\xi}$. Then, we have

$$|\hat{\xi}_{l+1} - \hat{\xi}_l| \leq \rho(2\epsilon^* h_l + \delta_l^*) \leq \rho(2\epsilon^* + \delta^*)h_l = ph_l.$$

Thus, (4.6) holds for all $m \geq 0$.

Now, we are ready to prove the main result. For $t \geq T^*$, let $m = \left[\frac{t-T^*}{t^*}\right]$ be the largest integer not greater than $\frac{t-T^*}{t^*}$, and define

$$\delta(t) = \delta_m^*, \quad \xi(t) = \hat{\xi}_m - \rho\delta_m^*\left(1 - e^{-\beta(t-T_m)}\right),$$

and

$$h(t) = h_m + 2\rho\delta_m^*\left(1 - e^{-\beta(t-T_m)}\right).$$

Then, $T_m = T^* + mt^* < t < T^* + (m+1)t^* = T_{m+1}$. In view of (4.6), one has

$$\begin{aligned}&\mathbf{U}\left(x+ct+\xi(t)\right) - \delta(t)\mathcal{U}^{\alpha}\left(x+ct+\xi(t)\right) \\ &\leq \mathbf{u}\left(x,t,\mathbf{u}_0\right) \\ &\leq \mathbf{U}\left(x+ct+\xi(t)+h_(t)\right) + \delta(t)\mathcal{U}^{\alpha}\left(x+ct+\xi(t)+h(t)\right)\end{aligned} \tag{4.7}$$

for all $t \geq T^*$ and $x \in \mathbb{R}$. Set $k := -\frac{1}{t^*}\ln(1-k^*)$ and $q =: e^{-\frac{t^*+T^*}{t^*}\ln(1-k^*)}$. Since $0 \leq m \leq \frac{t-T^*}{t^*} < m+1$, we have

$$(1-k^*)^m \leq (1-k^*)^{\frac{t-T^*}{t^*}-1} = qe^{-kt}.$$

Thus,

$$\delta(t) = \delta_m^* = (1-k^*)^m\delta^* \leq \delta^* qe^{-kt}, \tag{4.8}$$

and

$$h(t) \leq h_m + 2\rho\delta_m^* \leq (1+2\rho\delta^*)(1-k^*)^m \leq (1+2\rho\delta^*)qe^{-kt}. \tag{4.9}$$

It follows that for all $t' \geq t \geq T^*$, it holds

$$\begin{aligned}|\xi(t') - \xi(t)| &\leq \left|\hat{\xi}_n - \rho\delta_n^*\left(1-e^{-\beta(t-T_n)}\right) - \hat{\xi}_m + \rho\delta_m^*\left(1-e^{-\beta(t-T_m)}\right)\right| \\ &\leq |\hat{\xi}_n - \hat{\xi}_m| + \rho|\delta_n^* - \delta_m^*| + \rho\delta_m^*\left|e^{-\beta(t-T_n)} - e^{-\beta(t-T_m)}\right| \\ &\leq |\hat{\xi}_n - \hat{\xi}_m| + 4\rho\delta_m^* \leq \sum_{l=m}^{n-1} ph_l + 4\rho\delta_m^* \\ &= \left[\frac{p}{\delta_m^*}\sum_{l=0}^{n-m-1}(1-k^*)^l + 4\rho\right]\delta_m^* \\ &\leq \left(\frac{p}{k^*\delta^*} + 4\rho\right)\delta(t) \leq \left(\frac{p}{k^*} + 4\delta^*\rho\right)qe^{-kt},\end{aligned} \tag{4.10}$$

where $n = \left[\frac{t'-T^*}{t^*}\right]$. Obviously, $n \geq m$. It follows from (4.10) that $\xi_0 := \lim_{t\to\infty} \xi(t)$ is well-defined. Letting $t' \to \infty$, we obtain

$$|\xi_0 - \xi(t)| \leq \left(\frac{p}{k^*} + 4\delta^*\rho\right) q e^{-kt}, \quad t \geq T^*. \tag{4.11}$$

Since $\xi(t)$ and $h(t)$ are bounded, the asymptotic properties of $\mathbf{U}$ guarantee that

$$Q_1 := \sup_{\xi\in\mathbb{R}, t\geq 0, \tau_1,\tau_2\in[0,1]} \left\{ \frac{U_1'(\xi + \tau_1(\xi(t) - \xi_0) + \tau_2 h(t))}{U_2^{\alpha_1}(\xi + \xi_0)}, \frac{U_2'(\xi + \tau_1(\xi(t) - \xi_0) + \tau_2 h(t))}{U_2^{\alpha_2}(\xi + \xi_0)} \right\},$$

and

$$Q_2 := \sup_{\xi\in\mathbb{R}, t\geq 0} \left\{ \frac{U_1^{\alpha_1}(\xi + \xi(t) + h(t))}{U_2^{\alpha_1}(\xi + \xi_0)}, \frac{U_2^{\alpha_2}(\xi + \xi(t) + h(t))}{U_2^{\alpha_2}(\xi + \xi_0)} \right\}.$$

are well-defined and finite. Let $C_0 = \max\{Q_1, Q_2\}$. Then, (4.7) gives that

$$\left| \frac{u_i(x,t) - U_i(x + ct + \xi_0)}{U_2^{\alpha_i}(\xi + \xi_0)} \right| \leq C_0 \left(|\xi_0 - \xi(t)| + h(t) + \delta(t) \right). \tag{4.12}$$

Thus, (1.5) follows from (4.8)–(4.12). The proof is complete.

5. Conclusions

In this paper, we investigated the asymptotic stability of traveling fronts of a diffusion system with the BZ reaction. By the squeezing technique and the comparison principle, we proved that if the initial perturbation decays to zero at the space infinity, then the perturbed solution converges to the traveling front of the system (1.1) with an exponential rate as $t \to +\infty$. Due to the degeneracy of (1.1) at the equilibrium (0,0), the initial perturbation as $x \to -\infty$ was actually relaxed. It was only asked to decay to zero under an exponential weight. In fact, the degeneracy is caused by the second equation of (1.1). That is why we can derive the stability result in the form of (1.5), in which U_2 was taken to be the denominator at the left side of the inequality. In this point of view, we are able to present a little bit of contributions to the stability analysis of the BZ system.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest.

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